mu-differentiability of an internal function

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Definition *[Reeken, 1992]* Let E and F be normed spaces, $U \subseteq E$ open set and $f : {}^*U \to {}^*F$ an internal function. f is **m-differentiable** if

1. for all $a \in {}^\sigma U$ there exist $0 \approx \delta_a \in {}^*\mathbb{R}^+$ and a finite linear operator $Df_a \in {}^*L(E, F)$ such that, for all $x \in {}^*U$, there is some $\eta \approx 0$ with

$$\delta_a < |x - a| \approx 0 \Rightarrow f(x) - f(a) = Df_a(x - a) + |x - a|\eta$$

2. $f(ns({}^*U)) \subseteq ns({}^*F)$.

Theorem *[Schlesinger, 1997]* If E and F are finite dimensional and $f : {}^*K \to {}^*F$ an internal function, with K a compact set, then the following statements are equivalent:

1. f is S-continuous and m-differentiable;

2. There exists a differentiable standard function $g : K \to F$ with

$$\sup_{x \in {}^*K} |f(x) - g(x)| \approx 0.$$

Definition Let E and F be normed spaces, $U \subseteq E$ open set and $f : {}^*U \to {}^*F$ an internal function. f is **mu-differentiable** if

1. for each $a \in {}^\sigma U$ there exists a positive infinitesimal δ_a such that, for all $x \in \mu(a)$, there exists a finite linear operator $Df_x \in {}^*L(E, F)$ for which holds

$$\forall y \in \mu(a)\ |x - y| > \delta_a \Rightarrow f(x) - f(y) = Df_x(x - y) + |x - y|\eta$$

   for some $\eta \approx 0$.

2. $f(ns({}^*U)) \subseteq ns({}^*F)$.

Theorem Let $f : {}^*U \to {}^*F$ be a mu-differentiable function. Then, for all $x, y \in ns({}^*U)$ with $x \approx y$, we have

1. $f(x) \approx f(y)$;

2. if $d \in {}^*E$ with $|d| = 1$, $Df_x(d) \approx Df_y(d)$.

Theorem If E and F are finite dimensional and $f : {}^*U \to {}^*F$ an internal function, then:

1. If f is mu-differentiable then $st(f) : U \to F$ is a C^1 function, $Dst(f)_a = stDf_a$ for $a \in {}^\sigma U$ and
$$\forall a \in {}^\sigma U \ \exists \eta_0 \approx 0 \ \forall x \approx a \quad |f(x) - st(f)(x)| \leq \eta_0.$$
2. If there exists a C^1 standard function $g : U \to F$ with
$$\forall a \in {}^\sigma U \ \exists \eta_0 \approx 0 \ \forall x \approx a \quad |f(x) - g(x)| \leq \eta_0,$$
then f is mu-differentiable. The function $g = st(f)$.

Proof

(1) (...)

(2) Fix $a \in {}^{\sigma}U$ and let $\delta_a := \sqrt{\eta_0}$. Fix $x, y \in \mu(a)$ with $\delta_a < |x - y|$.

Since g is of class C^1 then

$$g(x) - g(y) = Dg_x(x - y) + |x - y|\eta$$

for some $\eta \approx 0$.

Define $\epsilon_1 := g(x) - f(x)$ and $\epsilon_2 := g(y) - f(y)$. Then

$$f(x) - f(y) = Dg_x(x - y) + |x - y|\eta + \epsilon_2 - \epsilon_1$$

and

$$\frac{|\epsilon_1 - \epsilon_2|}{|x - y|} \leq \frac{|\epsilon_1| + |\epsilon_2|}{|x - y|} \leq \frac{2\eta_0}{\sqrt{\eta_0}} \approx 0.$$

