Nonstandard Methods for Freimans Inverse Problems

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Appetizers:

Nonstandard proofs of some well-known theorems in combinatorial number theory

Let $(T, \leqslant_T)$ be a tree and T_α be the α-th level of T for every ordinal α. The height of T is the least ordinal α such that $T_\alpha = \emptyset$.

König's Lemma Suppose T is an infinite tree with height ω (the first infinite ordinal). Suppose T_n is finite for each $n \in \mathbb{N}$. Then T must have an infinite path.

Proof Let $T = \bigcup_{n \in \mathbb{N}} T_n$ where T_n is the n-th level of T. Then ${}^*T = \bigcup_{n \in {}^*\mathbb{N}} {}^*T_n$. Note that ${}^*T_n = T_n$ for every standard n. Let H be a hyperfinite integer and $t_H \in T_H$. Then $\{t \in T : t \leqslant_{{}^*T} t_H\}$ is an infinite path of T.

Ramsey's Theorem Given $f : [\mathbb{N}]^2 \mapsto 2$, then there is an infinite set $A \subseteq \mathbb{N}$ such that f is a constant function on $[A]^2$.

Proof Suppose the theorem is not true. Let H be a hyperfinite integer. For $i = 0, 1$ let $A_i \subseteq \mathbb{N}$ be maximal such that ${}^*f \restriction [A_i \cup \{H\}]^2 \equiv i$. By the transfer principle, there is $c \in \mathbb{N} \smallsetminus (A_0 \cup A_1)$ such that $f \restriction [A_i \cup \{c\}]^2 \equiv i$. Now ${}^*f(c, H) = i$ violates the maximality of A_i.

Background Music:

Freiman's Inverse Phenomenon

If $A+A$ is "small", then A must have some arithmetic structure.

Let *a.p.* be an abbreviation for "arithmetic progression" and *b.p.* be an abbreviation for the union of two arithmetic progressions I and J with the same difference such that $I+I$, $I+J$, and $J+J$ are pairwise disjoint.

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Let A be a finite subset of $\mathbb{N}$ and $|A| = k$. Suppose c is a constant independent of k. If

$$|A+A| \leqslant ck,$$

then c is called a doubling constant of A.

Freiman's "Great" Theorem Let $c \geqslant 2$ be a constant. There exists another constant c' such that if $|A| = k$ and

$$|A+A| \leqslant ck,$$

then A is a subset of a $\lfloor c-1 \rfloor$-dimensional arithmetic progression P and $|P| \leqslant c'k$.

Well-known Fact

For every finite A, $|A+A| \geqslant 2k-1$ and if $|A+A| = 2k-1$, then A is an *a.p.*

Freiman's "Little" Theorem Let $|A| = k$.

(1) If $k > 2$ and $|A+A| = 2k-1+b$ for $0 \leqslant b \leqslant k-3$, then A is a subset of an *a.p.* of length $k+b$.

(2) If $k > 6$ and $|A+A| = 3k-3$, then A is either a subset of an *a.p.* of length $2k-1$ or a *b.p.*

(3) If $k > 10$ and $|A+A| = 3k-2$, then A is either a subset of an *a.p.* of length $2k+1$ or a subset of a *b.p.* of (combined) length $k+1$.

Example (a) Let $A = [0, k-2] \cup \{k+b-1\}$ for $k > 2$ and $b < k-2$. Then $|A| = k$, $|A+A| = 2k-1+b$ and A is a subset of an *a.p.* of length $k+b$. Hence the upper bound of the length of *a.p.* containing A in (1) is optimal.

(b) Let $A = [0, k-3] \cup \{k-1, 2k-2\}$ for $k > 6$. Then $|A+A| = 3k-3$ and A is a subset of an *a.p.* of length $2k-1$. Note that A is not a subset of a *b.p.* of reasonable length. Let $A = [0, k-2] \cup \{k^2\}$. Then $|A| = k$, $|A+A| = 3k-3$, and A is a *b.p.* Note that A is not an *a.p.* of reasonable length.

