How to measure the size of sets

Vieri Benci

Dipartimento di Matematica Applicata “U. Dini”
Università di Pisa

28th May 2006

Work in collaboration with M. Di Nasso and M. Forti

Introduction

Definition

A Counting System is a triple $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ where:

- $\mathcal{W}$ is a nonempty class of sets which might have some structure and which is closed for the following operations:
  - **(a)** $A \in \mathcal{W}$ and $B \subset A \Rightarrow B \in \mathcal{W}$,
  - **(b)** $A, B \in \mathcal{W} \Rightarrow A \uplus B \in \mathcal{W}$,
  - **(c)** $A, B \in \mathcal{W} \Rightarrow A \times B \in \mathcal{W}$.
- $\mathcal{N}$ is a **linearly ordered** class whose elements will be called numbers (or $\mathfrak{s}$-numbers if we need to be more precise).

Definition

A Counting System is a triple $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ where:

- $\mathcal{W}$ is a nonempty class of sets which might have some structure and which is closed for the following operations:
  - **(a)** $A \in \mathcal{W}$ and $B \subset A \Rightarrow B \in \mathcal{W}$,
  - **(b)** $A, B \in \mathcal{W} \Rightarrow A \uplus B \in \mathcal{W}$,
  - **(c)** $A, B \in \mathcal{W} \Rightarrow A \times B \in \mathcal{W}$.
- $\mathcal{N}$ is a **linearly ordered** class whose elements will be called numbers (or $\mathfrak{s}$-numbers if we need to be more precise).

Introduction

Definition

A Counting System is a triple $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ where:

- $\mathcal{W}$ is a nonempty class of sets which might have some structure and which is closed for the following operations:
  - **(a)** $A \in \mathcal{W}$ and $B \subset A \Rightarrow B \in \mathcal{W}$,
  - **(b)** $A, B \in \mathcal{W} \Rightarrow A \uplus B \in \mathcal{W}$,
  - **(c)** $A, B \in \mathcal{W} \Rightarrow A \times B \in \mathcal{W}$.
- $\mathcal{N}$ is a **linearly ordered** class whose elements will be called numbers (or $\mathfrak{s}$-numbers if we need to be more precise).

Definition

A Counting System is a triple $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ where:

- $\mathcal{W}$ is a nonempty class of sets which might have some structure and which is closed for the following operations:
  - **(a)** $A \in \mathcal{W}$ and $B \subset A \Rightarrow B \in \mathcal{W}$,
  - **(b)** $A, B \in \mathcal{W} \Rightarrow A \uplus B \in \mathcal{W}$,
  - **(c)** $A, B \in \mathcal{W} \Rightarrow A \times B \in \mathcal{W}$.
- $\mathcal{N}$ is a **linearly ordered** class whose elements will be called numbers (or $\mathfrak{s}$-numbers if we need to be more precise).

Definition

- $\mathfrak{s} : \mathcal{W} \to \mathcal{N}$ is a **surjective** function which satisfies the following assumptions:
  - **(i) Unit principle:** If A and B are singleton, then $\mathfrak{s}(A) = \mathfrak{s}(B)$
  - **(ii) Monotonicity principle:** $A \subseteq B \Rightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B)$
  - **(iii) Union principle:** Suppose that $A \cap B = \varnothing$ and $A' \cap B' = \varnothing$; then, if
    $$\mathfrak{s}(A) = \mathfrak{s}(A') \quad e \quad \mathfrak{s}(B) = \mathfrak{s}(B')$$
    we have that
    $$\mathfrak{s}(A \uplus B) = \mathfrak{s}(A' \uplus B')$$
  - **(iv) Cartesian product principle:** If
    $$\mathfrak{s}(A) = \mathfrak{s}(A') \quad e \quad \mathfrak{s}(B) = \mathfrak{s}(B'),$$
    then
    $$\mathfrak{s}(A \times B) = \mathfrak{s}(A' \times B')$$

Definition

- $\mathfrak{s} : \mathcal{W} \to \mathcal{N}$ is a **surjective** function which satisfies the following assumptions:
  - **(i) Unit principle:** If A and B are singleton, then $\mathfrak{s}(A) = \mathfrak{s}(B)$
  - **(ii) Monotonicity principle:** $A \subseteq B \Rightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B)$
  - **(iii) Union principle:** Suppose that $A \cap B = \varnothing$ and $A' \cap B' = \varnothing$; then, if
  $$\mathfrak{s}(A) = \mathfrak{s}(A') \quad e \quad \mathfrak{s}(B) = \mathfrak{s}(B')$$
  we have that
  $$\mathfrak{s}(A \uplus B) = \mathfrak{s}(A' \uplus B')$$
  - **(iv) Cartesian product principle:** If
  $$\mathfrak{s}(A) = \mathfrak{s}(A') \quad e \quad \mathfrak{s}(B) = \mathfrak{s}(B'),$$
  then
  $$\mathfrak{s}(A \times B) = \mathfrak{s}(A' \times B')$$

