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Canard Solutions near a Degenerated Turning Point

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Singularly perturbed differential equation :

$$\varepsilon u' = \Psi(x, u, \alpha, \varepsilon) \qquad (1)$$

$x \in [-1, 1]$, u is a real function, $\alpha \in \mathbb{R}$, $\varepsilon \to 0$ positive

$\Rightarrow$ To simplify notations, $\varepsilon = \oslash$, fixed.

We are studying solutions of perturbed equations that are staying "near" the repulsive part of a slow curve.

Example : case of the equation

$$\varepsilon u' = x^3 u + \alpha + \varepsilon(u^2 + x)$$

PART 1 : Existence of canard solutions

(H1) (1) has a slow curve (α_0, u_0)

$$(\forall x, \Psi(x, u_0(x), \alpha_0, 0) = 0)$$

(H2) $\frac{\partial}{\partial u}\Psi(x, u_0(x), \alpha_0, 0)$ is $\begin{cases} < 0 \text{ if } x < 0 \\ > 0 \text{ if } x > 0 \end{cases}$

p : order of the zero $x = 0$ (p is ODD)

NOTE : *Similar study in the complex case*

(OVERSTABLE solutions)

Main difference :

- $\alpha \in \mathbb{C}^p$ in the complex case
- $\alpha \in \mathbb{R}$ in our case

Restriction of our study to the equations

$$\varepsilon u' = x^p u + \alpha x^L + \sum_i \alpha^{k_i} x^{l_i} + \varepsilon P(x, u, \alpha, \varepsilon) \quad (2)$$

with :

- $L < p$ even
- $k_i \geq 1$ and $l_i \geq L + 1$

THEOREM

"Locally" $\exists ! \alpha^*$ such that the equation

$$\begin{cases} \quad (2) \\ u(-1) = 0 = u(1) \end{cases}$$

has an unique solution $u^* \in \mathcal{C}([-1, 1], \mathbb{R})$ which is limited.

Solutions canard en des points tournants degeneres (*submitted*) [in french]

Demonstration :

Given (β, v), the linear equation

$$\begin{cases} \varepsilon u' = x^p u + \alpha x^L + \sum_i \alpha^{k_i} x^{l_i} + \varepsilon P(x, v, \beta, \varepsilon) \\ u(-1) = 0 = u(1) \end{cases}$$

has an unique solution $(\alpha, u) =: \Xi(\beta, v)$

$\Xi = \mathcal{I} \circ \wp$ with :

- $\wp(\beta, v)(x) := P(x, v(x), \beta, \varepsilon)$
- $\mathcal{I}$: linear operator such that $\mathcal{I}(w)$ solution of

$$\begin{cases} \varepsilon u' = x^p u + \alpha x^L + \sum_i \alpha^{k_i} x^{l_i} + \varepsilon w \\ u(-1) = 0 = u(1) \end{cases}$$

Ξ is a $(\pounds \varepsilon^{1/(p+1)})$-Lipschitz function

(α^*, u^*) : fixed point iteration of Ξ.

PART 2 : Asymptotic expansion

current work

- Ξ is a $(\pounds\varepsilon^{1/(p+1)})$-Lipschitz function
- $(\alpha^*, u^*) = \lim_{n\to+\infty}(\alpha_n, u_n)$

$\Rightarrow$

$$u^* = \sum_{n\geq 1} (u_n - u_{n-1})$$

where $\forall n$, $u_n - u_{n-1} = \pounds\varepsilon^{n/(p+1)}$.

Existence and **uniqueness** of an $\varepsilon^{1/(p+1)}$-asymptotic expansion for u^* ?

The "natural" $\varepsilon^{1/(p+1)}$-asymptotic expansion

$u^*(x) \approx \sum_k u_k(x)\varepsilon^{k/(p+1)}$, with u_k analytic in x ,

isn't suffisant :

0 can be a pole of the coefficients u_k.

We allow u_k to be analytic in x and in intermediary function(s) φ :

$$u^*(x) \approx \sum_k u_k(x, \varphi(x, \varepsilon))\varepsilon^{k/(p+1)}$$

Some possible choices for φ :

$$e^{-x^{p+1}/\varepsilon} \text{ , } (\mathcal{I}(x), ..., \mathcal{I}(x^p))$$

Note that :

$$||e^{-x^{p+1}/\varepsilon}|| = 1$$

$$||xe^{-x^{p+1}/\varepsilon}|| = \frac{e^{-1/(p+1)}}{(p+1)^{p+1}}\varepsilon^{1/(p+1)}$$

So, $x^i\varphi^j\varepsilon^{l/(p+1)}$ and $\varepsilon^{l/(p+1)}$ have possibly not the same "place" in the expansion.

We have to "ordered" the monomials

$$x^i \varphi^j \varepsilon^{l/(p+1)}$$

with respect to their estimates in $\varepsilon^{1/(p+1)}$.

$\Rightarrow$ Definition of an "order" :

$$\maltese(x^i \varphi^j \varepsilon^{l/(p+1)}) := \left(\frac{\ln \|x^i \varphi(x,\varepsilon)^j \varepsilon^{l/(p+1)}\|_x}{\ln \varepsilon} \right)^o$$

Set-up of a structure of gradued algebra $(\mathcal{A}_k)_k$ such that

$$\mathcal{A}_k = \mathrm{Vect}\{x^i \varphi^j \varepsilon^{l/(p+1)};\ \maltese(x^i \varphi^j \varepsilon^{l/(p+1)}) \leq k\}$$

$\mathcal{A}_k$ will be the set of the principal term with order k for the $\varepsilon^{1/(p+1)}$-asymptotic expansion of u^*.

Implementation in the case $\underline{p=0}$ ***:***

NOT a study of a canard solution !

Study of a limit layer with an attractive slow curve for $x \in [0,1]$.

$\varphi(x,\varepsilon) = e^{-x/\varepsilon}$, and $\maltese(x^i\varphi^j\varepsilon^l) = \begin{cases} i \text{ if } j=0 \\ i+l \text{ if } j>0 \end{cases}$

$$\begin{cases} \varepsilon u' = -u + \varepsilon P(x,u,\varepsilon) \\ u(0) = u_0 \neq 0 \end{cases}$$

$$u^*(x) \approx \sum_{i,l} a_{i,0,l} x^i \varepsilon^l + \sum_{i,l,j\geq 1} a_{i,j,l} \left(\frac{x}{\varepsilon}\right)^i \varphi(x,\varepsilon)^j \varepsilon^{l+i}$$

which is a particular form of a Combined Asymptotic Expansion :

$$u^*(x) \approx \sum_n \left(f_n(x) + g_n\left(\frac{x}{\varepsilon}\right)\right) \varepsilon^n$$

where for all n, f_n is analytic, and g_n is an exponentially decreasing function.

What about the "canard situation" case ?

\+ $p = 1$: Non degenerated turning point
→ No intermediary function needed
already countain in the overstable theory

\+ $p \geq 3$: degenerated turning point
→ Definition of adapted intermediary function done
* **if P is linear in u** :
study is done
* **if P is not necessary linear in u** :
problem to solve :
complete study of the interactions (multiplication+composition) of the intermediary functions.