

# Equilibrium in Continuous-Time Financial Markets: Endogenously Complete Markets

(with Roberto Raimondo,  
University of Melbourne)

Dedicated to the Memory of  
Gerard Debreu  
and  
Shizuo Kakutani

## General Equilibrium Theory:

- Theory of determination of prices in markets
- Given price vector describing the price of each commodity, the budget set of an agent is the the set of all consumption vectors s/he can afford
- Given a price vector, each agent purchases his/her demand, the vector of commodities that maximizes a utility function over his/her budget set
- An equilibrium price in a price vector such that total demand for all agents equals total supply of the commodities.
- Relatively mature theory

## Continuous-Time Finance:

- Time horizon  $[0, T]$
- Probability space  $\Omega$
- $J$  securities
- Price of securities is a stochastic process  $p_A$  taking values in  $\mathbf{R}_+^J$ , for example, geometric Brownian Motion, some other Itô Process, Lévy Process
- A trading strategy is a stochastic process taking values in  $\mathbf{R}^J$  which is integrable with respect to  $p_A$  which is “admissible” (roughly  $\int z dp_A$  is a martingale)
- Generally,  $p_A$  is taken as given; study pricing of derivative securities, such as options, given the exogenously specified  $p_A$ .

## General Equilibrium View of Continuous-Time Finance:

- Securities should be specified by their dividends in  $L^2([0, T] \times \Omega)$ , not by their prices
- $p_A$  should be determined as *equilibrium* pricing given the securities dividends, agents' endowments of goods, and agents' utility functions
- Given  $p_A$ , each admissible trading strategy for an agent determines a specific stochastic process of consumption
- Agent's budget set is the set of all consumption stochastic processes generated by admissible trading strategies
- Agent's demand for securities is that stochastic process which maximizes utility of consumption over the agent's budget set
- An equilibrium price is a stochastic process such that the demand for securities equals the supply of the securities almost everywhere.
- *Existence of equilibrium in continuous-time finance markets is biggest open problem in General Equilibrium Theory.* Two cases:
  - $J \geq K + 1$ ; markets are *potentially dynamically complete*
  - $J \leq K$ ; markets are necessarily dynamically incomplete.

We resolve the potentially dynamically complete case.

Paper closely related to Duffie-Zame (1989). Models are not identical; ours more general in some respects, theirs more general in others.

### **Common Features of the Models:**

1. Uncertainty specified in terms of a  $K$ -dimensional Brownian Motion  $\beta(t, \omega)$ .
2.  $J = K + 1$  securities traded in continuous time; one is usually a deterministic “bond.”
3. Securities characterized by their dividends.
4. Agents characterized by their endowments and utility functions.
5. Equilibrium is a set of securities price processes and trading strategies for the agents such that
  - demand for securities equals the supply of the securities at almost all nodes
  - consumptions generated by the trading strategies equal the sum of the endowments and dividends at almost all nodes
6. **Goal:** *Prove existence of equilibrium and show that equilibria of discrete-time approximations converge to equilibria of the continuous-time model.*

7. Existence of Price Equilibrium is established by fixed point arguments:

- Arrow-Debreu (1954): Finite set of goods and states, complete set of Arrow-Debreu contingent claims. Uses Kakutani's Fixed Point Theorem, an extension of Brouwer's Fixed Point Theorem to correspondences (set-valued functions) in  $\mathbf{R}^K$ .
- Duffie and Shafer (1985): For discrete versions of our model, equilibrium exists generically (except for a set of measure zero of endowments and dividends). Complex fixed point argument on the Grassmanian.
- Bewley, Mas-Colell, Aliprantis, Brown, Burkinshaw, Zame, others up through late 1980s: Equilibrium exists in these types of models if there is a complete set of Arrow-Debreu contingent claims (i.e. an infinite-dimensional set of securities). Functional analysis, plus a fixed point argument on the set of Pareto Optima (which is an  $I - 1$ -dimensional manifold in  $L^2$ ). No obvious way to combine Grassmanian approach with infinite-dimensional spaces like  $L^2$ , because incomplete markets equilibria are not Pareto Optimal.

