RELATIVE SET THEORY

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This is a report on a work in progress.
Partial results are in

(1) *Internally iterated ultrapowers*,
in: *Nonstandard Models of Arithmetic and Set Theory*, ed. by A. Enayat and R. Kossak, Contemporary Math. 361, AMS, Providence, R.I., 2004.

(2) *Stratified analysis?*,
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Hilbert:

We know sets before we know their elements.

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Elementary theory:

We work in **ZFC** extended by a new binary "precedence" predicate $\sqsubseteq$.
$y \sqsubseteq x$ reads "*y is accessible to x*".
We also write $y \in \mathbf{v}(x)$ for $y \sqsubseteq x$
and read it "*y is at level x*".

We postulate: (o) $x \in \mathbf{v}(x)$
(i) $y \in \mathbf{v}(x) \Rightarrow \mathbf{v}(y) \subseteq \mathbf{v}(x)$
(ii) $(\forall x)(\exists n \in \mathbf{N})(\mathbf{v}(x) = \mathbf{v}(n))$
(iii) $(\forall m, n \in \mathbf{N})(m \leq n \Rightarrow m \in \mathbf{v}(n))$
(iv) $(\forall m \in \mathbf{N})(\exists n \in \mathbf{N})(\mathbf{v}(m) \subset \mathbf{v}(n))$
(v) $\mathbf{v}(m) \subset \mathbf{v}(n) \Rightarrow (\exists k)(\mathbf{v}(m) \subset \mathbf{v}(k) \subset \mathbf{v}(n))$.

Transfer Principle. *If $x_1, \ldots, x_n \in \mathbf{v}(\alpha) \cap \mathbf{v}(\beta)$ then $\mathcal{P}(x_1, \ldots, x_n; \mathbf{v}(\alpha))$ iff $\mathcal{P}(x_1, \ldots, x_n; \mathbf{v}(\beta))$.*

The coarsest level containing $x_1, \ldots, x_n$ is
$\mathbf{v}(x_1, \ldots, x_n) = \mathbf{v}(\langle x_1, \ldots, x_n \rangle)$; hence
$\mathcal{P}(x_1, \ldots, x_n; \mathbf{v}(x_1, \ldots, x_n))$ iff $\mathcal{P}(x_1, \ldots, x_n; \mathbf{v}(\alpha))$
provided $x_1, \ldots, x_n \in \mathbf{v}(\alpha)$.

Predicates of the form $\mathcal{P}(x_1, \ldots, x_n; \mathbf{v}(x_1, \ldots, x_n))$ are called *acceptable.*
(Previously defined acceptable predicates may occur in $\mathcal{P}$.)

Definition Principle. *If $\mathcal{P}$ is acceptable then
$B := \{x \in A : \mathcal{P}(x, A, \overline{p}; \mathbf{v}(x, A, \overline{p}))\}$ is a set
and $B \in \mathbf{v}(A, \overline{p})$. Similarly, if $\mathcal{P}$ is acceptable and
$(\forall x \in A)(\exists! y)\mathcal{P}(x, y, A, \overline{p}; \mathbf{v}(x, A, \overline{p}))$ then
$F(x) = y \iff x \in A \wedge \mathcal{P}(x, y, A, \overline{p}; \mathbf{v}(x, y, A, \overline{p}))$
defines a function and $F \in \mathbf{v}(A, \overline{p})$.*

Definition.
(a) $x \in \mathbf{R}$ is *α-limited* iff $|x| < n$ for some n in $\mathbf{N} \cap \mathbf{v}(\alpha)$.
(b) $h \in \mathbf{R}$ is *α-infinitesimal* iff $h \neq 0$ and $|h| < \frac{1}{n}$ for all n in $\mathbf{N} \cap \mathbf{v}(\alpha)$.
(c) x is *α-infinitely close to y* iff $x - y$ is α-infinitesimal or 0. (Notation: $x \approx_\alpha y$.)

Standardization Principle for Real Numbers. *For every α-limited $x \in \mathbf{R}$ there is $r \in \mathbf{R} \cap \mathbf{v}(\alpha)$ such that $x \approx_\alpha r$.*

This r is unique; we call it the *α-shadow* of x and denote it $\mathbf{sh}_\alpha(x)$.