□

Corollary If $f : U \to F$ is a standard function then

$$f \text{ is of class } C^1 \Leftrightarrow f \text{ is mu-differentiable}$$

Theorem If $f : {}^*U \to {}^*F$ is a mu-differentiable function, then

- $\forall a \in {}^\sigma U \; \exists \delta \approx 0 \; \forall d \in {}^*E \; \exists L \in fin({}^*F) \; \forall x \in {}^*U$
  $$|d| = 1 \wedge \left[x \approx a \Rightarrow \frac{f(x+\delta d) - f(x)}{\delta} \approx L \right].$$
- $\forall x \in ns({}^*U) \; \exists \delta_x \approx 0 \; \exists Df_x \in {}^*L(E,F) \; \forall y \in {}^*U \; \exists \eta \approx 0$
  $|Df_x|$ is finite $\wedge \left[\delta_x < |x-y| \approx 0 \Rightarrow f(x) - f(y) = Df_x(x-y) + |x-y|\eta\right].$
- $\forall a \in {}^\sigma U \; \exists \delta_a \approx 0 \; \exists Df_a \in {}^*L(E,F) \; \forall x, y \in \mu(a) \; \exists \eta \approx 0$
  $|Df_a|$ is finite $\wedge \left[|x-y| > \delta_a \Rightarrow f(x) - f(y) = Df_a(x-y) + |x-y|\eta\right].$

Theorem If E and F are finite dimensional and $f : {}^*U \rightarrow {}^*F$ an internal function, then:

1. If f is k-times mu-differentiable then $st(f) : U \rightarrow F$ is a C^k function, $D^j st(f)_a = st D^j f_a$ for $j = 1, 2, \ldots, k$ and $a \in {}^\sigma U$. Furthermore
$$\forall j \in \{0, 1, \ldots, k-1\}\ \forall a \in {}^\sigma U\ \exists \eta_j \approx 0\ \forall x \approx a \quad |D^j f_x - D^j st(f)_x| \leq \eta_j.$$

2. If there exists a C^k standard function $g : U \rightarrow F$ with
$$\forall j \in \{0, 1, \ldots, k-1\}\ \forall a \in {}^\sigma U\ \exists \eta_j \approx 0\ \forall x \approx a \quad |D^j f_x - D^j g_x| \leq \eta_j$$
then f is k-times mu-differentiable and $g = st(f)$.

Taylor's Theorem. Let E and F be two standard finite dimensional normed spaces,U a standard open set and $f : {}^*U \rightarrow {}^*F$ a function k-times mu-differentiable, $k \in {}^\sigma\mathbb{N}$. Then,

1. for every $x \in ns({}^*U)$, there exists $\epsilon \approx 0$ such that, whenever $y \in {}^*U$ with $\epsilon < |y-x| \approx 0$, there exists $\eta \approx 0$ satisfying

$$f(y) = f(x)+Df_x(y-x)+\frac{1}{2!}D^2f_x(y-x)^{(2)}+...+\frac{1}{k!}D^kf_x(y-x)^{(k)}+|y-x|^k\eta.$$

2. for every $x \in ns({}^*U)$, there exists $\epsilon \approx 0$ such that, whenever $y \in {}^*U$ with $\epsilon < |y-x| \approx 0$, there exists $\eta \approx 0$ satisfying

$$f(y) = st(f)(x) + Dst(f)_x(y-x) + \frac{1}{2!}D^2st(f)_x(y-x)^{(2)} + \dots$$

$$+\frac{1}{k!}D^kst(f)_x(y-x)^{(k)} + |y-x|^k\eta.$$

Proof

(1) Define the sequence $(\epsilon_i)_{i=-1,\dots,k-1}$ by

- $f(y) = st(f)(y) + \epsilon_{-1}$, $(\epsilon_{-1} \leq \eta_0)$;
- $f(x) = st(f)(x) + \epsilon_0$, $(\epsilon_0 \leq \eta_0)$;
- $Df_x(y-x) = Dst(f)_x(y-x) + |y-x|\epsilon_1$, $(\epsilon_1 \leq \eta_1)$;
- ...
- $D^{k-1}f_x(y-x)^{(k-1)} = D^{k-1}st(f)_x(y-x)^{(k-1)} + |y-x|^{k-1}\epsilon_{k-1}$, $(\epsilon_{k-1} \leq \eta_{k-1})$;

Furthermore

$$D^k f_x \left(\frac{y-x}{|y-x|}\right)^{(k)} \approx D^k f_a \left(\frac{y-x}{|y-x|}\right)^{(k)} \approx D^k st(f)_a \left(\frac{y-x}{|y-x|}\right)^{(k)} \approx D^k st(f)_x \left(\frac{y-x}{|y-x|}\right)^{(k)},$$

so there exists $\epsilon_k \approx 0$ with

$$D^k f_x(y-x)^{(k)} = D^k st(f)_x(y-x)^{(k)} + |y-x|^k \epsilon_k.$$

Define $\epsilon = \max\{\eta_0^{\frac{1}{k+1}}, \eta_1^{\frac{1}{k}}, ..., \eta_{k-1}^{\frac{1}{2}}\}$ and take $y \in {}^*U$ with $\epsilon < |y-x| \approx 0$.

