Uniqueness for a second order gradient flow of elastic networks

Matteo Novaga and Paola Pozzi

Abstract In a previous work by the authors a second order gradient flow of the p-elastic energy for a planar theta-network of three curves with fixed lengths was considered and a weak solution of the flow was constructed by means of an implicit variational scheme. Long-time existence of the evolution and convergence to a critical point of the energy were shown. The purpose of this note is to prove uniqueness of the weak solution when $p = 2$.

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1 Introduction

In [1] we considered a second order gradient flow of the p-elastic energy for a planar theta-network of three curves with fixed lengths. We constructed a weak solution of the flow by means of an implicit variational scheme and showed long-time existence of the evolution and as well as convergence to a critical point of the energy. The purpose of this short note is to show uniqueness of the long-time weak solution when $p = 2$.

For the sake of conciseness we refer to [1] for motivation and a list of relevant references. Let us here briefly recall the setting and state our new contribution.

We consider a theta-network composed of three inextensible planar curves. Each curve $\gamma_i = \gamma_i(s)$ of fixed length $L_i > 0$, $i = 1, 2, 3$, is parametrized by

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arc-length s over the domain $\bar{I}_i = [0, L_i]$. Without loss of generality we may assume that

$$0 < L_3 \leq \min\{L_2, L_1\}.$$

Since the network is a theta- network, the three curves satisfy the constraint

$$\gamma_1(0) = \gamma_2(0) = \gamma_3(0), \qquad \gamma_1(L_1) = \gamma_2(L_2) = \gamma_3(L_3).$$

Let $T^i = T^i(s) = \gamma_i'(s) = (\cos\theta^i, \sin\theta^i)$ denote the unit tangent of the curve γ_i and let $\boldsymbol{\kappa}_i = \partial_s T^i$ be the curvature vector. Letting $p \in (1, +\infty)$, the p-elastic energy of the network is defined as

$$E_p(\Gamma) = \sum_{i=1}^{3} E_p(\gamma_i),$$

where

$$E_p(\gamma_i) := \frac{1}{p}\int_{I_i} |\boldsymbol{\kappa}_i|^p ds = \frac{1}{p}\int_{I_i} |\partial_s T^i|^p ds =: F_p(T^i).$$

In [1] we studied the L^2-gradient flow of the energy

$$F_p(\Gamma) := \sum_{i=1}^{3} F_p(T^i),$$

when expressed in terms of the angles θ^i corresponding to the tangent vectors T^i. This gave rise to a second order parabolic system.

The long-time existence result presented in [1] reads as follows: We let

$$H := \Big\{ \boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3) \in W^{1,p}(0, L_1) \times W^{1,p}(0, L_2) \times W^{1,p}(0, L_3) \,|\, \int_{I_1} (\cos\theta^1, \sin\theta^1) ds = \int_{I_2} (\cos\theta^2, \sin\theta^2) ds = \int_{I_3} (\cos\theta^3, \sin\theta^3) ds \Big\}$$

where note that the above constraint accounts for the fact that the theta-network should maintain its topology along the flow.

Theorem 1 *Let $\boldsymbol{\theta}_0 \in H$ and let $T > 0$. Assume that the lengths of the three curves are such that*

$$L_3 < \min\{L_1, L_2\}. \tag{1}$$

Then, there exist functions $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$, with $\theta^j \in L^\infty(0, T; W^{1,p}(I_j)) \cap H^1(0, T; L^2(I_j))$, and Lagrange multipliers $\lambda^1, \lambda^2, \mu^1, \mu^2 \in L^2(0, T)$ such that the following properties hold:

