ISOPERIMETRIC PLANAR CLUSTERS WITH INFINITELY MANY REGIONS

MATTEO NOVAGA, EMANUELE PAOLINI, EUGENE STEPANOV, AND VINCENZO MARIA TORTORELLI

ABSTRACT. An infinite cluster $\mathbf{E}$ in $\mathbb{R}^d$ is a sequence of disjoint measurable sets $E_k \subset \mathbb{R}^d$, $k \in \mathbb{N}$, called regions of the cluster. Given the volumes $a_k \geq 0$ of the regions E_k, a natural question is the existence of a cluster $\mathbf{E}$ which has finite and minimal perimeter $P(\mathbf{E})$ among all clusters with regions having such volumes. We prove that such a cluster exists in the planar case $d = 2$, for any choice of the areas a_k with $\sum \sqrt{a_k} < \infty$. We also show the existence of a bounded minimizer with the property $P(\mathbf{E}) = \mathcal{H}^1(\partial \mathbf{E})$, where $\partial \mathbf{E}$ denotes the measure theoretic boundary of the cluster. We also provide several examples of infinite isoperimetric clusters for anisotropic and fractional perimeters.

CONTENTS

1. INTRODUCTION

A finite *cluster* $\mathbf{E}$ is a sequence $\mathbf{E} = (E_1, \dots E_k, \dots, E_N)$ of measurable sets, such that $|E_k \cap E_j| = 0$ for $k \neq j$, where $|\cdot|$ denotes the Lebesgue measure (usually called *volume*). The sets E_j are called *regions* of the cluster $\mathbf{E}$ and $E_0 := \mathbb{R}^d \setminus \bigcup_{k=1}^\infty E_k$ is called *external region*. We denote the sequence of volumes of the regions of the cluster $\mathbf{E}$ as

$$\mathbf{m}(\mathbf{E}) := (|E_1|, |E_2|, \dots, |E_N|) \tag{1}$$

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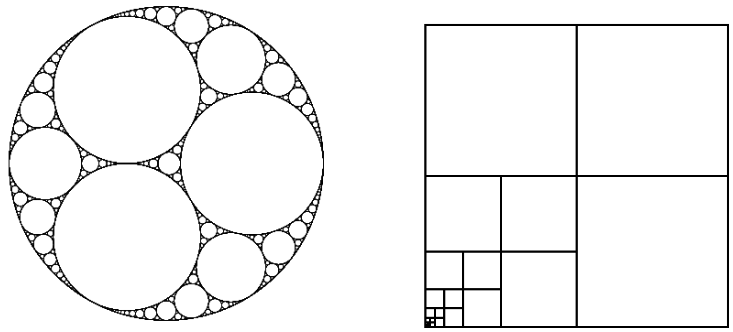

Figure 1. The Apollonian gasket, on the left-hand side, is a cluster with minimal fractional perimeter. On the right-hand side a similar construction with squares: this is a minimal cluster with respect to the perimeter induced by the Manhattan distance.

and we call *perimeter* of the cluster the quantity

$$P(\mathbf{E}) := \frac{1}{2}\left[P(E_0) + \sum_{k=1}^{N} P(E_k)\right], \tag{2}$$

where P is the Caccioppoli perimeter. A cluster $\mathbf{E}$ is called *minimal*, or *isoperimetric*, if

$$P(\mathbf{E}) = \min\left\{P(\mathbf{F})\colon \mathbf{m}(\mathbf{F}) = \mathbf{m}(\mathbf{E})\right\}.$$

In this paper we consider infinite clusters, i.e., infinite sequences $\mathbf{E} = (E_k)_{k\in\mathbb{N}}$ of essentially pairwise disjoint regions: $|E_j \cap E_i| = 0$ for $i \neq j$ (this can be interpreted as a model for a soap foam). Note that a finite cluster with N regions, can also be considered a particular case of an infinite cluster for example by posing $E_k = \emptyset$ for $k > N$. Clusters with infinitely many regions of equal areas were considered in [12], where it has been shown that the honeycomb cluster is the unique minimizer with respect to compact perturbations. Infinite clusters have been considered also in [17, 13, 3], where the variational curvature is prescribed, and in [23], where existence of generalized minimizers for both finite and infinite isoperimetric clusters has been proven in the general setting of homogeneous metric measure spaces.