Proposition.
(1) If $x, y \in \mathbf{R}$ are α-limited then $x + y, x - y, xy$ are α-limited.
(2) If h, k are α-infinitesimal and $x \in \mathbf{R}$ is α-limited then $h + k$, $h - k, xh$ are α-infinitesimal.
(3) $z \in \mathbf{R}$ is α-infinitesimal iff $\frac{1}{z}$ is α-unlimited.
(4) $\approx_\alpha$ is an equivalence relation.
If $x_1 \approx_\alpha y_1$ and $x_2 \approx_\alpha y_2$ then $x_1 + x_2 \approx_\alpha y_1 + y_2$.
If x_1, x_2 are α-limited then also $x_1 x_2 \approx_\alpha y_1 y_2$.

Proposition. Let $x, y \in \mathbf{R}$ be α-limited.
(0) x is α-infinitesimal iff $\mathbf{sh}_\alpha(x) = 0$.
(1) $x \leq y$ implies $\mathbf{sh}_\alpha(x) \leq \mathbf{sh}_\alpha(y)$.
(2) $\mathbf{sh}_\alpha(x + y) = \mathbf{sh}_\alpha(x) + \mathbf{sh}_\alpha(y)$.
(3) $\mathbf{sh}_\alpha(x - y) = \mathbf{sh}_\alpha(x) - \mathbf{sh}_\alpha(y)$.
(4) $\mathbf{sh}_\alpha(xy) = \mathbf{sh}_\alpha(x)\,\mathbf{sh}_\alpha(y)$.
(5) If y is not α-infinitesimal then $\mathbf{sh}_\alpha(\frac{x}{y}) = \frac{\mathbf{sh}_\alpha(x)}{\mathbf{sh}_\alpha(y)}$.

Proposition.

(a) If $x \in \mathbf{R}$ is α-infinitesimal and $\beta \sqsubseteq \alpha$ then x is β-infinitesimal.

(b) Every α-limited natural number is in $\mathbf{v}(\alpha)$.

(c) If y is α-infinitesimal then there is an α-infinitesimal x such that y is x-infinitesimal.

Example: CONTINUITY.

Definition. f is *continuous at* x iff
$y \approx_{\langle f,x\rangle} x$ implies $f(y) \approx_{\langle f,x\rangle} f(x)$.

Equivalently, f is continuous at x iff $y \approx_\alpha x$ implies $f(y) \approx_\alpha f(x)$, for *some* or *all* α such that $f, x \in \mathbf{v}(\alpha)$.

Definition.
f is *uniformly continuous* iff for all $x, y \in \operatorname{dom} f$,
$y \approx_f x$ implies $f(y) \approx_f f(x)$.

Let $\vec{s} := \langle s_n : n \in \mathbf{N}\rangle$ be an infinite sequence of reals.
$r \in \mathbf{R}$ is a *limit* of $\vec{s}$ iff $r = \mathbf{sh}_{\vec{s}}(s_n)$ for all $\vec{s}$-unlimited n.

Let $\vec{f} := \langle f_n : n \in \mathbf{N}\rangle$ be an infinite sequence of real valued functions with common domain $A \subseteq \mathbf{R}$.
$f_n \to f$ *pointwise* iff
for all $x \in A$ and all $\langle \vec{f}, x\rangle$-unlimited n, $f_n(x) \approx_{\langle \vec{f},x\rangle} f(x)$.
$f_n \to f$ *uniformly* iff
for all x and all $\vec{f}$-unlimited n, $f_n(x) \approx_{\vec{f}} f(x)$.

Proposition. *The limit of a uniformly convergent sequence of continuous functions is continuous.*

Proof. Let $f = \lim_{n\to\infty} f_n$; we note first that if $\vec{f} \in \mathbf{v}(\alpha)$ then also $f \in \mathbf{v}(\alpha)$, by Definition Principle. For $x, x' \in A$,
$|f(x') - f(x)| \leq$
$|f(x') - f_\nu(x')| + |f_\nu(x') - f_\nu(x)| + |f_\nu(x) - f(x)|$.
If $x' \approx_\alpha x$ then $x' \approx_\nu x$ for some α-unlimited ν. Now the middle term is ν-infinitesimal, by continuity of f_ν, hence also α-infinitesimal, and the other two are α-infinitesimal by definition of uniform convergence. So $f(x') \approx_\alpha f(x)$. $\square$

Proof of equivalence with the standard definition of continuity:

$\Rightarrow$: Given $\epsilon > 0$ fix α such that $f, x, \epsilon \in \mathbf{v}(\alpha)$. Let δ be α-infinitesimal. If $|y - x| < \delta$ then $y \approx_\alpha x$, so $f(y) \approx_\alpha f(x)$ and hence $|f(y) - f(x)| < \epsilon$.