(i) for any $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$ with $\varphi^j \in L^\infty(0,T; W^{1,p}(I_j))$, $j = 1,2,3$, there holds

$$
\begin{aligned}
0 = &\sum_{j=1}^{3} \int_0^T \int_{I_j} \partial_t \theta^j \, \varphi^j \, ds dt + \sum_{j=1}^{3} \int_0^T \int_{I_j} |\theta_s^j|^{p-2} \theta_s^j \cdot \varphi_s^j \, ds dt \\
&- \int_0^T (\lambda^1 - \mu^1) \int_{I_1} \sin(\theta^1) \, \varphi^1 ds dt + \int_0^T (\lambda^2 - \mu^2) \int_{I_1} \cos(\theta^1) \, \varphi^1 ds dt \\
&+ \int_0^T \lambda^1 \int_{I_2} \sin(\theta^2) \, \varphi^2 ds dt - \int_0^T \lambda^2 \int_{I_2} \cos(\theta^2) \, \varphi^2 ds dt \\
&- \int_0^T \mu^1 \int_{I_3} \sin(\theta^3) \, \varphi^3 ds dt + \int_0^T \mu^2 \int_{I_3} \cos(\theta^3) \, \varphi^3 ds dt \, ;
\end{aligned} \tag{2}
$$

(ii) the maps $|\partial_s \theta^j|^{p-2} \partial_s \theta^j$ belong to $L^\infty(0,T; L^{\frac{p}{p-1}}(I_j)) \cap L^2(0,T; H^1(I_j))$, $j = 1,2,3$, and satisfy

$$
(|\partial_s \theta^1|^{p-2} \partial_s \theta^1)_s = \theta_t^1 - (\lambda^1 - \mu^1) \sin \theta^1 + (\lambda^2 - \mu^2) \cos \theta^1, \tag{3}
$$

$$
(|\partial_s \theta^2|^{p-2} \partial_s \theta^2)_s = \theta_t^2 + \lambda^1 \sin \theta^2 - \lambda^2 \cos \theta^2, \tag{4}
$$

$$
(|\partial_s \theta^3|^{p-2} \partial_s \theta^3)_s = \theta_t^3 - \mu^1 \sin \theta^3 + \mu^2 \cos \theta^3, \tag{5}
$$

$$
\theta_s^j(0,t) = \theta_s^j(L_j, t) = 0, \ \text{for } j = 1,2,3 \text{ and for a.e. } t \in (0,T); \tag{6}
$$

(iii) for all $t \in [0,T]$, there holds

$$
\int_{I_1} (\cos \theta^1, \sin \theta^1) ds = \int_{I_2} (\cos \theta^2, \sin \theta^2) ds = \int_{I_3} (\cos \theta^3, \sin \theta^3) ds. \tag{7}
$$

Notice that the time $T > 0$ can be chosen arbitrarily, and hence Theorem 1 provides a long-time existence result.

The behavior of the solutions as $t \to +\infty$, the possible relaxation of condition (1), as well as the treatment of triods instead of theta-networks are discussed detail in [1].

Here we want to address the question of uniqueness of the above weak solution when $p = 2$. Our goal is to show the following statement.

Theorem 2 *Let the assumptions of Theorem 1 hold and let $p = 2$. Then the solution $(\boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ given in Theorem 1 is unique.*

Before providing the proof let us recall some important facts about the Lagrange multipliers and the solution given in Theorem 1. First of all by [1, Lemma 3.5] we have that

$$
\sup_{(0,T)} \|\partial_s \theta^j\|_{L^p(I_j)} \leq C, \qquad j = 1,2,3, \tag{8}
$$

where the constant C depends on the energy of the initial data and the choice of p. By [1, Proposition 3.9], we have also a uniform bound

$$|\boldsymbol{\lambda}(t)| + |\boldsymbol{\mu}(t)| \le C \tag{9}$$

for any $t \in (0,T)$, where $\boldsymbol{\lambda}(t) = (\lambda^1(t), \lambda^2(t))$, $\boldsymbol{\mu}(t) = (\mu^1(t), \mu^2(t))$. More precisely, the Lagrange multipliers solve the system

$$\boldsymbol{\lambda} \cdot A^2 + \boldsymbol{\mu} \cdot A^3 = G^3 - G^2 \tag{10}$$

$$-\boldsymbol{\lambda} \cdot (A^2 + A^1) + \boldsymbol{\mu} \cdot A^1 = G^2 - G^1 \tag{11}$$

for a.e. time $t \in (0,T)$ where A^i, $i = 1,2,3$, are the matrices

$$A^i = A^i(t) = \begin{pmatrix} \int_{I_i} \sin^2\theta^i ds & -\int_{I_i} \sin\theta^i \cos\theta^i ds \\ -\int_{I_i} \sin\theta^i \cos\theta^i ds & \int_{I_i} \cos^2\theta^i ds \end{pmatrix} =: A^i(\theta^i), \tag{12}$$

and G^i are the vectors

$$G^i = G^i(\theta^i) := \int_{I_i} |\partial_s \theta^i|^p (\cos\theta^i, \sin\theta^i) ds. \tag{13}$$