An interesting example of infinite cluster, detailed in Example 4.1 (see Figure 1) is the Apollonian packing of a circle (see [15]). In fact this cluster is composed by isoperimetric regions and hence should trivially have minimal perimeter among clusters with regions of the same areas. Actually, it turns out that this cluster has infinite perimeter and hence all clusters with same prescribed areas have infinite perimeter too. Note that very few explicit examples of minimal clusters are known [10, 26, 14, 25, 19]. Nevertheless, quite curiously, Apollonian packings give nontrivial examples of infinite isoperimetric clusters for fractional perimeters [4, 7, 6], as shown in Example 4.1. An even simpler example of an infinite isoperimetric planar cluster is given in Example 4.2 (see Figure 1 again) where the Caccioppoli perimeter is replaced by an anisotropic perimeter functional [16, 21, 22, 5, 8].

Our main result, Theorem 3.1, states that if $d = 2$ (planar case), given any sequence of positive numbers $\mathbf{a} = (a_1, a_2, \dots, a_k, \dots)$ such that $\sum_{k=0}^{\infty} \sqrt{a_k} < +\infty$, there exists a minimal cluster $\mathbf{E}$ in $\mathbb{R}^2$ with $\mathbf{m}(\mathbf{E}) = \mathbf{a}$. The assumption on $\mathbf{a}$ is necessary to have at least a competitor cluster with finite perimeter. The proof relies on two facts which are only available in the planar case: the isodiametric inequality for connected sets and the semicontinuity of the length of connected sets (Gołąb theorem).

2. Notation and preliminaries

2.1. **Perimeters and boundaries.** For a set $E \subset \mathbb{R}^d$ with finite perimeter one can define the *reduced boundary* $\partial^* E$ which is the set of boundary points x where the outer normal vector $\nu_E(x)$ can be defined. One has $D1_E = \nu_E \cdot \mathcal{H}^{d-1} \llcorner \partial^* E$ where 1_E is the characteristic function of E and $D1_E$ is its distributional derivative (the latter is a vector valued measure and its total variation is customarily denoted by $|D1_E|$). The measure theoretic boundary of a measurable set E is defined by

$$\partial E := \{x \in \mathbb{R}^d \colon 0 < |E \cap B_\rho(x)| < |B_\rho(x)| \quad \text{for all } \rho > 0\}.$$

The corresponding notions for clusters can be defined as follows:

$$\begin{aligned}\partial^* \mathbf{E} &:= \bigcup_{k=1}^{+\infty} \bigcup_{j=0}^{k-1} \partial^* E_k \cap \partial^* E_j, \\ \partial \mathbf{E} &:= \big\{x \in \mathbb{R}^d \colon 0 < |E_k \cap B_\rho(x)| < |B_\rho(x)| \\ &\qquad \text{for all } \rho > 0 \text{ and some } k = k(\rho, x) \in \mathbb{N}\big\}.\end{aligned}$$

Clearly $\partial^* \mathbf{E} \subseteq \partial \mathbf{E}$ because given $x \in \partial^* \mathbf{E}$ there exists k such that $x \in \partial^* E_k$, while $\partial E_k \subseteq \partial \mathbf{E}$ for all k. Also it is easy to check that $\partial \mathbf{E}$ is closed (and it is the closure of the union of all the measure theoretic boundaries ∂E_k). Moreover the following result holds true.

Proposition 2.1. *If* $\mathbf{E}$ *is a cluster with finite perimeter, then* $P(\mathbf{E}) = \mathcal{H}^{d-1}(\partial^* \mathbf{E})$.

Proof. Consider the sets X_n, for $1 \le n \le \infty$, defined by

$$X_n := \big\{x \in \mathbb{R}^d \colon \#\{k \in \mathbb{N} \colon x \in \partial^* E_k\} = n\big\}$$

(notice that $k = 0 \in \mathbb{N}$, the external region, is included in the count). It is clear that $X_n = \emptyset$ for all $n \ge 3$ because in every point of $\partial^* E_k$ there is an approximate tangent hyper-plane which can only be shared by two regions.

