On a possible set of axioms for the external numbers of Nonstandard Analysis

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Sorites (*Soros*=heap) Paradox

Would you describe a single grain of wheat as a heap? No. Would you describe two grains of wheat as a heap? No. ... You must admit the presence of a heap sooner or later, so where do you draw the line?

- *Ars negligendi longa vita brevis*

(Van der Corput)

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- An external number is the algebraic sum of a real number and a neutrix.
- The symbol $\mathfrak{N}$ represents the set of all neutrices and the symbol $\mathbb{E}$ represents the set of all external numbers.

Definitions (2)

- The **sum** and the **product** in $\mathfrak{N}$ are defined by:
- '+': $\mathfrak{N} \times \mathfrak{N} \to \mathfrak{N}$

$$(A, B) \;\to\; A + B = \{a + b \mid (a, b) \in A \times B\}$$

- '.': $\mathfrak{N} \times \mathfrak{N} \to \mathfrak{N}$

$$(A, B) \;\to\; AB = \{ab \mid (a, b) \in A \times B\}$$

Definitions (3)

- Given two external numbers $\boldsymbol{x=a+M}$, $\boldsymbol{y=b+N}$ *we* define their **sum** by:
- '+': $\mathbb{E} \times \mathbb{E} \to \mathbb{E}$

$$(x,y) \rightarrow x+y=(a+b)+M \cup N$$

- And their **product** by:
- '.': $\mathbb{E} \times \mathbb{E} \to \mathbb{E}$

$$(x,y) \rightarrow x.y=(ab)+MN \cup aN \cup bM$$

Axioms

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