Definition

- $\mathfrak{s} : \mathcal{W} \to \mathcal{N}$ is a **surjective** function which satisfies the following assumptions:
  - **(i) Unit principle:** If A and B are singleton, then $\mathfrak{s}(A) = \mathfrak{s}(B)$
  - **(ii) Monotonicity principle:** $A \subseteq B \Rightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B)$
  - **(iii) Union principle:** Suppose that $A \cap B = \varnothing$ and $A' \cap B' = \varnothing$; then, if

$$\mathfrak{s}(A) = \mathfrak{s}(A') \quad e \quad \mathfrak{s}(B) = \mathfrak{s}(B')$$

we have that

$$\mathfrak{s}(A \uplus B) = \mathfrak{s}(A' \uplus B')$$

  - **(iv) Cartesian product principle:** If

$$\mathfrak{s}(A) = \mathfrak{s}(A') \quad e \quad \mathfrak{s}(B) = \mathfrak{s}(B'),$$

then

$$\mathfrak{s}(A \times B) = \mathfrak{s}(A' \times B')$$

Definition

- $\mathfrak{s} : \mathcal{W} \to \mathcal{N}$ is a **surjective** function which satisfies the following assumptions:
  - **(i) Unit principle:** If A and B are singleton, then $\mathfrak{s}(A) = \mathfrak{s}(B)$
  - **(ii) Monotonicity principle:** $A \subseteq B \Rightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B)$
  - **(iii) Union principle:** Suppose that $A \cap B = \varnothing$ and $A' \cap B' = \varnothing$; then, if

$$\mathfrak{s}(A) = \mathfrak{s}(A') \ \ e \ \ \mathfrak{s}(B) = \mathfrak{s}(B')$$

we have that

$$\mathfrak{s}(A \uplus B) = \mathfrak{s}(A' \uplus B')$$

  - **(iv) Cartesian product principle:** If

$$\mathfrak{s}(A) = \mathfrak{s}(A') \ \ e \ \ \mathfrak{s}(B) = \mathfrak{s}(B'),$$

then

$$\mathfrak{s}(A \times B) = \mathfrak{s}(A' \times B')$$

Definition

- $\mathfrak{s} : \mathcal{W} \rightarrow \mathcal{N}$ is a **surjective** function which satisfies the following assumptions:
  - **(i) Unit principle:** If A and B are singleton, then $\mathfrak{s}(A) = \mathfrak{s}(B)$
  - **(ii) Monotonicity principle:** $A \subseteq B \Rightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B)$
  - **(iii) Union principle:** Suppose that $A \cap B = \varnothing$ and $A' \cap B' = \varnothing$; then, if

    $$\mathfrak{s}(A) = \mathfrak{s}(A') \;\; e \;\; \mathfrak{s}(B) = \mathfrak{s}(B')$$

    we have that

    $$\mathfrak{s}(A \uplus B) = \mathfrak{s}(A' \uplus B')$$
  - **(iv) Cartesian product principle:** If

    $$\mathfrak{s}(A) = \mathfrak{s}(A') \;\; e \;\; \mathfrak{s}(B) = \mathfrak{s}(B'),$$

    then

    $$\mathfrak{s}(A \times B) = \mathfrak{s}(A' \times B')$$

Definition

- $\mathfrak{s} : \mathcal{W} \to \mathcal{N}$ is a **surjective** function which satisfies the following assumptions:
  - **(i) Unit principle:** If A and B are singleton, then $\mathfrak{s}(A) = \mathfrak{s}(B)$
  - **(ii) Monotonicity principle:** $A \subseteq B \Rightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B)$
  - **(iii) Union principle:** Suppose that $A \cap B = \varnothing$ and $A' \cap B' = \varnothing$; then, if
  $$\mathfrak{s}(A) = \mathfrak{s}(A') \;\; e \;\; \mathfrak{s}(B) = \mathfrak{s}(B')$$
  we have that
  $$\mathfrak{s}(A \uplus B) = \mathfrak{s}(A' \uplus B')$$
  - **(iv) Cartesian product principle:** If
  $$\mathfrak{s}(A) = \mathfrak{s}(A') \;\; e \;\; \mathfrak{s}(B) = \mathfrak{s}(B'),$$
  then
  $$\mathfrak{s}(A \times B) = \mathfrak{s}(A' \times B')$$

The number $\mathfrak{s}(A)$ is called **size** of A.

Example

$$(\mathbf{Fin}, |\cdot|, \mathbb{N})$$

where

- $\mathbf{Fin}$ is the class of finite sets
- $|\cdot|$ is the "number of elements" of a set
- $\mathbb{N}$ is the set of natural numbers

The counting system $(\mathbf{Fin}, |\cdot|, \mathbb{N})$ is ruled by two general principles:

The counting system $(\mathbf{Fin}, |\cdot|, \mathbb{N})$ is ruled by two general principles:

AP **- Aristotle's Principle**.

