QUANTIFIERS IN LIMITS

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An important advantage of the nonstandard approach to calculus is that it eliminates two quantifiers in the definition of a limit. The standard definition of $\lim_{z\to\infty} F(z) = \infty$ is an $\forall\exists\forall$ sentence. But Abraham Robinson showed that in the nonstandard setting, this is equivalent to the one quantifier statement that $F(z)$ is infinite for all infinite z.

In the standard setting, the number of quantifier blocks needed to define the limit depends on the underlying structure $\mathcal{M}$ in which one is working. Given a first order structure $\mathcal{M}$ with an ordering, we add a new function symbol F to the vocabulary of $\mathcal{M}$ and ask for the minimum number of quantifier blocks needed to define the class of structures $(\mathcal{M}, F)$ in which $\lim_{z\to\infty} F(z) = \infty$ holds.

Our main results show that in the standard setting the limit cannot be defined with fewer than three quantifier blocks, provided that the underlying structure $\mathcal{M}$ is not too powerful. In the cases that $\mathcal{M}$ is countable or saturated, the limit cannot be defined by an $\exists\forall\exists$ sentence. In the case that $\mathcal{M}$ is an o-minimal expansion of the real ordered field, the problem is open for $\exists\forall\exists$, but we show that the limit cannot be defined by a Boolean combination of $\forall\exists$ sentences.

In the standard setting, there are also structures $\mathcal{M}$ which are so powerful that the limit can be defined in both two-quantifier forms. We show that there is no structure $\mathcal{M}$ over which the limit can be defined by a Boolean combination of universal sentences.

These results clarify the statement that nonstandard analysis reduces the quantifier complexity of the limit concept.

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