As discussed in [1] condition (1) yields not only the solvability of the above system, but also the bound

$$|\boldsymbol{\lambda}(t)| + |\boldsymbol{\mu}(t)| \le C \left(|G^3 - G^2| + |G^2 - G^1| \right) \tag{14}$$

which is crucial for the analysis. The above constants C appearing in (9) and (14) depend on the initial data, initial energy, the length of the three curves, but not on time (see [1, Lemma 2.5 and Proposition 3.9] for more details).

2 Proof of uniqueness

Here we provide the proof of Theorem 2. Let the assumptions of Theorem 1 hold and let $p = 2$. Moreover let $\boldsymbol{\theta} = (\theta^1, \theta^2, \theta^3)$ and $\hat{\boldsymbol{\theta}} = (\hat\theta^1, \hat\theta^2, \hat\theta^3)$ with Lagrange multipliers (λ^1, λ^2), (μ^1, μ^2) respectively $(\hat\lambda^1, \hat\lambda^2)$, $(\hat\mu^1, \hat\mu^2)$ be two solutions to the same initial data $\boldsymbol{\theta}_0 \in H$ and satisfying (2). Taking the difference of the two weak formulations tested with $\varphi = (\varphi^1, \varphi^2, \varphi^3)$, $\varphi^j = (\theta^j - \hat\theta^j)\eta_\epsilon$, $j = 1,2,3$, where $\eta_\epsilon \in C^\infty([0,T],[0,1])$ is such that $\eta_\epsilon(t) = 1$ for $t \in [0,\tau]$, $\eta_\epsilon(t) = 0$ for $t \in [\tau+\epsilon, T]$, $0 < \epsilon < T - \tau$, we obtain after sending $\epsilon \to 0$ the following equation

$$
\begin{aligned}
0 = \sum_{j=1}^{3}\int_0^\tau \int_{I_j} (\partial_t\theta^j - \partial_t\hat\theta^j)\,(\theta^j - \hat\theta^j) ds dt + \sum_{j=1}^{3}\int_0^\tau \int_{I_j} |(\theta^j_s - \hat\theta^j_s)|^2\, ds dt \\
+\Big\{ -\int_0^\tau (\lambda^1 - \mu^1) \int_{I_1} \sin(\theta^1)\,(\theta^1 - \hat\theta^1) ds dt \\
+\int_0^\tau (\lambda^2 - \mu^2) \int_{I_1} \cos(\theta^1)\,(\theta^1 - \hat\theta^1) ds dt \\
-\Big(-\int_0^\tau (\hat\lambda^1 - \hat\mu^1) \int_{I_1} \sin(\hat\theta^1)\,(\theta^1 - \hat\theta^1) ds dt \\
+\int_0^\tau (\hat\lambda^2 - \hat\mu^2) \int_{I_1} \cos(\hat\theta^1)\,(\theta^1 - \hat\theta^1) ds dt\Big) \\
+\int_0^\tau \lambda^1 \int_{I_2} \sin(\theta^2)\,(\theta^2 - \hat\theta^2) ds dt - \int_0^\tau \lambda^2 \int_{I_2} \cos(\theta^2)\,(\theta^2 - \hat\theta^2) ds dt \\
-\Big(\int_0^\tau \hat\lambda^1 \int_{I_2} \sin(\hat\theta^2)\,(\theta^2 - \hat\theta^2) ds dt - \int_0^\tau \hat\lambda^2 \int_{I_2} \cos(\hat\theta^2)\,(\theta^2 - \hat\theta^2) ds dt\Big) \\
-\int_0^\tau \mu^1 \int_{I_3} \sin(\theta^3)\,(\theta^3 - \hat\theta^3) ds dt + \int_0^\tau \mu^2 \int_{I_3} \cos(\theta^3)\,(\theta^3 - \hat\theta^3) ds dt \\
-\Big(-\int_0^\tau \hat\mu^1 \int_{I_3} \sin(\hat\theta^3)\,(\theta^3 - \hat\theta^3) ds dt + \int_0^\tau \hat\mu^2 \int_{I_3} \cos(\hat\theta^3)\,(\theta^3 - \hat\theta^3) ds dt\Big)\Big\}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\sum_{j=1}^{3}\frac{1}{2}\|(\theta^j - \hat\theta^j)(\tau)\|^2_{L^2(I_j)} + \sum_{j=1}^{3}\int_0^\tau \|(\theta^j_s - \hat\theta^j_s)(t)\|^2_{L^2(I_j)} dt \qquad (15) \\
= \sum_{j=1}^{3}\frac{1}{2}\|(\theta^j - \hat\theta^j)(0)\|^2_{L^2(I_j)} - \{\ldots\},
\end{aligned}
$$