If A is a proper subset of B then $\mathfrak{s}(A) < \mathfrak{s}(B)$,

The counting system $(\mathbf{Fin}, |\cdot|, \mathbb{N})$ is ruled by two general principles:

AP - **Aristotle's Principle**.

If A is a proper subset of B then $\mathfrak{s}(A) < \mathfrak{s}(B)$,

and

CP - **Cantor's Principle**

$\mathfrak{s}(A) = \mathfrak{s}(B)$ if and only if A is in 1–1 correspondence with B.

The problem is to extend the operation of counting to a larger class of sets which contains some infinite sets; we would like to extend the counting system in such a way that the Aristotle and the Cantor Principles remain valid.

The problem is to extend the operation of counting to a larger class of sets which contains some infinite sets; we would like to extend the counting system in such a way that the Aristotle and the Cantor Principles remain valid.

This is not possible, in fact

Theorem

A counting system $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ *satisfies the Cantor and the Aristotle principles if and only if* $\mathcal{W} \subset \mathbf{Fin}$ *and* $\mathcal{N} = \mathbb{N}$.

However, if we weaken one of these two principles, it is possible to get counting systems that lead to interesting theories

However, if we weaken one of these two principles, it is possible to get counting systems that lead to interesting theories

Definition

A counting system $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ is called Cantorian if satisfies the Cantor principle CP and the weak Aristotle principle

- (Weak Aristotle's Principle): If A is a proper subset of B then $\mathfrak{s}(A) \leq \mathfrak{s}(B)$.

However, if we weaken one of these two principles, it is possible to get counting systems that lead to interesting theories

Definition

A counting system $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ is called Cantorian if satisfies the Cantor principle CP and the weak Aristotle principle

- (Weak Aristotle's Principle): If A is a proper subset of B then $\mathfrak{s}(A) \leq \mathfrak{s}(B)$.

Definition

A counting system $(\mathcal{W}, \mathfrak{s}, \mathcal{N})$ is called Aristotelian if it satisfies the Aristotle principle AP and the weak Cantor principle

- (Weak Cantor's Principle): If $\mathfrak{s}(A) = \mathfrak{s}(B)$, then A is in 1–1 correspondence with B.

Cantorian Counting Systems:

Essentially there is only one Cantorian Counting System

Cantorian Counting Systems:

Essentially there is only one Cantorian Counting System

CARDINAL NUMBERS

$$(\mathbf{Set}, |\cdot|, \mathbf{Card})$$

where

- $\mathbf{Set}$ is the class of all sets
- $|\cdot|$ is the cardinality of a set
- $\mathbf{Card}$ is the class of cardinal numbers

Ordinal Numbers:

ORDINAL NUMBERS

$$(\mathbf{Woset},\ ord,\ \mathbf{Ord})$$

where

- **Woset** is the class of well ordered sets
- *ord* is the order type of a set
- **Ord** is the class of cardinal numbers

Ordinal Numbers:

ORDINAL NUMBERS

$$(\mathbf{Woset},\ ord,\ \mathbf{Ord})$$

where

- **Woset** is the class of well ordered sets
- *ord* is the order type of a set
- **Ord** is the class of cardinal numbers

The Ordinal Numbers form a Counting System which does not satisfy the Cantor Principle, nor the Aristotle principle; however they are a bridge between the Cantorian and the Aristotelian counting theories.

Aristotelian Counting Systems:

Definition

A Numerosity System is an Aristotelian Counting System $(\mathcal{W}, \mathfrak{n}, \mathcal{N})$ such that

$$\mathcal{N} \subset \mathcal{R}^{+} \cup \{0\}$$

where $\mathcal{R}$ is an ordered field.

Aristotelian Counting Systems:

Definition

A Numerosity System is an Aristotelian Counting System $(\mathcal{W}, \mathfrak{n}, \mathcal{N})$ such that

$$\mathcal{N} \subset \mathcal{R}^+ \cup \{0\}$$

where $\mathcal{R}$ is an ordered field.

The number $\mathfrak{n}(A)$ is called **numerosity** of A and $\mathfrak{n}$ is called numerosity function.
Thus a numerosity function is a measure of the size of a set which satisfies good algebraic properties.

The class of labelled sets

Next we define a class of sets suitable for a numerosity theory:

The class of labelled sets

Next we define a class of sets suitable for a numerosity theory:

Definition

A labelled set $\mathbf{A}$ is a pair (A, ℓ) where A is a set and

$$\ell : A \to \mathbf{Ord}$$

is an application such that $\forall \gamma \in \mathbf{Ord}$, the set $\ell^{-1}(\gamma)$ is finite. The class of labelled sets will be denoted by $\mathbf{Lset}$

Well ordered set vs. labelled sets

When you "count" the elements of a set by an ordinal number, you order the elements of a set in a "long line" without empty spaces.

