ASYMPTOTIC BEHAVIOR OF ATTRACTORS FOR INHOMOGENEOUS ALLEN-CAHN EQUATIONS

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Abstract. We consider front propagation problems for forced mean curvature flows with a transport term and their phase field variants that take place in stratified media, i.e., heterogeneous media whose characteristics do not vary in one direction. We provide a convergence result relating asymptotic in time front propagation in the diffuse interface case to that in the sharp interface case, for suitably balanced nonlinearities of Allen-Cahn type. Our results generalize previous results for forced Allen-Cahn equations.

1. Introduction

In this paper, we consider the following equation:

$$\varepsilon u_t = \varepsilon \Delta u + \frac{1}{\varepsilon}\mu^2(x) f(u) + a(x,u), \tag{1.1}$$

where $\varepsilon > 0$, $f(u)$ is a balanced bistable nonlinearity, with stable equilibria 0 and 1, $\mu(x)$ is a positive function that is bounded away from zero and infinit, and $a(x,u)$ is a bounded function with $a(x,0) = 0$ for every x. The reaction term on the right-hand side of (1.1) is a perturbation of a spatially modulated Allen-Cahn type balanced nonlinearity (see [2]). Such an equation may arise, e.g., in modeling the dynamics of two co-existing phases in a phase transition with non-conserved order parameter in a medium with periodically varying properties. When $\varepsilon \ll 1$, the variations of the properties are weak and slowly changing in space (for a more detailed discussion, see [14]).

We are interested in the behavior of the solutions of (1.1) when ε is a small parameter. On formal asymptotic grounds (see [23, appendix A], [24]), the dynamics governed by (1.1) with some fixed initial condition is expected to converge as $\varepsilon \to 0$ to a forced mean curvature flow with an extra transport term. The presence of the forcing term is due to the fact that we are perturbing the balanced nonlinearity $\mu^2(x)f(u)$ with an unbalancing higher order term $\varepsilon a(x,u)$. The presence of the transport term is due to the x-dependance of the lower order term $\mu^2(x)f(u)$ in the nonlinearity. More precisely, for each (x,t) fixed the solution $u(x,t)$ is expected to converge to either 0 or 1 everywhere except for an $(n-1)$-dimensional evolving hypersurface $\Gamma(t) \subset \mathbb{R}^n$ separating the regions where $u = 0$ and $u = 1$ in the limit and whose equation of motion reads

$$V(x) = -\kappa(x) + \frac{g(x)}{c_W \mu(x)} - \nu(x) \cdot \frac{\nabla \mu(x)}{\mu(x)}, \tag{1.2}$$

where $g(x) = \int_0^1 a(x,u)du$. Here $V(x)$ is the velocity in the direction of the outward normal $\nu(x)$ (i.e., pointing into the region where $u = 0$ in the limit) at a given point $x \in \Gamma(t)$, κ is the sum of the principal curvatures (positive if the limit set where $u = 1$ is convex), and

$$c_W := \int_0^1 \sqrt{2W(u)}\, du, \qquad W(u) := -\int_0^u f(s)\, ds, \tag{1.3}$$

where we defined the double-well potential W associated with f, which is nonnegative and whose only zeros are $u = 0$ and $u = 1$. Such a result was rigorously established by Barles and Souganidis, interpreting (1.2) in the viscosity sense [8] (see also [7], and [1] for rigorous leading order asymptotic formulas). Note that, since the above mentioned results are local in space and time, they are not suitable for making conclusions about the behavior as $t \to +\infty$ of solutions of (1.1) for $\varepsilon \ll 1$, via the analysis of (1.2).

We focus on reaction-diffusion equations and mean curvature flows in infinite cylinders that describe the so-called stratified media. By a cylinder, we mean a set $\Sigma = \Omega \times \mathbb{R} \subset \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^{n-1}$ is a bounded domain with sufficiently smooth boundary. Stratified media are fibered media along the cylinder, i.e., media whose features do not change along the cylinder axis, and this property can be characterized by the dependence of the nonlinearity for reaction-diffusion equations and of the forcing and transport terms for mean curvature flows only on the transverse coordinate of the cylinder. All the results obtained in this paper remain valid in the periodic setting (i.e., when Ω is an $(n-1)$-dimensional parallelogram with periodic boundary conditions), so we do not treat this case separately.

Our approach to the problem is variational: it stems from the basic observation that a rather wide class of reaction-diffusion systems may be viewed as gradient flows in exponentially weighted spaces when looked at in a moving reference frame (see [19], [20]). Analogously, also the considered forced mean curvature flows can be interpreted as gradient flows of appropriate geometric functionals, given by perimeter type functionals plus volume terms (see [13], [6]).

The purpose of this paper is to study the long-time behavior of solutions of (1.1) for $\varepsilon \ll 1$ via the analysis of traveling wave solutions to (1.2). Our aim is to characterize the long time limit of fronts in (1.1) invading the $u = 0$ equilibrium in terms of uniformly translating graphs solving (1.2). In particular we provide a convergence result, relating the asymptotic propagation speeds and the shape of the long time limit of the fronts for (1.1) to those for (1.2) in the spirit of Γ-convergence (as is done for stationary fronts in [18]). In this paper we generalize previous results obtained in [14], for nonlinearities of the form $f(u) + \varepsilon a(x,u)$ (i.e without x-dependance on the lower order term of the nonlinearity). In our case, with the addition of x-dependance also on the lower order term of the nonlinearity, we obtain a different sharp interface limit, where also a transport term is appearing, so the arguments used in [14] have to be appropriately adapted.

1.1. **Notations.** Throughout the paper H^1, BV, L^p, C^k, C^k_c, $C^{k,\alpha}$ denote the usual spaces of Sobolev functions, functions of bounded variation, Lebesgue functions, continuous functions with k continuous derivatives, k-times continuously differentiable functions with compact support, continuously differentiable functions with Hölder-continuous derivatives of order k for $\alpha \in (0,1)$ (or Lipschitz-continuous when $\alpha = 1$), respectively. For a point $x \in \Sigma$ in the cylinder $\Sigma = \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^{n-1}$, we always write $x = (y, z)$, where $y \in \Omega$ is the transverse coordinate and $z \in \mathbb{R}$ is the coordinate along the cylinder axis. The symbol $B(x, r)$ stands for the open ball in $\mathbb{R}^n$ with radius r centered at x, and for a set A the symbols $\overline{A}$, $|A|$ and χ_A always denote the closure of A, the Lebesgue measure of A and the characteristic function of A, respectively. We also use the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

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2. Problem formulation and standing assumptions

We assume that Ω is a bounded domain with boundary of class C^2 and $2 \le n \le 7$ (of course, the physically relevant cases correspond to $n = 3$ and $n = 2$). We set $\Sigma := \Omega \times \mathbb{R}$, and in Σ we consider the family of singularly perturbed reaction-diffusion equations for $u = u(x,t) \in \mathbb{R}$, with parameter $\varepsilon > 0$:

$$\varepsilon u_t = \varepsilon \Delta u + \frac{1}{\varepsilon}\mu^2(y) f(u) + a(y,u) \qquad (x,t) \in \Sigma \times (0, +\infty), \tag{2.1}$$

with initial datum $u(x, 0) = u_0(x) \ge 0$ and Neumann boundary conditions on $\partial\Sigma$. For simplicity, we also assume that $a(x, u)$ does not depend on ε. Nevertheless the results remain valid after perturbing a with terms that can be controlled by $C\varepsilon u$ for some $C > 0$ independent of ε.