We claim that $\mathcal{H}^{d-1}(X_1) = 0$. To this aim suppose by contradiction that $\mathcal{H}^{d-1}(X_1) > 0$. Then there exists a $j \in \mathbb{N}$ such that

$$\left|D1_{E_j}\right|(X_1) = \mathcal{H}^{d-1}(X_1 \cap \partial^* E_j) > 0,$$

because X_1 is contained in the countable union $\cup_{j=0}^{\infty} X_1 \cap \partial^* E_j$. Hence there is a subset $A \subset X_1 \cap \partial^* E_j$ such that $D1_{E_j}(A) \neq \mathbf{0}$. Notice that $\sum_{k=0}^{\infty} 1_{E_k} = 1$, hence also $\sum_k D1_{E_k} = \mathbf{0}$. Since $D1_{E_j}(A) \neq \mathbf{0}$ there must exist at least another index $k \neq j$ such that $D1_{E_k}(A) \neq \mathbf{0}$, hence $\mathcal{H}^{d-1}(A \cap \partial^* E_k) > 0$. But then

$$\emptyset \neq A \cap \partial^* E_k \subset X_1 \cap \partial^* E_j \cap \partial^* E_k, \quad j \neq k,$$

contrary to the definition of X_1, which proves the claim.

In conclusion, the union of all reduced boundaries $\partial^* E_k$ is contained in X_2 up to a $\mathcal{H}^{d-1}$-negligible set. Hence

$$\begin{aligned} P(\mathbf{E}) &= \frac{1}{2}\sum_{k=0}^{+\infty} P(E_k) = \frac{1}{2}\sum_{k=0}^{+\infty} \mathcal{H}^{d-1}(\partial^* E_k \cap X_2) = \\ &= \frac{1}{2}\sum_{k=0}^{+\infty}\sum_{j\neq k} \mathcal{H}^{d-1}(\partial^* E_k \cap \partial^* E_j) = \sum_{k=0}^{+\infty}\sum_{j=k+1}^{+\infty} \mathcal{H}^{d-1}(\partial^* E_k \cap \partial^* E_j) = \\ &= \mathcal{H}^{d-1}\left(\bigsqcup_{k=0}^{+\infty}\bigsqcup_{j=k+1}^{+\infty} \partial^* E_k \cap \partial^* E_j\right) = \mathcal{H}^{d-1}(\partial^* \mathbf{E}) \end{aligned}$$

as claimed. □

2.2. **Auxiliary results.** In the following theorem we collect known existence and regularity results for *finite* minimal clusters from [20, 18].

Theorem 2.2 (existence and regularity of planar N-clusters)**.** *Let $a_1, a_2, \dots, a_N$ be given positive real numbers. Then there exists a minmal N-cluster $\mathbf{E} = (E_1, \dots E_N)$ in $\mathbb{R}^s$, with $|E_k| = a_k$ for $k = 1, \dots, N$. If $\mathbf{E}$ is a minimal N-cluster and $d = 2$, then $\partial\mathbf{E}$ is a pathwise connected set composed by circular arcs or line segments joining in triples at a finite number of vertices. Moreover in this case $P(\mathbf{E}) = \mathcal{H}^1(\partial\mathbf{E})$.*

The statement below gives isodiametric inequality for planar finite clusters,

Proposition 2.3 (diameter estimate)**.** *If $\mathbf{E}$ is an N-cluster in $\mathbb{R}^2$ and $\partial\mathbf{E}$ is pathwise connected, then*

$$\operatorname{diam} \partial\mathbf{E} \le P(\mathbf{E}).$$

Proof. Since $\partial\mathbf{E}$ is pathwise connected, given any two points $x, y \in \partial\mathbf{E}$ we find that $|x - y| \le \mathcal{H}^1(\partial\mathbf{E}) = P(\mathbf{E})$. □

Another ingredient will be the following statement on cluster truncation,

Proposition 2.4 (cluster truncation)**.** *Let $\mathbf{E} = (E_1, \dots, E_k, \dots)$ be a (finite or infinite) cluster and let $T_N\mathbf{E}$ be the N-cluster $(E_1, \dots, E_N)$. Then*

$$P(T_N\mathbf{E}) \le P(\mathbf{E}).$$

Proof. For measurable sets E, F the inequality

$$P(E \cup F) + P(E \cap F) \le P(E) + P(F)$$

holds, hence if $|E \cap F| = 0$, one has

$$P(E) = P((E \cup F) \cap (\mathbb{R}^d \setminus F)) \le P(E \cup F) + P(F).$$

It follows that

$$
\begin{aligned}
2P(T_N\mathbf{E}) &= \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{n} E_k\right) \\
&\le \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) + P\left(\bigcup_{k=n+1}^{\infty} E_k\right) \\
&\le \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) + \sum_{k=n+1}^{\infty} P(E_k) \\
&= \sum_{k=1}^{\infty} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) = 2P(\mathbf{E})
\end{aligned}
$$

as claimed. □

Lemma 2.5. *Let E be a measurable set and Ω an open connected set. If $\partial E \cap \Omega = \emptyset$, then either $|\Omega \cap E| = 0$ or $|\Omega \setminus E| = 0$.*

