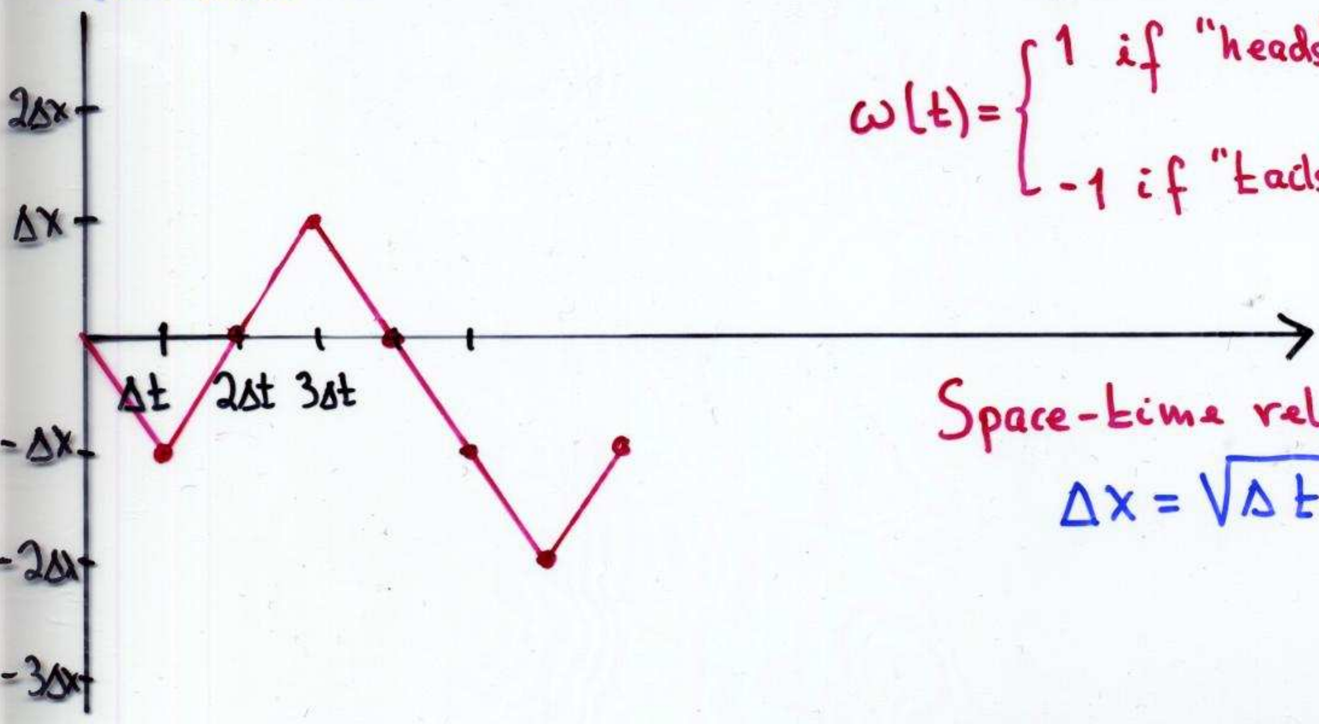
NONLINEAR STOCHASTIC INTEGRALS FOR HYPERFINITE LÉVY PROCESSES

(http://folk.uio.no/lindstro)

1. Stochastic integration

Anderson's Brownian motion: $\Delta t \approx 0$, $\Delta x \approx 0$



$$\omega(t) = \begin{cases} 1 & \text{if "heads" at time } t \\ -1 & \text{if "tails" at time } t \end{cases}$$

Space-time relationship:

$$\Delta x = \sqrt{\Delta t}$$

Formal definition: $B : \Omega \times T \to {}^*\mathbb{R}$ given by

$$B(\omega, t) = \sum_{s=0}^{t-\Delta t} \omega(s)\sqrt{\Delta t}$$

If $X(\omega,t)$ is another stochastic process, the <u>stochastic</u> <u>integral</u> $\int X\,dB$ is defined by:

$$\int_0^t X\,dB = \sum_{s<t} X(\omega,s)\,\Delta B(\omega,s)$$

In order to get a decent theory, one "has to" require that X is <u>nonanticipating</u>, i.e.

