Mountain Pass Theorems without Palais-Smale conditions

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Outline

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3. Mountain Pass Theorems without Palais-Smale conditions

Definition: Let E be a real Banach space. We say that $f \in C^1(E, \mathbb{R})$ satisfies the **Palais-Smale condition** ((PS) for short) if for all sequence $(u_n)_{n\in\mathbb{N}}$ in E,

$$(f(u_n))_{n\in\mathbb{N}} \text{ is bounded and } \lim_{n\to\infty} f'(u_n) = 0$$

$$\Downarrow$$

$$(u_n)_{n\in\mathbb{N}} \text{ has a convergent subsequence.}$$

Mountain Pass Theorem of Ambrosetti-Rabinowitz (1973): Let E be a real Banach space and $f \in C^1(E, \mathbb{R})$. Suppose that

1. there exist $x_1, x_2 \in E$ and $r \in \mathbb{R}^+$ such that $\| x_1 - x_2 \| > r$ and
$$k_0 := \max\{f(x_1), f(x_2)\} < \inf_{\|y - x_1\| = r} f(y);$$
2. $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$;
3. f satisfies **(PS)**.

Then $k_1 > k_0$ and k_1 is a critical value of f.

Lemma 1: Suppose H is a real Hilbert space with norm $\| \cdot \|$ and let U be an open subset of H. Let $f \in C^1(U, \mathbb{R})$ and $x \in ns({}^\star U)$. If $f'(x) \not\approx 0$, then for every $0 < \varepsilon \approx 0$, the following inequality holds:

$$f(x - \varepsilon f'(x)) < f(x) - \varepsilon \frac{\| f'(x) \|^2}{2}.$$

Lemma 2: Let H be a real Hilbert space with norm $\|\cdot\|$. Suppose that $f \in C^1(H,\mathbb{R})$ satisfies the mountain pass geometry with respect to x_1 and x_2. Let

$$\Gamma := \{\gamma \in C([0,1],H) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$$

and

$$k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)).$$

Then

$$\forall \gamma \in {}^\star\Gamma \left[\left[\gamma({}^\star[0,1]) \subseteq ns({}^\star H) \ \wedge \ \max_{t\in{}^\star[0,1]} f(\gamma(t)) \approx k_1 \right] \right.$$

$$\left. \Rightarrow \ \exists t_0 \in {}^\star[0,1] \left[f(\gamma(t_0)) \approx k_1 \wedge \ \| f'(\gamma(t_0)) \| \approx 0 \right] \right].$$

Proof: Take $\gamma \in {}^\star\Gamma$ such that

$$\gamma({}^\star[0,1]) \subseteq ns({}^\star H) \quad \wedge \quad k_2 := \max_{t \in {}^\star[0,1]} f(\gamma(t)) \approx k_1$$

and let $k_0 := \max\{f(x_1), f(x_2)\}$.

Then

$$k_0 \quad < \quad k_1 \quad \leq \quad k_2 \quad \approx \quad k_1.$$

Define

$$U := \{t \in {}^\star[0,1] \ : \ k_1 \leq f(\gamma(t)) \leq k_2\}$$

and

$$d \ := \ \min\{\|f'(\gamma(t))\| \ : \ t \in U\}.$$

Suppose that $d \not\approx 0$.

Define

$$V := \left\{ \, t \in {}^{\star}[0,1] \;\; : \;\; \|f'(\gamma(t))\| > \frac{d}{2} \, \right\}$$

and

$$W := (\, {}^{\star}[0,1] \setminus V) \cup \{0,1\}.$$

Note that $U \subseteq V$, V is *open and W and U are *closed.

Moreover,

$$U \cap W = \emptyset.$$

Take $u \in {}^\star C([0,1],[0,1])$ such that

$$u(W) = \{0\} \quad \text{and} \quad u(U) = \{1\}.$$

Choose b such that

$$0 \;\leq \frac{2(k_2 - k_1)}{d^2} \; < b \; \approx 0$$

and define $\eta : {}^\star[0,1] \longrightarrow [0,b]$ by

$$\eta(t) := bu(t).$$

Define

$$\gamma_\eta(t) := \gamma(t) - \eta(t) f'(\gamma(t)).$$

Note that $\gamma_\eta \in {}^\star\Gamma$.

We will prove

$$\forall t \in^{\star} [0,1] \quad f(\gamma_\eta(t)) \; < \; k_1. \qquad \textit{Contradiction!}$$

If $t \in W$, **then**

$$f(\gamma_\eta(t)) = f\Big(\gamma(t) - \eta(t) f'(\gamma(t))\Big) = f(\gamma(t)) < k_1,$$

because $\eta(t) = 0$ **and** $t \notin U$.

