A GENERAL FATOU LEMMA

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SETUP

Let Ω be a non-empty internal set,
$\mathcal{A}_0$ an internal algebra on Ω, and
$\mathcal{A}$ the σ-algebra generated by $\mathcal{A}_0$.

Let J be a finite or countably infinite set.
$\forall j \in J$, let $(\Omega, \mathcal{A}_0, \mu_{0j})$ and $(\Omega, \mathcal{A}, \mu_j)$ be internal and Loeb probability spaces.

From these generate $\bar{\mu}$ so that $\forall j$, $\mu_j << \bar{\mu}$.
We may assume $\mathcal{A}$ is $\bar{\mu}$-complete.

Let Y be a separable Banach lattice, and
X is its dual Banach space with
the natural dual order (denoted by $\leq$) and
lattice norm (i.e., $|x| \leq |z| \Rightarrow \|x\| \leq \|z\|$).

Let P be any probability measure on $(\Omega, \mathcal{A})$.

Definition. A sequence $\{g_n\}_{n=1}^{\infty}$ of functions from $(\Omega, \mathcal{A}, P)$ to X is said to be weak* P-tight, if for any $\varepsilon > 0$, there exists a weak* compact set K in X such that for all $n \in \mathbb{N}$, $P(g_n^{-1}(K)) > 1 - \varepsilon$.

Definition. For each $x \in X$, $y \in Y$, the value of the linear functional x at y will be denoted by $\langle x, y \rangle$. A function f from $(\Omega, \mathcal{A}, P)$ to X is said to be Gelfand P-integrable if for each $y \in Y$, the real-valued function $\langle f(\cdot), y \rangle$ is integrable on $(\Omega, \mathcal{A}, P)$.

Proposition. If $f : (\Omega, \mathcal{A}, P) \longmapsto X$ is Gelfand P-integrable, then there is a unique $x \in X$ such that $\langle x, y \rangle = \int_\Omega \langle f(\omega), y \rangle \; P(d\omega)$ for all $y \in Y$.
(That element x, called the Gelfand integral, will be denoted by $\int_\Omega f \; dP$.)

Proof. (Well-known): Let $T(y)$ be the element of $L^1(P)$ given by $\omega \longmapsto \langle f(\omega), y \rangle$. By Closed Graph Theorem, $\|T\| < \infty$, so

$$\left| \int_\Omega \langle f(\omega), y \rangle \; P(d\omega) \right| \leq \int_\Omega |\langle f(\omega), y \rangle| \; P(d\omega)$$
$$\leq \|T\| \, \|y\| \, . \; \square$$

Simplifying Assumption: $\exists$ an increasing (perhaps constant) sequence $y_m \geq 0$ in Y with $\lim_{m\to\infty} \langle x, y_m \rangle = \|x\| \; \forall x \geq 0$ in X.

The assumption is valid when $X = \ell^1$ or $X = \mathcal{M}(S)$, the space of finite, signed Borel measures on a second-countable, locally compact Hausdorff space S.

The main result, stated here for a sequence of functions $g_n \geq 0$, is generalized with the assumption that each $n \in \mathbb{N}$, $g_n \geq f_n$ where the sequence $\langle f_n \rangle$ has appropriate properties.

Theorem. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of nonnegative functions from Ω to X.

Suppose $\forall j \in J$, each function g_n is Gelfand integrable on $(\Omega, \mathcal{A}, \mu_j)$,
and the Gelfand integrals $\int_\Omega g_n d\mu_j$
have a weak* limit $a_j \in X$ as $n \to \infty$.

Then $\exists g : \Omega \mapsto X$ such that

1. for $\bar{\mu}$-a.e. $\omega \in \Omega$, $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^{\infty}$,

2. the function g is Gelfand μ_j-integrable with $\int_\Omega g d\mu_j \leq a_j$ for each $j \in J$;

3. the integral $\int_\Omega \langle g, y\rangle d\mu_j = \langle a_j, y\rangle$ for any $y \in Y$ and $j \in J$ for which $\{\langle g_n, y\rangle\}_{n=1}^{\infty}$ is uniformly μ_j-integrable;

4. In particular, $\int_\Omega g \; d\mu_j = a_j$ for any $j \in J$ for which the sequence $\{\|g_n\|\}_{n=1}^{\infty}$ is uniformly μ_j-integrable.

Corollary. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{A}$-measurable functions from Ω to a complete separable metric space Z.

Assume $\forall j \in J$, $\{\mu_j f_n^{-1}\}_{n=1}^{\infty}$ converges weakly to a Borel probability measure ν_j.

Then, there is an $\mathcal{A}$-measurable function f from Ω to Z such that $f(\omega)$ is a limit point of $\{f_n(\omega)\}_{n=1}^{\infty}$ for $\bar{\mu}$-a.e. $\omega \in \Omega$, and $\mu_j f^{-1} = \nu_j$ for each $j \in J$.

