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Nonstandard Averaging and Signal Processing

E.Benoît Université de La Rochelle

IST framework.

Notations :

$\mathbf{R}$ for the (hyper)reals

$\underline{\mathbf{R}}$ for the standard reals (external set)

$\pounds$ for the limited real numbers

Point of view for applications :

The *natural* objects are modelised by internal elements. In all the talk, f (the signal) will be a given internal function.

Aim of the talk :

Revisit averaging theory for application to signal processing.

M. Fliess, a specialist in control theory and signal processing, hopes that averaging can give new methods to study noise in signal processing.

I - Averaging

C. Reder (1985), P.Cartier and Y. Perrin (1995)

A. Robinson, P. Loeb, etc... in *ANS-language

T : a (hyper)finite set.

m : a measure on it : $m : T \to \mathbf{R}^+$

d : a distance on it : $d : T \times T \to \mathbf{R}^+$

For internal $A \subset T$, we write $m(A) := \sum_{t \in A} m(t)$.

Internal subset A is rare iff $m(A) \simeq 0$.

External subset A is rare iff

$$\forall^{st} \varepsilon > 0 \quad \exists U \subset T \quad m(U) < \varepsilon$$

σ-additivity : if $(A_n), (n \in \underline{\mathbf{N}})$ is an external sequence of external rare sets, then $\bigcup_{n \in \underline{\mathbf{N}}} A_n$ is rare.

For f in $\mathbf{R}^T$ and internal A, we write

$$\int_A f\,dm \;:=\; \sum_{t\in A} f(t)m(t)$$

Problem (C. Reder):

Define (if it is possible) an external function $\tilde{f} : X \in \underline{\mathbf{R}}$ such that

$$\tilde{f}(t) \;\simeq\; \frac{1}{|\mathit{hal}(t)|}\int_{|\mathit{hal}(t)|} f\,dm$$

If T is included in some standard set E, then $\tilde{f}$ would be a standard function on E.

Examples :

for $T \subset \mathbf{R}$, $m(t_k) = t_{k+1} - t_k = dt$, and usual distance :

if f is S-continuous, then $\tilde{f} = {}^{\circ}f$.

if $f(t) = \sin(\omega t)$, $\omega \simeq \infty$, then $\tilde{f} = 0$.

if f =Heaviside, then $\tilde{f}$ can not be defined on 0.

if $f(t) = \pm 1$ with independent random variables, then $\tilde{f} = 0$ almost surely.

Cartier-Perrin article :

$$S(T) \ := \ \left\{ f \in \mathbf{R}^T \ , \ \int |f| dm = \pounds \right\}$$

$$SL^1(T) \ := \ \left\{ f \in S(T) \ , \ m(A) \simeq 0 \Rightarrow \int_A f\, dm \simeq 0 \right\}$$

Theorem 1 (Radon-Nykodym: Let f be in $S(T)$. Then there exist g and k such that $f = g + k$, $g \in SL^1(T)$ and $k = 0$ almost everywhere.

The proof is *constructive*: if λ is infinitely large, but small enough (in an other level in RIST axiomatic ?), $g = f\chi_{|f|<\lambda}$ is convenient.

At this point only we introduce metric d and topology.

$$L^1(T) := \Big\{ f \in SL^1(T) \ , \ \exists^{ext} A \text{ rare} \ , \\ f \ S\text{-continuous on } T - A \Big\}$$

$A \subset T$ is quadrable iff $\mathit{hal}(A) \cap \mathit{hal}(T - A)$ is rare.

A function h is quickly oscillating iff it is in $SL^1(T)$ and for all quadrable set A we have $\int_A h \, dm \simeq 0$.

Examples : $h(t_k) = (-1)^k$,
$h(t_k) = \sin(\omega t_k)$ (ω unlimited, $\omega \not\simeq 0 \bmod 2\pi/dt$)

Theorem 2 : Let f be in $SL^1(T)$. Then there exist g and h such that $f = g + h$, $g \in L^1(T)$ and h is quickly oscillating.

The idea of the proof is interesting because it shows that the studied notions persist if we replace T by a subset of it. It explain why g is the average of f.

Let $\mathcal{P}$ a partition of T. We define $E^{\mathcal{P}}(f)$ by $E^{\mathcal{P}}(f)(t) = \frac{1}{m(A)}\int_A f dm$ where A is the atom of $\mathcal{P}$ containing t.

We say that f_n is a martingale of f if $f_n = E^{\mathcal{P}_n}$ where $\mathcal{P}_n$ is a family of partitions such that

• The partition $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$.

• For all limited n, every subset of limited diameter in T is covered by a limited number of atoms of $\mathcal{P}_n$.

• For all limited n, all atoms of $\mathcal{P}_n$ are quadrable.

• For all unlimited n, all atoms of $\mathcal{P}_n$ have infinitesimal diameter.