Well ordered set vs. labelled sets

When you "count" the elements of a set by an ordinal number, you order the elements of a set in a "long line" without empty spaces.

When you "count" the elements of a set by an numerosity function, you order the elements of a set in in a "long line" of finite piles and you allow to have empty spaces.

Well ordered set vs. labelled sets

When you "count" the elements of a set by an ordinal number, you order the elements of a set in a "long line" without empty spaces.

When you "count" the elements of a set by an numerosity function, you order the elements of a set in in a "long line" of finite piles and you allow to have empty spaces.

Namely the notion of labelled set is an obvious extension of the notion of well ordered set.

The well ordered sets have a natural labelling given by their ordering. The ordinal numbers have the labelling given by the identity

$$\ell(x) = x, \quad \forall x \in \mathbf{Ord}.$$

Thus

$$E \subset \mathbf{Ord} \Rightarrow E \in \mathbf{Lset}$$

The well ordered sets have a natural labelling given by their ordering. The ordinal numbers have the labelling given by the identity

$$\ell(x) = x, \quad \forall x \in \mathbf{Ord}.$$

Thus

$$E \subset \mathbf{Ord} \Rightarrow E \in \mathbf{Lset}$$

The class of labelled sets whose label is less than $\Omega \in \mathbf{Ord}$ will be denoted by $\mathcal{W}(\Omega)$.

The well ordered sets have a natural labelling given by their ordering. The ordinal numbers have the labelling given by the identity

$$\ell(x) = x, \quad \forall x \in \mathbf{Ord}.$$

Thus

$$E \subset \mathbf{Ord} \Rightarrow E \in \mathbf{Lset}$$

The class of labelled sets whose label is less than $\Omega \in \mathbf{Ord}$ will be denoted by $\mathcal{W}(\Omega)$.

$\mathcal{W}(\omega)$ is called the class of Natural Labelled Set since ω can be identified with the set of natural numbers $\mathbb{N}$.

Numerosity of Natural Labelled Sets

Definition

To every natural labelled set $\mathbf{A} = (A, \ell)$, we associate the **counting function**

$$\varphi_{\mathbf{A}} : \omega \rightarrow \mathbb{N}$$

defined as follows

$$\varphi_{\mathbf{A}}(n) = |\{x \in A \mid \ell(x) \leq n\}| \,. \tag{1}$$

Theorem

Let $\mathbb{N}^*$ be a model of the hypernatural numbers constructed over a selective ultrafilter $\mathcal{U}$ and let

$$\mathfrak{n} : \mathcal{W}(\omega) \to \mathbb{N}^* = \mathbb{N}^\omega / \mathcal{U}$$

be a function defined as follows

$$\mathfrak{n}(\mathbf{A}) = [\varphi_{\mathbf{A}}]_{\mathcal{U}} . \tag{2}$$

Then, $(\mathcal{W}(\omega), \mathfrak{n}, \mathbb{N}^*)$ *is a Numerosity System.*

Remark

Nonstandard models of $\mathbb{N}$ arise in a natural way from numerosity theories.

Remark

Numerosity theories select special kinds of ultrafilters ; for example the above theorem is true if $\mathcal{U}$ is a *selective* ultrafilter. Otherwise we would have $\mathcal{N} \subset \mathbb{N}^*$.

Remark

Numerosity theories might have some foundational intrest; in fact the existence of selective ultrafilters cannot be proved in ZFC (but it can be proved in ZFC+CH).

Remark

Nonstandard models of $\mathbb{N}$ arise in a natural way from numerosity theories.

Remark

Numerosity theories select special kinds of ultrafilters ; for example the above theorem is true if $\mathcal{U}$ is a *selective* ultrafilter. Otherwise we would have $\mathcal{N} \subset \mathbb{N}^*$.

Remark

Numerosity theories might have some foundational intrest; in fact the existence of selective ultrafilters cannot be proved in ZFC (but it can be proved in ZFC+CH).

Remark

Nonstandard models of $\mathbb{N}$ arise in a natural way from numerosity theories.

Remark

Numerosity theories select special kinds of ultrafilters ; for example the above theorem is true if $\mathcal{U}$ is a *selective* ultrafilter. Otherwise we would have $\mathcal{N} \subset \mathbb{N}^*$.

Remark

Numerosity theories might have some foundational intrest; in fact the existence of selective ultrafilters cannot be proved in ZFC (but it can be proved in ZFC+CH).

Now the problem consists in extending the notion of numerosity to any labelled set A such that $\ell(A) \subset \Omega$ where Ω is an arbitrarily large ordinal number.

Now the problem consists in extending the notion of numerosity to any labelled set A such that $\ell(A) \subset \Omega$ where Ω is an arbitrarily large ordinal number.