$\Leftarrow$: Fix α such that $f, x \in \mathbf{v}(\alpha)$. Let $x' \in \operatorname{dom} f, x' \approx_\alpha x$; we have to show that $f(x') \approx_\alpha f(x)$ Given $\epsilon \in \mathbf{v}(\alpha), \epsilon > 0$, there exists δ such that
(*) $(\forall y \in \operatorname{dom} f)(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$.
We take one such δ and fix β so that $f, x, \epsilon, \delta \in \mathbf{v}(\beta)$. Then there exists $\delta \in \mathbf{v}(\beta)$ such that (*); hence by Transfer, there exists $\delta \in \mathbf{v}(\alpha)$ such that (*). As $|x' - x|$ is α-infinitesimal, we have $|x' - x| < \delta$, hence $|f(x') - f(x)| < \epsilon$. This is true for all $\epsilon \in \mathbf{v}(\alpha)$, proving $f(x') \approx_\alpha f(x)$. $\square$

Example: DERIVATIVE.

Definition.
f is *differentiable* at x iff there is an $\langle f,x\rangle$-standard $L \in \mathbf{R}$ such that $\frac{f(x+h)-f(x)}{h} - L$ is $\langle f,x\rangle$-infinitesimal, for all $\langle f,x\rangle$-infinitesimal $h \neq 0$.
If this is the case, $f'(x) := L = \mathbf{sh}_{\langle f,x\rangle}\left(\frac{f(x+h)-f(x)}{h}\right)$.

Proposition. *If f is differentiable at x then f is continuous at x.*

Proof By definition, for any $\langle f,x\rangle$-infinitesimal h, $f(x+h) - f(x) = Lh + kh$ where k is $\langle f,x\rangle$-infinitesimal. This value is $\langle f,x\rangle$-infinitesimal. $\square$

Proposition. (l'Hôpital Rule)
If $\lim_{x\to a}|g(x)| = \infty$ *and* $lim_{x\to a}\frac{f'(x)}{g'(x)} = d \in \mathbf{R}$ *then* $lim_{x\to a}\frac{f(x)}{g(x)} = d.$

Proof (after Benninghofen and Richter). We can assume that $a = 0$ (replace x by $x - a$). Fix α so that $f, g, d \in \mathbf{v}(\alpha)$. Let x be α-infinitesimal and y be x-infinitesimal. By Cauchy's Theorem, there is η between x and y (hence, η is α-infinitesimal) such that $\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(\eta)}{g'(\eta)} \approx_\alpha d$. Now factor

$$d \approx_\alpha \frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f(y)-f(x)}{g(y)} \times \frac{g(y)}{g(y)-g(x)} = (\frac{f(y)}{g(y)} - \frac{f(x)}{g(y)})(1 - \frac{g(x)}{g(y)})^{-1}$$

and observe that $\frac{f(x)}{g(y)} \approx_\alpha 0$, $\frac{g(x)}{g(y)} \approx_\alpha 0$.

($\lim_{x\to 0}|g(x)| = \infty$ implies that for all α-infinitesimal z, $g(z)$ is α-unlimited. By transfer to x-level, for all x-infinitesimal z, $g(z)$ is x-unlimited. As y is x-infinitesimal, $\frac{f(x)}{g(y)}$ and $\frac{g(x)}{g(y)}$ are x-infinitesimal.)

It follows that the first factor is α-infinitely close to $\frac{f(y)}{g(y)}$ and the second to 1. From properties of infinitesimals we conclude that $\frac{f(y)}{g(y)} \approx_\alpha d$.

Every α-infinitesimal y is x-infinitesimal for *some* α-infinitesimal x. Hence $\frac{f(y)}{g(y)} \approx_\alpha d$ holds for *every* α-infinitesimal y, and we are done. $\square$

FRIST:
Language: $\in$, $\sqsubseteq$ (binary).