Since $st(f)$ is of class C^k, then

$$st(f)(y) = st(f)(x) + Dst(f)_x(y-x) + \frac{1}{2!}D^2st(f)_x(y-x)^{(2)} + ... + \frac{1}{k!}D^kst(f)_x(y-x)^{(k)} + |y-x|^k\eta$$

Consequently

$$f(y) = f(x) + Df_x(y-x) + \frac{1}{2!}D^2f_x(y-x)^{(2)} + ... + \frac{1}{k!}D^kf_x(y-x)^{(k)} + |y-x|^k\eta +$$

$$+\epsilon_{-1} - \epsilon_0 - |y-x|\epsilon_1 - |y-x|^2\epsilon_2 - ... - |y-x|^{k-1}\epsilon_{k-1} - |y-x|^k\epsilon_k$$

and

$$\frac{|\epsilon_{-1}|}{|y-x|^k} + \frac{|\epsilon_0|}{|y-x|^k} + \frac{|\epsilon_1|}{|y-x|^{k-1}} + \frac{|\epsilon_2|}{|y-x|^{k-2}} + \ldots + \frac{|\epsilon_{k-1}|}{|y-x|} \leq$$

$$\leq \frac{\eta_0}{\eta_0^{\frac{k}{k+1}}} + \frac{\eta_0}{\eta_0^{\frac{k}{k+1}}} + \frac{\eta_1}{\eta_1^{\frac{k-1}{k}}} + \frac{\eta_2}{\eta_2^{\frac{k-2}{k-1}}} + ... + \frac{\eta_{k-1}}{\eta_{k-1}^{\frac{1}{2}}} \approx 0.$$

(2) Define $\epsilon := \eta_0^{\frac{1}{k+1}}$, (...)

$\square$

Chain Rule. Let g, f be two m-differentiable functions at a and $g(a)$, respectively, where a and $g(a)$ are two standards. If Dg_a is invertible and $|(Dg_a)^{-1}|$ is finite, then $f \circ g$ is m-differentiable at a and $D(f \circ g)_a = Df_{g(a)} Dg_a$

Proof Take $\delta :=$ max$\{\delta_a, 2\delta_{g(a)}|(Dg_a)^{-1}|\}$ and choose x with $\delta < |x-a| \approx 0$. Then

$$0 \approx |g(x) - g(a)| = |Dg_a(x-a) + |x-a|\eta_1| > 2\delta_{g(a)}|(Dg_a)^{-1}| \left| Dg_a \left(\frac{x-a}{|x-a|} \right) + \eta_1 \right| > \delta_{g(a)}$$

and

$$f(g(x)) - f(g(a)) = Df_{g(a)}(g(x) - g(a)) + |g(x) - g(a)|\eta_2 = Df_{g(a)} Dg_a(x-a) + |x-a|\eta$$

for some $\eta \approx 0$.

□

Mean Value Theorem. Let $f : {}^*U \to {}^*\mathbb{R}$ be a mu-differentiable function with U open and convex. Then, for all $x, y \in ns({}^*U)$ with $|x-y| > \delta_a$, where $a := st(x)$

$$\exists c \in [x,y] \;\; f(x) - f(y) = Df_c(x-y) + |x-y|\eta$$

for some $\eta \approx 0$.

Proof Define an hyper-finite sequence $(x_n)_{n\in I}$ in the following way:

Let $x_1 = x$ and fix $\delta_1 \approx 0$ with, for all $z \in {}^*U$:

$$\delta_1 < |z - x_1| \approx 0 \Rightarrow f(z) - f(x_1) = Df_{x_1}(z - x_1) + |z - x_1|\eta_1.$$

Let $x_2 = x_1 + 2\delta_1 \frac{y-x}{|y-x|}$ and fix $0 \approx \delta_2 > \delta_1$ with, for all $z \in {}^*U$:

$$\delta_2 < |z - x_2| \approx 0 \Rightarrow f(z) - f(x_2) = Df_{x_2}(z - x_2) + |z - x_2|\eta_2$$

and take $x_3 = x_2 + 2\delta_2 \frac{y-x}{|y-x|}$.