Corollary. A simplified version of our theorem holds for functions taking values in $\mathbb{R}^p$, where the norm of each $x = (x^1, \ldots, x^p)$ in $\mathbb{R}^p$ is given by $\Sigma_{i=1}^{p} \left| x^i \right|$.

For a more general theorem, the following consequence of the Simplifying Assumption about X must be added to the hypotheses.

Claim. $\forall j \in J$, $\{g_n\}_{n=1}^{\infty}$ is weak* μ_j-tight.

Proof. By an argument of H. Lotz using the Monotone Convergence Theorem,
$\forall j \in J, \forall n \in \mathbb{N}, \left\| \int_\Omega g_n(\omega) d\mu_j \right\|$
$$= \lim_{m\to\infty} \left\langle \int_\Omega g_n(\omega) d\mu_j, y_m \right\rangle$$
$$= \lim_{m\to\infty} \int_\Omega \langle g_n(\omega), y_m \rangle \, d\mu_j$$
$$= \int_\Omega \lim_{m\to\infty} \langle g_n(\omega), y_m \rangle \, d\mu_j = \int_\Omega \| g_n(\omega) \| \, d\mu_j.$$
The Gelfand integrals $\int_\Omega g_n d\mu_j$ converge in the weak*-topology, so by the Uniform Boundedness Principle $\exists M_j > 0$ such that $\forall n \in \mathbb{N}, \left\| \int_\Omega g_n \, d\mu_j \right\| \leq M_j$. Since

$$\forall n, k \in \mathbb{N}, \quad \int_{\{\|g_n(\omega)\| \geq k\}} \|g_n\| \, d\mu_j \leq M_j,$$

$$\mu_j \left(\{ \omega \in \Omega : \| g_n (\omega) \| \geq k \} \right) \leq M_j / k. \ \square$$

EXAMPLES

We have an example showing that even for a single measure μ, there may be no function g if μ is Lebesgue measure on $[0,1]$.

Here, we let $X = \ell^1$. An example of Liapounoff constructs an $h : [0,1] \to \ell^1$ such that for no $E \subset [0,1]$ is it true that for coordinate-wise integration,
$\int_E h(t)\, dt = \frac{1}{2} \int_{[0,1]} h(t)\, dt.$

We use the Liapounoff Theorem and $\forall n$ the first n components of h, to construct a sequence $g_n \geq 0$ satisfying the conditions of our theorem, but g can not exist by the Liapounoff example.

A modification of this first example shows that the corollary, even for $\mathbb{R}^2$, can fail when the measures μ_j are multiples of Lebesgue measure on $[0,1]$.

Lemma 1. Let X be a standard, separable metric space with metric ρ and the Borel σ-algebra $\mathcal{B}$. Fix $x_0 \in X$.

Let P_0 be an internal probability measure on $(\Omega, \mathcal{A}_0)$ with Loeb space $(\Omega, \mathcal{A}, P)$.

Let h be an internal, measurable map from $(\Omega, \mathcal{A}_0)$ to $({}^*X, {}^*\mathcal{B})$.

Let ν be the internal probability measure on $({}^*X, {}^*\mathcal{B})$ such that $\nu = P_0 h^{-1}$.

Fix a standard tight probability measure γ on $(X, \mathcal{B})$ such that ${}^*\gamma \simeq \nu$ in the nonstandard extension of the topology of weak convergence of Borel measures on X.

Then the standard part ${}^\circ h(\omega)$ exists for P-almost all $\omega \in \Omega$ (where $h(\omega)$ is not near-standard, set ${}^\circ h(\omega) = x_0$). This function ${}^\circ h$ is measurable, and $\gamma = P({}^\circ h)^{-1}$.

Proof. For every standard, bounded, continuous real-valued f on X,

$$\int_{^*X} {}^*f\, d\nu \simeq \int_{^*X} {}^*f\, d^*\gamma = \int_X f\, d\gamma.$$

Let $K_0 = \varnothing$, and $\forall n \in \mathbb{N}$, let $K_n \supseteq K_{n-1}$ be compact in X with $\gamma(K_n) > 1 - \frac{1}{2n}$. $\forall j \in \mathbb{N}$,

$$V_n^j := \{x \in X : \rho(x, K_n) < \frac{1}{j}\}$$

has the property that $\nu({}^*V_n^j) > 1 - \frac{1}{n}$, whence $\exists H \in {}^*\mathbb{N}_\infty$, with $\nu(V_n^H) > 1 - \frac{1}{n}$.

Now the monad $m\left(K_n\right) := \cap_{j\in\mathbb{N}} {}^*V_n^j$, and

$$h^{-1}\left[m(K_n)\right] = h^{-1}\left[\cap_{j\in\mathbb{N}} {}^*V_n^j\right] = \cap_{j\in\mathbb{N}} h^{-1}\left[{}^*V_n^j\right]$$

is measurable and $P\left(h^{-1}\left[m(K_n)\right]\right) \geq 1 - \frac{1}{n}$.

The standard part ${}^\circ h$ is defined on $h^{-1}[m(K_n)]$, is measurable there, and takes values in K_n.