Proof. Notice that $\Omega \setminus \partial E = A_0 \cup A_1$, where

$$
\begin{aligned}
A_0 &:= \{x \in \Omega\colon\ |B_\rho(x) \cap E| = 0 \quad \text{for some } \rho > 0\}\,, \\
A_1 &:= \{x \in \Omega\colon\ |B_\rho(x) \setminus E| = 0 \quad \text{for some } \rho > 0\}\,.
\end{aligned}
$$

It is clear that A_0 and A_1 are open disjoint sets, and if $\partial E \cap \Omega = \emptyset$ their union is the whole set Ω. If Ω is connected, then either A_0 or A_1 is equal to Ω which means that either $|\Omega \cap E| = 0$ or $|\Omega \setminus E| = 0$. □

3. Main result

The statement below provides existence of infinite planar isoperimetric clusters.

Theorem 3.1 (existence)**.** *Let $\mathbf{a} = (a_1, \dots, a_k, \dots)$ be a sequence of nonnegative numbers such that $\sum_{k=1}^{\infty} \sqrt{a_k} < \infty$. Then there exists a minimal cluster $\mathbf{E}$ in $\mathbb{R}^2$, with $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ satisfying additionally*

$$
\bigcup_{k=1}^{\infty} E_k \text{ is bounded,} \tag{3}
$$

$$
\partial\mathbf{E} \text{ is pathwise connected,} \tag{4}
$$

$$
\mathcal{H}^1(\partial\mathbf{E} \setminus \partial^*\mathbf{E}) = 0. \tag{5}
$$

Remark 3.2. In view of (5) and Proposition 2.1, for the minimal cluster provided by the above Theorem 3.1, one has

$$
P(\mathbf{E}) = \mathcal{H}^1(\partial\mathbf{E}) = \mathcal{H}^1(\partial^*\mathbf{E}). \tag{6}
$$

Of course there exists a set with finite perimeter E such that $P(E) < \mathcal{H}^1(\partial E)$ hence (6) is false for general, non minimal, clusters.

It is interesting to note that, as shown in example 4.3, there exists a finite cluster $\mathbf{E}$ satisfying (6), for which one does not have $P(E_k) = \mathcal{H}^1(\partial E_k)$ for all k. It would be interesting to see whether these equalities hold for minimal clusters.

Proof. Let $\bar{p} := 2\sqrt{\pi}\sum_{k=1}^{\infty}\sqrt{a_k} < +\infty$, and

$$p := \inf\{P(\mathbf{E})\colon \mathbf{E} \text{ cluster in } \mathbb{R}^2 \text{ with } |E_k| = a_k,\ k = 1, 2, \dots, n, \dots\},$$
$$p_n := \inf\{P(\mathbf{E})\colon \mathbf{E}\ n\text{-cluster in } \mathbb{R}^2 \text{ with } |E_k| = a_k,\ k = 1, \dots, n\}$$

so that a cluster $\mathbf{E}$ with measures $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ is minimal if and only if $P(\mathbf{E}) = p$, while an n-cluster $\mathbf{E}$ with measures $|E_k| = a_k$ for $k = 1, \dots, n$ is minimal if and only if $P(\mathbf{E}) = p_n$.

If $\mathbf{E}$ is a competitor for p, then $T_n\mathbf{E}$ is a competitor for p_n and, by Proposition 2.4, one has $P(T_n\mathbf{E}) \le P(\mathbf{E})$. Hence $p_n \le p$. Moreover one can build a competitor for p which is composed by circular disjoint regions $(B_1, \dots, B_j, \dots)$, where B_j are disjoint balls of radii $\sqrt{\frac{a_j}{\pi}}$, to find that $p \le \bar{p} < +\infty$.