The main difficulty is to extend the notion of numerosity function

$$\varphi_{\mathbf{A}} : \omega \to \mathbb{N},\ \varphi_{\mathbf{A}}(n) = |\{x \in A \mid \ell(x) < n\}|$$

to a function

$$\varphi_{\mathbf{A}} : \Omega \to \mathbb{N}$$

Clearly an immediate generalization does not work.

We shall overcome this difficulty introducing a new order relation. Given two ordinals $x, y \in \mathbf{Ord}$, using the Cantor normal form they can be written as follows:

$$x = \sum_{i=0}^{N} \omega^{\gamma_i} x_i;$$
$$y = \sum_{i=0}^{N} \omega^{\gamma_i} y_i;$$

$$x_i, y_i \in \omega; \; \gamma_i \in \mathbf{Ord}$$

We set

$$x \vee y \quad : \quad = \sum_{i=0}^{N} \omega^{\gamma_i} \cdot \max\ \{x_i, y_i\}$$

and

$$x \wedge y \quad : \quad = \sum_{i=0}^{N} \omega^{\gamma_i} \cdot \min\ \{x_i, y_i\}$$

In this way, **Ord** is equipped with a lattice structure. Now, we can introduce a partial order relation "$\sqsubseteq$" which exploits this lattice structure:

$$x \sqsubseteq y :\Leftrightarrow x = x \wedge y \Leftrightarrow y = x \vee y.$$

Thus, given the two ordinals (19), we have that

$$x \sqsubseteq y :\Leftrightarrow x_i \leq y_i, \; i = 1, .., N$$

We define the sum of two labelled sets $\mathbf{A}_1 = (A_1, \ell_1)$ and $\mathbf{A}_2 = (A_2, \ell_2)$, as follows:

$$\mathbf{A}_1 \uplus \mathbf{A}_2 = (A_1 \cup A_2,\ \ell)$$

where $\uplus$ denotes their union and the labelling ℓ is defined as follows:

$$\ell(x) = \begin{cases} \ell_1(x) & if\ x \in A_1 \\ \ell_2(x) & if\ x \in A_2 \end{cases} \tag{3}$$

The product between two labelled sets $[A_1, \ell_1]$ and $[A_2, \ell_2]$, is defined as follows as follows:

$$\mathbf{A}_1 \times \mathbf{A}_2 = (A_1 \times A_2,\ \ell(x_1, x_2))$$

where

$$\ell_1(x_1, x_2) = \ell_1(x_1) \vee \ell_2(x_2)$$

Thus the class $\mathcal{W}(\Omega)$ is closed for union and cartesian product.

Using this order relation it is possible to generalize the notion of counting function as follows: if $\mathbf{A} \in \mathcal{W}(\Omega)$

$$\varphi_{\mathbf{A}} : \Omega \to \mathbb{N}$$

is defined as follows

$$\varphi_{\mathbf{A}}(\gamma) = |\{x \in A \mid \ell(x) \sqsubseteq \gamma\}| \,. \tag{4}$$

Theorem

There is a numerosity system

$$\{\mathcal{W}(\Omega), num, \mathcal{N}(\Omega)\}$$

such that

$$\mathcal{N}(\Omega) \subset \mathbb{R}^{\circledast}(\Omega)$$

where

$$\mathbb{R}^{\circledast}(\Omega) = \mathbb{R}^{\Omega}/\mathcal{U}$$

and

$$num(\mathbf{A}) = [\varphi_{\mathbf{A}}]_{\mathcal{U}}.$$

Theorem

There is a numerosity system

$$\{\mathcal{W}(\Omega), num, \mathcal{N}(\Omega)\}$$

such that

$$\mathcal{N}(\Omega) \subset \mathbb{R}^{\circledast}(\Omega)$$

where

$$\mathbb{R}^{\circledast}(\Omega) = \mathbb{R}^{\Omega}/\mathcal{U}$$

and

$$num(\mathbf{A}) = [\varphi_{\mathbf{A}}]_{\mathcal{U}}.$$

The main technicality of the proof consists in constructing a suitable ultrafilter $\mathcal{U}$.

Ordinal numbers and numerosities

The ordinal numbers, are not an Aristotelian Counting System since they violate the Aristotle principle, nevertheless they satisfy good arithmetic properties with respect to the natural operations $\oplus$ and $\otimes$.

$$\xi = \sum_{j=0}^{n} \omega^{\beta_j} a_j; \ \zeta = \sum_{j=0}^{n} \omega^{\beta_j} b_j$$

$$\begin{aligned} \xi \oplus \zeta &= \sum_{j=0}^{n} \omega^{\beta_j} (a_j + b_j) \\ \xi \otimes \zeta &= \sum_{i,j=0}^{n} \omega^{\beta_i \oplus \beta_j} a_i b_j \end{aligned}$$

Thus they must be strictly related to a numerosity theory.