We associate to f and a the potentials

$$W(u) := -\int_0^u f(s)\, ds, \qquad G(y,u) := \int_0^u a(y,s)\, ds. \tag{2.2}$$

We now state our assumptions on the functions μ, a and f. Let $\alpha \in (0, 1]$.

Assumption 1. $\mu \in C^{1,\alpha}(\overline{\Omega})$, $\mu(y) > 0$ *for all* $y \in \overline{\Omega}$.

Assumption 2. $a \in C^\alpha_{loc}(\overline{\Omega} \times \mathbb{R})$, $a_u \in C^\alpha_{loc}(\overline{\Omega} \times \mathbb{R})$, $a(\cdot, 0) = 0$.

Assumption 3. $f \in C^{1,\alpha}_{loc}(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $W(1) = W(0) = 0$, $W(u) > 0$ for all $u \ne 0, 1$, and $\liminf\limits_{|u|\to\infty} W(u) > 0$.

Assumption 3 implies that $W(u)$ is a balanced non-degenerate double-well potential (as a model function one could think of $W(u) = \frac{1}{4}u^2(1-u)^2$). However, we do not require that f has only one other zero, which is located in $(0, 1)$, as is usually done in the literature. Instead, we only assume that $u = 0$ and $u = 1$ have the same value of W, and that W is greater for all other values of u, including at infinity. Note that by Assumptions 2 and 3 there exists $C, \delta_0 > 0$, depending only on μ, f and a, such that for every $\varepsilon \leq C^{-1}\delta_0$

$$\varepsilon^{-1}\mu^2(y)W(u) - G(y,u) \geq 0 \qquad \forall (y,u) \in \overline{\Omega} \times \left(\mathbb{R}\backslash(1 - C\sqrt{\varepsilon}, 1 + C\sqrt{\varepsilon})\right), \tag{2.3}$$

and

$$\varepsilon^{-1}\mu^2(y)W(\cdot) - G(y,\cdot) \text{ is increasing on } [1 + C\varepsilon, 1 + \delta_0] \qquad \forall y \in \overline{\Omega}. \tag{2.4}$$

Remark 2.1. Observe that, if the initial datum u_0 satisfies $0 \leq u_0(x) \leq 1 + \delta$ for some $\delta \in (0, \delta_0)$ and all $x \in \Sigma$, then by the maximum principle and (2.4) we have $0 \leq u(x,t) \leq 1 + \delta$ for all $(x,t) \in \Sigma \times [0, +\infty)$ and all $\varepsilon \leq C^{-1}\delta$.

We consider a family of measurable sets $S(t) \subseteq \Sigma$ with regular boundary, such that $\Gamma(t) = \partial S(t)$ evolves according to (1.2) with forcing term

$$g(y) := \int_0^1 a(y,s)ds = G(y,1). \tag{2.5}$$

associate to this flow the following quasilinear parabolic problem for $h = h(y,t) \in \mathbb{R}$ in Ω which describes the motion of the set $\{(y,z) \in \Sigma : z = h(y,t)\}$ according to (1.2):

$$h_t = \sqrt{1 + |\nabla h|^2}\left(\nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right) + \frac{g}{c_W \mu}\right) + \nabla h \cdot \nabla \log \mu \quad \text{in } \Omega \times (0, +\infty)\,, \tag{2.6}$$

with initial datum $h(y,0) = h_0(y)$, and Neumann boundary conditions on $\partial\Omega$. In particular, the subgraph $S(t) = \{(y,z) \in \Sigma : z < h(y,t)\}$ of the solution of (2.6) coincides with the family of sets evolving according to (1.2), with initial datum $S_0 = \{(y,z) \in \Sigma : z < h_0(y)\}$.

We recall the definition of the perimeter with weight μ of a Lebesgue measurable set $A \subseteq \Omega$ relative to Ω (see, e.g., [3], [5]): let

$$\mathrm{Per}_\mu(A,\Omega) = \sup\left\{\int_A \left(\nabla \cdot \phi(y) + \phi \cdot \nabla \log \mu\right)\mu(y)dy : \ \phi \in C_c^1(\Omega; \mathbb{R}^{n-1}), \ |\phi| \leq 1\right\}. \tag{2.7}$$

When $\mu \equiv 1$, we recover the standard definition of perimeter. We also recall the classical isoperimetric inequality [3]:

Proposition 2.2. *There exists $C_\Omega > 0$, depending on n and Ω, such that*

$$\mathrm{Per}(A,\Omega) \geq C_\Omega |A|^{\frac{n-2}{n-1}} \tag{2.8}$$

for all $A \subseteq \Omega$ of finite perimeter and such that $|A| \leq \frac{1}{2}|\Omega|$.

Remark 2.3. Notice that $C_\Omega = 2$ if $n = 2$.

Assumption 4. Let $g \in C^\alpha(\overline{\Omega})$. Then there exists $A \subseteq \Omega$ such that

$$\int_A g(y)dy \ > c_W \,\mathrm{Per}_\mu(A,\Omega). \tag{2.9}$$

This assumption basically ensures that the trivial state $u = 0$ is energetically less favorable for ε sufficiently small, resulting in the existence of the invasion fronts.

Remark 2.4. Notice that (2.9) implies, in particular, that $\sup_\Omega g > 0$, and is automatically satisfied if

$$\int_\Omega g(y)dy > 0. \tag{2.10}$$

Throughout the rest of the paper Assumptions 1–4 are always taken to be satisfied, with g defined by (2.5).

3. Traveling waves in the diffuse interface case

In this section we review some results on existence and stability of traveling wave solutions of the reaction-diffusion equation (2.1) for $\varepsilon > 0$, in the form of fronts invading the equilibrium $u = 0$ from above. Our main references are [20], [21] and [14].

We look for solutions to (2.1) of the form $u(x,t) = \bar{u}(y, z - ct)$, where $c > 0$, $\bar{u} > 0$,

$$\bar{u}(\cdot, z) \to 0 \qquad \text{uniformly as } z \to +\infty, \tag{3.1}$$

and $\bar{u} \in C^2(\Sigma) \cap C^1(\overline{\Sigma}) \cap L^\infty(\Sigma)$ solves

$$\varepsilon\Delta\bar{u} + c\varepsilon\bar{u}_z + \frac{1}{\varepsilon}\mu^2(y)f(\bar{u}) + a(y,\bar{u}) = 0 \qquad (y,z) \in \Sigma, \tag{3.2}$$

with Neumann boundary conditions $\nu \cdot \nabla\bar{u} = 0$ on $\partial\Sigma$. The constant c is referred to as the speed of the wave, and $\bar{u}$ the profile.