8. What did Duffie and Zame do?
- (a) Verify that an Arrow-Debreu equilibrium exists. Induce securities prices from the Arrow-Debreu prices.
  - (b) *Under the endogenous assumption that the induced securities prices satisfy a dynamic spanning condition* (the  $(K + 1) \times K$  Jacobian of securities prices with respect to the Brownian motion components has rank  $K$  for almost all  $(t, \omega)$ ), securities are dynamically complete (can replicate any Arrow-Debreu security by continuous trading of given securities).
  - (c) Show that dynamic completeness implies that the securities prices induced by the Arrow are *equilibrium* prices in the *securities* market.
  - (d) Did not tackle convergence of discrete to continuous.
9. Obvious approach for existence: Show that for a generic (open dense, residual, or relatively prevalent) set of primitives, the spanning condition holds for the endogenously determined Arrow-Debreu equilibrium prices.
- (a) Nobody's done it.
  - (b) We tried and failed.

## 10. Our Approach (Summary for Nonstandard Analysts)

- Start with a continuous-time, Brownian Motion-based model  $\beta$  taking values in  $\mathbf{R}^K$
- Discretize this to a model based on a hyperfinite random walk  $\hat{\beta}$  with the property that each node has  $K+1$  successor nodes
- Apply Duffie-Shafer and the Transfer Principle to show that the hyperfinite model has a hyperfinite equilibrium price  $\hat{p}_A$  and trading strategies  $\hat{z}_1, \dots, \hat{z}_I$  (you can choose the right discretization to lie in the generic set, hence equilibrium is universal for the continuous-time model)
- Work hard to show that the hyperfinite equilibrium

$$(\hat{p}_a, \hat{z}_1, \dots, \hat{z}_I)$$

is sufficiently nicely behaved that it is a lifting in a strong sense of a candidate continuous-time equilibrium. Issues:

- Each agent's optimal trading strategy solves a dynamic programming problem with respect to the (to be determined) equilibrium price. In order to extract a candidate continuous-time equilibrium trading strategy, need to show the optimal strategy doesn't chatter, but in fact it *does* chatter with respect to certain non-equilibrium prices, including price processes with the same standard part as  $\hat{p}_A$ .
- In order to extract a candidate continuous-time equilibrium price, need to show that the hyperfinite equilibrium price is nicely behaved. Indeed, to show that the optimal

trading strategies don't chatter, need to show that

$$\hat{p}_A(t, \omega) \text{ is an } SC^1 \text{ function of } (t, \hat{\beta}(t, \omega))$$

- The optimal trading strategies are determined *simultaneously* with the equilibrium price by a fixed point argument.
- Show that the endogenous Duffie-Zame spanning condition is satisfied. Issues:
  - At each node  $(t, \omega)$  in the event tree, the securities prices at the successor nodes give rise to a determinant  $D(t, \omega)$ .
  - The condition for dynamically complete markets in the hyperfinite model is

$$\forall_{t, \omega} D(t, \omega) \neq 0 \tag{1}$$

- The condition for dynamically complete markets in the continuous time model is

$${}^\circ D(t, \omega) \neq 0 \text{ Loeb almost surely} \tag{2}$$

- Since the standard part of an infinitesimal is zero, Equation (1) does not imply Equation (2).
- We show that Equation (2) arises from the way information is revealed in the random walk filtration.
- Show the candidate continuous-time equilibrium is in fact an equilibrium (like Brown-Robinson, but more work because of setting). Uses fact that continuous-time stochastic integral is standard part of hyperfinite stochastic integral.
- Show that equilibria of the discretizations converge to equilibria of the continuous-time economy (routine Nonstandard Analysis).



