Iterated Ultrapowers and Automorphisms

Ali Enayat

Pisa, May 2006

Our story begins with:

- *Question* (Häsenjäger): Does PA have a model with a nontrivial automorphism?

- *Answer* (Ehrenfeucht and Mostowski): Yes, indeed given any first order theory T with an infinite model $\mathfrak{M} \models T$, and any linear order $\mathbb{L}$, there is a model $\mathfrak{M}_{\mathbb{L}}$ of T such that

$$Aut(\mathbb{L}) \hookrightarrow Aut(\mathfrak{M}_{\mathbb{L}}).$$

- *Corollaries:*

(a) PA, RCF, and ZFC have models with rich automorphism groups.

(b) Nonstandard models of analysis with rich automorphism groups exist.

The EM Theorem via Iterated Ultrapowers (1)

- Gaifman saw a radically different proof of the EM Theorem: iterate the ultrapower construction along a prescribed linear order.

- Suppose

  (a) $\mathfrak{M} = (M, \cdots)$ is a structure,

  (b) $\mathcal{U}$ is an ultrafilter over $\mathcal{P}(\mathbb{N})$, and

  (c) $\mathbb{L}$ is a linear order.

  we wish to describe the $\mathbb{L}$-iterated ultrapower

$$\mathfrak{M}^* := \prod_{\mathcal{U},\mathbb{L}} \mathfrak{M}.$$

The EM Theorem via Iterated Ultrapowers, Continued (2)

- A key definition (reminiscent of Fubini):

$$\mathcal{U}^2 := \{X \subseteq \mathbb{N}^2 : \{a \in \mathbb{N} : \overbrace{\{b \in \mathbb{N} : (a, b) \in X\}}^{(X)_a} \in \mathcal{U}\} \in \mathcal{U}.$$

- More generally, for each $n \in \mathbb{N}^+$:

$$\mathcal{U}^{n+1} := \{X \subseteq \mathbb{N}^{n+1} : \{a \in \mathbb{N} : (X)_a \in \mathcal{U}^n\} \in \mathcal{U}\},$$

  where

$$(X)_a := \{(b_1, \cdots, b_n) : (a, b_1, \cdots, b_n) \in X\}$$

- Let Υ be the set of terms τ of the form

$$f(l_1, \cdots, l_n),$$

where $n \in \mathbb{N}^+$, $f : \mathbb{N}^n \to M$ and

$$(l_1, \cdots, l_n) \in [\mathbb{L}]^n.$$

- The universe M^* of $\mathfrak{M}^*$ consists of equivalence classes $\{[\tau] : \tau \in \Upsilon\}$, where the equivalence relation $\sim$ at work is defined as follows: given $f(l_1, \cdots, l_r)$ and $g(l_1^{'}, \cdots, l_s^{'})$ from Υ, first suppose that

$$\left(l_1, \cdots, l_r, l_1^{'}, \cdots, l_s^{'}\right) \in [\mathbb{L}]^{r+s};$$

let $p := r + s$, and define: $f(l_1, \cdots, l_r) \sim g(l_1^{'}, \cdots, l_s^{'})$ iff:

$$\{(i_1, \cdots, i_p) \in \mathbb{N}^p : f(i_1, \cdots, i_r) = g(i_{r+1}, \cdots, i_p)\} \in \mathcal{U}^p.$$

More generally:

- Given $f(l_1, \cdots, l_r)$ and $g(l_1^{'}, \cdots, l_s^{'})$ from Υ, let

$$P := \{l_1, \cdots, l_r\} \cup \{l_1^{'}, \cdots, l_s^{'}\}, \quad p := |P|,$$

  and relabel the elements of P in increasing order as $\bar{l}_1 < \cdots < \bar{l}_p$. This relabelling gives rise to increasing sequences $(j_1, j_2, \cdots, j_r)$ and $(k_1, k_2, \cdots, k_s)$ of indices between 1 and p such that

$$l_1 = \bar{l}_{j_1}, l_2 = \bar{l}_{j_2}, \cdots, l_r = \bar{l}_{j_r}$$

  and

$$l_1^{'} = \bar{l}_{k_1}, l_2^{'} = \bar{l}_{k_2}, \cdots, l_s^{'} = \bar{l}_{k_s}.$$

  Then define: $f(l_1, \cdots, l_r) \sim g(l_1^{'}, \cdots, l_s^{'})$ iff

$$\{(i_1, \cdots, i_p) \in \mathbb{N}^p : f(i_{j_1}, \cdots, i_{j_r}) = g(i_{k_1}, \cdots, i_{k_s})\} \in \mathcal{U}^p.$$