Note that existence and qualitative properties of traveling fronts in a variety of settings have been extensively studied, starting with the classical work of Berestycki and Nirenberg [10] who analyzed traveling fronts $\bar{u}$ connecting zero with some equilibrium $v > 0$ (i.e. which satisfies also $\lim_{z\to-\infty}\bar{u}(\cdot,z) = v$ uniformly).

By an equilibrium for (2.1), we mean a function $v : \overline{\Omega} \to \mathbb{R}$ which solves

$$\varepsilon\Delta v + \frac{1}{\varepsilon}\mu^2(y)f(v) + a(y,v) = 0 \qquad y \in \Omega, \tag{3.3}$$

with $\nu \cdot \nabla v = 0$ on $\partial\Omega$. Note that every critical point of the functional

$$E^\varepsilon(v) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla v|^2 + \mu^2(y)\frac{W(v)}{\varepsilon} - G(y,v)\right)dy \qquad v \in H^1(\Omega) \cap L^\infty(\Omega) \tag{3.4}$$

is an equilibrium for the reaction-diffusion equation (2.1). Moreover, we will consider stable equilibria, according to the following definition.

Definition 3.1. *A function $v \in H^1(\Omega) \cap L^\infty(\Omega)$ is a stable critical point of E^ε if it is a critical point of the functional and the second variation of E^ε is nonnegative, i.e.*

$$\int_\Omega \left(\varepsilon|\nabla\phi|^2 + \left(\varepsilon^{-1}\mu^2(y)W''(v) - G_{uu}(y,v)\right)\phi^2\right)dy \geq 0 \qquad \forall\phi \in H^1(\Omega)\,. \tag{3.5}$$

Moreover, v is a nondegenerate stable critical point of E^ε if strict inequality holds in (3.5).

By Assumption 3, $v = 0$ is a non-degenerate stable critical point of the functional E^ε for every ε sufficiently small. Indeed, defining

$$\nu_0^\varepsilon := \min_{\int_\Omega \phi^2 = 1} \int_\Omega \Big(\varepsilon |\nabla \phi|^2 + \left(\varepsilon^{-1} \mu^2(y) W''(0) - G_{uu}(y,0) \right) \phi^2 \Big) dy, \tag{3.6}$$

observe that there exists $\varepsilon_0 > 0$, depending on $W''(0)$, $\inf \mu$ and $\|a_u(\cdot, 0)\|_\infty$, such that

$$\nu_0^\varepsilon > 0 \qquad \text{for all } \varepsilon < \varepsilon_0. \tag{3.7}$$

We point out that existence of the considered solutions (i.e. a traveling front connecting 0 and a prescribed equilibrium $v > 0$) is not guaranteed in general. In particular, we have to impose some condition on g assuring the existence of non-trivial positive equilibria, with negative energy, as in the following proposition.

Proposition 3.2. *Under Assumptions 1–4, there exist positive constants ε_0 and C such that for all $\varepsilon < \varepsilon_0$ there exists $v_\varepsilon^0 \in H^1(\Omega)$ such that $0 \le v_\varepsilon^0 \le 1$ and $E^\varepsilon(v_\varepsilon^0) < 0$.*

Proof. By [18] we have that

$$\Gamma - \lim_{\varepsilon \to 0} E^\varepsilon(u) = \begin{cases} E^0(A) := c_W \mathrm{Per}_\mu(A, \Omega) - \int_A g \, dy & \text{if } u = \chi_A, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.8}$$

where the convergence is understood in the sense of Γ-convergence in $L^1(\Omega)$. So the proof follows the same argument as in [14, Proposition 3.4]. $\square$

Following the variational approach to front propagation problems [20] (see also [17, 19, 21, 22]), for every $c > 0$ we associate to the reaction-diffusion equation in (2.1) the energy functional (for fixed $\varepsilon > 0$)

$$\Phi_c^\varepsilon(u) = \int_\Sigma e^{cz} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} \mu^2(y) W(u) - G(y, u) \right) dx. \tag{3.9}$$

This functional is naturally defined on $H_c^1(\Sigma) \cap L^\infty(\Sigma)$, where $H_c^1(\Sigma)$ is an exponentially weighted Sobolev space with the norm

$$\|u\|^2_{H_c^1(\Sigma)} = \int_\Sigma e^{cz} (|\nabla u|^2 + |u|^2) \, dx. \tag{3.10}$$

Furthermore, the functional Φ_c^ε is differentiable in $H_c^1(\Sigma) \cap L^\infty(\Sigma)$, and its critical points satisfy the traveling wave equation (3.2) [17, 20].

Remark 3.3. Following [10, Section 4] (see also [20, Theorem 3.3(iii)]) one can show that every traveling wave (c, u) to (2.1) satisfying (3.1) belongs to $H_c^1(\Sigma)$ and is a critical point to Φ_c^ε.

Critical points of Φ_c^ε and, in particular, minimizers of Φ_c^ε play an important role for the long-time behavior of the solutions of the initial value problem associated with (2.1) in the case of front-like initial data. Indeed, in [22] it is proved under generic assumptions on the nonlinearity (see also [19,20]) that the non-trivial minimizers of Φ_c^ε over $H_c^1(\Sigma)$ are selected as long-time attractors for the initial value problem associated to (2.1) with front-like initial data. Also, in [20] it was proved under minimal assumptions on the nonlinearity that the speed of the leading edge of the solution is determined by the unique value of $c^\dagger_\varepsilon > 0$ for which $\Phi^\varepsilon_{c^\dagger_\varepsilon}$ has a non-trivial minimizer. Appropriate assumptions to guarantee existence of minimizers of Φ^ε_c were given in [20]. In our case these conditions are verified for every ε sufficiently small.

Theorem 3.4. *Under Assumptions 1–4, there exist positive constants ε_0, C and M, depending on f, μ, a and Ω, such that for all $0 < \varepsilon < \varepsilon_0$ there exists a unique $0 < c^\dagger_\varepsilon \le M$ such that*

i) $\Phi^\varepsilon_{c^\dagger_\varepsilon}$ admits a non-trivial minimizer $\bar u_\varepsilon \in H^1_{c^\dagger_\varepsilon} \cap L^\infty(\Sigma)$ which satisfies

$$\sup\left\{z \in \mathbb{R} \mid \sup_{y\in\Omega} \bar u_\varepsilon(y,z) > \frac{1}{2}\right\} = 0. \tag{3.11}$$

ii) $\bar u_\varepsilon \in C^2(\Sigma) \cap C^1(\overline{\Sigma}) \cap W^{1,\infty}(\Sigma)$, and $(c^\dagger_\varepsilon, \bar u_\varepsilon)$ is a traveling wave solution to (2.1).