The numerosity function provides a natural embedding

$$num : \Omega \to \mathcal{N}(\Omega) \subset \mathbb{R}^{\circledast}(\Omega) \quad (5)$$

which associates to each Von Neumann ordinal number $\gamma \in \Omega$ its numerosity $\hat{\gamma} = num(\gamma)$.

The numerosity function provides a natural embedding

$$num : \Omega \to \mathcal{N}(\Omega) \subset \mathbb{R}^{\circledast}(\Omega) \quad (5)$$

which associates to each Von Neumann ordinal number $\gamma \in \Omega$ its numerosity $\hat{\gamma} = num(\gamma)$.

Theorem

If $\beta, \gamma \in \Omega$, then

- $num(\beta \oplus \gamma) = \hat{\beta} + \hat{\gamma}$
- $num(\beta \otimes \gamma) = \hat{\beta} \cdot \hat{\gamma}$

Moreover, we have that

Theorem

If $\xi \in \mathcal{N}(\Omega)$, *then there exist* $E \subset Ord$, *such that*

$$\xi = num(E)$$

Moreover, we have that

Theorem

If $\xi \in \mathcal{N}(\Omega)$, then there exist $E \subset Ord$, such that

$$\xi = num(E)$$

So we have that

$$\mathcal{N}(\Omega) = \mathcal{B}\left(\overline{\Omega}\right) / \approx$$

where $\overline{\Omega}$ is a suitable ordinal number larger than Ω and $\mathcal{B}\left(\overline{\Omega}\right)$ is the family of bounded subsets of $\overline{\Omega}$.

Infinite sums

We would like to give a meaning to infinite sums of the type

$$\sum_{j\in\Omega}\xi_j, \quad \xi_j \in \mathbb{R}^{\circledast}(\Omega) \tag{6}$$

and to have that

$$num(E) = \sum_{j\in\Omega}|E_j| \tag{7}$$

where $E \in \mathcal{W}(\Omega)$ and $E_j = \{x \in E : \ell(x) = j\}$.

Proposition

There exists an operator $\sigma : \mathbb{R}^{\circledast}(\Omega) \rightarrow \mathfrak{F}(\Omega, \mathbb{R})$ *such that*

- *(it is a ring homomorphism) for every* $\xi, \eta \in \mathbb{R}^{\circledast}(\Omega)$,

$$\begin{aligned} \sigma(\xi + \eta) &= \sigma(\xi) + \sigma(\eta); \\ \sigma(\xi \cdot \eta) &= \sigma(\xi) \cdot \sigma(\eta); \end{aligned}$$

- *(it is a ring section)* $J_\Omega \circ \sigma = identity$ *where*

$$J_\Omega : \mathfrak{F}(\Omega, \mathbb{R}) \rightarrow \mathbb{R}^{\circledast}(\Omega)$$

is the ring homomorphism defined by

$$J_\Omega(\varphi) = [\varphi]_{\mathcal{U}}.$$

Now for every $j \in \Omega$, we define

$$\delta_j : \Omega \longrightarrow \mathbb{N}$$

$$\delta_j(x) = \begin{cases} 1 & if \;\; x \sqsupseteq j \\ 0 & otherwise \end{cases}$$

Proposition

For every $j \in \Omega$, $J_\Omega(\delta_j) = 1$.

Now for every $j \in \Omega$, we define

$$\delta_j : \Omega \longrightarrow \mathbb{N}$$

$$\delta_j(x) = \begin{cases} 1 & if \;\; x \sqsupseteq j \\ 0 & otherwise \end{cases}$$

Proposition

For every $j \in \Omega$, $J_\Omega(\delta_j) = 1$.

Also this proposition is a consequence of the ultrafilter which we have chosen

Now, in order to simplify the notation, when no ambiguity is possible, for any $\xi \in \mathbb{R}^{\circledast}(\Omega)$, we set

$$\xi(x) = [\sigma(\xi)](x);$$

Now, in order to simplify the notation, when no ambiguity is possible, for any $\xi \in \mathbb{R}^{\circledast}(\Omega)$, we set

$$\xi(x) = [\sigma(\xi)](x);$$

By the above definitions, we have that $\xi = J_{\Omega}(\xi\delta_j)$ and hence, if $I \subset \Omega$ is a finite set,

$$\sum_{j \in I} \xi_j = J_{\Omega}\left(\sum_{j \in I} \xi_j \delta_j\right)$$

This fact suggests to generalize this equation to the case in which I is infinite:

Definition

Given $\xi_j \in \mathbb{R}^{\circledast}(\Omega)$, $I \subseteq \Omega$,we set

$$\sum_{j \in I} \xi_j = J_\Omega \left(\sum_{j \in I} \xi_j \delta_j \right) \tag{8}$$

This fact suggests to generalize this equation to the case in which I is infinite:

Definition

Given $\xi_j \in \mathbb{R}^{\circledast}(\Omega)$, $I \subseteq \Omega$,we set

$$\sum_{j\in I} \xi_j = J_\Omega \left(\sum_{j\in I} \xi_j \delta_j \right) \quad (8)$$

Equation (8) makes sense, in fact, for any $x \in \Omega$

$$\sum_{j\in I} \xi_j(x)\,\delta_j(x) = \sum_{j\in I;\ j\sqsubseteq x} \xi_j(x)$$

and this is a finite sum since the set of j's $\sqsubseteq x$ is finite.