$\mathbb{S}_\alpha := \mathbf{v}(\alpha) = \{x : x \sqsubseteq \alpha\}$; in particular $\mathbb{S} := \mathbb{S}_0$.
$x \sqsubseteq_\alpha y \equiv (x \sqsubseteq \alpha \ \wedge \ y \sqsubseteq \alpha) \ \vee \ x \sqsubseteq y$.

Let φ be any $\in$-$\sqsubseteq$-formula; φ^α denotes the formula obtained from φ by replacing each occurence of $\sqsubseteq$ by $\sqsubseteq_\alpha$.

Axioms:
ZFC (Separation and Replacement for $\in$-formulas only).

Stratification: $\sqsubseteq$ is a dense linear preordering with a least element 0 and no greatest element.

Boundedness: $(\forall x)(\exists A \in \mathbb{S}_0)(x \in A)$

Transfer: For any α, $(\forall \overline{x} \in \mathbb{S}_0)(\varphi^0(\overline{x}) \ \Leftrightarrow \ \varphi^\alpha(\overline{x}))$.

Standardization:
$(\forall \overline{x})(\forall x \in \mathbb{S}_0)\ (\exists y \in \mathbb{S}_0)\ (\forall z \in \mathbb{S}_0)$
$(z \in y \ \Leftrightarrow \ z \in x \ \wedge \ \varphi^0(z, x, \overline{x}))$.

Idealization:
For any $0 \sqsubset \alpha$, any $A, B \in \mathbb{S}_0$ and any $\overline{x}$,
$(\forall a \in A^{\text{fin}} \cap \mathbb{S}_0)(\exists x \in B)(\forall y \in a)\ \varphi^\alpha(x, y, \overline{x})$
$\Leftrightarrow \ (\exists x \in B)(\forall y \in A \cap \mathbb{S}_0)\ \varphi^\alpha(x, y, \overline{x})$.

In these axioms φ can be any $\in$-$\sqsubseteq$-formula, not just an $\in$-formula as usual. 0 can be replaced by any $\beta \sqsubseteq \alpha$: **FRIST** is *fully relativized.*

Theorem. **FRIST** *is a conservative extension of* **ZFC**. *In fact*, **FRIST** *has a standard core interpretation in* **ZFC**.

Example: LEBESGUE MEASURE on $[0, 1]$.

$\mathcal{B}$ is the algebra generated by all left-closed right-open intervals.
$l([a, b)) = b - a$ for $a < b$.
$l(b) = \sum_{k=1}^{n} l(I_k)$ if $b = \bigcup_{k=1}^{n} I_k \in \mathcal{B}$ and the I_k are mutually disjoint.

Proposition. *Let* $X \subseteq [0, 1]$, $X \in \mathbf{v}(\alpha)$, *and* $\alpha \sqsubset \beta$.
X *is* Lebesgue measurable *iff there exist* $b_1, b_2 \in \mathcal{B}$ *such that* $b_1 \subseteq \mathbf{sh}_\beta^{-1}(X) \subseteq b_2$ *and* $l(b_2) - l(b_1)$ *is* α*-infinitesimal.*
$\mathbf{sh}_\alpha(l(b_1)) = \mathbf{sh}_\alpha(l(b_2))$ *is the Lebesgue measure of* X.

Example: HIGHER DERIVATIVES.

We assume that $f, x \in \mathbf{v}(\alpha)$ and $f'(y)$ exists for all $y \approx_\alpha x$.

If $f''(x) = L$ exists, then $L \approx_\alpha \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$ holds for all $h \approx_\alpha 0,\ h \neq 0$. However, the converse of this statement is false; existence of $L \in \mathbf{R} \cap \mathbf{v}(\alpha)$ with the above property does not imply that $f''(x)$ exists.

Proposition. *Assume that $f, x \in \mathbf{v}(\alpha)$ and $f'(y)$ exists for all $y \approx_\alpha x$. Then $f''(x)$ exists iff there is a $L \in \mathbf{R} \cap \mathbf{v}(\alpha)$ such that*

$$L \approx_\alpha \frac{f(x+h_0+h_1)-f(x+h_0)-f(x+h_1)+f(x)}{h_0 h_1}$$

for all $h_0 \approx_\alpha 0,\ h_1 \approx_{h_0} 0,\ h_0, h_1 \neq 0$.
If this is the case, $f''(x) = L$.