## 11. Key Facts about Real Analytic Functions:

- If an analytic function  $f$  is zero on a set of positive measure, then  $f$  is identically zero.
- *Analytic Implicit Function Theorem*: In the usual statement of the Implicit Function Theorem, if the given function is analytic, the implicitly defined function is also analytic.
- Integrals of analytic functions are analytic
- If  $G : \{T\} \times \mathbf{R}^K \rightarrow \mathbf{R}$  is measurable and satisfies the growth condition

$$|G(T, x)| \leq r + e^{r|x|}$$

then

$$F(t, \beta) = E(G(T, \beta(T)) | \beta(t) = \beta)$$

is analytic on  $(0, T) \times \mathbf{R}^K$ .

## 12. Key to Establishing Regularity Properties of the Hyperfinite Equilibrium:

- The hyperfinite equilibrium is Pareto Optimal (no agent can be made better off without making someone else worse off). This implies that the hyperfinite equilibrium individual consumptions are given by an implicit function of the social consumption at each node.
- In our model, consumptions and dividends at times  $t \in [0, T)$  represent flows, while they represent a lump at time  $t = T$ .
- Assume the endowments and preferences of the agents and the dividends of the securities are analytic functions of  $(t, \beta(t, \omega))$  satisfying a growth condition for  $t \in [0, T)$ . The

Analytic Implicit Function Theorem implies the individual consumptions are analytic in  $(t, \beta(t, \omega))$ .

- Assume the endowments and preferences of the agents at time  $T$  are given by separate analytic functions of  $\beta(T, \omega)$  satisfying a growth condition. The Analytic Implicit Function Theorem implies the individual consumptions are analytic in  $\beta(T, \omega)$ .
- Securities prices are expectations of securities dividends times marginal utility of consumption. Since consumption and utility are analytic, marginal utility is analytic.
- Since we want to allow derivatives such as options, and shares of limited liability companies, to be among the basic securities, we are reluctant to assume that dividends at time  $T$  are analytic. So instead, assume the dividends at time  $T$  are measurable, locally Lipschitz (with the Lipschitz constant satisfying a growth condition),  $C^1$  on an open set in  $\mathbf{R}^K$ , and satisfy the Nondegeneracy Condition: for some  $x \in V$ ,

$$G_0(T, x) > 0 \text{ and rank } \begin{pmatrix} \frac{\partial(G_1/G_0)}{\partial\beta} \Big|_{(T,x)} \\ \vdots \\ \frac{\partial(G_J/G_0)}{\partial\beta} \Big|_{(T,x)} \end{pmatrix} = K$$

This implies that  ${}^\circ\hat{D}(t, \hat{\beta})$  is analytic on  $(0, T) \times \mathbf{R}^K$ ,  $SC^1$  on  $(0, T] \times V$ . The Nondegeneracy Condition implies that  ${}^\circ\hat{D}(T, x) \neq 0$  for  $x \in V$ , so  ${}^\circ\hat{D}(t, \hat{\beta}(t, \omega))$  is non-zero on a set of positive measure. Since it is analytic, it is nonzero except on a set of measure zero, which proves the Duffie-Zame spanning condition.

**Theorem 1** *The continuous-time finance model just described has an equilibrium, which is Pareto optimal. Let  $p_A$ ,  $p_c$ ,  $c_i$ , and  $z_i$  denote the equilibrium securities prices, consumption prices, consumptions, and trading strategies.*