Freiman's $3k-3+b$ Conjecture There is a $K>0$ such that if $|A|=k>K$ and $|A+A|=3k-3+b$ for $0\leqslant b\leqslant \frac{k}{3}-3$, then A is either a subset of an *a.p.* of length $2k-1+2b$ or a subset of a *b.p.* of length $k+b$.

It can be shown that the upper bound of the length of the *a.p.* and the upper bound of the length of the *b.p.* containing A in the conjecture above are optimal.

Example Let $k=3n$, $m>2n$, and $A=[0,n-1]\cup[m,m+n-1]\cup[2m,2m+n-1]$. Then $|A|=k$,

$$|A+A|=10n-5=3k-3+\frac{k}{3}-2,$$

and A is neither a subset of an *a.p.* of reasonable length nor a subset of a *b.p.* of reasonable length.

For an infinite set $A\subseteq\mathbb{N}$ let $A(n)=|A\cap[1,n]|$. The lower asymptotic density of A is defined by

$$\underline{d}(A)=\liminf_{n\to\infty}\frac{A(n)}{n}.$$

Kneser's Theorem Let $A,B\subseteq\mathbb{N}$ be infinite. If $\underline{d}(A+B)<\underline{d}(A)+\underline{d}(B)$, then there exists a $g>0$ and $F,F'\subseteq[0,g-1]$ such that

(1) $A\subseteq F+g\mathbb{N}$ and $B\subseteq F'+g\mathbb{N}$ and

(2) $\underline{d}(A)+\underline{d}(B)>\frac{|F|+|F'|}{g}-\frac{1}{g}$.

Remark The theorem above indicates that if $B = A$ and $\underline{d}(A + A) < 2\underline{d}(A)$, then A is a large subset of the union of $|F|$–*a.p.*'s of difference g.

Note that in Freiman's theorems, the structure can be pinpointed only when the doubling constant is $\leqslant 3$. However, if A is infinite, then either A has nice arithmetic structure or $\underline{d}(A + A) \geqslant 2\underline{d}(A)$, which implies that we can find an increasing sequence a_n such that

$$(A + A)(2a_n) \geqslant 4A(a_n) - \epsilon$$

for any arbitrary $\epsilon > 0$.

Main Course:

Nonstandard Cuts

An infinite proper initial segment U of ${}^*\mathbb{N}$ is called a cut if $U+U \subseteq U$ (or U is a convex additive semi-group of non-negative integers). We often write $n < U$ for $n \in U$ and write $n > U$ for $n \in {}^*\mathbb{N} \setminus U$.

Example $\mathbb{N}$ is the smallest cut. Let H be a hyperfinite integer. Then

$$U_H = \bigcap_{n \in \mathbb{N}} [0, \frac{H}{n}]$$

is the largest cut below H.

Note that if $x < U_H$, then $\frac{x}{H} \approx 0$ and if $x > U_H$, then $\frac{x}{H} > \epsilon$ for some standard positive ϵ.

Proposition Let U be a cut and $A \subseteq {}^*\mathbb{N}$ be internal.

(1) Suppose $g \in \mathbb{N}$ and $G \subseteq [0, g-1]$. If $A \cap U \subseteq G + g\,{}^*\mathbb{N}$, then there is $H > U$ such that $A \cap [0, H] \subseteq G + g\,{}^*\mathbb{N}$.

(2) Let $\alpha \in \mathbb{R}$. Suppose for every $x \in U$ there is $y \in A$ with $x < y < U$ such that $\frac{(A+A)(y)}{y} \geqslant \alpha$. Then there is a $H > U$ in A such that $\frac{(A+A)(H)}{H} \geqslant \alpha$.

Proof The proposition follows from the fact that a set definable by a first-order formula with internal parameters is internal.

Lower U-Density of A

Let U be a cut and $A \subseteq {}^*\mathbb{N}$ be internal. The lower U-density of A is defined by

$$\underline{d}_U(A) = \sup\left\{\inf\left\{st\left(\frac{A(x)}{x}\right) : m < x < U\right\} : m < U\right\}.$$

Note that for $A \subseteq \mathbb{N}$ we have $\underline{d}_{\mathbb{N}}({}^*A) = \underline{d}(A)$.