For each $n \ge 1$ consider a minimal n-cluster $\mathbf{F}^n$ with $|F_k^n| = a_k$ for $k \le n$, $F_k^n := \emptyset$ for $k > n$ so that $P(\mathbf{F}^n) = p_n$. Hence, by Proposition 2.3, up to translations we might and shall suppose that all the regions F_k^n of all the clusters $\mathbf{F}^n$ are contained in a ball of radius $\bar{p}$. In fact:

$$\bar{p} \ge p \ge \sup_n p_n = \sup_n P(\mathbf{F}^n) \ge \sup_n \operatorname{diam} \partial\mathbf{F}^n.$$

Up to a subsequence we can hence assume that the first regions F_1^n converge to a set E_1 in the sense that their characteristic functions $\mathbf{1}_{F_1^n}$ converge to the characteristic function $\mathbf{1}_{E_1}$ in the Lebesgue space $L^1(\mathbb{R}^2)$ (we call this convergence L^1 convergence of sets). Analogously, up to a sub-subsequence also the second regions F_2^n converge in L^1 sense to a set E_2, and in this way we define inductively the sets E_k for all $k \ge 1$. Then there exists a diagonal subsequence with indices n_j such that for all k one has $F_k^{n_j} \to E_k$ in L^1 for all $k \ge 1$ as $j \to +\infty$.

Consider the cluster $\mathbf{E}$ with the components E_k defined above. By continuity we have $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ because $F_k^{n_j} \to E_k$ in L^1 as $j \to \infty$ and $\left|F_k^{n_j}\right| = a_k$ for all j. We claim that the union of all the regions of $\mathbf{F}^{n_j}$ also converges to the union of all the regions of $\mathbf{E}$. For all $\varepsilon > 0$ take N such that $\sum_{k=N+1}^{\infty} a_k \le \varepsilon$ and notice that

$$\left(\bigcup_{k=1}^{\infty} E_k\right) \triangle \left(\bigcup_{k=1}^{\infty} F_k^{n_j}\right) \subseteq \bigcup_{k=1}^{N} \left(E_k \triangle F_k^{n_j}\right) \cup \bigcup_{k=N+1}^{\infty} E_k \cup \bigcup_{k=N+1}^{\infty} F_k^{n_j}.$$

Hence

$$\limsup_j \left|\bigcup_{k=1}^{\infty} E_k \triangle \bigcup_{k=1}^{\infty} F_k^{n_j}\right| \le \lim_j \sum_{k=1}^{N} \left|E_k \triangle F_k^{n_j}\right| + 2\varepsilon = 2\varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the claim.

By lower semicontinuity of perimeter:

$$P(E_k) \le \liminf_{j\to+\infty} P(F_k^{n_j}) \qquad \text{and} \qquad P\left(\bigcup_{k=1}^{+\infty} E_k\right) \le \liminf_{j\to+\infty} P\left(\bigcup_{k=1}^{+\infty} F_k^{n_j}\right)$$

and hence $P(\mathbf{E}) \le \liminf_j P(\mathbf{F}^{n_j}) \le p$ proving that $\mathbf{E}$ is actually a minimal cluster. Since all the regions F_k^n are equi-bounded we obtain (3).

We are going to prove (5). By Theorem 2.2 the minimal n-cluster $\mathbf{F}^n$ has a measure theoretic boundary $\partial\mathbf{F}^n$ which is a compact and connected set such that $P(\mathbf{F}^n) = \mathcal{H}^1(\partial\mathbf{F}^n)$. Up to a subsequence, the compact sets $\partial\mathbf{F}^{n_j}$, being uniformly bounded, converge with respect to the Hausdorff distance, to a compact set K. Without loss of generality suppose n_j is labeling this new subsequence.

We claim that $\partial\mathbf{E} \subseteq K$. In fact for any given $x \in \partial\mathbf{E}$ and any $\rho > 0$ there exists $k = k(\rho)$ such that $B_\rho(x) \cap E_k$ and $B_\rho(x) \setminus E_k$ both have positive measure. Since $\left|B_\rho(x) \cap F_k^{n_j}\right| \to |B_\rho(x) \cap E_k| > 0$ and $\left|B_\rho(x) \setminus F_k^{n_j}\right| \to |B_\rho(x) \setminus E_k| > 0$ for $j = j(\rho)$ sufficiently large by Lemma 2.5 there is a point $x_k^j \in B_\rho(x) \cap \partial F_k^{n_j}$. As $\rho \to 0$ the sequence x_k^j converges to x and since $\partial F_k^{n_j} \subseteq \partial\mathbf{F}^{n_j}$ we conclude that $x \in K$.

