SOME NONSTANDARD NOTES ON THE FIXED POINT PROPERTY IN THE PLANE

STEVEN C. LETH

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Note: This is a slightly simplified (there were several hand-drawn diagrams and accompanying explanation used in the talk that are not reproduced here) and somewhat corrected version of the slides for my talk given at NSM2006 in Pisa. The format has also been changed to improve readability.

The motivation for the results is the following open question:

Does every non-separating plane continuum have the fixed point property? i.e. is it true that if $E \subset \mathbb{R}^2$ is compact and connected with connected complement, and $f : E \to E$ is continuous then f has a fixed point? An outline of a simple (or simplistic) approach to the problem:

Definition 1. *We will write ∂A for the boundary of a set A, and $\overline{A}$ for the closure of A.*

We will write $B(a, r)$ for the open ball about a of radius r.

We will write $\mathcal{C}(a, A)$ for the connected component of A containing a.

Proposition 1. *With E as in the statement of the theorem, for all $\delta > 0$ there exists a set D homeomorphic to the disk such that $E \subset D$ and every point of D is within δ of a point in E (thus E is a countable intersection of sets homeomorphic to the disk).*

The proposition is well known and the standard proof is not difficult. However the nonstandard proof below is especially simple.

Proof. Let $\delta > 0$ be standard and let $\zeta > 0$ be infinitesimal. Let K be a (*) finite union of closed ζ- balls that cover E, and let

$$D = \{p \in \ {}^*\mathbb{R}^2 : p \notin \mathcal{C}(q_0,\ {}^*\mathbb{R}^2 - K)$$

Then there can be no δ-ball contained in D that does not intersect E, for if so it is easy to see by taking standard parts that E would disconnect the plane. The boundary of D (which is also the *outer boundary* of K) is a finite union of arcs of circles, and it is easy to see that D is homeomorphic to the disk. □

Now let $\epsilon > 0$ be infinitesimal. We will attempt to use the Brouwer Fixed Point Theorem to approach the open question in the most straightforward possible manner.

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As we did in the proof of proposition 1 we let K_1 be a finite union of closed ϵ-balls that cover ${}^*E$ and define a set H_1 by:

$$H_1 = \{p \in {}^*\mathbb{R}^2 : p \notin \mathcal{C}(q_0, {}^*\mathbb{R}^2 - K_1)$$

Thus, H_1 consists of all points "enclosed" by K_1.

Let $f : E \to E$ be continuous, and extend f continuously to all of $\mathbb{R}^2$.

We let the infinitesimal δ_1 be small enough that if we define:

$$H' = \{p \in {}^*\mathbb{R}^2 : p \notin \mathcal{C}(q_0, {}^*\mathbb{R}^2 - \cup_{q \in {}^*E} B(q, \delta_1)\}$$

then $f(H') \subset H_1$. This is possible by proposition 1 and the continuity of f.

We are interested in sets of the following type: let U be a bounded connected component of ${}^*\mathbb{R}^2 - {}^*E - B(a, \epsilon)$ for some $a \in {}^*\mathbb{R}^2$, and

$$Y = \partial(U) \cup \left\{ p : p \in \mathcal{C}(q, {}^*E \cap \overline{B(a, \epsilon)}) \text{ for some } q \in \partial(U) \cap \overline{B(a, \epsilon)} \right\}$$

(OR there may be a union of two balls $B(a_1, \epsilon) \cup B(a_2, \epsilon)$ everywhere in the above definitions).

Now about each point b of ${}^*E$ we will define an infinitesimal δ_b^1 as follows: if b is in some set Y as defined above, and $f(b) \notin Y$ then δ_b^1 is such that

$$0 < \delta_b^1 \leq \delta_1 \text{ and } f(B(b, \delta_b^1)) \cap (Y \cup U) = \varnothing.$$

(we note that such a Y is closed). For all other points of ${}^*E$ we let $\delta_b^1 = \delta_1$.

We now let K_2 be a finite union of closed δ_b^1-balls that cover ${}^*E$ and

$$H_2 = \{p \in {}^*\mathbb{R}^2 : p \notin \mathcal{C}(q_0, {}^*\mathbb{R}^2 - K_2).$$

We can now define a mapping g from H_1 to H_2 in such a way that g is the identity on H_n and $g(p) - p$ is infinitesimal for all p not in some set Y as above. The mapping $f \circ g$ now takes H_2 to H_2 and must have a fixed point by the Brouwer Fixed Point Theorem. Taking standard parts, we obtain a fixed point for f, unless both p and $f(p)$ are in some set Y as above.