iii) $0 < \bar u_\varepsilon \le 1 + C\varepsilon < \frac{3}{2}$, $(\bar u_\varepsilon)_z < 0$ in $\overline{\Sigma}$, and

$$\lim_{z\to+\infty} \bar u_\varepsilon(\cdot, z) = 0 \qquad \lim_{z\to-\infty} \bar u_\varepsilon(\cdot, z) = v_\varepsilon \qquad \text{in } C^1(\overline{\Omega}),$$

where v_ε is a stable critical point of E^ε in (3.4) with $E^\varepsilon(v_\varepsilon) < 0$.

iv) $\Phi^\varepsilon_{c^\dagger_\varepsilon}(\bar u_\varepsilon) = 0$, and all non-trivial minimizers of $\Phi^\varepsilon_{c^\dagger_\varepsilon}$ are translates of $\bar u_\varepsilon$ along z.

v) Given $\delta \in (0,1)$ and $\bar x \in \overline{\Sigma}$, there exist $C, \bar r, \bar\varepsilon > 0$ depending only on W, G, μ, Ω and δ such that, for every $\varepsilon \in (0, \bar\varepsilon)$ and $r \in (\varepsilon, \bar r)$, there holds

$$\bar u_\varepsilon(\bar x) \ge \delta \quad \Rightarrow \quad \int_{B(\bar x, r)\cap\Sigma} \bar u_\varepsilon^2 \, dx \ge C r^n, \tag{3.12}$$

$$\bar u_\varepsilon(\bar x) \le 1 - \delta \quad \Rightarrow \quad \int_{B(\bar x, r)\cap\Sigma} (1 - \bar u_\varepsilon)^2 \, dx \ge C r^n. \tag{3.13}$$

Proof. For the proof we refer to [14, Thm 3.5, Prop. 3.7 and Prop. 3.8]. □

4. Traveling waves in the sharp interface case

In this section we consider the front propagation problem in the cylinder Σ, for the forced mean curvature flow with transport term (1.2). Also in this case, we are interested in traveling wave solutions with positive speed. We review and generalize some results contained in [14] (see also [13]).

Definition 4.1 (Traveling waves)**.** *A traveling wave for the forced mean curvature flow is a pair (c, ψ), where $c > 0$ is the speed of the wave and the graph of the function $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is the profile of the wave, such that $h(y,t) = \psi(y) + ct$ solves (2.6).*

Observe that to prove existence of a traveling wave solution it is sufficient to determine $c > 0$ such that the equation in Ω

$$-\nabla \cdot \left(\frac{\nabla\psi}{\sqrt{1 + |\nabla\psi|^2}}\right) = \frac{1}{c_W \mu(y)} g(y) - \frac{c}{\sqrt{1 + |\nabla\psi|^2}} + \frac{\nabla\psi}{\sqrt{1 + |\nabla\psi|^2}} \cdot \nabla \log \mu(y), \tag{4.1}$$

with Neumann boundary condition $\nu \cdot \nabla\psi = 0$ on $\partial\Omega$, admits a classical solution. The graph of this solution will be the profile of the traveling wave.

Following the variational approach for the forced mean curvature flow (see [13], [14]), for $c > 0$ we consider the family of exponentially weighted area-type functionals

$$F_c(\psi) = \int_\Omega e^{c\psi(y)} \left(c_W \mu(y) \sqrt{1 + |\nabla\psi(y)|^2} - \frac{g(y)}{c} \right) dy, \tag{4.2}$$

among all $\psi \in C^1(\overline{\Omega})$. Note that if ψ is bounded and is a critical point of the functional F_c, then it is a solution to (4.1). Here ψ defines the graph $z = \psi(y)$ that represents the sharp interface front. The functional F_c has a well-known geometric characterization. Let us introduce the following exponentially weighted perimeter for $S \subseteq \Sigma$:

$$\mathrm{Per}_{c,\mu}(S, \Sigma) := \sup \left\{ \int_S \left[\nabla \cdot \phi + (c\hat{z} + \nabla \log \mu) \cdot \phi \right] \mu(y) e^{cz} dx : \phi \in C^1_c(\Sigma; \mathbb{R}^n),\ |\phi| \leq 1 \right\}. \tag{4.3}$$

We then define the following geometric functional on measurable sets $S \subset \Sigma$ with the weighted volume $\int_S e^{cz}\mu(y)dx < \infty$:

$$\mathcal{F}_c(S) := c_W \,\mathrm{Per}_{c,\mu}(S, \Sigma) - \int_S e^{cz}\, g(y)\, dx, \tag{4.4}$$

where we noted that by Assumption 1 the last term in (4.4) is well-defined in the considered class of sets. After the change of variable $\zeta(y) := \frac{e^{c\psi(y)}}{c} \geq 0$, the functional F_c is equivalent to

$$G_c(\zeta) = \int_\Omega \Big(c_W \mu(y) \sqrt{c^2\zeta^2(y) + |\nabla\zeta(y)|^2} - g(y)\zeta(y) \Big) dy, \tag{4.5}$$

in the sense that $F_c(\psi) = G_c(\zeta)$ for all $\zeta \in C^1(\overline{\Omega})$ [13]. Since the functional G_c is naturally defined on $BV(\Omega)$ as the lower-semicontinuous relaxation, we introduce the following generalization to the notion of a traveling wave for (2.6).

Definition 4.2 (Generalized traveling waves)**.** *A generalized traveling wave for the forced mean curvature flow is a pair (c, ψ), where $c > 0$ is the speed of the wave, and $\psi = \frac{1}{c} \ln c\zeta$ is the profile of the wave, where $\zeta \in BV(\Omega)$ is a non-negative critical point of G_c, not identically equal to zero.*

We define $\omega \subset \Omega$ to be the interior of the support of ζ. By standard regularity of minimizers of perimeter-type functionals [3], we have that ψ solves (4.1) classically in ω with $\nu \cdot \nabla\psi = 0$ on $\partial\omega \cap \partial\Omega$ and, therefore, we have that $h(y, t) = \psi(y) + ct$ solves (2.6) in ω with Neumann boundary conditions on $\partial\omega \cap \partial\Omega$. In particular, if $\omega = \Omega$, then the above definition implies that (c, ψ) is a traveling wave in the sense of Definition 4.1. In general, however, ω may differ from Ω by a set of positive measure, in which case the traveling wave profile ψ obeys the following kind of boundary condition:

$$\lim_{y \to \bar{y}} \psi(y) = -\infty \qquad \forall\, \bar{y} \in \partial\omega \cap \Omega. \tag{4.6}$$

In this situation a generalized traveling wave may have the form of one or several “fingers” invading the cylinder from left to right with speed c.

The next proposition explains the relation between Assumption 4 and the minimization problems associated with functionals G_c and, hence, F_c.