Proposition. *Assume that $n, f, x \in \mathbf{v}(\alpha)$ and $f^{(n-1)}(y)$ exists for all $y \approx_\alpha x$. Then $f^{(n)}(x)$ exists iff there is $L \in \mathbf{R} \cap \mathbf{v}(\alpha)$ such that*

$$L \ \approx_\alpha \ \frac{1}{h_0 \ldots h_{n-1}} \sum_i \ (-1)^{i_0+\ldots+i_{n-1}} \ f(x + h^{i_0} + \ldots + h^{i_{n-1}})$$

for all $\langle h_0, \ldots, h_{n-1}\rangle$, where $i = \langle i_0, \ldots, i_{n-1}\rangle \in \{0,1\}^n$,
$h^{i_k} := h_k$ if $i_k = 0$, $h^{i_k} := 0$ if $i_k = 1$;
$h_0 \approx_\alpha 0$, $h_k \approx_{h_{k-1}} 0$ for $0 < k < n$, and all $h_k \neq 0$.
If this is the case, $f^{(n)}(x) = L$.

This proposition implies existence of "*strongly decreasing*" sequences of infinitesimals of any finite length n : $\langle h_0, \ldots, h_{n-1}\rangle$ where each h_k is h_{k-1}-infinitesimal.

BST:

Language: $\in$ (binary), **st** (unary).

$\mathbb{S} := \{\, x \mid \mathbf{st}\, x \,\}, \quad \mathbb{I} := \{\, x \mid x = x \,\}.$

If φ is an $\in$-formula, $\varphi^{\mathbb{S}}$ is the formula obtained from φ by replacing each subformula of the form $(\exists x)\,\psi$ by $(\exists^{\mathbf{st}} x)\,\psi$, and each subformula of the form $(\forall x)\,\psi$ by $(\forall^{\mathbf{st}} x)\,\psi$.
A^{fin} is the set of all finite subsets of A.

Axioms of BST:

ZFC: $\varphi^{\mathbb{S}}$ where φ is any axiom of ZFC (Separation and Replacement for $\in$-formulas only).

Boundedness: $(\forall x)(\exists A \in \mathbb{S})(x \in A)$.

Transfer: $(\forall \overline{x} \in \mathbb{S})(\varphi^{\mathbb{S}}(\overline{x}) \;\Leftrightarrow\; \varphi(\overline{x}))$
where $\varphi(\overline{x})$ is any $\in$-formula.

Standardization:
$(\forall \overline{x})(\forall x \in \mathbb{S})(\exists y \in \mathbb{S})(\forall z \in \mathbb{S})$
$(z \in y \;\Leftrightarrow\; z \in x \;\wedge\; \varphi(z, x, \overline{x}))$
where $\varphi(z, x, \overline{x})$ is any $\in$-**st**-formula.

Idealization:
For any $A, B \in \mathbb{S}$ and any $\overline{x}$,
$(\forall a \in A^{\text{fin}} \cap \mathbb{S})(\exists x \in B)(\forall y \in a)\,\varphi(x, y, \overline{x}) \;\Leftrightarrow$
$(\exists x \in B)(\forall y \in A \cap \mathbb{S})\,\varphi(x, y, \overline{x})]$
where $\varphi(x, y, \overline{x})$ is any $\in$-formula.

Theorem (see the book of Kanovei and Reeken).
BST *is a conservative extension of* **ZFC**.
In fact, **BST** *has a standard core interpretation in* **ZFC**.

We use letters U, V to denote ultrafilters.
If U is an ultrafilter, $I_U := \bigcup U$.
If $I_U \cap (\mathbb{I} \times \mathbb{I}) \in U$ then $\pi(U)$ denotes the *projection* of U onto the domain of I_U; i.e., for $A \subseteq \operatorname{dom} I_U$, $A \in \pi(U) \Leftrightarrow \{\langle a, b\rangle \in I_U\} \mid a \in A\} \in U$; $\pi(U)$ is an ultrafilter.

For a standard ultrafilter U, $x \,\mathfrak{M}\, U$ denotes that $x \in \bigcap(U \cap \mathbb{S})$ (x belongs to the *monad* of U).