If $t \in U$, then

$$\begin{aligned} f(\gamma_\eta(t)) &= f(\gamma(t) - bf'(\gamma(t))) \\ &< f(\gamma(t)) - b\frac{\| f'(\gamma(t)) \|^2}{2} \qquad \text{(by Lemma 1)} \\ &\leq f(\gamma(t)) - b\frac{d^2}{2} \\ &< f(\gamma(t)) - (k_2 - k_1) \\ &\leq k_1. \end{aligned}$$

If $t \in V \setminus U$, Lemma 1 and the definition of U imply

$$f(\gamma_\eta(t)) \leq f(\gamma(t)) - \eta(t)\frac{\| f'(\gamma(t)) \|^2}{2} < k_1.$$

Therefore, $d \approx 0$.

Hence, there exists $t_0 \in U$ such that $\| f'(\gamma(t_0)) \| \approx 0$, that is,

$$\exists t_0 \in {}^\star[0,1] \left(f(\gamma(t_0)) \approx k_1 \wedge \| f'(\gamma(t_0)) \| \approx 0 \right). \qquad \square$$

Mountain Pass Theorem without Palais-Smale conditions in finite dimension: Let E be a finite dimensional real Banach space, $x_1, x_2 \in E$ and $f \in C^1(E, \mathbb{R})$. Suppose that

1. f satisfies the mountain pass geometry with respect to x_1 and x_2;
2. $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t))$;
3. there exists $s \in \mathbb{R}^+$ such that $\| x_2 - x_1 \| < s$ and if $\| x - x_1 \| \geq s$ then $f(x) < k_1$.

Then k_1 is a critical value of f.

Proof: Let $\gamma_0 \in {}^\star\Gamma$ be such that $k_1 \leq \max_{t\in {}^\star[0,1]} f(\gamma_0(t)) \approx k_1$.

We may assume that $\gamma_0({}^\star[0,1]) \subseteq \overline{\mathrm{B}}_s(x_1)$ and, since E is finite dimensional, $\gamma_0({}^\star[0,1]) \subseteq ns({}^\star E)$.

Therefore, by Lemma 2, there exists $t_0 \in {}^\star[0,1]$ such that

$$f(\gamma_0(t_0)) \approx k_1 \quad \wedge \quad \| f'(\gamma_0(t_0)) \| \approx 0.$$

The continuity of f and f' shows that $st(\gamma_0(t_0))$ is a critical point with value k_1. $\square$

Example: Let $h(x,y) = [1-(x^2+y^2)]\exp^{-(x^2+y^2)}\arctan(x^2+y^2)$ for all $(x,y)\in\mathbb{R}^2$.

Clearly h is a C^1 functional, $h(0,0)=0$, $h(1,0)=0$,

$$\inf_{\|(x,y)\|=\frac{1}{2}} h(x,y) > k_0 := \max\{h(0,0),h(1,0)\} = 0$$

and

$$(x,y)\notin \mathrm{B}_2(0,0) \Rightarrow h(x,y) < 0 < k_1.$$

However, h does not satisfy (PS) condition:

$$(h(n,n))_{n\in\mathbb{N}} \text{ is bounded} \quad \wedge \quad \frac{\partial h}{\partial x}(n,n)\to 0 \quad \wedge \quad \frac{\partial h}{\partial y}(n,n)\to 0$$

$((n,n))_{n\in\mathbb{N}}$ does not contain a convergent subsequence.

Mountain Pass Theorem without Palais-Smale conditions in Hilbert spaces: Let H be a real Hilbert space, $x_1, x_2 \in H$ and $f \in C^1(H, \mathbb{R})$. Suppose that

1. f satisfies the mountain pass geometry with respect to x_1 and x_2;
2. $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = x_1 \wedge \gamma(1) = x_2\}$ and $k_1 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$;
3. $\exists \gamma \in {}^\star\Gamma \; [\; \gamma({}^\star[0,1]) \subseteq ns({}^\star H) \;\wedge\; \max_{t\in{}^\star[0,1]} f(\gamma(t)) \approx k_1 \;]$.

Then k_1 is a critical value of f.

Example: Let $h(x,y) = [1-(x^2+y^2)]\exp^{-(x^2+y^2)}\arctan(x^2+y^2)$ for all $(x,y)\in\mathbb{R}^2$.

We saw that this function satisfies the mountain pass geometry with respect to $(0,0)$ and $(1,0)$ and

$$(x,y)\notin \mathrm{B}_2(0,0) \Rightarrow h(x,y) < 0.$$

Since $k_1 > 0$, there exists $\gamma \in {}^\star\Gamma$ such that

$$\max_{t\in^\star[0,1]} h(\gamma(t)) \approx k_1 \quad \wedge \quad \gamma(^\star[0,1]) \subseteq \mathrm{B}_2(0,0) \subseteq ns(^\star\mathbb{R}^2).$$

Hence, h satisfies all the conditions of our theorem but do not satisfy (PS) condition.

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