Proposition 4.3. *Let Assumption 4 hold. Then there exists a unique $c^\dagger > 0$ such that*

i)
$$\frac{\int_\Omega g\,dy}{c_W \int_\Omega \mu\,dy} \le c^\dagger \le \frac{1}{c_W}\sup_\Omega \left(\frac{g}{\mu}\right).$$

ii) If $0 < c < c^\dagger$, then $\inf\{G_c(\zeta)\ :\ \zeta \in BV(\Omega),\ \zeta \ge 0\} = -\infty$.

iii) If $c > c^\dagger$, then $\inf\{G_c(\zeta)\ :\ \zeta \in BV(\Omega),\ \zeta \ge 0\} = 0$, and $G_c(\zeta) > 0$ for every non-trivial $\zeta \ge 0$.

iv) If $c = c^\dagger$, then there exists a non-trivial $\bar\zeta \ge 0$, with $\bar\zeta \in BV(\Omega)$, such that $G_c(\bar\zeta) = \inf\{G_c(\zeta)\ :\ \zeta \in BV(\Omega),\ \zeta \ge 0\} = 0$.

Proof. The result follows from [13, Proposition 3.1 and Corollary 3.2] (see also [21, Proposition 4.1]). □

Note that the same argument as in [13, Proposition 3.4, Lemma 3.5] gives that for all $\zeta \ge 0$ such that $\zeta \in BV(\Omega)$ we have

$$G_c(\zeta) = \mathcal{F}_c(S_\psi) \tag{4.7}$$

where $S_\psi = \{(y,z) \in \Omega \times \mathbb{R}\ :\ z < \psi(y)\}$ is the subgraph of $\psi = \frac{1}{c}\ln c\zeta$. Moreover if $\zeta \ge 0$ is a non trivial minimizer of G_c, then the subgraph S_ψ of ψ is a minimizer, under compact perturbations, of the functional $\mathcal{F}_c$ defined in (4.4). So, from Proposition 4.3 we obtain the following result about existence of generalized traveling waves (see [14], [13]).

Theorem 4.4 (Existence of generalized traveling waves)**.** *Let Assumption 4 hold. Then there exists a unique $c^\dagger > 0$, which coincides with the one in Proposition 4.3, such that:*

i) There exist a function $\psi : \Omega \to [-\infty, \infty)$ such that $(c^\dagger, \psi)$ is a generalized traveling wave for the forced mean curvature flow and the set $S_\psi := \{(y,z) \in \Sigma \mid z < \psi(y)\}$ is a minimizer of $\mathcal{F}_{c^\dagger}$.

ii) The set $\omega := \{\psi > -\infty\}$ is open and satisfies $E^0(\omega) < 0$, where E^0 is defined in (3.8). Moreover, $\omega \times \mathbb{R}$ is a minimizer of $\mathcal{F}_{c^\dagger}$ under compact perturbations, and there exist $r_0 > 0$ and $\lambda > 0$ such for all $\bar x \in \overline{\omega} \times \mathbb{R}$, all $\bar x' \in \overline{\Sigma \backslash \omega \times \mathbb{R}}$ and all $r \in (0, r_0)$ the following density estimates hold:

$$|\omega \times \mathbb{R} \cap B(\bar x, r)| \ \ge\ \lambda\, r^{n-1}, \tag{4.8}$$
$$|(\Sigma \setminus \omega \times \mathbb{R}) \cap B(\bar x', r)| \ \ge\ \lambda\, r^{n-1}. \tag{4.9}$$

iii) $\psi \in C^2(\omega)$ and it is unique up to additive constants on every connected component of ω, in the following sense: there exists a number $k \in \mathbb{N}$ and functions $\psi_i : \Omega \to [-\infty, \infty)$ for each $i = 1, \dots, k$ such that $\omega_i := \{\psi_i > -\infty\} \neq \varnothing$ are open, connected and disjoint, $\psi_i \in C^2(\omega_i)$ and $\psi = \ln\left(\sum_{i=1}^k e^{\psi_i + k_i}\right)$, for some $k_i \in [-\infty, \infty)$.

iv) *∂S_ψ is a hypersurface of class C^2 uniformly in Σ, and $\partial\omega$ is a C^2 solution to the prescribed curvature problem*

$$c_W \kappa = \frac{g}{\mu} - c_W \nabla \log \mu \cdot \nu_{\partial\omega} \qquad \text{on } \partial\omega \cap \Omega, \tag{4.10}$$

where κ is the sum of the principal curvatures of $\partial\omega \cap \Omega$, with Neumann boundary conditions $\nu_{\partial\omega} \cdot \nu_{\partial\Omega} = 0$ at $\partial\omega \cap \partial\Omega$.

Proof. For the proof we refer to [14, Lemma 4.4, Theorem 4.8]. □

We now introduce an additional assumption, under which stronger conclusions about the convergence of fronts can be made.

Assumption 5. Let $g \in C^\alpha(\overline{\Omega})$ and assume that (2.10) holds. Then $\Omega \times \mathbb{R}$ is the unique minimizer of $\mathcal{F}_{c^\dagger}$ under compact perturbations among sets $S = \omega \times \mathbb{R}$ with $\omega \subseteq \Omega$ and $c^\dagger := \inf\{c > 0 : \inf \mathcal{F}_c \geq 0\} \in (0, \infty)$.

Clearly, Assumption 5 is quite implicit. In Proposition 4.6 we give some sufficient conditions for it, first in the two-dimensional case and then in every dimension.

This result of Theorem 4.4 can be strengthened under Assumption 5, as follows.

Theorem 4.5 (Existence, uniqueness and stability of traveling waves)**.** *Let Assumption 5 hold. Then there exists a unique $c^\dagger > 0$, such that:*

i) *there exists a unique $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $\max_{y\in\overline{\Omega}} \psi(y) = 0$, and $(c^\dagger, \psi)$ is a traveling wave for the forced mean curvature flow* (2.6). *Moreover ψ is the unique minimizer of the functional $F_{c^\dagger}$ over $C^1(\overline{\Omega})$, up to additive constants,*

ii) *$S = \{(y, z) \in \Sigma : z < \psi(y)\}$ is the unique minimizer of $\mathcal{F}_{c^\dagger}$ up to translations in z.*

iii)

$$h(\cdot, t) - c^\dagger t - k \longrightarrow \psi \qquad \text{in } C^{1,\alpha}(\overline{\Omega}), \text{ as } t \to +\infty$$

where $h(y, t)$ is the unique solution to (2.6) *with Neumann boundary conditions and initial datum $h(y, 0) = h_0(y) \in C(\overline{\Omega})$ and $k \in \mathbb{R}$ is constant, depending on h_0.*

Proof. For the proof we refer to [14, Theorem 4.10, Theorem 4.11]. □

As Assumption 5 is quite implicit, in the following proposition we list some sufficient conditions for it to hold.