The sets $\partial\mathbf{F}^n$ are connected, hence, by the classical Gołąb theorem on semicontinuity of one-dimensional Hausdorff measure over sequences of connected sets (see [2, theorem 4.4.17] or [24, theorem 3.3] for its most general statement and a complete proof), one has

$$\mathcal{H}^1(K) \leq \liminf_n \mathcal{H}^1(\partial\mathbf{F}^n)$$

and K is itself connected. Summing up and using Proposition 2.1

$$\begin{aligned} P(\mathbf{E}) &= \mathcal{H}^1(\partial^*\mathbf{E}) \leq \mathcal{H}^1(\partial\mathbf{E}) \leq \mathcal{H}^1(K) \\ &\leq \liminf_n P(\mathbf{F}^n) \leq \limsup_n p_n \leq p \leq P(\mathbf{E}) \end{aligned} \tag{7}$$

hence $\mathcal{H}^1(\partial^*\mathbf{E}) = \mathcal{H}^1(\partial\mathbf{E}) = \mathcal{H}^1(K)$, $p_n \to p$ and (5) follows.

Finally, to prove that $\partial\mathbf{E}$ is connected, it is enough to show $\partial\mathbf{E} = K$. We already know that $\partial\mathbf{E} \subseteq K$ so we suppose by contradiction that there exists $x \in K \setminus \partial\mathbf{E}$. Take any $y \in K$. The set K is arcwise connected by rectifiable arcs, since it is a compact connected set of finite one-dimensional Hausdorff measure (see e.g. [9, lemma 3.11] or [2, theorem 4.4.7]), in other words, there exists an injective continuous curve $\gamma\colon [0,1] \to K$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since $\partial\mathbf{E}$ is closed in K there is a small $\varepsilon > 0$ such that $\gamma([0,\varepsilon]) \subset K \setminus \partial\mathbf{E}$ and hence $\mathcal{H}^1(K \setminus \partial\mathbf{E}) > 0$ contrary to $\mathcal{H}^1(K) = \mathcal{H}^1(\partial\mathbf{E})$; this contractictions shows the last claim and hence concludes the proof. □

4. Some examples

We collect here some interesting examples of infinite planar clusters.

Example 4.1 (Apollonian packing). A cluster $\mathbf{E}$, as depicted in Figure 1, can be constructed so that each region $E_k = B_{r_k}(x_k)$, $k \neq 0$, is a ball contained in the ball $B_1 = \mathbb{R}^2 \setminus E_0$. The balls can be choosen to be pairwise disjoint and such that the measure of $B_1 \setminus \bigcup_{k=1}^\infty E_k = 0$ (see [15]).

Clearly such a cluster must be minimal because each region E_k has the minimal possible perimeter among sets with the given area and the same is true for the complement of the exterior region E_0 which is their union. The boundary $\partial\mathbf{E}$ of such a cluster is the *residual set*, i.e. the set of zero measure which remains when the balls E_k are removed from the large ball $\overline{B_1}$:

$$\partial\mathbf{E} = \bigcup_{k=0}^{+\infty} \partial B_{r_k}(x_k) = \overline{B}_1 \setminus \bigcup_{k=1}^{+\infty} B_{r_k}(x_k). \tag{8}$$

Unfortunately the residual set of such a cluster has Hausdorff dimension $d > 1$ (see [13]) and hence the cluster $\mathbf{E}$ cannot have finite perimeter.

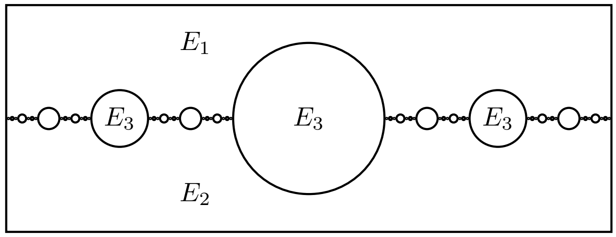

FIGURE 2. An example of a cluster $\mathbf{E}$ with finite perimeter such that $P(\mathbf{E}) = \mathcal{H}^1(\partial \mathbf{E})$ but $P(E_3) < \mathcal{H}^1(\partial E_3)$.