It is for this reason that the following proposition seems to be helpful.

Proposition 2. *Let $E \subset \mathbb{R}^2$ be compact, connected and have a connected complement in the plane. Let f be a continuous function from E to E with no fixed point. Given $\varepsilon > 0$, let U be a bounded connected component of ${}^*\mathbb{R}^2 - {}^*E - B(a, \epsilon)$ for some $a \in {}^*\mathbb{R}^2$, and*

$$Y = \partial(U) \cup \left\{ p : p \in \mathcal{C}(q, {}^*E \cap \overline{B(a, \epsilon)}) \textit{ for some } q \in \partial(U) \cap \overline{B(a, \epsilon)} \right\}$$

Then for all $k \in \mathbb{N}$ there does not exist a set of points $a_1, a_2, \ldots a_k \in {}^*\mathbb{R}^2 - \mathbb{R}^2$ such that each a_i and $f(a_i)$ is in Y, and $f(a_1) \approx a_2, f(a_2) \approx a_3, \ldots f(a_k) \approx a_1$.

A very rough outline of the proof is as follows: We assume for the sake of contradiction that such an E, f, k, U, Y, a_i exist as in the statement of the theorem. We note there must exist a real number $d > 0$ such that for all $p \in E$, $\|f(p) - p\| > d$. There are two cases to consider.

Case i) ${}^*st(Y) \cap Y = \varnothing$

Case ii) ${}^*st(Y) \cap Y \neq \varnothing$

Proposition 3. *If p is in ${}^*st(Y) \cap Y$ then $st(p) \in Y$*

Thus, case ii) is equivalent to the case that there exists a standard point in Y. It can be shown that if there is more than one such standard point, then there are infinitely many (and, in fact, there must be an infinite connected component of Y that is standard in this case). We thus have:

Case iia) There exists exactly one standard point in Y.

Case iib) There exists an infinite connected component of Y that is standard.

In each of the cases above it is possible to define a sequence of polygonal "boxes" $B_1, B_2, ...B_m$, with m finite, such that every a_i and every $f(a_i)$ is contained in one of the boxes, every two boxes with successive indices share one common side, and sides of all the boxes not shared by another box are line segments completely contained in ${}^*\mathbb{R}^2 - {}^*E$. Furthermore, the only side of B_1 that is not completely contained in ${}^*\mathbb{R}^2 - {}^*E$ is the side it shares with B_2 (the analogous statement for B_m will not hold), and the maximum total distance between any two points in each B_i is noninfinitesimal but small compared to d.

The way in which the B_i are constructed varies significantly among the cases i), iia) and iib). For example, in case i) we make use of the fact that there is a polygonal path in P in ${}^*\mathbb{R}^2 - {}^*E$ starting at some external point (well away from E) that is within an infinititesimal distance of each of the a_i and "separates" ${}^*st(Y)$ from Y, i.e. is such that for every point y in Y and corresponding nearest point s in ${}^*st(Y)$ there is a point in P nearer to y than s and nearer to s than y. The power of the nonstandard approach comes from the fact that there must be infinitely many standard polygonal paths in $\mathbb{R}^2 - E$ approaching such a P (as well as infinitely many connected components of $\mathbb{R}^2 - E -$ (small standard balls) close to U, etc.).

Now, the assumption that $f(a_1) \approx a_2, f(a_2) \approx a_3, ...f(a_k) \approx a_1$ implies that at least one a_i is such that it is contained in a higher-numbered box than $f(a_i)$.

Proposition 4. *Assuming that $Y \cap {}^*E$ is connected, if a_i is contained in a higher-numbered box than $f(a_i)$, then every point of Y in that box is mapped to a lower-numbered box.*

Although there are additional complications in the proof, letting j be the index of the box containing a_i, we begin by using the fact that the connected component of $Y \cap B_j$ that contains a_i must all map to points in lower numbered boxes, for otherwise some element maps inside B_j, contradicting the fact that all points in B_j are less than d apart.

The set Y is always connected, but it is possible that $Y \cap {}^*E$ is not, and this case is somewhat more difficult. Such a Y must be contained in a Y' that is defined in a slightly modified way from the sets used here. Thus, essentially, this case can be reduced to the case in which $Y \cap {}^*E$ is connected.

Department of Mathematical Sciences, University of Northern Colorado, Greeley, CO 80639
E-mail address: `steven.leth@unco.edu`