The next theorem describes the main properties satisfied by the infinite sum:

Theorem

The infinite sum satisfies the following properties:

- *(i) (finite associative property)* $\sum_{j\in I}\xi_j + \sum_{j\in I}\zeta_j = \sum_{j\in I}(\xi_j + \zeta_j)$
- *(ii) (distributive property)* $\zeta\sum_{j\in I}\xi_j = \sum_{j\in I}\zeta\xi_j$
- *(iii) (partial sum) if* $r_j \in \mathbb{R}$, *then* $\sum_{j\in\omega^\gamma} r_j = J_\Omega(S)$ *where where*

$$S(x) := \sum_{j\sqsubseteq x} r_j$$

is a "partial sum".

The next theorem describes the main properties satisfied by the infinite sum:

Theorem

The infinite sum satisfies the following properties:

- *(i) (finite associative property)* $\sum_{j\in I}\xi_j + \sum_{j\in I}\zeta_j = \sum_{j\in I}(\xi_j + \zeta_j)$
- *(ii) (distributive property)* $\zeta\sum_{j\in I}\xi_j = \sum_{j\in I}\zeta\xi_j$
- *(iii) (partial sum) if* $r_j \in \mathbb{R}$, *then* $\sum_{j\in\omega^\gamma} r_j = J_\Omega(S)$ *where where*

$$S(x) := \sum_{j\sqsubseteq x} r_j$$

is a "partial sum".

The next theorem describes the main properties satisfied by the infinite sum:

Theorem

The infinite sum satisfies the following properties:

- *(i) (finite associative property)* $\sum_{j\in I}\xi_j + \sum_{j\in I}\zeta_j = \sum_{j\in I}(\xi_j + \zeta_j)$
- *(ii) (distributive property)* $\zeta\sum_{j\in I}\xi_j = \sum_{j\in I}\zeta\xi_j$
- *(iii) (partial sum) if* $r_j \in \mathbb{R}$, *then* $\sum_{j\in\omega^\gamma} r_j = J_\Omega(S)$ *where where*

$$S(x) := \sum_{j\sqsubseteq x} r_j$$

is a "partial sum".

The next theorem describes the main properties satisfied by the infinite sum:

Theorem

The infinite sum satisfies the following properties:

- *(i) (finite associative property)* $\sum_{j\in I}\xi_j + \sum_{j\in I}\zeta_j = \sum_{j\in I}(\xi_j + \zeta_j)$
- *(ii) (distributive property)* $\zeta\sum_{j\in I}\xi_j = \sum_{j\in I}\zeta\xi_j$
- *(iii) (partial sum) if* $r_j \in \mathbb{R}$, *then* $\sum_{j\in\omega^\gamma} r_j = J_\Omega(S)$ *where where*

$$S(x) := \sum_{j\sqsubseteq x} r_j$$

is a "partial sum".

Theorem

- *(iv) (hyperfinite sum) if* $r_j \in \mathbb{R}, j \in \omega$, *then* $\sum_{j\in\omega} r_j = \sum_{j=0}^{\hat{\omega}} r_j$ *where*

$$\sum_{j=0}^{\hat{\omega}} r_j$$

  is the usual hyperfinite sum

- *(v) (finite permutation) let* $\pi : \omega^\gamma \to \omega^\gamma$ *be a permutation of a finite number of points; then*

$$\sum_{j\in\omega^\gamma} \xi_j = \sum_{j\in\omega^\gamma} \xi_{\pi(j)}$$

Theorem

- *(iv) (hyperfinite sum) if* $r_j \in \mathbb{R}, j \in \omega$, *then* $\sum_{j\in\omega} r_j = \sum_{j=0}^{\hat{\omega}} r_j$ *where*

$$\sum_{j=0}^{\hat{\omega}} r_j$$

  is the usual hyperfinite sum

- *(v) (finite permutation) let* $\pi : \omega^\gamma \to \omega^\gamma$ *be a permutation of a finite number of points; then*

$$\sum_{j\in\omega^\gamma} \xi_j = \sum_{j\in\omega^\gamma} \xi_{\pi(j)}$$

Theorem

- *(vi) (translation of indices) if, for $j \in \omega^\gamma$, $\zeta_j = \xi_{\omega^\gamma \beta + j}$, then*

$$\sum_{j \in \omega^\gamma} \xi_j = \sum_{j=\omega^\gamma \beta}^{\omega^\gamma(\beta+1)} \zeta_j$$