Therefore, ${}^\circ h$ defines a measurable mapping from $\cup_n h^{-1}[m(K_n)]$ to $\cup_n K_n$, and

$$P\left(\cup_n h^{-1}[m(K_n)]\right) = 1.$$

Set ${}^\circ h = x_0$ on $\Omega \diagdown \cup_n h^{-1}[m(K_n)]$.

With this extension, ${}^\circ h$ is a measurable mapping defined on $(\Omega, \mathcal{A}, P)$.

Finally, given a bounded, continuous, real-valued function f on X,

$$\begin{aligned}
\int_X f \, dP({}^\circ h)^{-1} &= \int_\Omega f \circ {}^\circ h \, dP \\
&= \int_\Omega \mathrm{st}\left({}^* f \circ h\right) \, dP \\
&\simeq \int_\Omega {}^* f \circ h \, dP_0 = \int_{{}^* X} {}^* f \, d\nu \\
&\simeq \int_{{}^* X} {}^* f \, d{}^*\gamma = \int_X f \, d\gamma.
\end{aligned}$$

It follows that $\gamma = P\left({}^\circ h\right)^{-1}$ on X. $\Box$

Lemma 2. Let (X, ρ) be a separable metric space with the Borel σ-algebra $\mathcal{B}$.

Let P_0 be an internal probability measure on $(\Omega, \mathcal{A}_0)$ with Loeb space $(\Omega, \mathcal{A}, P)$.

Fix an internal sequence $\{h_n : n \in {}^*\mathbb{N}\}$ of measurable maps from $(\Omega, \mathcal{A}_0)$ to $({}^*X, {}^*\mathcal{B})$.

Fix a nonempty compact $K \subseteq X$.

Then $\exists H \in {}^*\mathbb{N}_\infty$ and a P-null set $S \subset \Omega$ such that

if $n \leq H$ in ${}^*\mathbb{N}_\infty$, while $\omega \notin S$,
and $h_n(\omega)$ has standard part in K,

then for any standard $\varepsilon > 0$, there are infinitely many limited $k \in \mathbb{N}$ for which ${}^*\rho(h_k(\omega), h_n(\omega)) < \varepsilon$.

Proof. Given $l \in \mathbb{N}$ cover K with n_l open balls of radius $1/l$. Let $B(l,j)$ denote the nonstandard extension of the j^{th} ball.

For each $i \in {}^*\mathbb{N}$, set

$$A_i(l,j) := \{\omega \in \Omega : h_i(\omega) \notin B(l,j)\}.$$

$\forall k \in \mathbb{N}$, choose $m_k(l,j) \in {}^*\mathbb{N}_\infty$ so that

$$P\left(\cap_{i=k}^{m_k(l,j)} A_i(l,j)\right) = P\left(\cap_{i=k, i\in\mathbb{N}}^{\infty} A_i(l,j)\right).$$

Set

$$S_k(l,j) := \left(\cap_{i=k, i\in\mathbb{N}}^{\infty} A_i(l,j)\right) \setminus \cap_{i=k}^{m_k(l,j)} A_i(l,j).$$

Fix $H \in {}^*\mathbb{N}_\infty$ with $H \le m_k(l,j)$ $\forall l \in \mathbb{N}$, $\forall j \le n_l$, and $\forall k \in \mathbb{N}$.

Let S be the P-null set formed by the union of the set $S_k(l,j)$ $\forall l \in \mathbb{N}$, $\forall j \le n_l$, $\forall k \in \mathbb{N}$.

Fix $n \in {}^*\mathbb{N}_\infty$ with $n \leq H$, and suppose $\mathrm{st}\,(h_n(\omega)) \in K$ but $\exists l \in \mathbb{N}$ for which there are at most finitely many limited $k \in \mathbb{N}$ for which ${}^*\rho(h_k(\omega), h_n(\omega)) < 2/l$.

Then for some $j \leq n_l$, $h_n(\omega) \in B(l, j)$, and by assumption there is a limited $k \in \mathbb{N}$ such that for all limited $i \geq k$, $h_i(\omega) \notin B(l, j)$. It follows that $\omega \in S_k(l, j) \subseteq S$. $\Box$

Idea of Parts of Theorem's Proof.
Replace sequence $\{g_n\}$ with a subsequence so $\forall j \in J$, $\mu_j {g_n}^{-1}$ converges weakly to γ_j.

Lift and extend $\{g_n\}$ to $\{h_n\}$ and work with measures $\mu_{0j} {h_n}^{-1}$. Use Lemma 1 to show $\exists H \in {}^*\mathbb{N}_\infty$ so $g(\omega) := ({}^\circ h_H)(\omega)$ exists for $\overline{\mu}$-a.e. $\omega \in \Omega$ and $\gamma_j = \mu_j ({}^\circ h_H)^{-1}$ $\forall j \in J$.

Use Lemma 2 to show that for $\bar{\mu}$-a.e. $\omega \in \Omega$, $g(\omega)$ is a weak* limit point of $\{g_n(\omega)\}_{n=1}^{\infty}$.