$$X(\omega,s) = X(\omega',s)$$

whenever $\omega(r)=\omega'(r)$ for all $r\le s$ (X can't look into the future)

<u>Extension to higher dimensions:</u>

$$B(\omega,t) = \begin{pmatrix} B_1(\omega,t) \\ B_2(\omega,t) \\ \vdots \\ B_d(\omega,t) \end{pmatrix}$$

independent versions of Anderson's process

$$\int_0^t X\,dB = \sum_{s=0}^{t} \underbrace{X(\omega,s)}_{\substack{\text{values in} \\ (m\times d)\text{-matrices} \\ \text{over } {}^*\mathbb{R}}}\ \underbrace{\Delta B(\omega,s)}_{\text{values in } {}^*\mathbb{R}^d}$$

Hence the increments of the integral process

$$I(\omega,t) = \int_0^t X\, dB = \sum_{s<t} X(\omega,s)\, \Delta B(\omega,s)$$

are produced by letting matrices $X(\omega,s)$ act on the increments $\Delta B(\omega,s)$ of the original process, i.e. by applying linear functions to the increments $\Delta B(\omega,s)$.

QUESTION: What happens if we let nonlinear functions act on the increments?

But first we need to extend the class of integrator processes.

2. Hyperfinite Lévy processes

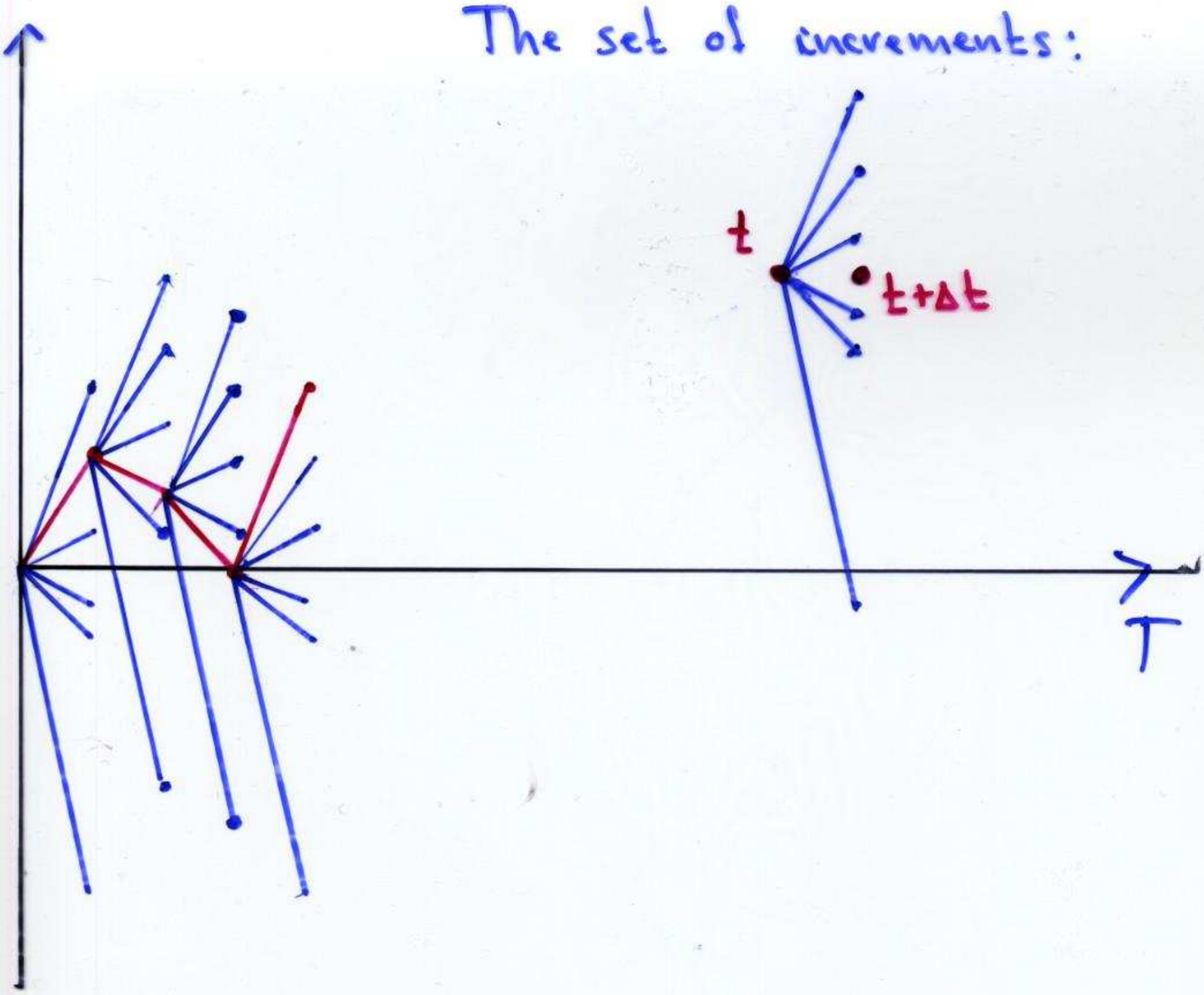
In Anderson's case, the process has only two optcons at each time t: it may go up $\sqrt{\Delta t}$ with probability $1/2$ or down $\sqrt{\Delta t}$ with probability $1/2$.

