# Mathematics in the Hyperfinite World

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# Harmonic analysis on finite abelian groups

- $G$ a finite abelian group
- Dual group $\widehat{G} = \mathrm{Hom}(G, \mathbb{S}^1)$
- $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

- Pontrjagin Duality:
- $G \simeq \widehat{\widehat{G}}$
- $g \longmapsto \kappa_g : \widehat{G} \to \mathbb{S}^1$ where $\kappa_g(\chi) = \chi(g)$

- The Haar integral $I(f) = \Delta \sum_{g \in G} f(g)$.

- The Fourier transform: $F_\Delta : \mathbb{C}^G \to \mathbb{C}^{\widehat{G}}$
- $F_\Delta(f)(\chi) = \Delta \sum_{g \in G} f(g)\overline{\chi(g)}$,
- $F_\Delta^{-1}(\varphi)(g) = \frac{1}{|G|\Delta} \sum \varphi(\chi)\chi(g)$.

# Harmonic analysis on the nonstandard hulls of hyperfinite abelian groups

- $G$ - a hyperfinite abelian group;
- $G_b \subseteq G$ a $\sigma$-subgroup;
- $G_0 \subseteq G_b$ a $\pi$-subgroup.

- Topology on $G^{\#} = G_b/G_0$
- For $A \subseteq G_0$ put $i(A) = \{a \in A \mid a + G_0 \subseteq A\}$.
- $\mathcal{T} = \{i(F)^{\#} \mid G_0 \subseteq F \subseteq G_b$ and $F$ is internal$\}$. - a base of neighborhoods of zero.

- Proposition

  *The topology $\mathcal{T}$ is locally compact iff for any internal set $F \supset G_0$ and for any internal set $B \subseteq G_b$ there exists standardly finite set $K \subseteq B$ such that $B \subseteq K + F$.*

- Corollary

  *1). For every internal set $F \subseteq G_b$ the set $F^{\#}$ is compact.*
  *2). Every compact set $K \subseteq G^{\#}$ is contained in some such $F^{\#}$.*

- Corollary

  *$K \subseteq G^{\#}$ is a compact open subgroup iff $K = H^{\#}$, where $H \supset G_0$ is an internal subgroup of $G_b$.*

  - If a locally compact group $H$ is topologically isomorphic to $G^{\#}$, then we say that the triple $(G, G_b, G_0)$ represents the $H$
  - $C_0(G^{\#})$ the set of all continuous functions with compact support on $G^{\#}$
  - $C_0(G)$ the set of all internal $S$-continuous functions, whose support is contained in $G_b$.

- Proposition

  *A function $f \in C_0(G^{\#})$ iff there exists an internal function $\varphi \in C_0(G)$ such that $\mathrm{supp}\varphi \subseteq G_b$ and and for every $g \in G_b$ holds*

$$f(g^{\#}) = {}^{\circ}\varphi(g).$$

  *In this case we denote $f$ by $\varphi^{\#}$.*

# Haar integral on $G^{\#}$

- A positive hyperreal number $\Delta$ is a *normalizing multiplier* (n.m.) if for every internal set $F$, $G_0 \subseteq F \subseteq G_b$, holds $^{\circ}(\Delta \cdot |F|) < +\infty$.
- If $\Delta$ is an n.m., then a hyperreal number $\Delta_1$ is an n.m. iff $0 < {}^{\circ}\left(\frac{\Delta_1}{\Delta}\right) < +\infty$.

- Theorem

  *If $\Delta$ is an n.m., then the functional $\mathcal{I}$ on $C_0(G^{\#})$ defined for every $\varphi \in C_0(G)$ by the formula*

$$\mathcal{I}(\varphi^{\#}) = {}^{\circ}I_{\Delta}(f),$$

  *is the Haar integral on $G^{\#}$.*

# Dual group $\widehat{G^{\#}}$

- $\widehat{G}$ – (internal) group dual to $G$;
- $\widehat{G}_b = \{\chi \in \widehat{G} \mid \chi \restriction G_0 \approx 1\}$;
- $\widehat{G}_0 = \{\chi \in \widehat{G} \mid \chi \restriction G_b \approx 1\}$;
- $\widehat{G}^{\#} = \widehat{G}_b / \widehat{G}_0$.
- $\alpha^{\#} \in \widehat{G}^{\#} \longmapsto \psi(\alpha^{\#}) \in \widehat{G^{\#}}$, $\alpha \in \widehat{G}_b$;
- $\psi(\alpha^{\#})(g^{\#}) = {}^{\circ}\alpha(g)$.