Proposition. (Andreev and H.) (*Back and Forth Lemma*)
(a) $(\forall x)(\forall U \in \mathbb{S})[x \,\mathfrak{M}\, U \Rightarrow (\forall y)(\exists V \in \mathbb{S})\ (\pi(V) = U \wedge \langle x, y\rangle \,\mathfrak{M}\, V)]$
(b) $(\forall U \in \mathbb{S})(\forall x)[x \,\mathfrak{M}\, U \Rightarrow (\forall V \in \mathbb{S})(\pi(V) = U \Rightarrow (\exists y)\langle x, y\rangle \,\mathfrak{M}\, V)]$.

Underlying this lemma is the existence of an isomorphism between $(\mathbb{V}^I/U)^{\mathbb{S}}$, the ultraproduct of the universe modulo U constructed inside $\mathbb{S}$, and $\mathbb{S}[x] := \{f(x) : f \in \mathbb{S}\}$ for $x \,\mathfrak{M}\, U$, given by $f \mapsto f(x)$ (for $f \in \mathbb{S}$, $\operatorname{dom} f = I_U$), and the fact that these isomorphisms "fit together" in a natural way.

Corollary. (*Normal Form Theorem*, or *Reduction to* $\Sigma_2^{\mathbf{st}}$ *Formulas.*) *There is an effective procedure that assigns to each* $\in$-**st**-*formula* $\varphi(\overline{x})$ *an* $\in$-*formula* $\varphi^m(U)$ *so that, for all* $\overline{x}$, $\varphi(\overline{x}) \Leftrightarrow (\exists U \in \mathbb{S})(\langle \overline{x}\rangle \,\mathfrak{M}\, U \wedge \varphi^m(U)) \Leftrightarrow (\forall U \in \mathbb{S})(\langle \overline{x}\rangle \,\mathfrak{M}\, U \rightarrow \varphi^m(U))$.

Kanovei and Reeken used Reduction to $\Sigma_2^{\mathbf{st}}$ to prove that Collection for arbitrary $\in$-**st**-formulas holds in **BST**.

Corollary. *Any two countable models of* **BST** *with the same standard core are isomorphic.*

Definition: $U \sim V \iff U \cap V$ is an ultrafilter.

Definition (**Stratified ultrafilters** over A):
$\gamma_0 A := A$;
$\gamma_\xi A := \gamma_{<\xi} A \cup \{U : U$ is non-principal over $\gamma_{<\xi} A$ and $U \sim V$ does not hold for any $V \in \gamma_{<\xi} A\}$.

Definition (**FRIST**):
Let $x \in A \in \mathbb{S}$. A **standardizer** for x over A is a sequence $\overrightarrow{u} := \langle u_i : i \leq \nu \rangle$ where $\nu \in \omega$ and
i) each u_i is a stratified ultrafilter over A;
ii) $u_0 \in \mathbb{S}$, $u_\nu = x$;
iii) $u_i \sqsubset u_{i+1}$ for $i < \nu$;
iv) if $u_i \sqsubseteq \alpha \sqsubset u_{i+1}$ then $u_{i+1} \in \bigcap(u_i \cap \mathbb{S}_\alpha)$.

Theorem. *In the interpretation for* **FRIST** *constructed in ref.* (1), *for any* $x \in A \in \mathbb{S}$ *there is a unique standardizer* $\overrightarrow{u}_A$ *for* x over A. *The universe* $\mathbb{S}[\overrightarrow{u}_A]$ *is independent of* A; *we denote it* $\mathbb{S}[[x]]$.

Definition (**FRIST**): $x \,\mathfrak{M}\, U$ denotes that $U \in \mathbb{S}$ is a stratified ultrafilter over A and there is a standardizer $\overrightarrow{u}_A$ for x over A with $u_0 = U$.

Theorem. *The Back and Forth Lemma holds in the interpretation for* **FRIST** *constructed in ref.* (1).

Corollary. *Any two countable models of* **GRIST** = "**FRIST + The Back and Forth Lemma**" *with the same standard core are isomorphic.*

Corollary. (**GRIST**) *Collection for* $\in$-$\sqsubseteq$-*formulas fails.*

Repeated ultrapowers:

$\mathbb{V}^I/U \models$ " $k(U)$ is an ultrafilter over $k(I)$ "
(k is the canonical embedding of $\mathbb{V}$ into $\mathbb{V}^I/U$)

Observation:
$[\mathbb{V}^{k(I)}/k(U)]^{\mathbb{V}^I/U}$ is isomorphic to $\mathbb{V}^{I\times I}/U\otimes U$ where
$X \in U\otimes U \equiv \{i_0 \in I : \{i_1 \in I : \langle i_0, i_1\rangle \in X\} \in U\} \in U$.