The EM Theorem via Iterated Ultrapowers (5)

- We can also use the previous relabelling to define the operations and relations of $\mathfrak{M}^*$ as follows, e.g.,

$$[f(l_1, \cdots, l_r)] \odot^{\mathfrak{M}^*} [g(l_1', \cdots, l_s')] := [v(\bar{l}_1, \cdots, \bar{l}_p)]$$

where $v : \mathbb{N}^n \to M$ by

$$v\,(i_1, \cdots, i_p) := f(i_{j_1}, \cdots, i_{j_r}) \odot^{\mathfrak{M}} g(i_{k_1}, \cdots, i_{k_s});$$

$[f(l_1, \cdots, l_r)] \lhd^{\mathfrak{M}^*} [g(l_1', \cdots, l_s')]$ iff

$$\{(i_1, \cdots, i_p) \in \mathbb{N}^p : f(i_{j_1}, \cdots, i_{j_r}) \lhd^{\mathfrak{M}^*} g(i_{k_1}, \cdots, i_{k_s})\} \in \mathcal{U}^p.$$

The EM Theorem via Iterated Ultrapowers (6)

- For $m \in M$, let c_m be the constant m-function on $\mathbb{N}$, i.e., $c_m : N \to \{m\}$. For any $l \in \mathbb{L}$, we can identify the element $[c_m(l)]$ with m.
- We shall also identify $[id(l)]$ with l, where $id : \mathbb{N} \to \mathbb{N}$ is the identity function (WLOG $\mathbb{N} \subseteq M$).
- Therefore $M \cup \mathbb{L}$ can be viewed as a subset of M^*.

- *Theorem.* For every formula $\varphi(x_1, \cdots, x_n)$, and every $(l_1, \cdots, l_n) \in [\mathbb{L}]^n$:

$$\mathfrak{M}^* \models \varphi(l_1, l_2, \cdots, l_n) \Longleftrightarrow$$

$$\{(i_1, \cdots, i_n) \in \mathbb{N}^n : \mathfrak{M} \models \varphi(i_1, \cdots, i_n)\} \in \mathcal{U}^n.$$

The EM Theorem via Iterated Ultrapowers (7)

- *Corollary 1.* $\mathfrak{M} \prec \mathfrak{M}^*$, and $\mathbb{L}$ is a set of order indiscernibles in $\mathfrak{M}^*$.

- *Corollary 2.* Every automorphism j of $\mathbb{L}$ lifts to an automorphism $\hat{j}$ of $\mathfrak{M}^*$ via

$$\hat{j}([f(l_1, \cdots, l_n)]) = [f(j(l_1), \cdots, j(l_n))].$$

  Moreover, the map

$$j \mapsto \hat{j}$$

is a *group embedding* of $Aut(\mathbb{L})$ into $Aut(\mathfrak{M}^*)$.

Skolem-Gaifman Ultrapowers (1)

- If $\mathfrak{M}$ has definable Skolem functions, then we can form the *Skolem ultrapower*

$$\prod_{\mathcal{F},\mathcal{U}} \mathfrak{M}$$

as follows:

(a) Suppose $\mathcal{B}$ is the Boolean algebra of parametrically definable subsets of M, and $\mathcal{U}$ is an ultrafilter over $\mathcal{B}$.

(b) Let $\mathcal{F}$ be the family of functions from M into M that are parametrically definable in $\mathfrak{M}$.

(c) The universe of the $\mathfrak{M}^*$ is

$$\{[f] : f \in \mathcal{F}\},$$

where

$$f \sim g \Longleftrightarrow \{m \in M : f(m) = g(m)\} \in \mathcal{U}$$

- *Theorem* (MacDowell-Specker) Every model of PA has an elementary end extension.

  Proof: for an appropriate choice of $\mathcal{U}$,

$$\mathfrak{M} \prec_e \prod_{\mathcal{F},\mathcal{U}} \mathfrak{M}.$$

- For models of some Skolemized theories, such as PA, the process of ultrapower formation can be iterated along any linear order.
- For each parametrically definable $X \subseteq M$, and $m \in M$,

$$(X)_m = \{x \in M : \langle m, x \rangle \in X\}.$$

- $\mathcal{U}$ is an *iterable* ultrafilter over $\mathcal{B}$ if for every definable $X \subseteq M$, $\{m \in M : (X)_m \in \mathcal{U}\}$.