- Proposition

  *The mapping $\psi : \widehat{G}^{\#} \to \psi(\widehat{G}^{\#}) \subseteq \widehat{G^{\#}}$ is a topological isomorphism.*

## Theorem

*1). Suppose that there exits an internal subgroup $K \subseteq G_b$, $G_0 \subseteq K$. Then the following statements hold.*

a). $\psi(\widehat{G}^{\#}) = \widehat{G^{\#}}$, *thus* $\widehat{G}^{\#}$ *is canonically isomorphic to* $\widehat{G^{\#}}$.

b). *The hyperreal number* $\widehat{D} = (|G|\Delta)^{-1}$ *is a normalizing multiplier for* $\widehat{G}$

c). *Let* $f \in L_1(G^{\#})$ *and* $\varphi$ *be an S-integrable lifting of* $f$. *Then the Fourier transform on* $G$ $F_\Delta(\varphi)$ *is an S-continuous function on* $\widehat{G}$ *and the linear operator* $\mathcal{F} : L_1(G^{\#}) \to C(\widehat{G^{\#}})$ *defined by the formula*

$$\mathcal{F}(f) = F_\Delta(\varphi)^{\#}$$

*is the Fourier transform on* $G^{\#}$. *The operator defined in the similar way by* $F_\Delta^{-1}$ *is the inverse Fourier transform on* $\widehat{G^{\#}}$.

- **Theorem**
  *For every locally compact group $H$ there exists a triple $(G, G_b, G_0)$ representing $H$ that satisfies the statements a) – c) of the first part of the theorem.*

- **Definition**
  *We say that a hyperfinite group $G$ approximates a locally compact group $H$ if there exist an internal injective map $j : G \to {}^*H$ that satisfies the following conditions:*
  1. $\forall\, h \in H \exists\, g \in G\ (j(g) \approx h)$;
  2. $\forall\, g_1, g_2 \in j^{-1}(ns({}^*H))\ \ (j(g_1 \pm g_2) \approx j(g_1) \pm j(g_2))$.

  *In this case we say that the pair $(G, j)$ is a hyperfinite approximation of $H$.*

  - $(G, j) \longmapsto (G, G_b, G_0)$;
  - $G_b = \{g \in G \mid j(g) \in \mathrm{ns}(H)\}, \qquad G_0 = \{g \in G \mid j(g) \approx 0\}$.

## Hyperfinite representations of locally compact non-commutative groups

- $G$ – a non-commutative hyperfinite group.
- $G_b$– a $\sigma$-subgroup, $G_0 \subseteq G_b$ – a $\pi$-subgroup, which is normal in $G_b$.
- $G^{\#} = G_b/G_0$.
- For $A \subseteq G$ put $i(A) = \{a \in G\ aG_0 \subseteq A\}$.
- $\mathcal{T} = \{i(F)^{\#} \mid G_0 \subseteq F \subseteq G_b$ and $F$ is internal$\}$ form a base of a topology on $G^{\#}$.

- Proposition

  *The topology $\mathcal{T}$ is locally compact iff for any internal set $F \supset G_0$ and for any internal set $B \subseteq G_b$ there exists standardly finite set $K \subseteq B$ such that $B \subseteq K \cdot F$.*

- Corollary

  *1). For every internal set $F \subseteq G_b$ the set $F^{\#}$ is compact.*
  *2). Every compact set $K \subseteq G^{\#}$ is contained in some such $F^{\#}$.*

- Theorem

  *If $\Delta$ is a normalizing multiplier, then the positive functional $\mathcal{I}$ on $C_0(G^{\#})$ defined by the formula $\mathcal{I}(f^{\#}) = {}^{\circ}(\Delta \sum_{g \in G} f(g))$ is left and right Haar integral.*

- Corollary

  *The group $G^{\#}$ is unimodular.*

- Definition

  *A locally compact group $H$ is weakly approximable by finite groups if there exists a triple $(G, G_b, G_0)$ representing $H$. The group $H$ is strongly approximable by finite groups if has a hyperfinite approximation*

  .

- Theorem

  *A compact Lie group $H$ is strongly approximable by finite groups iff it has arbitrary dense finite subgroups.*

## Definition

*We say that a groupoid $(Q, \circ)$ is a quasigroup if for an arbitrary $a, b \in Q$ each of the equations $a \circ x = b$ and $x \circ a = b$ has a unique solution. If it holds only for the first (second) equation, then we say that $(Q, \circ)$ is a left (right) quasigroup.*

- $(Q, \circ)$ a hyperfinite groupoid,
- $Q_b \subseteq Q$ a $\sigma$-subgroupoid,
- $\rho$ a $\pi$-equivalence relation on $Q$, that is a congruence relation on $Q_b$.
- For $A \subseteq Q_b$ put $i(A) = \{q \in Q_b \mid \rho(q) \subseteq A\}$.