- *if we define  $\Sigma = \frac{\partial p_A}{\partial \beta}$  to be the volatility matrix of the securities prices,  $\Sigma_i = z_i \Sigma$  to be the volatility matrix of agent  $i$ 's trading strategy, and  $W_i = z_i \cdot p_A$  to be agent  $i$ 's securities wealth, then each  $\Sigma_i$  and  $W_i$  has a continuous version (still denoted  $\Sigma_i$  and  $W_i$ );*
- *$p_A, c_1, \dots, c_I, W_1, \dots, W_I \in L^2$ ;  $\Sigma, \Sigma_1, \dots, \Sigma_I \in \mathcal{H}^2$ ;  $z_1, \dots, z_I \in \mathcal{H}^2(\gamma)$ ;  $c_i$  is uniformly bounded below, and  $p_C$  is uniformly bounded above;*
- *$p_A, \Sigma, \Sigma_1, \dots, \Sigma_I, W_1, \dots, W_I$  are functions of  $\mathcal{I}(t, \omega)$  which are continuous on  $[0, T] \times \mathbf{R}^K$  and analytic on  $(0, T) \times \mathbf{R}^K$ ; moreover,  $p_A(T, \cdot)$  and  $\Sigma(T, \cdot)$  are analytic functions of  $\beta(T, \omega)$ ;*
- *$p_c$  and  $c_i$  are given separately over  $[0, T) \times \mathbf{R}^k$  and  $\{T\} \times \mathbf{R}^k$  as functions of  $\mathcal{I}(t, \omega)$ ; the functions over  $[0, T) \times \mathbf{R}^k$  are continuous on  $[0, T) \times \mathbf{R}^k$ , have a continuous extension to  $[0, T] \times \mathbf{R}^k$ , and are analytic on  $(0, T) \times \mathbf{R}^k$ ; the functions over  $\{T\} \times \mathbf{R}^k$  are analytic in  $\beta(T, \omega)$ ;*
- *there is a open set  $B$  of full measure zero in  $(0, T) \times \Omega$  and analytic functions  $Z_1, \dots, Z_I$  with domain  $B$  such that  $z_i(t, \omega) = Z_i(\mathcal{I}(t, \omega))$  whenever  $\mathcal{I}(t, \omega) \in B$ ;*
- *the equilibrium prices are effectively dynamically complete: any integrable consumption process which is adapted to the Brownian filtration can be replicated by a unique admissible trading strategy.*

**Theorem 2** *Given a continuous-time economy satisfying the assumptions of Theorem 1, let  $\hat{p}_{nA}$ ,  $\hat{p}_{nc}$ ,  $\hat{c}_{ni}$  and  $\hat{z}_{ni}$  denote any equilibrium securities prices, consumption prices, consumptions and trading strategies of the discretized sequence of economies just described. Then there are equilibrium securities prices  $p_{nA}$ , consumption prices  $p_{nc}$ , consumptions  $c_{ni}$  and trading strategies  $z_{ni}$  of the continuous-time economy satisfying*

- *if we define  $\Sigma_n = \frac{\partial p_{nA}}{\partial \beta}$ ,  $\Sigma_{ni} = z_{ni}\Sigma_n$ , and  $W_{ni} = z_{ni} \cdot p_{nA}$ , then each  $\Sigma_{ni}$  and  $W_{ni}$  has a continuous version (still denoted  $\Sigma_{ni}$  and  $W_{ni}$ );*
- *$p_{nA}, c_{n1}, \dots, c_{nI}, W_{n1}, \dots, W_{nI} \in L^2$ ,  $\Sigma_n, \Sigma_{n1}, \dots, \Sigma_{nI} \in \mathcal{H}^2$ ;  $z_{n1}, \dots, z_{nI} \in \mathcal{H}^2(\gamma)$ ;  $c_{n1}, \dots, c_{nI}$  are uniformly bounded below and the  $p_{nc}$  are uniformly bounded above;*
- *$p_{nA}, \Sigma_n, \Sigma_{n1}, \dots, \Sigma_{nI}$  and  $W_{n1}, \dots, W_{nI}$  are functions of  $\mathcal{I}(t, \omega)$  which are continuous on  $[0, T] \times \mathbf{R}^K$  and analytic on  $(0, T) \times \mathbf{R}^K$ ; moreover,  $p_{nA}(T, \cdot)$  and  $\Sigma_{nA}(T, \cdot)$  are analytic on  $\mathbf{R}^K$ ;*
- *$p_{nc}$  and  $c_{ni}$  are given separately over  $[0, T) \times \mathbf{R}^K$  and  $\{T\} \times \mathbf{R}^k$  as functions of  $\mathcal{I}(t, \omega)$ ; the functions over  $[0, T) \times \mathbf{R}^K$  are continuous on  $[0, T) \times \mathbf{R}^K$ , have a continuous extension to  $[0, T] \times \mathbf{R}^K$ , and are analytic on  $(0, T) \times \mathbf{R}^K$ ; the functions over  $\{T\} \times \mathbf{R}^K$  are analytic in  $\beta(T, \omega)$ ;*
- *There are open sets  $B_n \subset (0, T) \times \mathbf{R}^K$  of full measure and analytic functions  $Z_{n1}, \dots, Z_{nI}$  with domain  $B_n$  such that  $z_{ni}(t, \omega) = Z_{ni}(\mathcal{I}(t, \omega))$  whenever  $\mathcal{I}(t, \omega) \in B_n$ ;*