Let us extend the possibilities:

Fix a hyperfinite set $A \subseteq {}^*\mathbb{R}^d$ of <u>increments</u> and a corresponding set $\{p_a\}_{a\in A}$ of <u>transition probabilities</u> such that $\sum_{a\in A} p_a = 1$ (and $p_a \geq 0$ for all $a \in A$)

The idea is that at each instant $t \in T$, the process will choose its next increment to be a with probability p_a (independent of its past)

The set of increments:



Formally: An internal process $L : \Omega \times T \to {}^*\mathbb{R}^d$ is a <u>hyperfinite</u> <u>random</u> <u>walk</u> with increments A and transition probabilities $\{p_a\}_{a \in A}$ if:

(i) $L(0) = 0$

(ii) The increments $\Delta L(0), \Delta L(\Delta t), \Delta L(2\Delta t), \ldots$ are *-independent

(iii) $P[\Delta L(\omega, t) = a] = p_a$ for all $t \in T$ and all $a \in A$.

So far we have no size restrictions, and a hyperfinite random walk may jump around in the infinite far. We want the process to stay finite for all finite times in the following sense:

Definition: A hyperfinite random walk L is called a hyperfinite Lévy process if there is a set $\Omega' \subset \Omega$ of Loeb measure 1 such that $L(\omega, t)$ is finite for all $\omega \in \Omega'$ and all finite $t \in T$.

(There are better criteria available!)

3. Splitting infinitesimals

In many situations it would be convenient to separate the infinitesimal increments of L from the noninfinitesimal, but it is impossible to do this in an internal way.

It is, however, possible to find infinitesimals η which are so large that infinitesimals increments larger than η only contribute to the process in a negligable way. Such infinitesimals η are called splitting infinitesimals.

Formal definition: $\eta \approx 0$ is a splitting infinitesimal if

$$S\text{-}\lim_{b \downarrow 0} \left(\frac{1}{\Delta t} \sum_{\eta \le |a| < b} |a|^2 p_a \right) = 0$$

4. Nonlinear stochastic integrals

We want to define stochastic integrals

$$I(\omega,t) = \sum_{s\leq t} \varphi(\omega, \Delta L_s, s)$$

where φ acts nonlinearly on ΔL_s

We need some conditions on φ:

(i) $\varphi(\omega, 0, s) = 0$

(ii) φ is nonanticipating in the sense that $\varphi(\omega, x, s) = \varphi(\omega', x, s)$ if $\omega(r) = \omega'(r)$ for all $r \leq s$.

(iii) Integrability and differentiability conditions on φ

The strategy is to decompose $I(\omega,t)$ into simpler parts which we know how to handle.