From now on we are only interested in the cut of the form U_H for some hyperfinite integer H. When H is clearly given, we will drop the subscript H and simply write U for U_H.

Key Lemma Let H be hyperfinite and $A \subseteq [0, H]$ be internal such that $0 \in A$ and $0 < \underline{d}_U(A) = \alpha < \frac{2}{3}$. Then one of the following is true.

(1) There is $g > 1$ such that $A \cap U \subseteq gU$.

(2) There is $g > 1$ and $a \in [1, g-1]$ with $2a \neq g$ such that $A \cap U \subseteq gU \cup (a + gU)$.

(3) There is a positive standard real ϵ such that for every $x < U$, there is $y \in A$ with $x < y < U$ and

$$\frac{(A+A)(2y)}{2y} \geqslant \frac{3}{2}\alpha + \epsilon.$$

The lemma above is a weak version of Kneser's theorem for U.

Theorem There exist $K \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$, such that if $|A| = k > K$ and $|A + A| = 3k - 3 + b$ for $0 \leqslant b \leqslant \epsilon k$, then A is either a subset of an *a.p.* of length $2k - 1 + 2b$ or A is a subset of a *b.p.* of length $k + b$.

Idea of Proof

First Step Suppose the theorem is not true. For each $n \in \mathbb{N}$ there is a counter-example A_n with $|A_n| = k_n > n$ and

$$|A_n + A_n| - 3k_n - 3 < \frac{1}{n} k_n.$$

Let N be a hyperfinite integer such that $A = A_N$ is a counter-example of the theorem with $|A|$ being hyperfinite and $\frac{|A+A|}{|A|} \approx 3$. Without loss of generality we can assume

$$0 = \min A, \;\; H = \max A, \;\; \gcd(A) = 1,$$

$$\text{and } st\left(\frac{|A|}{H+1}\right) = \alpha > 0.$$

Note that $\alpha \leqslant \frac{1}{2}$. We can also assume that α is the least number such that there is a hyperfinite counter-example of the theorem $A \subseteq [0, H]$ for some hyperfinite number H with $0, H \in A$, $\gcd(A) = 1$, and $st\left(\frac{|A|}{H+1}\right) = \alpha$.

Second Step Let $\beta = \underline{d}_U(A)$.

Case 1: $\beta \geqslant \frac{2}{3}$. Then there is $N > U$ in A such that $[0, N] \subseteq A + A$ and $st\left(\frac{A(N,H)}{H-N+1}\right) < \alpha$.

Case 2: $0 < \beta < \frac{2}{3}$. Then either (a) $A + A$ has nice arithmetic structure in $[0, N]$ for some $N > U$ in A or (b) $(A + A)(2N) \geqslant (3 + \epsilon)A(N)$ for some $N > U$ in A. If (a) is true, then we have a pan-handle to start a tedious verification process that A has the desired structure, which is not assumed to have. If (b) is true, then we can derive the conclusion that $\frac{|A+A|}{|A|} \geqslant 3 + \epsilon$ for some positive standard ϵ, which again leads to a contradiction.

Case 3: $\beta = 0$. Then we can consider the lower U-density of A from the right end of an interval $[0, N]$ for some $N > U$ in A instead.

Work in Progress

Improved Key Lemma Let H be hyperfinite, $A \subseteq [0, H]$ be internal, $0 \in A$, and $0 < \underline{d}_U(A) = \alpha < \frac{3}{8}$. Then one of the following is true.

(1) There is $g > 0$ such that $A \cap U \subseteq gU$.

(2) There is $g > 0$ and $a \in [1, g]$ such that $|\{0, a\} + \{0, a\}| = 3 \bmod g$ and

$$A \cap U \subseteq \{0, a\} + gU.$$

(3) There is $g > 0$ and $a, b \in [1, g-1]$ such that $|\{0, a, b\} + \{0, a, b\}| = 5 \bmod g$ and

$$A \cap U \subseteq \{0, a, b\} + gU.$$

(4) For every $x < U$ there is $y \in A$ with $x < y < U$ such that

$$(A + A)(2y) \geqslant \left(\frac{10}{3} + \epsilon\right) A(y).$$

Conjecture (hope to be settled soon) There is $K \in \mathbb{N}$ such that for any $0 < c < \frac{1}{3}$ and any A with $|A| = k > K$ and

$$|A + A| = 3k - 3 + b$$

for $0 \leqslant b \leqslant ck$, A is either a subset of an *a.p.* of length $2k - 1 + 2b$ or a subset of a *b.p.* of length $k + b$.