Proposition 4.6. *Let* (2.10) *hold and let C_Ω be the relative isoperimetric constant of Ω (see* (2.8)*). Then Assumption 5 holds if one of the following conditions is verified:*

i) *there is no embedded hypersurface $\partial\omega \subseteq \Omega$ which solves the prescribed curvature problem*

$$c_W \mu\, \kappa - g + c_W \nabla\mu \cdot \nu = 0 \qquad \partial\omega \cap \Omega\,, \tag{4.11}$$

with Neumann boundary conditions $\nu_{\partial\omega} \cdot \nu_{\partial\Omega} = 0$ on $\partial\omega \cap \partial\Omega$.

ii) *$n = 2$ and $g(y) > c_W|\mu'|$ on Ω.*

iii) $\inf_\Omega(g - c_W|\nabla\mu|) \le 0$ *and*

$$\sup_\Omega\left(\frac{g + c_W|\nabla\mu|}{\mu}\right) - \inf_\Omega\left(\frac{g - c_W|\nabla\mu|}{\mu}\right) < c_W C_\Omega 2^{\frac{1}{n-1}}|\Omega|^{-\frac{1}{n-1}}.$$

iv) $\inf_\Omega(g - c_W|\nabla\mu|) \ge 0$ *on* Ω *and*

$$\sup_\Omega\left(\frac{g + c_W|\nabla\mu|}{\mu}\right) < c_W C_\Omega 2^{\frac{1}{n-1}}|\Omega|^{-\frac{1}{n-1}}.$$

Proof. We start proving (i) and (ii). Let $\omega \times \mathbb{R}$ a local minimizer of $\mathcal{F}_c$. So $\partial\omega$ is a solution to (4.11), with Neumann boundary conditions, which is the Euler-Lagrange equation for $\mathcal{F}_{c^\dagger}$ (the regularity of $\partial\omega \times \mathbb{R}$ is a consequence of the classical regularity theory for minimal surfaces with prescribed mean curvature). When $n = 2$, the prescribed mean curvature problem (4.11) reads $g(y) = \pm c_W\mu'(y)$ for some $y \in \Omega$.

Now we prove the other items. Let ψ be a solution to (4.1), with maximal support $\omega \subset \Omega$. This means that every other solution to (4.1) has support $\omega' \subset \omega$ (see (iii) in Theorem 4.4). Integrating (4.1) over ω and recalling that $\psi(x) \to -\infty$ as $\mathrm{dist}(x;\partial\omega) \to 0$ we obtain

$$\mathrm{Per}(\omega,\Omega) = \int_\omega \left(\frac{g(y)}{c_W\mu(y)} - \frac{c}{\sqrt{1+|\nabla\psi(y)|^2}} + \frac{\nabla\log\mu(y)\cdot(\nabla\psi(y))}{\sqrt{1+|\nabla\psi(y)|^2}}\right)dy \tag{4.12}$$

where $\mathrm{Per}(\omega,\Omega)$ is the standard perimeter relative to Ω (as defined in (2.7), with weight $\mu \equiv 1$, see also [3]). (4.12) gives

$$c_W \mathrm{Per}(\omega,\Omega) \le \left[\sup_\Omega\left(\frac{g + c_W|\nabla\mu|}{\mu}\right)\right]|\omega|. \tag{4.13}$$

Let $\zeta = \frac{e^{c^\dagger\psi}}{c^\dagger}$. By Proposition 4.3, we get that ζ is a minimizer of $G_{c^\dagger}$. Observe that $G_{c^\dagger}$ in (4.5) is a convex, lower semicontinuous functional on $L^2(\Omega)$ ($G_{c^\dagger}(u) = +\infty$ if $u \in L^2(\Omega) \setminus BV(\Omega)$). So, by the general theory of subdifferentials in [4, Ch.w 6] there exist a vector field $\xi_\zeta = \xi : \Omega \to \mathbb{R}^n$, with $|\xi| \le 1$ and $\mathrm{div}(\xi) \in L^2(\Omega)$, and a function $h_\zeta = h : \Omega \to \mathbb{R}$, with $0 \le h \le 1$, such that

$$\int_\Omega \left[-c_W\mu(\mathrm{div}\,\xi(y) + \nabla\log\mu\cdot\xi) + c^\dagger c_W\mu(y)h(y) - g(y)\right](\chi - \zeta)dy \ge 0 \qquad \text{in } \Omega, \tag{4.14}$$

for all $\chi \in BV(\Omega)$ such that $\chi \ge 0$. Moreover, for all $y \in \omega$,

$$\begin{aligned} h(y) &= \frac{c^\dagger\zeta}{\sqrt{(c^\dagger)^2\zeta^2(y) + |\nabla\zeta(y)|^2}} \\ \xi(y) &= \frac{\nabla\zeta(y)}{\sqrt{(c^\dagger)^2\zeta^2(y) + |\nabla\zeta(y)|^2}}. \end{aligned}$$

If we apply inequality (4.14) to $\chi(y) = \zeta(y) + \frac{\chi_A(y)}{\mu(y)}$, where $A \subseteq \Omega$ is a set of finite perimeter, we obtain

$$\text{Per}(A, \Omega) + \int_A \left(-\nabla \log \mu \cdot \xi + c^\dagger h(y) - \frac{g(y)}{c_W \mu(y)} \right) dy \geq 0 \,. \tag{4.15}$$

In particular, (4.12) and (4.15) imply that ω is a minimum for the functional

$$\mathcal{G}(A) = \text{Per}_1(A, \Omega) + \int_A \left(-\nabla \log \mu \cdot \xi + c^\dagger h(y) - \frac{g(y)}{c_W \mu(y)} \right) dy \,. \qquad A \subseteq \Omega.$$

This gives that $\int_\Omega \left(-\nabla \log \mu \cdot \xi + c^\dagger h(y) - \frac{g(y)}{c_W \mu(y)} \right) dy \geq 0$ and so, by (4.12), by (i) in Proposition 4.3 and by the definition of h, ξ, we get

$$\begin{aligned} \text{Per}(\omega, \Omega) &\leq \int_{\Omega \setminus \omega} \left(-\nabla \log \mu \cdot \xi + c^\dagger h(y) - \frac{g(y)}{c_W \mu(y)} \right) dy \\ &\leq \frac{1}{c_W} \left(\sup_\Omega \frac{g}{\mu} - \inf_\Omega \frac{g - c_W |\nabla \mu|}{\mu} \right) |\Omega \setminus \omega|. \end{aligned} \tag{4.16}$$

Observe that by the isoperimetric inequality (2.8) and (4.16), either $|\Omega \setminus \omega| > \frac{1}{2}|\Omega|$ or

$$\frac{1}{c_W} \left(\sup_\Omega \frac{g}{\mu} - \inf_\Omega \frac{g - c_W |\nabla \mu|}{\mu} \right) \frac{|\Omega|^{\frac{1}{n-1}}}{2^{\frac{1}{n-1}}} \geq \frac{1}{c_W} \left(\sup_\Omega \frac{g}{\mu} - \inf_\Omega \frac{g - c_W |\nabla \mu|}{\mu} \right) |\Omega \setminus \omega|^{\frac{1}{n-1}} \geq C_\Omega. \tag{4.17}$$