where the first term in the right-hand side vanishes since $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ have the same initial data. The terms in the bracket $\{\ldots\}$ are made up of differences that are estimated in a similar way. We give here in a exemplary manner the treatment of the term

$$
J := \int_0^\tau \lambda^1 \int_{I_2} \sin(\theta^2)\,(\theta^2 - \hat\theta^2) ds dt - \int_0^\tau \hat\lambda^1 \int_{I_2} \sin(\hat\theta^2)\,(\theta^2 - \hat\theta^2) ds dt.
$$

First of all notice that

$$\begin{aligned}|J| &\leq \left|\int_0^\tau (\lambda^1 - \hat\lambda^1) \int_{I_2} \sin(\theta^2)\,(\theta^2 - \hat\theta^2) ds dt\right| \\ &+ \left|\int_0^\tau \hat\lambda^1 \int_{I_2} (\sin(\hat\theta^2) - \sin(\theta^2))\,(\theta^2 - \hat\theta^2) ds dt\right| \\ &\leq C\int_0^\tau |\lambda^1(t) - \hat\lambda^1(t)|\,\|(\theta^2 - \hat\theta^2)(t)\|_{L^2(I_2)} dt \\ &\qquad + C\int_0^\tau \|(\theta^2 - \hat\theta^2)(t)\|^2_{L^2(I_2)} dt\end{aligned} \tag{16}$$

where we have used the mean value theorem and the bound (9) in the last inequality.

To estimate the difference in the Lagrange multipliers we recall that they fulfill the system (10), (11) for almost every time. Subtraction of the corresponding equations yield

$$\begin{aligned}(\boldsymbol\lambda - \hat{\boldsymbol\lambda}) \cdot A^2 + (\boldsymbol\mu - \hat{\boldsymbol\mu}) \cdot A^3 &= rhs1 \\ -(\boldsymbol\lambda - \hat{\boldsymbol\lambda}) \cdot (A^2 + A^1) + (\boldsymbol\mu - \hat{\boldsymbol\mu}) \cdot A^1 &= rhs2\end{aligned}$$

where

$$\begin{aligned}rhs1 &= G^3 - \hat G^3 - (G^2 - \hat G^2) + \hat{\boldsymbol\lambda}(\hat A^2 - A^2) + \hat{\boldsymbol\mu}(\hat A^3 - A^3) \\ rhs2 &= G^2 - \hat G^2 - (G^1 - \hat G^1) + \hat{\boldsymbol\lambda}(A^2 + A^1 - \hat A^2 - \hat A^1) + \hat{\boldsymbol\mu}(\hat A^1 - A^1).\end{aligned}$$

Similarly to (14) we obtain

$$|\boldsymbol\lambda - \hat{\boldsymbol\lambda}| + |\boldsymbol\mu - \hat{\boldsymbol\mu}| \leq C(|rhs1| + |rhs2|).$$