- *(vii) (infinite associative property) for any $\gamma \in \Omega$, we have*

$$\sum_{j \in \Omega} \xi_j = \sum_{\beta \in \Omega} \left(\sum_{j=\omega^\gamma \beta}^{\omega^\gamma(\beta+1)} \zeta_j \right)$$

Theorem

- *(vi) (translation of indices) if, for* $j \in \omega^\gamma$, $\zeta_j = \xi_{\omega^\gamma \beta + j}$, *then*

$$\sum_{j \in \omega^\gamma} \xi_j = \sum_{j=\omega^\gamma \beta}^{\omega^\gamma(\beta+1)} \zeta_j$$

- *(vii) (infinite associative property) for any* $\gamma \in \Omega$, *we have*

$$\sum_{j \in \Omega} \xi_j = \sum_{\beta \in \Omega} \left(\sum_{j=\omega^\gamma \beta}^{\omega^\gamma(\beta+1)} \zeta_j \right)$$

The product principle

In set theory and hence in Counting Systems, the idea of product arises from the idea of Cartesian Product. However in elementary Arithmetic the product $m \cdot n$ is thought as the sum of m terms equal to n.

The product principle

In set theory and hence in Counting Systems, the idea of product arises from the idea of Cartesian Product. However in elementary Arithmetic the product $m \cdot n$ is thought as the sum of m terms equal to n.

Thus the most general idea of product of two sets F and E is the following one: we suppose to have a family of sets E_j, $j \in F$, pairwise disjoint, and equinumerous to a set E; we would like to have

$$num\,(F) \cdot num\,(E) = num\left(\bigcup_{j \in F} E_j\right) \quad (9)$$

We may assume that $E, F \subset \Omega \in \mathbf{Ord}$. Then if we assume that

$$\ell(E) \subset \omega^{\gamma}, \text{ for a fixed } \gamma \in \mathbf{Ord}$$

and we set

$$E_j = \{\omega^{\gamma} j + x : x \in E\}, \quad j \in F.$$

We may assume that $E, F \subset \Omega \in \mathbf{Ord}$. Then if we assume that

$$\ell(E) \subset \omega^\gamma, \text{ for a fixed } \gamma \in \mathbf{Ord}$$

and we set

$$E_j = \{\omega^\gamma j + x : x \in E\}, \quad j \in F.$$

Then

$$\forall j \in F, \ num(E_j) = num(E)$$

and

$$num(F) \cdot num(E) = num\left(\bigcup_{j \in F} E_j\right)$$

holds.

Exponentiation

We can also define an "exponentiation" between labelled sets. Given a function $f: E \to \gamma,\ \gamma \in \mathbf{Ord}$, the support of f is defined as follows:

$$\mathfrak{supp}(f) = \{x \in E : f(x) \neq 0\}$$

Exponentiation

We can also define an "exponentiation" between labelled sets. Given a function $f : E \to \gamma,\ \gamma \in \mathbf{Ord}$, the support of f is defined as follows:

$$\mathfrak{supp}(f) = \{x \in E : f(x) \neq 0\}$$

Definition

Given a labelled set $\mathbf{A} = (A, \ell)$ and a function $f : A \to \gamma,\ \gamma \in \mathbf{Ord}$, we set

$$\gamma^{\mathbf{A}} = \{f \in \mathfrak{F}(A, \gamma) : \mathfrak{supp}(f) \text{ is finite}\}$$

Moreover, for any $f \in \gamma^{\mathbf{A}}$, we set

$$\ell(f) = \bigvee \{\ell(x, f(x)) : x \in \mathfrak{supp}(f)\}$$

Exponentiation

We can also define an "exponentiation" between labelled sets. Given a function $f : E \to \gamma,\ \gamma \in \mathbf{Ord}$, the support of f is defined as follows:

$$\mathfrak{supp}(f) = \{x \in E : f(x) \neq 0\}$$

Definition

Given a labelled set $\mathbf{A} = (A, \ell)$ and a function $f : A \to \gamma,\ \gamma \in \mathbf{Ord}$, we set

$$\gamma^{\mathbf{A}} = \{f \in \mathfrak{F}(A, \gamma) : \mathfrak{supp}(f) \text{ is finite}\}$$

Moreover, for any $f \in \gamma^{\mathbf{A}}$, we set

$$\ell(f) = \bigvee \{\ell(x, f(x)) : x \in \mathfrak{supp}(f)\}$$

In particular, we have that

$$2^{\mathbf{A}} \cong \mathcal{P}_{fin}(\mathbf{A})$$

Theorem

If $\gamma \in \mathbf{Ord}$ *and* $E \in \mathbf{Lset}$

$$num\left(\gamma^{E}\right) = num\left(\gamma\right)^{num(E)}$$

and

$$num\left(\mathcal{P}_{fin}\left(E\right)\right) = 2^{num(E)}$$

The end

Thank you for your attention!