Skolem-Gaifman Ultrapowers (3)

- Theorem (Gaifman) If $\mathcal{U}$ is iterable, and $\mathbb{L}$ is a linear order, then

$$\mathfrak{M} \prec_{e,cons} \prod_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}.$$

- *Theorem* (Gaifman). For an appropriate choice of iterable $\mathcal{U}$,

  (a) $Aut(\prod_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}; M) \cong Aut(\mathbb{L})$.

  (b) $\prod_{\mathcal{F},\mathcal{U},\mathbb{L}} \mathfrak{M}$ has an automorphism j such that

$$\mathit{fix}(j) = M.$$

- *Theorem* (Schmerl). Suppose $G \leq Aut(\mathbb{L})$ for some linear order $\mathbb{L}$.

  (a) $G \cong Aut(\mathfrak{M})$ for some $\mathfrak{M} \models PA$.

  (b) $G \cong Aut(\mathbb{F})$ for some ordered field $\mathbb{F}$.

Automorphisms of Countable Recursively Saturated Models of PA (1)

- A cut I of $\mathfrak{M} \models PA$ is an initial segment of M with no last element.
- For a cut I of $\mathfrak{M}$, $SSy_I(\mathfrak{M})$ is the collection of sets of the form $X \cap I$, where X is parametrically definable in $\mathfrak{M}$.
- *I is strong in* $\mathfrak{M}$ iff $(\mathbf{I}, SSy_I(\mathfrak{M})) \models ACA_0$.
- $\mathfrak{M}$ is *recursively saturated* if for every $\mathbf{m} \in M$, every recursive finitely realizable type over $(\mathfrak{M}, \mathbf{m})$ is realized in $\mathfrak{M}$.
- For $j \in Aut(\mathfrak{M})$,

$$I_{fix}(j) := \{x \in dom(j) : \forall y \leq x \; j(y) = y\},$$

$$\mathit{fix}(j) := \{x \in M : j(x) = x\}$$

Automorphisms of Countable Recursively Saturated Models of PA (2)

Suppose $\mathfrak{M} \models PA$ is ctble, rec. sat., and I is a cut of $\mathfrak{M}$.

- *Theorem* (Smoryński) $I = I_{fix}(j)$ for some $j \in Aut(\mathfrak{M})$ iff I is closed under exponentiation.

- *Theorem* (Kaye-Kossak-Kotlarski) $I = \mathit{fix}(j)$ for some $j \in Aut(\mathfrak{M})$ iff I is a strong elementary submodel of $\mathfrak{M}$.

Automorphisms of Countable Recursively Saturated Models of PA (3)

- *Theorem* (Kaye-Kossak-Kotlarski)

$$\overbrace{\mathfrak{M} is arithmetically saturated}^{\mathbb{N} is strong in \mathfrak{M}} \quad \text{iff} \quad \text{for some } j \in Aut(\mathfrak{M}),$$

$$\overbrace{\text{fix}(j) is the collection of definable elements of \mathfrak{M}}^{j is maximal}.$$

- *Theorem* (Schmerl) $Aut(\mathbb{Q}) \hookrightarrow Aut(\mathfrak{M})$.

Automorphisms of Countable Recursively Saturated Models of PA (4)

- *Theorem* (E). If I is a closed under exponentiation, then there is a group embedding

$$j \mapsto \hat{\jmath}$$

from $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ such that:

(a) $I_{fix}(\hat{\jmath}) = I$ for every nontrivial $j \in Aut(\mathbb{Q})$;

(b) $\mathrm{fix}(\hat{\jmath}) \cong \mathfrak{M}$ for every fixed point free $j \in Aut(\mathbb{Q})$.

- *Idea of the proof:* Fix $c \in M \backslash I$, let $\overline{c} := \{x \in M : x < c\}$, $\mathcal{B} := \mathcal{P}^{\mathfrak{M}}(\overline{c})$, and $\mathcal{F}$ be the family of functions from $(c)^n \to M$ that are coded in $\mathfrak{M}$. For an appropriate choice of $\mathcal{U}$,

$$\mathfrak{M} \cong \prod_{\mathcal{F},\mathcal{U},\mathbb{Q}} \mathfrak{M} over I.$$

This sort of iteration was implicitly considered by Mills and Paris.

Automorphisms of Countable Recursively Saturated Models of PA (5)

- A new type of iteration that subsumes both Gaifman and Paris-Mills iteration: starting with

$$\boxed{I \subseteq_e \mathfrak{M} \preceq \mathfrak{N}, with I \subseteq_{strong} \mathfrak{N},}$$

(a) $\mathcal{F} = \{f \restriction I^n : f$ par. definable in $\mathfrak{N}\}$;
(b) $\mathcal{B} := SSy_I(\mathfrak{N})$;
(c) $\mathcal{U}$ an appropriate ultrafilter over $\mathcal{B}$.