The existence of martingales needs some additional hypothesis of local compacity:
For every appreciable r, every subset of T with limited diameter can be covered by a limited number of subsets of diameter less than r.

Let f be a function in $SL^1(T)$. Let f_n a martingale of f. Then one can prove that if n is unlimited but small enough (in an intermediate level in RIST axiomatic ?), f_n is in $L^1(T)$ and $f - f_n$ is quickly oscillating.

The decomposition $f = g + h$ is almost unique, i.e. if $f = g_1 + h_1 = g_2 + h_2$ with g_1, g_2 in $L^1(T)$ and with h_1 and h_2 quickly oscillating, then $g_1 \simeq g_2$ and $h_1 \simeq h_2$ almost everywhere.

Conclusion

If $\int |f| dm = £$, there exist g, h, k such that

- $f = g + h + k$
- $g \in L^1$ i.e. g is S-continuous on the complementary of a rare set and $\int_A f \, dm \simeq 0$ on every set A of infinitesimal measure.
- h is quickly oscillating
- $k = 0$ almost everywhere.

II - Signal processing

A signal is the output of a physical instrument. He pretends to measure some physical quantity. It is often digital i.e. discrete.

Let us give $T = \{t_1, t_2, \ldots, t_N\}$ the instants of measure. They are not known exactly. Let us give also a weight $m(t_k)$ at all these instants. We could choose $m(t_k) = t_{k+1} - t_k$, but we have to fix m even if the instants are not known.

The operational calculus is very common in the community of automaticians. With Laplace transform, all the computations on functions of t are replaced by computations on functions of s (the adjoint variable). The operational calculus is very well adapted for two reasons : the linear autonomous differential operators are replaced by rational operators, and the frequence are directly readable : a frequence of the signal $f(t)$ is the imaginay part of a pole of the Laplace transform $F(s)$.

M. Fliess has developed a new algebraic theory in the operational calculus. It is based on differential extension of differentiable fields. For example, a parameter can be estimated sometimes as the solution of an equation in the differential field.

I will know present transformations in frequence domain of the Cartier-Perrin theorems.

Let us give $T = \{t_1, t_2, \ldots, t_N\}$ an increasing sequence of real positive numbers. Let us give a measure m on T. The distance is the usual distance.

We assume : for all k, *hal*(t_k) is a rare set.

For a function f element of $\mathbf{R}^T$, we define the Laplace transform F by

$$F(s) \; = \; \sum_{t \in T} f(t) \, e^{-st} m(t)$$

The function F (as an internal function on $\mathbb{C}$) is analytic. The limit of F is 0 when $\Re\mathrm{e}(s)$ tends to infinity. The derivative of F is the Laplace transform of $-tf$.

Proposition (Callot) : If F is analytic and limited in the S-interior of a standard domain D, then there exists a standard analytic function ${}^{\circ}F$ defined on D with $F(x) \simeq {}^{\circ}F(x)$ for all x in the S-interior of D.

Proposition 1: If $f \in S(T)$ then $F(s)$ is limited for $\Re\mathrm{e}(s) \geq 0$.

Obvious : $|F(s)| \leq \int |f(t)| dm$

Then there exists a standard function (unique) ${}^{\circ}F$ analytic in the half-plane $\Re\mathrm{e}(s) > 0$ with ${}^{\circ}F(s) \simeq F(s)$ while $\Re\mathrm{e}(s) \underset{\not\simeq}{>} 0$. The following questions concern the equivalence between properties of the functions f and ${}^{\circ}F$ even if the t_k are not regular.

Proposition 2 (EB): If f is quickly oscillating then
$F(s) \simeq 0$ while $\Re\mathrm{e}(s) > 0$ and $\frac{\Im\mathrm{m}(s)}{\Re\mathrm{e}(s)}$ limited.

Corollary : ${}^{\circ}F = 0$.

Example : $f(t) = \sin \omega t$. When ω is limited, the classical Laplace transform is $F(s) = \frac{\omega}{s^2+\omega^2}$. When ω is unlimited, this function $F(s)$ satisfies the proposition 2. We will show that our discrete Laplace transform has also this property.

Proof:

- The set $\{t_0, t_1, \ldots, t_k\}$ is quadrable.
- Define the "primitive" $g(t_k) = \int_{\{t_1,\ldots,t_k\}} f\, dm$.
- $g \simeq 0$. Indeed, f is quickly oscillating.
- Lemma (integration by parts) :

$$F(s) \;=\; g(t_N)e^{-st_N} + \sum_{k=1}^{N-1} g(t_k)\left(e^{-st_k} - e^{-st_{k+1}}\right)$$

- By classical majorations one can prove that

$$\sum_{k=1}^{N-1} \left|e^{-st_k} - e^{-st_{k+1}}\right| \;=\; \pounds$$

while $\Re e(s) > 0$ and $\frac{\Im m(s)}{\Re e(s)}$ limited, even if the repartition of the t_k is not regular.