However we can consider the fractional (non local) perimeter P_s defined by

$$P_s(E) = \int_E \int_{\mathbb{R}^2 \setminus E} \frac{1}{|x-y|^{2+s}} \, dx \, dy$$

to define the corresponding non local perimeter $P_s(\mathbf{E})$ of the cluster $\mathbf{E}$ by means of definition (2) with P_s in place of P. If r_k is the radius of the k-th disk of the cluster it turns out (see [3]) that the infimum of all α, such that the series $\sum_k r_k^\alpha$ converges, is equal to d, the Hausdorff dimension of $\partial \mathbf{E}$. Since $d \in (1, 2)$ for all $s < 2 - d$ we have

$$\sum_k r_k^{2-s} < +\infty$$

and since $P_s(B_r) = C \cdot r^{2-s}$ (with $0 < C < +\infty$) we obtain $P_s(\mathbf{E}) < +\infty$ for such s. It is well known (see [11]) that the solution to the fractional isoperimetric problem is given by balls, hence $\mathbf{E}$ provides an example of an infinite minimal cluster with respect to the fractional perimeter P_s.

Example 4.2 (Anisotropic isoperimetric packing). We can find a similar example if we consider an anisotropic perimeter such that the isoperimetric problem has the square (instead of the circle) as a solution. If ϕ is any norm on $\mathbb{R}^2$ one can define the perimeter P_ϕ which is the relaxation of the following functional defined on regular sets $E \subset \mathbb{R}^2$:

$$P_\phi(E) = \int_{\partial E} \phi(\nu_E(x)) \, d\mathcal{H}^1(x)$$

where $\nu_E(x)$ is the exterior unit normal vector to ∂E in x. If $\phi(x, y) = |x| + |y|$ (the *Manhattan* norm) it is well known that the P_ϕ-minimal set with prescribed area (i.e. the *Wulff shape*) is a square with sides parallel to the coordinated axes (which is the ball for the dual norm). It is then easy to construct an infinite cluster $\mathbf{E} = (E_1, \ldots, E_k \ldots)$ where each E_k is a square and also the union of all such squares is a square, see figure 1. By iterating such a construction it is not difficult to realize that given any sequence a_k, $k = 1, \ldots, n, \ldots$ of numbers such that their sum is equal to 1 and each number is a power of $\frac{1}{4}$ it is possible to find a cluster $\mathbf{E}$ with $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ such that each E_k is a square and the union $\bigcup_k E_k$ is the unit square.

Example 4.3 (Cantor circles). See Figure 2 and [1, example 2 pag. 59]. Take a rectangle R divided in two by segmente S on its axis. Let C be a Cantor set with positive measure constructed on S. Consider the set E_3 which is the union of the balls with diameter on the intervals composing the complementary set $S \setminus C$. Let E_1

and E_2 be the two connected components of $R \setminus \overline{E_3}$. It turns out that the 3-cluster $\mathbf{E} = (E_1, E_2, E_3)$ has finite perimeter and the perimeter of $\mathbf{E}$ is represented by the Hausdorff measure of the boundary:

$$P(\mathbf{E}) = \mathcal{H}^1(\partial \mathbf{E}).$$

However the same is not true for each region. In fact the boundary ∂E_3 of the region E_3 includes C and hence

$$P(E_3) < \mathcal{H}^1(\partial E_3).$$

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(Matteo Novaga, Emanuele Paolini, Vincenzo Tortorelli) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I-56127, Pisa

Email address, Matteo Novaga: `matteo.novaga@unipi.it`

Email address, Emanuele Paolini: `emanuele.paolini@unipi.it`

(Eugene Stepanov) Scuola Normale Superiore, Piazza dei Cavalieri 6, Pisa, Italy and St.Petersburg Branch of the Steklov Mathematical Institute of the Russian Academy of Sciences, St.Petersburg, Russia and Faculty of Mathematics, Higher School of Economics, Moscow

Email address, Eugene Stepanov: `stepanov.eugene@gmail.com`

Email address, Vincenzo Maria Tortorelli: `tortorelli@dm.unipi.it`