- *Theorem* (E). Suppose $\mathfrak{M}$ is arithmetically saturated. There is a group embedding

$$j \mapsto \hat{\jmath}$$

from $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ such that $\hat{\jmath}$ is maximal for every fixed point free $j \in Aut(\mathbb{Q})$.

Automorphisms of Countable Recursively Saturated Models of PA (6)

- *Conjecture* (Schmerl). Suppose $\mathfrak{M}$ is arithmetically saturated, and $\mathfrak{M}_0 \prec \mathfrak{M}$. Then $\mathit{fix}(j) \cong \mathfrak{M}_0$ for some $j \in Aut(\mathfrak{M})$.

- *Theorem* (Kossak) Every countable model of PA is isomorphic to some $\mathit{fix}(j)$, for some $j \in Aut(\mathfrak{M})$, and some countable arithmetically saturated model $\mathfrak{M}$.

- *Theorem* (Kossak) The cardinality of

$$\{\, \mathit{fix}(j) : j \in Aut(\mathfrak{M})\} \;/ \cong$$

  is either $2^{\aleph_0}$ or 1, depending on whether $\mathfrak{M}$ is arithmetically saturated or not.

- *Theorem* (E). Suppose $\mathfrak{M}_0 \prec \mathfrak{M}$, and $\mathfrak{M}$ is arithmetically saturated. There are $\mathfrak{M}_1 \prec \mathfrak{M}$ with $\mathfrak{M}_0 \cong \mathfrak{M}_1$, and an embedding $j \mapsto \hat{j}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$, such that $\mathrm{fix}(\hat{\jmath}) = \mathfrak{M}_1$ for every fixed point free $j \in Aut(\mathbb{Q})$.

- Suppose I is a proper cut of $\mathfrak{M}$. A subset X of M is *I-coded* in $\mathfrak{M}$, if for some $c \in M$, $X = \{(c)_i : i \in I\}$, and for all distinct i and j in I, $(c)_i \neq (c)_j$.
- I is I-coded in $\mathfrak{M}$.
- The collection of definable elements of $\mathfrak{M}$ is $\mathbb{N}$-coded in $\mathfrak{M}$.
- *Theorem* Suppose $I \subseteq_{strong} \mathfrak{M}$, $\mathfrak{M}_0 \prec \mathfrak{M}$ and M_0 is I-coded in $\mathfrak{M}$. Then,

  (a) There is an embedding $j \mapsto \hat{\jmath}$ of $Aut(\mathbb{Q})$ into $Aut(\mathfrak{M})$ such that $\text{fix}(\hat{\jmath}) = M_0$ for every fixed point free $j \in Aut(\mathbb{Q})$;

  (b) Moreover, if j is expansive on $\mathbb{Q}$, then $\hat{\jmath}$ is expansive on $M \backslash \overline{M_0}$.

Automorphisms and Foundations (1)

- Strong foundational axiomatic systems can be characterized in terms of the fixed point sets of automorphisms of models of weak foundational systems.

- The above phenomenon sheds light on the close relationship between orthodox foundational systems, and the Quine-Jensen system NFU of set theory with a universal set.

- *Weak arithmetical system:*

  I-Δ_0 (bounded arithmetic).

- *Strong arithmetical systems*:

  $I\Delta_0 + Exp + B\Sigma_1$,

  WKL_0^*,

  PA,

  ACA_0,

  $Z_2 + \Pi^1_\infty$-DC.

Automorphisms and Foundations (2)

- *Weak set theoretical system:* Set theories no stronger than KP (Kripke-Platek).

- *Strong set theoretical systems:*

  KP^{Power},

  $ZFC + \Phi$,

  GBC + "**Ord** is w. compact",

  KMC + "**Ord** is w. compact"$+\Pi^1_\infty$-DC.

- *Theorem* (E). The following are equivalent for a model $\mathfrak{M}$ of the language of arithmetic:

  (a) $M = fix(j)$ for some $j \in Aut(\mathfrak{M}^*)$, where $\mathfrak{M} \subset_e \mathfrak{M}^* \models I\text{-}\Delta_0$.

  (b) $\mathfrak{M} \models PA$.

- *Theorem* (E). The following are equivalent for a model $\mathfrak{M}$ of the language of arithmetic:

  (a) $M = I_{fix}(j)$ for some $j \in Aut(\mathfrak{M}^*)$, where $\mathfrak{M} \subset_e \mathfrak{M}^* \models I\text{-}\Delta_0$.