More generally, let
$\bigotimes_0 U :=$ the principal ultrafilter over $\{0\}$;
$\bigotimes_1 U := U$;
$\bigotimes_{n+1} U := U \otimes (\bigotimes_n U)$.

For $X \subseteq I^{n+1}$, $\quad X \in \bigotimes_{n+1} U \;\Leftrightarrow$
$\{i_0 \in I : \{\langle i_1, \dots, i_n\rangle : \langle i_0, i_1, \dots, i_n\rangle \in X\} \in \bigotimes_n U\} \in U$.

$\varphi : I_2 \to I_1$ is a *morphism* of U_2 to U_1 iff
$(\forall X \in U_1)(\varphi^{-1}[X] \in U_2)$.
Every morphism φ induces an elementary embedding
$\varphi^* : \mathbb{V}^{I_1}/U_1 \to \mathbb{V}^{I_2}/U_2$ defined by $\varphi^*(f) = f\circ\varphi$.

For $0 \le \ell \le n$, $\;\pi_{\ell,n}$ is the projection of I^n onto I^ℓ:
$\pi_{\ell,n}(\langle i_0, \dots, i_{n-1}\rangle) = \langle i_0, \dots, i_{\ell-1}\rangle$.
Then $\pi_{\ell,n} : \bigotimes_n U \to \bigotimes_\ell U$ is a morphism of ultrafilters, so
$\pi^*_{\ell,n} : \mathbb{V}^{I^\ell}/\bigotimes_\ell U \to \mathbb{V}^{I^n}/\bigotimes_n U$ is an elem. embedding.

Proposition. (*Factoring Lemma*)
For $0 \le \ell \le n$
$\mathbb{V}^{I^n}/\bigotimes_n U \cong [\, \mathbb{V}^{\pi^*_{0,\ell}(I^{n-\ell})}/\bigotimes_{n-\ell} \pi^*_{0,\ell}(U)\,]^{\mathbb{V}^{I^\ell}/\otimes_\ell U}$.

Iterated ultrapowers:

The system $\langle \pi^*_{\ell,n} : \ell \leq n \in \omega \rangle$ has a direct limit $({}^*\mathbb{V}^U_\omega, =^*, \in^*)$, which elementarily extends each $\mathbb{V}^{I^n} / \bigotimes_n U$.

Iterated ultrapowers (Gaifman and Kunen)
(*iteration with finite support*):
ω can be replaced by any linear ordering $(\Lambda, \leq)$.

Note: If U is NOT countably complete then ${}^*\mathbb{V}^U_\omega$ is NOT isomorphic to $[\,{}^*\mathbb{V}^{k(U)}_{k(\omega \setminus 1)}\,]^{\mathbb{V}^I/U}$, i.e., the Factoring Lemma for the direct limit fails at stage 1. (*Reason*: $k(\omega)$ is not well-founded and it has cofinality $> \omega$.)

Observation:
Ultrapowers can be repeated into transfinite!

Assume U is over $I = \omega$ and let $U_n := \bigotimes_n U$. Then we can define an ultrafilter W over $I^{<\omega}$ (Rudin-Frolík sum) by:
$A \in W \;\Leftrightarrow\; \{n \in I : \{t \in I^n : \langle n \rangle \frown t \in A\} \in U_n\} \in U$.
($\langle n \rangle := \{\langle 0, n \rangle\}$.)

Let $\bar{U} := \langle U_n : n \in \omega \rangle$, $\nu := \langle n : n \in \omega \rangle$.

$\mathbb{V}^I/U \models$ " $\bar{U}$ is an ultrafilter over $k(I)^\nu$; $\bar{U} = \bigotimes_\nu k(U)$ ".

Factoring Lemma: $\mathbb{V}^{I^{<\omega}}/W \cong [\,\mathbb{V}^{k(I)^\nu}/\bar{U}\,]^{\mathbb{V}^I/U}$.

"*Iteration with *-finite support*": **Internally iterated ultrapowers** are obtained by allowing arbitrary transfinite repetitions in the Gaifman-Kunen construction.