In particular, if

$$\frac{1}{c_W} \left(\sup_\Omega \frac{g}{\mu} - \inf_\Omega \frac{g - c_W |\nabla \mu|}{\mu} \right) < C_\Omega 2^{\frac{1}{n-1}} |\Omega|^{-\frac{1}{n-1}} \tag{4.18}$$

then necessarily $|\omega| \leq \frac{1}{2}|\Omega|$ and so by (2.8) and (4.13) we obtain

$$\sup_\Omega \left(\frac{g + c_W |\nabla \mu|}{\mu} \right) \geq c_W \frac{\text{Per}(\omega, \Omega)}{|\omega|} \geq c_W C_\Omega 2^{\frac{1}{n-1}} |\Omega|^{-\frac{1}{n-1}}. \tag{4.19}$$

We start proving (iii). Note that if $\inf_\Omega \frac{g - c_W |\nabla \mu|}{\mu} \leq 0$,

$$\sup_\Omega \left(\frac{g + c_W |\nabla \mu|}{\mu} \right) \leq \sup_\Omega \left(\frac{g + c_W |\nabla \mu|}{\mu} \right) - \inf_\Omega \left(\frac{g - c_W |\nabla \mu|}{\mu} \right).$$

So if

$$\sup_\Omega \left(\frac{g + c_W |\nabla \mu|}{\mu} \right) - \inf_\Omega \left(\frac{g - c_W |\nabla \mu|}{\mu} \right) < c_W C_\Omega 2^{\frac{1}{n-1}} |\Omega|^{-\frac{1}{n-1}},$$

neither (4.17) nor (4.19) are verified. Hence $\omega = \Omega$ and (iii) is proved.

We prove (iv). If $\inf_\Omega (g - c_W |\nabla \mu|) \geq 0$ and $\sup_\Omega \left(\frac{g + c_W |\nabla \mu|}{\mu} \right) < C_\Omega c_W 2^{\frac{1}{n-1}} |\Omega|^{-\frac{1}{n-1}}$, then neither (4.17) nor (4.19) are verified, and (iv) is proved. $\square$

5. Asymptotic behavior as $\varepsilon \to 0$

We now state our main result.

Theorem 5.1. *Let Assumptions 1–4 hold. Let $c_\varepsilon^\dagger$, $\bar{u}_\varepsilon$ and v_ε be as in Theorem 3.4, and let $c^\dagger$ be as in Theorem 4.4.*

i) *There holds*

$$\lim_{\varepsilon \to 0} c_\varepsilon^\dagger = c^\dagger.$$

ii) *For every sequence $\varepsilon_n \to 0$ there exists a subsequence (not relabeled) and an open set $S \subset \Sigma$ such that*

$$\bar{u}_{\varepsilon_n} \to \chi_S \qquad \text{in } L^1_{loc}(\Sigma),$$

where S is a non-trivial minimizer of $\mathcal{F}_{c^\dagger}$ satisfying $S \subseteq \Omega \times (-\infty, 0)$ and $\partial S \cap (\overline{\Omega} \times \{0\}) \neq \varnothing$. Moreover,

$$\bar{u}_{\varepsilon_n} \to \chi_S \qquad \text{locally uniformly on } \overline{\Sigma} \setminus \partial S,$$

and for every $\theta \in (0,1)$ the level sets $\{\bar{u}_{\varepsilon_n} = \theta\}$ converge to ∂S locally uniformly in the Hausdorff sense.

iii) *If also Assumption 5 holds, then S is the unique minimizer of $\mathcal{F}_{c^\dagger}$ from Theorem 4.5 satisfying $S \subseteq \Omega \times (-\infty, 0)$ and $\partial S \cap (\overline{\Omega} \times \{0\}) \neq \varnothing$. Moreover*

$$v_\varepsilon \to 1 \qquad \text{uniformly in } \overline{\Omega}.$$

Proof. The proof follows exactly the same arguments as in the proof of [14, Theorem 5.3]. The only modification that is required is in the proof of Step 1, whose details we present below. We shall prove that

$$\liminf_{\varepsilon \to 0} c_\varepsilon^\dagger \geq c^\dagger.$$

We adapt the standard Modica-Mortola construction of a recovery sequence [18] to the situation, in which an extra weight μ is present.

Let S_ψ be as in Theorem 4.4. Then the hypersurface ∂S_ψ is of class C^2 uniformly in Σ, and

$$c_W \operatorname{Per}_{c^\dagger,\mu}(S_\psi, \Sigma) = \int_{S_\psi} e^{c^\dagger z} g(y)\,dx. \tag{5.1}$$

We consider d_{S_ψ} to be the signed distance function from ∂S_ψ, i.e.,

$$d_{S_\psi}(x) := \operatorname{dist}(x, \Sigma \setminus S_\psi) - \operatorname{dist}(x, S_\psi)$$

and $\gamma : \mathbb{R} \to \mathbb{R}$ to be the unique solution to $\gamma' = \sqrt{2W(\gamma)}$ with $\gamma(0) = \frac{1}{2}$. Note that γ is monotonically increasing and connects the two equilibria 0 and 1 at infinity. Furthermore, under Assumption 3 there exist $A, B > 0$ such that (see, e.g., [12]; for a complete proof, see [9, Lemma 2.2])

$$\gamma(s) \leq \min\left(1, Ae^{Bs}\right) \quad \text{and} \quad \gamma'(s) \leq Ae^{-B|s|} \qquad \forall s \in \mathbb{R}. \tag{5.2}$$

Therefore, defining

$$u_\varepsilon := \gamma\left(\frac{\mu\, d_{S_\psi}}{\varepsilon}\right),$$

we have that $u_\varepsilon \in H^1_{c^\dagger}(\Sigma)$ for all ε sufficiently small, and $u_\varepsilon \to \chi_{S_\psi}$ as $\varepsilon \to 0$ in $L^1_{loc}(\Sigma)$.

Differentiating u_ε, we get

$$\begin{aligned}|\nabla u_\varepsilon| &= \gamma'\left(\frac{\mu\, d_{S_\psi}}{\varepsilon}\right)\left|\mu\frac{\nabla d_{S_\psi}}{\varepsilon} + \nabla\mu\frac{d_{S_\psi}}{\varepsilon}\right| \\ &\le \frac{1}{\varepsilon}\gamma'\left(\frac{\mu\, d_{S_\psi}}{\varepsilon}\right)\left(\mu + |d_{S_\psi}\nabla\mu|\right) \\ &= \frac{\mu}{\varepsilon}\sqrt{2W(u_\varepsilon)} + \frac{1}{\varepsilon}\gamma'\left(\frac{\mu\, d_{S_\psi}}{\varepsilon}\right)|d_{S_\psi}\nabla\mu|.\end{aligned} \tag{5.3}$$