  (b) $\mathfrak{M} \models I\Delta_0 + Exp + B\Sigma_1$,

  where $Exp := \forall x \exists y\ 2^x = y$, and $B\Sigma_1(\mathcal{L})$ is the scheme consisting of the universal closure of formulae of the form

$$[\forall x < a \exists y \overbrace{\varphi(x,y)}^{\Delta_0}] \rightarrow [\exists z \forall x < a \exists y < z \varphi(x,y)].$$

- *Theorem* (E). The following two conditions are equivalent for a countable model $(\mathfrak{M}, \mathcal{A})$ of the language of second order arithmetic:

  (a) $\mathfrak{M} = I_{fix}(j)$ for some nontrivial $j \in Aut(\mathfrak{M}^*)$, $\mathfrak{M}^* \models I\Delta_0$ and $\mathcal{A} = SSy_M(\mathfrak{M}^*)$.

  (b) $(\mathfrak{M}, \mathcal{A}) \models WKL_0^*$.

- WKL_0^* is a weakening of the well-known subsystem WKL_0 of second order arithmetic in which the Σ_1^0-induction scheme is replaced by $I\Delta_0 + Exp$.

- WKL_0^* was introduced by Simpson and Smith who proved that $I\Delta_0 + Exp + B\Sigma_1$ is the first order part of WKL_0^* (in contrast to WKL_0, whose first order part is $I\Sigma_1$).

- Suppose $\mathfrak{M} \subseteq \mathfrak{M}^* \models I\Delta_0$. An automorphism j of $\mathfrak{M}^*$ is *M-amenable* if $M = fix(j)$, and for every formula $\varphi(x, j)$ in the language $\mathcal{L}_A \cup \{j\}$, possibly with suppressed parameters from M^*,

$$\{m \in M : (\mathfrak{M}^*, j) \models \varphi(m, j)\} \in SSy_M(\mathfrak{M}^*).$$

- *Theorem* (E). *If* $\mathfrak{M} \subseteq_e \mathfrak{M}^* \models I\Delta_0$, and $j \in Aut(\mathfrak{M}^*)$ is M-amenable, then

$$(\mathfrak{M}^*, SSy_M(\mathfrak{M}^*)) \models Z_2.$$

- *Theorem* (E). Suppose $(\mathfrak{M}, \mathcal{A})$ is a countable model of $Z_2 + \Pi^1_\infty$-DC. There exists an e.e.e. $\mathfrak{M}^*$ of $\mathfrak{M}$ that has an M-amenable automorphism j such that $SSy_M(\mathfrak{M}^*) = \mathcal{A}$, where Π^1_∞-DC is the scheme of formulas of the form

$$\forall n\ \forall X\ \exists Y\ \theta(n, X, Y) \rightarrow$$

$$[\forall X\ \exists Z\ (X = (Z)_0 \text{ and } \forall n\ \theta(n, (Z)_n, (Z)_{n+1}))].$$

- $EST(\mathcal{L})$ [Elementary Set Theory] is obtained from the usual axiomatization of $ZFC(\mathcal{L})$ by deleting Power Set and $\Sigma_\infty(\mathcal{L})$-Replacement, and adding $\Delta_0(\mathcal{L})$-Separation.

- GW [Global Well-ordering] is the axiom expressing "$\lhd$ well-orders the universe".

- GW^* is the strengthening of GW obtained by adding the following two axioms to GW:

  (a) $\forall x \forall y (x \in y \rightarrow x \lhd y)$;

  (b) $\forall x \exists y \forall z (z \in y \longleftrightarrow z \lhd x)$.

- $\Phi := \{\exists\kappa(\kappa$ is n-Mahlo and V_κ is a Σ_n-elementary submodel of $\mathbf{V}) : n \in \omega\}$.

- *Theorem* (E). The following are equivalent for a model $\mathfrak{M}$ of the language $\mathcal{L} = \{\in, \lhd\}$.

  (a) $M = fix(j)$ for some $j \in Aut(\mathfrak{M}^*)$, where $\mathfrak{M} \subset_\lhd \mathfrak{M}^* \models EST(\mathcal{L}) + GW^*$.

  (b) $\mathfrak{M} \models ZFC + \Phi$.

$$\boxed{\frac{I\text{-}\Delta_0}{PA} \quad \sim \quad \frac{EST(\mathcal{L})+GW^*}{ZFC+\Phi}}$$