## Theorem

*If $Q$ is a left quasigroup and $\Delta$ is a normalizing multiplier, then the positive functional $\mathcal{I}$ on $C_0(Q^\#)$ defined by the formula*

$$\mathcal{I}(f^\#) = {}^\circ\left(\Delta \sum_{q \in Q} f(q)\right)$$

*is left invariant. If $Q$ is a quasigroup, then $\mathcal{I}(f)$ is right invariant*

### Theorem

*1) Every locally compact group is strongly approximable by finite left quasigroups.*

*2) A locally compact group is unimodular iff it is strongly approximable by finite quasigroups*

# Discrete groups

- The topology on $Q^{\#}$ is discrete iff $\rho$ is the equality relation.
- A discrete group $G$ is weakly approximable by a hyperfinite groupoid $Q$ if it is isomorphic to a $\sigma$-subgroupoid of $Q$.
- The group $G$ is strongly approximable by the hyperfinite groupoid $Q$ iff there exists an internal injective map $j : Q \to {}^*G$ such that $j \restriction j^{-1}(G)$ is a homomorphism.

## Theorem

*A discrete group $G$ is amenable iff there exists a hyperfinite set $H$, $G \subseteq H \subseteq {}^*G$, and a binary operation $\circ : H \times H \to H$ that satisfy the following conditions:*

1. $(H, \circ)$ *is a left quasigroup;*
2. $G$ *is a subgroup of the left quasigroup* $(H, \circ)$, *i.e.* $\forall a, b \in G \; a \cdot b = a \circ b$.
3. $\forall a \in G$
$$\frac{|\{h \in H \mid a \cdot h = a \circ h\}|}{|H|} \approx 1$$

.

## Definition

*A discrete group $G$ is sofic iff there exists a hyperfinite set $H$, $G \subseteq H$, and a binary operation $\circ : H \times H \to H$ that satisfy the following conditions:*

1. $(H, \circ)$ *is a left quasigroup;*
2. $G$ *is a subgroup of the left quasigroup* $(H, \circ)$, *i.e.* $\forall a, b \in G \ a \cdot b = a \circ b$.
3. $\forall a, b \in G$

$$\frac{|\{h \in H \mid (a \cdot b) \circ h = a \circ (b \cdot h\}|}{|H|} \approx 1$$

.

Theorem

*(Elek, Szabo) Let $N$ be an infinite hyperreal number and $S_N$ an internal group of permutations of the set $\{1, \ldots, N\}$. Consider its $\pi$ normal subgroup*

$$S_N^{(0)} = \{\alpha \in S_N \mid \frac{\{n \leq N \mid \alpha(n) = n\}|}{N} \approx 1\}.$$

*Then $S(N) = S_N / S_N^{(0)}$ is a simple sofic group. Moreover, a group $G$ is sofic iff it is isomorphic to a subgroup of the group $S(N)$ for some infinite $N$.*

# Hyperfinite representations of topological universal algebras

- $\theta$ a finite signature that contains only functional symbols,
- $\mathcal{A} = \langle A, \theta \rangle$ a hyperfinite algebra of the signature $\theta$.
- $\mathcal{A}_b = \langle A_b, \theta \rangle$ - $\sigma$-subalgebra of $\mathcal{A}$
- $\rho$ a $\pi$-equivalence relation on $A$, that is a congruence relation on $A_b$.
- $a, b \in A$: $\alpha \approx \beta \rightleftharpoons \langle a, b \rangle \in \rho$.
- $\varphi(x_1, \ldots, x_n)$ a first order formula of the signature $\theta$.
- $\varphi_{\approx}$ the formula obtained from $\varphi$ by replacing of every subformula $t_1 = t_2$ by the formula $t_1 \approx t_2$, $t_1, t_2$ are $\theta$-terms.