In ref. (1), interpretations for **GRIST** in **ZFC** are constructed using internally iterated ultrapowers of $\mathbb{V}$.

External sets:

Given an ultrapower $\mathbb{V}^I/U = (\mathbb{V}^I, =_U, \in_U)$, one can build a cumulative universe $\mathbb{E}_U$ over this structure and extend $=_U$ and $\in_U$ to it so that this **completed ultrapower** $(\mathbb{E}_U, =_U, \in_U)$ satisfies **ZFC**$^-$ (**ZFC** minus Regularity).

In the construction of ref.(1) *ultrapowers can be replaced by completed ultrapowers.*

The last two slides outline the theory of the resulting structure.

RST:
Language: $\in$ (ternary).
$x \in^w y$ reads "*x belongs to y relative to w*".

It is possible that $x \in^w y$ and $x \notin^{w'} y$,
but we want some stability.

Definition: $x \in y$ iff $(\exists w)(x \in^w y)$

Axioms: $\varnothing$, $\{x, y\}$ exist.

Definition: x *is w-internal* iff $(\exists y)(x \in^w y)$.
Notation: $\mathbb{I}_w(x)$.

Definition: y *is w-standard* iff $y = \varnothing \vee (\exists x)(x \in^w y)$.
Notation: $\mathbb{S}_w(y)$.

Axioms:
$\mathbb{S}_w(w)$
$\mathbb{S}_w(y) \Rightarrow \mathbb{I}_w(y)$
$\mathbb{S}_{\{x,y\}}(x)$, $\mathbb{S}_{\{x,y\}}(y)$, $\mathbb{S}_w(x) \wedge \mathbb{S}_w(y) \Rightarrow \mathbb{S}_w(\{x, y\})$
$\mathbb{S}_w(x) \Rightarrow (\mathbb{S}_x(z) \Rightarrow \mathbb{S}_w(z))$
$\mathbb{I}_w(x) \Rightarrow (\mathbb{I}_x(z) \Rightarrow \mathbb{I}_w(z))$
$(\mathbb{I}_w(x) \wedge \mathbb{S}_w(y) \wedge x \in y) \Rightarrow x \in^w y$

Definition: $x \sqsubseteq_w y$ iff $\mathbb{I}_w(x) \wedge \mathbb{I}_w(y) \wedge \mathbb{S}_{\{y,w\}}(x)$.

Axioms: $\varphi^{(\mathbb{I}_w, \sqsubseteq_w)}$ where φ is any axiom of **GRIST**.

Axiom:
$(\exists! W)(\forall x, y)(x \sqsubseteq_w y \;\Leftrightarrow\; (\mathbb{S}_W(\langle x, y\rangle) \;\wedge\; \langle x, y\rangle \in W))$.

It follows that
$(\exists! A)(\forall x)(\mathbb{S}_w(x) \;\Leftrightarrow\; \mathbb{S}_W(x) \wedge x \in A)$ Notation: $A = \mathbb{S}_w$.
$(\exists! B)(\forall x)(\mathbb{I}_w(x) \;\Leftrightarrow\; \mathbb{S}_W(x) \wedge x \in B)$ Notation: $B = \mathbb{I}_w$.

Note: It is necessary to carefully distinguish between $x \in \mathbb{S}_w$ and $\mathbb{S}_w(x)$. $\mathbb{S}_w$ and $\mathbb{I}_w$ are sets in $\mathbb{S}_W$.
In **RST** there is no need for classes!
$\mathbb{S}_W$ can serve as the external universe for $\mathbb{I}_w$. It contains all collections definable in $(\mathbb{I}_w, \sqsubseteq_w)$ and satisfies **ZFC**$^-$.

Definition: $\mathbb{I}_w^W(x)$ iff $\mathbb{I}_W(x) \;\wedge\; (\exists y)(\mathbb{S}_w(y) \;\wedge\; x \in y)$.

Axioms: $(\forall \overline{x})(\mathbb{I}_w(\overline{x}) \Rightarrow (\varphi^{(\mathbb{I}_w,\, \sqsubseteq_w)}(\overline{x}) \;\Leftrightarrow\; \varphi^{(\mathbb{I}_w^W,\, \sqsubseteq_W \restriction \mathbb{I}_w^W)}(\overline{x})))$
where φ is any $\in$-$\sqsubseteq$-formula.

Work on a "complete" axiomatization is in progress.