Dessert:

Applications to Upper Asymptotic Density Problems

The upper asymptotic density of an infinite $A \subseteq \mathbb{N}$ is defined by

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{A(n)}{n}.$$

What will be the structure of A when $\overline{d}(A + A)$ is small?

Fact Suppose $0 \in A$, $\gcd(A) = 1$, and $\overline{d}(A) = \alpha \leqslant \frac{1}{2}$. Then $\overline{d}(A + A) \geqslant \frac{3}{2}\alpha$.

Example (1) If $A = 10\mathbb{N} \cup (3 + 10\mathbb{N})$, then $\overline{d}(A + A) = \frac{3}{2}\overline{d}(A)$.

(2) If

$$A = \bigcup_{n \in \mathbb{N}} [(1 - \alpha)2^{2^n}, 2^{2^n}]$$

for $\alpha \leqslant \frac{1}{2}$, then $\overline{d}(A) = \alpha$ and $\overline{d}(A + A) = \frac{3}{2}\alpha$.

Theorem Let $A \subseteq \mathbb{N}$ be such that $0 \in A$, $\gcd(A) = 1$, $0 < \overline{d}(A) = \alpha < \frac{1}{2}$ and $\overline{d}(A+A) = \frac{3}{2}\alpha$. Then one of the following is true.

(1) There is $g > 1$ and $a \in [1, g-1]$ such that $A \subseteq g\mathbb{N} \cup (a + g\mathbb{N})$ and $\frac{2}{g} = \alpha$.

(2) For any increasing sequence a_n of positive integers with $\lim_{n\to\infty} \frac{A(a_n)}{a_n} = \alpha$ there exist $0 \leqslant c_n \leqslant b_n \leqslant a_n$ such that

$$\lim_{n\to\infty} \frac{c_n}{a_n} = 0,$$

$$\lim_{n\to\infty} \frac{A(b_n, a_n)}{a_n - b_n + 1} = 1, \text{ and}$$

$A \cap [c_n + 1, b_n - 1] = \emptyset$ for each $n \in \mathbb{N}$.

There are also results about the structure of A when $\overline{d}(A) \geqslant \frac{1}{2}$ and $\overline{d}(A+A)$ achieves the least possible value. But they are less interesting.

Bordes' Theorem Let $0 \in A$ and $\gcd(A) = 1$. There is $\alpha_0 > 0$ such that if $\overline{d}(A) = \alpha \leqslant \alpha_0$ and $\overline{d}(A+A) = \sigma \overline{d}(A)$ for some σ with $\frac{3}{2} \leqslant \sigma < \frac{5}{3}$, then one of the following is true.

(1) There is $g > 1$ and $a \in [1, g-1]$ such that $A \subseteq g\mathbb{N} \cup (a + g\mathbb{N})$ and $\alpha \geqslant \frac{6}{(4\sigma-3)g}$.

(2) There are sequences $0 \leqslant c_n \leqslant b_n \leqslant a_n$ such that a_n is increasing, $\lim_{n\to\infty} \frac{A(a_n)}{a_n} = \alpha$, $A \cap [c_n+1, b_n-1] = \emptyset$ for every $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{c_n}{a_n - b_n} \leqslant \frac{2\sigma - 3}{2\sigma - 2} \left(\frac{1}{2\sigma - 2} - \alpha \right)^{-1},$$

and

$$\lim_{n\to\infty} \frac{A(b_n, a_n)}{a_n - b_n + 1} \geqslant \left(\frac{1}{2\sigma - 2} + \left(\frac{1}{2\sigma - 2} - \alpha \right) \lim_{n\to\infty} \frac{c_n}{a_n - b_n} \right).$$

We hope to prove a common generalization of both theorems soon.

Fortune Cookie Reading:

Questo non e' un pollo alla Cantonese.