## Proposition

*For every* $a_1, \ldots a_n \in A_b$

$$\mathcal{A}^{\#} \models \varphi(a_1^{\#}, \ldots, a_n^{\#}) \Longleftrightarrow \mathcal{A}_b \models \varphi_{\approx}(a_1, \ldots a_n).$$

# Hyperfinite representations of reals

- The floating point representation of reals:

$$\alpha = \pm 10^p \times 0.a_1 a_2 \dots, \tag{1}$$

  $p \in \mathbb{Z}$, $0 \leq a_n \leq 9$, $a_1 \neq 0$.
- $P, Q$ hypernatural numbers;
- $A_{PQ}$ the hyperfinite set of all reals of the form (1), where $|p| \leq P$ and the mantissa contains no more than $Q$ decimal digits.
- $\oplus$, $\otimes$ binary operations on $A_{PQ}$, $*$ stands for either $+$ or $\times$
- $\alpha, \beta \in A_{PQ}$: $\alpha * \beta = \pm 10^r \times 0.c_1 c_2 \dots$ .

$$\alpha \circledast \beta = \begin{cases} \pm 10^r \times 0.c_1 c_2 \dots c_Q & \text{if } |r| \le P, \\ \pm 10^P \times 0.\underbrace{99\dots9}_{Q \text{ digits}} & \text{if } r > P, \\ 0 & \text{if } r < -P. \end{cases}$$

- $\mathcal{A}_{PQ}$ the algebra $\langle A_{PQ}, \oplus, \otimes \rangle$
- $(A_{PQ})_b$ consists of all finite hyperreal numbers from $A_{PQ}$
- $\rho$ a restriction of the relation $\approx$ on $\mathbb{R}$ to $A_{PQ}$.
- Then $\mathcal{A}^{\#}_{PQ} \simeq \mathbb{R}$.

## example

$$\begin{aligned} 5x - 7y + 8z &= b \\ 3x - ay + 4z &= 5 \\ ax + 4y - bz &= 2 \end{aligned} \tag{2}$$

Infinitely many solutions iff

$$f(b) = b^4 - 25b^3 + 260b^2 - 2856b + 4288 = 0 \tag{3}$$

and $a$ is found by the formula $p(a, b)$:

$$a = -\frac{21}{29} + \frac{3}{464}b^3 + \frac{5}{464}b^2 - \frac{19}{232}b \tag{4}$$

General solution of the system (2):

$$\begin{aligned} x &= 10 - b + t\left(\tfrac{245}{29} - \tfrac{19}{116}b + \tfrac{5}{232}b^2 + \tfrac{3}{232}b^3\right) \\ y &= t \\ z &= -\tfrac{25}{4} + \tfrac{3}{4}b + t\left(\tfrac{357}{5}8 + \tfrac{95}{928}b - \tfrac{25}{1856}b^2 - \tfrac{15}{1856}b^3\right) \end{aligned} \tag{5}$$

$\Phi(x, y, z, a, b)$ the conjunction of equations of the system (2),
$\Psi(x, y, z, b, t)$ the conjunction of formulas in (5).
Formula $\Gamma$:

$$\forall\, a, b\ (p(a, b) \wedge f(b) = 0 \rightarrow (\exists\, x_1, y_1, z_1, x_2, y_2, z_2((x_1 \neq x_2 \vee y_1 \neq y_2 \vee z$$
$$\wedge \Phi(x_1, y_1, z_1, a, b) \wedge \Phi(x_2, y_2, z_2, a, b))$$
$$\wedge \forall\, x, y, z\, (\Phi(x, y, z, a, b) \rightarrow \exists t \Psi(x, y, z, b, t)))$$

Formula $\Gamma^{(1)}$:

$$\forall\, a, b, a_1, b_1\ (a_1 = a_2 \wedge b_1 = b_2 \wedge p(a_1, b_1) \wedge f(b_1) = 0$$
$$\wedge p(a_2, b_2) \wedge f(b_2) = 0 \rightarrow (\exists\, x_1, y_1, z_1, x_2, y_2, z_2$$
$$((x_1 \neq x_2 \vee y_1 \neq y_2 \vee z_1 \neq z_2) \wedge \Phi(x_1, y_1, z_1, a, b) \wedge \Phi(x_2, y_2, z_2, a, b)),$$

Formula $\Gamma^{(2)}$:

$$\forall, a, b, x, y, z\, (p(a, b) \wedge f(b) = 0 \wedge \Phi(x, y, z, a, b) \rightarrow \exists\, t\, \Psi(x, y, z, b, t)).$$

- $a, b$ with 10 digits: $x = 2.885016341,\ y = 0.6249221609,\ z = -1.038737628,$
- $a, b$ with 12 digits: $x = 1.83282895579,\ y = 0.747271181171,\ z = -0.274065119805,$
- $a, b$ with 15 digits: $x = 1.61877806403204,\ y = 0.772161155406311,\ z = -0.118504584584998824.$

### Theorem

*There does not exist a topological hyperfinite triple* $(\mathcal{A}, \mathcal{A}_b, \rho)$ *such that* $\mathcal{A}$ *and* $\mathcal{A}_b$ *are hyperfinite associative rings and* $\mathcal{A}^{\#}$ *is a locally compact field.*

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in K,\ a \neq 0 \right\}$$

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