Repeating the process, we obtain a sequence $\{x_n | 1 \leq n \leq N+1\}$ which satisfies the conditions

- $x_1 = x$;
- $x_{n+1} = x_n + 2\delta_n \frac{y-x}{|y-x|}$, $\delta_n \approx 0$ and $\delta_n > \delta_1$, $n = 1, \ldots, N$;
- $f(x_{n+1}) - f(x_n) = Df_{x_n}(x_{n+1} - x_n) + |x_{n+1} - x_n|\eta_n$, for some $\eta_n \approx 0$, $n = 1, \ldots, N$;
- $x_{N+1} = y$ (if not, choose $0 \approx \delta > \delta_N$ with $x_N + 2\delta \frac{y-x}{|y-x|} = y$).

Then

$$f(x) - f(y) = \sum_{n=1}^{N} (f(x_n) - f(x_{n+1})) = \sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1}) + \sum_{n=1}^{N} |x_n - x_{n+1}|\eta_n$$

and

$$\frac{\left| \sum_{n=1}^{N} |x_n - x_{n+1}|\eta_n \right|}{|x - y|} \approx 0.$$

We will prove now that there exists $c \in [x, y]$ such that

$$Df_c\left(\frac{x-y}{|x-y|}\right) \approx \frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{|x-y|}.$$

Letting $d := \frac{x-y}{|x-y|}$, it is true that

$$\frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{|x-y|} = \frac{\sum_{n=1}^{N} Df_{x_n}(x_n - x_{n+1})}{\sum_{n=1}^{N} |x_n - x_{n+1}|} = \frac{\sum_{n=1}^{N} 2\delta_n Df_{x_n}(d)}{\sum_{n=1}^{N} 2\delta_n}.$$

Choosing $m, M \in \{x_1, ..., x_N\}$ with

$$Df_m(d) = \min_{1 \le n \le N} Df_{x_n}(d) \;\; \& \;\; Df_M(d) = \max_{1 \le n \le N} Df_{x_n}(d),$$

we get

$$Df_m(d) \le \frac{\sum_{n=1}^{N} 2\delta_n Df_{x_n}(d)}{\sum_{n=1}^{N} 2\delta_n} \le Df_M(d).$$

So, there exists $c \in [m, M] \subseteq [x, y]$ with

$$Df_c(d) \approx \frac{\sum_{n=1}^{N} 2\delta_n Df_{x_n}(d)}{\sum_{n=1}^{N} 2\delta_n}. \qquad \square$$

Norm Mean Value Theorem Let $f : {}^*U \to {}^*F$ be a mu-differentiable function with U open and convex. Then, for all $x, y \in ns({}^*U)$ with $|x - y| > \delta_a$, where $a := st(x)$

$$\exists c \in [x, y] \ \ |f(x) - f(y)| \leq |Df_c(x - y)| + |x - y|\eta$$

for some $\eta \approx 0$.

Inverse Mapping Theorem Let $f : {}^*U \to {}^*F$ be a mu-differentiable function and $a \in {}^\sigma U$. If Df_a is invertible and $|(Df_a)^{-1}|$ is finite, then there exists a standard neighborhood V of a such that

$$\forall x, y \in {}^\sigma V \ x \neq y \Rightarrow f(x) \neq f(y).$$

Proof Let

$$A := \left\{ \epsilon \in {}^*\mathbb{R}^+ | \ \forall x, y \in B_\epsilon(a) \ \ |x - y| > \delta_a \Rightarrow f(x) \neq f(y) \right\}.$$

Then A contains all positive infinitesimal numbers since, for $\epsilon \approx 0^+$ and $x, y \in B_\epsilon(a)$ with $|x - y| > \delta_a$

$$\frac{f(x) - f(y)}{|x - y|} \approx Df_a \left(\frac{x - y}{|x - y|} \right).$$

But

$$\left| Df_a \left(\frac{x - y}{|x - y|} \right) \right| \geq \frac{1}{|(Df_a)^{-1}|} \not\approx 0$$

and therefore $f(x) \neq f(y)$. By Cauchy's Principle there exists $\epsilon \in {}^\sigma\mathbb{R}$ with $\epsilon \in A$. Define $V := B_\epsilon(a)$. The proof follows.

$\square$

Reference

- Schlesinger, K., *Generalized manifolds*, Addison Wesley Longman, 1997.