Let us calculate! We choose a splitting infinitesimal η:

$$I(\omega,t) = \sum_{s<t} \varphi(\omega, \Delta L_s, s) =$$

$$\sum_{s<t} \varphi(\omega, \Delta L_s^{>\eta}, s) + \sum_{s<t} \varphi(\omega, \Delta L_s^{\leq\eta}, s)$$

$$= \sum_{s<t} \left\{ \varphi(\omega, \Delta L_s^{>\eta}, s) - \nabla\varphi(\omega,0,s)\cdot\Delta L_s^{>\eta} \right\}$$

$$+ \sum_{s<t} \varphi(\omega, \Delta L_s^{\leq\eta}, s) + \sum_{s<t} \nabla\varphi(\omega,0,s)\cdot\Delta L_s^{>\eta}$$

$$\approx \sum_{s<t} \left\{ \varphi(\omega, \Delta L_s^{>\eta}, s) - \nabla\varphi(\omega,0,s)\cdot\Delta L_s^{>\eta} \right\}$$

$$+ \sum_{s<t} \left\{ \overset{=0}{\varphi(\omega,0,s)} + \nabla\varphi(\omega,0,s)\cdot\Delta L_s^{\leq\eta} + \frac{1}{2}\sum_{i,j} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(\omega,0,s)\,\Delta L_i^{\leq\eta}(s)\,\Delta L_j^{\leq\eta}(s) \right\}$$

$$+ \sum_{s<t} \nabla\varphi(\omega,0,s)\,\Delta L_s^{>\eta}$$

$$= \sum_{s<t} \left\{ \varphi(\omega, \Delta L_s^{>\eta}, s) - \nabla\varphi(\omega,0,s)\,\Delta L_s^{>\eta} \right\}$$

$$+ \underbrace{\sum_{s<t} \nabla\varphi(\omega,0,s)\,\Delta L_s}_{\text{ordinary stochastic integral}} + \frac{1}{2}\sum_{s<t}\sum_{i,j} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(\omega,0,s)\,\Delta L_i^{\leq\eta}(s)\,\Delta L_j^{\leq\eta}(s)$$

We thus have:

$$I(\omega,t) = \sum_{s<t} \{\varphi(\omega, \Delta L_s^{>\eta}, s) - \nabla\varphi(\omega,0,s)\,\Delta L_s^{>\eta}\}$$

$$+ \sum_{s<t} \nabla\varphi(\omega,0,s)\,\Delta L_s + \frac{1}{2}\sum_{s<t}\sum_{ij} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(\omega,0,s)\,\Delta L_i^{\leq\eta}(s)\,\Delta L_j^{\leq\eta}(s)$$

and need to control the first and the last term on the right.

First term: Using Taylor's formula and the definition of splitting infinitesimal, we can prove that this term is finite and equal to

$$S\text{-}\lim_{b\downarrow 0}{}^{\circ}\Big(\sum_{s<t}\{\varphi(\omega,\Delta L_s^{>b}, s) - \nabla\varphi(\omega,0,s)\,\Delta L_s^{>b}\}\Big)$$

Hence it makes "standard sense".

Last term: Using martingale theory, we can prove that it is finite and infinitely close to

$$\frac{1}{2}\sum_{i,j} C_{ij} \sum_{s=0}^{t} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(\omega,0,s)\,\Delta t$$

where is the covariance matrix

$$C_{ij} = \frac{1}{\Delta t} E\left[\Delta L_i^{\leq\eta}(t)\,\Delta L_j^{\leq\eta}(t)\right]$$

Thus we end up with

$$I(\omega,t) = \sum_{s=0}^{t} \varphi(\omega, \Delta L_s, s) \approx$$

$$\approx \sum_{s<t} \left\{ \varphi(\omega, \Delta L_s^{>\eta}, s) - \nabla\varphi(\omega,0,s)\, \Delta L_s^{>\eta} \right\}$$

$$+ \int_0^t \nabla\varphi(\omega,0,s)\, dL_s + \frac{1}{2} \sum_{i,j} c_{ij} \int_0^t \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(\omega,0,s)\, ds$$

We may think of this a Sum Formula for infinitely many, mostly infinitesimal terms $\varphi(\omega, \Delta L_s, s)$.

This suggests the idea of a corresponding Product Formula for $\prod_{s<t} \psi(\omega, \Delta L_s, s)$ where $\psi(\omega, \Delta L_s, s)$ is (mostly) infinitely close to 1.