Analogously

$$\begin{aligned}|\nabla u_\varepsilon| &\ge \frac{1}{\varepsilon}\gamma'\left(\frac{\mu\, d_{S_\psi}}{\varepsilon}\right)\left(\mu - |d_{S_\psi}\nabla\mu|\right) \\ &= \frac{\mu}{\varepsilon}\sqrt{2W(u_\varepsilon)} - \frac{1}{\varepsilon}\gamma'\left(\frac{\mu\, d_{S_\psi}}{\varepsilon}\right)|d_{S_\psi}\nabla\mu|.\end{aligned} \tag{5.4}$$

Hence, from (5.3) and (5.4) we obtain

$$\left|\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \mu\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right| \le \frac{|\nabla\mu|}{\sqrt{2\varepsilon}}\gamma'\left(\frac{\mu\, d_{S_\psi}}{\varepsilon}\right)|d_{S_\psi}|. \tag{5.5}$$

Recalling (5.2) again, we observe that $|s\gamma'(s)| \le AB^{-1}e^{-\frac{1}{2}B|s|}$ for all $s \in \mathbb{R}$, and, therefore, from (5.5) we obtain

$$\left(\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \mu\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right)^2 \le \frac{\varepsilon|\nabla\mu|^2A^2}{2\mu^2B^2}\exp\left(-\frac{B\mu\,|d_{S_\psi}|}{\varepsilon}\right) \le C\varepsilon\min\left(1, e^{-2c^\dagger z}\right),$$

for some $C = C(\mu, W) > 0$ and ε sufficiently small. Thus, completing the square we obtain

$$\int_\Sigma\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \mu^2(y)\frac{W(u_\varepsilon)}{\varepsilon}\right)e^{c^\dagger z}dx \le \int_\Sigma\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\mu(y)e^{c^\dagger z}dx + C\varepsilon,$$

for ε sufficiently small. It then follows

$$\begin{aligned}\Phi^\varepsilon_{c^\dagger}(u_\varepsilon) &\le \int_\Sigma\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\mu(y)e^{c^\dagger z}dx - \int_\Sigma e^{c^\dagger z}G(y,u_\varepsilon)dx + C\varepsilon \\ &= \int_\Sigma|\nabla\phi(u_\varepsilon)|\mu(y)e^{c^\dagger z}dx - \int_\Sigma e^{c^\dagger z}G(y,u_\varepsilon)dx + C\varepsilon,\end{aligned}$$

where

$$\phi(u) := \int_0^u\sqrt{2W(s)}ds. \tag{5.6}$$

Recalling the definition of g in (2.5) and observing that $\phi(u_\varepsilon) \to c_W \chi_{S_\psi}$ uniformly in the set $\{|d_{S_\psi}| \geq \delta\}$ for any $\delta > 0$ as $\varepsilon \to 0$, we can apply the Co-area Formula [3] and obtain from (5.1), as $\varepsilon \to 0$,

$$\begin{aligned} \Phi^\varepsilon_{c^\dagger}(u_\varepsilon) &= \int_0^\infty \mathrm{Per}_{c^\dagger,\mu}(\{\phi(u_\varepsilon) > t\}, \Sigma)dt - \int_\Sigma e^{c^\dagger z} G(y, u_\varepsilon)dx + C\varepsilon \\ &\to c_W \mathrm{Per}_{c^\dagger,\mu}(S_\psi, \Sigma) - \int_{S_\psi} e^{c^\dagger z} g(y)dx = 0. \end{aligned} \tag{5.7}$$

Assume now by contradiction that there exists a sequence of $c^\dagger_\varepsilon$ converging to a constant $c < c^\dagger$. A simple computation (see [14, Lemma 5.2]) gives

$$\Phi^\varepsilon_{c^\dagger}(u_\varepsilon) \geq \frac{(c^\dagger)^2 - (c^\dagger_\varepsilon)^2}{(c^\dagger)^2} \int_\Sigma e^{c^\dagger z} \frac{\varepsilon}{2} |(u_\varepsilon)_z|^2 dx, \tag{5.8}$$

and observe that by the definition of u_ε and the regularity of ∂S_ψ, we get that

$$|\nabla \phi(u_\varepsilon)| = \varepsilon |\nabla u_\varepsilon|^2 \leq 2\varepsilon |(u_\varepsilon)_z|^2,$$

in a ball $B(x, r)$ for some $r > 0$, where $x = (y, z) \in \partial S_\psi$ and $y \in \overline{\Omega}$ is a point at which ψ attains its maximum. Combining these two facts yields

$$\begin{aligned} \Phi^\varepsilon_{c^\dagger}(u_\varepsilon) &\geq \frac{(c^\dagger)^2 - (c^\dagger_\varepsilon)^2}{4(c^\dagger)^2} \int_{\Sigma \cap B(x,r)} \frac{\mu(y)}{\sup_\Omega \mu} e^{c^\dagger z} |\nabla \phi(u_\varepsilon)| dx \\ &\to \frac{(c^\dagger)^2 - c^2}{4(c^\dagger)^2 \sup_\Omega \mu} c_W \mathrm{Per}_{c^\dagger,\mu}(S_\psi, \Sigma \cap B(x, r)) > 0, \end{aligned}$$

which then contradicts (5.7). □

Theorem 5.1 implies that the level sets of solutions of the initial value problem with general front-like linitial data for $\varepsilon \ll 1$ asymptotically spread with average speed that approaches $c^\dagger$ as $\varepsilon \to 0$. Moreover, under the stronger assumption 5, we can show that the long-time limit of such solutions converges, as $\varepsilon \to 0$, to a traveling wave solution to (2.6) moving with speed $c^\dagger$. For the proof we refer to [14].

Corollary 5.2. *Let Assumptions 1–4 hold. Let $\delta > 0$ be such that*

$$(1 - u)f(u) > 0 \qquad \textit{for all} \qquad u \in [1 - \delta, 1) \cup (1, 1 + \delta],$$

let $u^\varepsilon_0 \in W^{1,\infty}(\Sigma) \cap L^2_{c^\dagger_\varepsilon}(\Sigma)$ be such that

$$0 \leq u^\varepsilon_0 \leq 1 + \delta \qquad \textit{and} \qquad \liminf_{z \to -\infty} u^\varepsilon_0(y, z) \geq 1 - \delta \textit{ uniformly in } \Omega, \tag{5.9}$$

and let u^ε be the solution of (2.1) with initial datum u^ε_0.

i) *For every $\theta \in (0, 1)$*

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{\sup\{z \in \mathbb{R} : \ u^\varepsilon(y, z, t) > \theta \textit{ for some } y \in \Omega\}}{t} = c^\dagger. \tag{5.10}$$

ii) *If also Assumption 5 holds, then there exists $R_\infty \in \mathbb{R}$ such that, for all $M > 0$,*

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \|u^\varepsilon(y, z - c_\varepsilon^\dagger t - R_\infty, t) - \chi_{S_\psi}(y,z)\|_{L^1(\Sigma_M)} = 0\,, \tag{5.11}$$

where ψ is given by Theorem 4.5. Moreover, the convergence as $\varepsilon \to 0$ after passing to the limit $t \to \infty$ is locally uniform in $\overline{\Sigma}\backslash\partial S_\psi$.

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