and such that

$$\|\hat{p}_{nA} - p_{nA} \circ \hat{\mathcal{I}}\|_2 \rightarrow 0 \quad (3)$$

$$\max_{t \in \mathcal{I}_n} |\hat{p}_{nA}(t, \cdot) - p_{nA}(\hat{\mathcal{I}}(t, \cdot))| \rightarrow 0 \text{ in probability} \quad (4)$$

$$\|\hat{p}_{nc} - p_{nc} \circ \hat{\mathcal{I}}\|_2 \rightarrow 0 \quad (5)$$

$$\max_{t \in \mathcal{I}_n} |\hat{p}_{nc}(t, \cdot) - p_{nc}(\hat{\mathcal{I}}(t, \cdot))| \rightarrow 0 \text{ in probability} \quad (6)$$

$$\|\hat{\sigma}_n - \Sigma_n \circ \hat{\mathcal{I}}\|_2 \rightarrow 0 \quad (7)$$

$$\max_{t \in \mathcal{I}_n} |\hat{\sigma}_n(t, \cdot) - \Sigma_n(\hat{\mathcal{I}}(t, \cdot))| \rightarrow 0 \text{ in probability} \quad (8)$$

$$\|\hat{c}_{ni} - c_{ni} \circ \hat{\mathcal{I}}\|_2 \rightarrow 0 \quad (9)$$

$$\max_{t \in \mathcal{I}_n} |\hat{c}_{ni}(t, \cdot) - c_{ni}(\hat{\mathcal{I}}(t, \cdot))| \rightarrow 0 \text{ in probability} \quad (10)$$

$$|\hat{U}_{ni}(\hat{c}_{ni}) - U_{ni}(c_{ni})| \rightarrow 0 \quad (11)$$

$$|\hat{z}_{ni} - z_{ni} \circ \hat{\mathcal{I}}| \rightarrow 0 \text{ in probability} \quad (12)$$

$$\|\hat{\sigma}_{ni} - \Sigma_{ni} \circ \hat{\mathcal{I}}\|_2 \rightarrow 0 \quad (13)$$

$$\max_{t \in \mathcal{I}_n} |\hat{\sigma}_{ni}(t, \cdot) - \Sigma_{ni}(\hat{\mathcal{I}}(t, \cdot))| \rightarrow 0 \text{ in probability} \quad (14)$$

$$\|\hat{z}_{ni} \cdot \hat{p}_{An} - W_{ni} \circ \hat{\mathcal{I}}\|_2 \rightarrow 0 \quad (15)$$

$$\max_{t \in \mathcal{I}_n} |\hat{z}_{ni}(t, \cdot) \cdot \hat{p}_{An} - W_{ni}(\hat{\mathcal{I}}(t, \cdot))| \rightarrow 0 \text{ in probability} \quad (16)$$