5. The Product Formula

Assume that $\psi(\omega, 0, s) = 1$. To find an expression for $\prod_{s \le t} \psi(\omega, \Delta L_s, s)$, we write

$$\prod_{s<t} \psi(\omega, \Delta L_s, s) = e^{\sum_{s<t} \ln(\psi(\omega, \Delta L_s, s))}$$

and apply our previous work to the function $\varphi(\omega, \Delta L_s, s) = \ln(\psi(\omega, \Delta L_s, s))$. We get

$$\prod_{s<t} \psi(\omega, \Delta L_s, s) \approx$$

$$\approx \prod_{s<t} \left\{ \psi(s, \Delta L_s^{\geq 1}, s)\, e^{-\nabla\psi(\omega, 0, s) \cdot \Delta L_s^{\geq 1}} \right\} \times$$

$$\times\, e^{\int_0^t \nabla\psi(\omega, 0, s)\, dL_s + \frac{1}{2} \sum_{i,j} C_{ij} \int_0^t \left[\frac{\partial^2 \psi}{\partial x_i \partial x_j} - \frac{\partial \psi}{\partial x_i}\frac{\partial \psi}{\partial x_j}\right](\omega, 0, s)\, ds}$$

6. Geometric Lévy processes

A geometric Lévy process is a solution of a stochastic differential equation

$$dX_t = X_t \cdot \Gamma(\omega, dL_t, t)$$

In our terms

$$\Delta X_t = X_t \cdot \Gamma(\omega, \Delta L_t, t)$$

or, equivalently,

$$X_{t+\Delta t} = X_t \left(1 + \Gamma(\omega, \Delta L_t, t)\right)$$

By induction

$$X_t = X_0 \prod_{s<t} \left(1 + \Gamma(\omega, \Delta L_s, s)\right)$$

Using the product formula with $\varphi(\omega, \Delta L_s, s) = 1 + \Gamma(\omega, \Delta L_s, s)$, we get

$$X_t \approx X_0 \prod_{s<t} \left(1 + \Gamma(\omega, \Delta L_s^{>1}, s)\right) e^{-\Gamma_x(\omega, 0, s)\Delta L_s^{>1}}$$

$$\times\, e^{\int_0^t \Gamma_x(\omega, 0, s)\, dL_s + \frac{c^2}{2}\int_0^t (\Gamma_{xx} - \Gamma_x^2)(\omega, 0, s)\, ds}$$

which extends previous standard results.

7. Transforming increments

Given a hyperfinite Lévy process with increments A and transition probabilities $\{p_a\}_{a\in A}$, we can create a new one by applying a function φ to the elements in A. The new process will have increments $\{\varphi(a)\}_{a\in A}$ and transition probabilities p_a. It is given by

$$ {}^{\varphi}L(\omega,t) = \sum_{s<t} \varphi(\Delta L_s) \approx $$

$$ \approx \nabla\varphi(0)\cdot L(t) + \frac{t}{2}\sum_{i,j} C^{\eta}_{ij} \frac{\partial^2\varphi}{\partial x_i \partial x_j}(0) $$

$$ + \sum_{s=0}^{t} \left\{ \varphi\left(\Delta L^{>\eta}_s(\omega,s)\right) - \nabla\varphi(0)\,\Delta L^{>\eta}(\omega,s) \right\} $$

8. Transforming probabilities

We may also change a hyperfinite Lévy process by keeping the increments and changing the transition probabilities. If the new probabilities q_a are given by

$$q_a = \psi(a)\, p_a$$

we need

$$\sum_{a \in A} \psi(a)\, p_a = 1$$

The density of the new measure Q with respect to the old measure P on the timeline $T_t = \{ s \in T : s < t \}$ is

$$D(\omega, t) = \prod_{s=0}^{t} \psi(\Delta L(\omega, s))$$

Applying the product formula one gets:

$$D(\omega,t) = \prod_{s<t} \varphi(\Delta L^{>\eta}(\omega,s))\, e^{-\nabla\varphi(0)\,\Delta L^{>\eta}(\omega,s)}$$

$$\times\, e^{\lambda t + \nabla\varphi(0)\cdot L(\omega,t) + \frac{t}{2}\sum_{i,j} c_{ij}\left[\frac{\partial^2\varphi}{\partial x_i \partial x_j}(0) - \frac{\partial\varphi}{\partial x_i}(0)\frac{\partial\varphi}{\partial x_j}(0)\right]}$$

Where $\lambda := \frac{\varphi(0)-1}{\Delta t}$ is finite.

There are similar (but more complex) applications to equivalent martingale measures.