Again we show exemplary the treatment of a few terms in the evaluation of $|rhs1| + |rhs2|$, since all remaining ones are estimated in a similar way. We have using the mean value theorem that

$$\begin{aligned}|G^3 - \hat G^3| &= \left|\int_{I_3} |\partial_s\theta^3|^2(\cos\theta^3, \sin\theta^3) ds - \int_{I_3} |\partial_s\hat\theta^3|^2(\cos\hat\theta^3, \sin\hat\theta^3) ds\right| \\ &\leq \left|\int_{I_3} (|\partial_s\theta^3|^2 - |\partial_s\hat\theta^3|^2)(\cos\theta^3, \sin\theta^3) ds\right| \\ &+ \left|\int_{I_3} |\partial_s\hat\theta^3|^2(\cos\hat\theta^3 - \cos\theta^3, \sin\hat\theta^3 - \sin\theta^3) ds\right| \\ &\leq C\|(\theta^3_s - \hat\theta^3_s)\|_{L_2(I_3)}\|(\theta^3_s + \hat\theta^3_s)\|_{L_2(I_3)} + C\|\hat\theta^3_s\|^2_{L_2(I_3)}\|\theta^3 - \hat\theta^3\|_{L^\infty(I_3)}.\end{aligned}$$

Using (8) and embedding theory yields

$$|G^3 - \hat G^3| \leq C\|(\theta^3_s - \hat\theta^3_s)\|_{L_2(I_3)} + C\|(\theta^3 - \hat\theta^3)\|_{L_2(I_3)}.$$

Next observe that by (9) and the mean value theorem we can compute

$$|\hat{\boldsymbol{\lambda}}(\hat{A}^2 - A^2)| \leq C \int_{I_2} |\theta^2 - \hat{\theta}^2| ds \leq C \|(\theta^2 - \hat{\theta}^2)\|_{L_2(I_2)}.$$

With similar argument as depicted above we therefore infer that

$$|\boldsymbol{\lambda}(t) - \hat{\boldsymbol{\lambda}}(t)| + |\boldsymbol{\mu}(t) - \hat{\boldsymbol{\mu}}(t)| \leq C \sum_{j=1}^{3} \Big(\|(\theta_s^j - \hat{\theta}_s^j)(t)\|_{L_2(I_j)} + \|(\theta^j - \hat{\theta}^j)(t)\|_{L_2(I_j)} \Big) \tag{17}$$

for almost every time $t \in (0, T)$. Using this estimate in (16) for the evaluation of the term J we obtain by means of a ϵ-Young inequality

$$|J| \leq \epsilon \sum_{j=1}^{3} \int_0^\tau \|(\theta_s^j - \hat{\theta}_s^j)(t)\|_{L_2(I_j)}^2 dt + C_\epsilon \sum_{j=1}^{3} \int_0^\tau \|(\theta^j - \hat{\theta}^j)(t)\|_{L_2(I_j)}^2 dt.$$

Going back to (15) and treating all remaining terms in the bracket $\{\dots\}$ in an analogous way we finally infer

$$\sum_{j=1}^{3} \frac{1}{2} \|(\theta^j - \hat{\theta}^j)(\tau)\|_{L^2(I_j)}^2 + \sum_{j=1}^{3} \int_0^\tau \|(\theta_s^j - \hat{\theta}_s^j)(t)\|_{L^2(I_j)}^2 dt$$
$$\leq \epsilon \sum_{j=1}^{3} \int_0^\tau \|(\theta_s^j - \hat{\theta}_s^j)(t)\|_{L_2(I_j)}^2 dt + C_\epsilon \sum_{j=1}^{3} \int_0^\tau \|(\theta^j - \hat{\theta}^j)(t)\|_{L_2(I_j)}^2 dt.$$

Choosing ϵ sufficiently small yields

$$\sum_{j=1}^{3} \|(\theta^j - \hat{\theta}^j)(\tau)\|_{L^2(I_j)}^2 \leq C \sum_{j=1}^{3} \int_0^\tau \|(\theta^j - \hat{\theta}^j)(t)\|_{L_2(I_j)}^2 dt$$

for any $\tau \in (0, T)$. A Gronwall argument gives $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ and hence by (17) also equality of the Lagrange multipliers, as claimed.

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