Gradient theory of phase transitions with a rapidly oscillating forcing term

Nicolas Dirr
Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22-26
D-04103 Leipzig, Germany
ndirr@mis.mpi.de
and
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY, UK
N.Dirr@maths.bath.ac.uk

Marcello Lucia
Mathematics Department
The City University of New York, CSI
Staten Island NY 10314, USA
lucia@mail.csi.cuny.edu

Matteo Novaga
Dipartimento di Matematica
Università di Pisa
Largo B. Pontecorvo 5, 56127 Pisa, Italy
novaga@dm.unipi.it

Abstract

We consider the $\Gamma-$limit of a family of functionals which model the interaction of a material that undergoes phase transition with a rapidly oscillating conservative vector field. These functionals consist of a gradient term, a double-well potential and a vector field. The scaling is such that all three terms scale in the same way and the frequency of the vector field is equal to the interface thickness. Difficulties arise from the fact that the two global minimizers of the functionals are nonconstant and converge only in the weak L^2-topology.

Mathematics Subject Classification (2000): 35B27, 49J45.

Keywords: Phase transitions, homogenization, Γ-convergence.

1 Introduction

Given a bounded Lipschitz domain Ω of $\mathbb{R}^n$ and $0 < \varepsilon \ll 1$, we study the singular limit as $\varepsilon \to 0$ of a family of functionals $G_\varepsilon : H^1(\Omega) \to \mathbb{R}$ which consist of three competing parts:

- An "interaction term" penalizing spatial changes in u;
- a double-well potential W, i.e. a non-convex function which has exactly two minimizers, for simplicity $+1$ and -1;

- and a term coupling u to a rapidly oscillating vector field $\nabla v\left(\frac{\cdot}{\varepsilon^\alpha}\right)$, where $\alpha \in (0,1]$ and $v \in W^{1,\infty}(\mathbb{R}^N)$ is a periodic function with cell domain $Q := (-1/2, 1/2)^N$.

Such a family of functionals is given by :

$$G_\varepsilon(u) := \int_\Omega \left[\varepsilon|\nabla u(x)|^2 + \frac{W(u(x))}{\varepsilon} + \left\langle \nabla v\left(\frac{x}{\varepsilon^\alpha}\right), \nabla u(x)\right\rangle\right] \mathrm{d}x, \quad u \in H^1(\Omega). \tag{1.1}$$

On a mesoscopic scale, i.e. a scale that is much larger than the atomistic scale but smaller than the macroscopic one, these functionals model the free energy of a physical system which undergoes a phase transition. The parameter ε is the ratio between meso-and macroscale, and the limit $\varepsilon \to 0$ yields an effective description on the macroscale, which is the scale of interest. In this setting, the function $u : \Omega \to \mathbb{R}$ in the functional (1.1) is the "order parameter" that describes to what extent a material confined in the domain Ω is in one of two "pure" phases, called henceforth the "+" or "−" phase. Examples for such systems are, among others, highly anisotropic magnetic materials (the two pure phases are "all spins up/down") or two-component alloys (u would be the concentration of one of the two components). The mean-field free energy of states with spatially constant order parameter at sufficiently low temperatures is given by a double-well potential $W : \mathbb{R} \to [0, \infty)$, a function having exactly two global minimizers ± 1, and convex in a neighborhood of each minimizer. In different spatial regions the system may be close to different phases. Such oscillations are penalized by an interaction term, which is due to some exchange energy in the underlying atomistic picture.

If only double-well and interaction energy are of interest one can consider the following functional:

$$M_\varepsilon(u) := \int_\Omega \left[\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right] \mathrm{d}x, \qquad u \in H^1(\Omega). \tag{1.2}$$

If the system is prevented from staying close to $+1$ or to -1 everywhere (for example by a volume constraint), then the transition layer (roughly speaking the set separating the positive and negative regions), will formally be of order ε, therefore ε corresponds to the interface thickness on the macroscopic scale. Because of the double-well potential, it is well known that sequences whose energy are uniformly bounded converge for $\varepsilon \to 0$ to ± 1 almost everywhere. The effective free energy functional on the macroscale can then be obtained by considering the limit of the functionals M_ε as $\varepsilon \to 0$. The notion of Γ-convergence introduced by De Giorgi (see [7, 6]) provides a suitable mathematical framework to study the limit behavior of a family of functionals. In [13, 14] Modica and Mortola characterized the Γ-convergence of the family (1.2) by showing

$$\big(\Gamma - \lim_{\varepsilon \to 0} M_\varepsilon\big)(u) := \begin{cases} c_W \int_\Omega |Du| & \text{if } u \in BV\big(\Omega; \{\pm 1\}\big), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.3}$$

where $BV\big(\Omega; \{\pm 1\}\big)$ denotes the space of functions having bounded variations taking values ± 1 a.e., $\int_\Omega |Du|$ denotes the total variation of the Radon measure Du (which coincides with the perimeter $\mathcal{P}(\{u > 0\}, \Omega)$), and

$$c_W := 2 \int_{-1}^{1} \sqrt{W(t)}\, \mathrm{d}t.$$

This paper paper addresses the convergence of the modified energies defined in (1.1) which consists in adding a rapidly oscillating gradient vector field to the Modica-Mortola functional (1.2). When $v \in H^2(\Omega)$ and has vanishing normal derivative on $\partial\Omega$ the last term in (1.1) can be written as

$$-\int_\Omega \frac{1}{\varepsilon^\alpha} \Delta v \left(\frac{x}{\varepsilon^\alpha}\right) u(x) \; \mathrm{d}x \,,$$

and the functional reduces to

$$G_\varepsilon(u) := \int_\Omega \left[\varepsilon |\nabla u(x)|^2 + \frac{W(u(x))}{\varepsilon} - \frac{1}{\varepsilon^\alpha} g\left(\frac{x}{\varepsilon^\alpha}\right) u(x) \right] \mathrm{d}x, \qquad u \in H^1(\Omega), \tag{1.4}$$

where $g := \Delta v$ is a function of average zero on $(-1/2, 1/2)^N$ and periodic. In such a case this adding term gives rise to a fast oscillating functions with high amplitude, that can model strong oscillating spatial inhomogeneities. For $\alpha \in [0, 1)$ the asymptotic limit (for $\varepsilon \to 0$) of this class of functionals (1.4) has been discussed in [8], where we explained how in this case the mesoscopic can be linked to the macroscopic scale. Here we continue this discussion for the more general class of functionals (1.1) with emphasize on the case $\alpha = 1$.

To understand the limit behavior as $\varepsilon \to 0$ of the family (1.1) we have to take into account two limits which occur simultaneously:

- The singular limit $\varepsilon \to 0$ as in [13, 14],
- and the "homogenization" of the rapidly oscillating field.

When $\alpha = 0$, the resulting functional G_ε consists of adding to (1.2) a continuous linear functional (independent of ε). Hence the results in [13, 14] implies that

$$\big(\Gamma - \lim_{\varepsilon \to 0} G_\varepsilon\big)(u) = c_W \int_\Omega |Du| + \int_\Omega \big\langle \nabla v(x), Du(x) \big\rangle, \quad \forall u \in BV\big(\Omega; \{\pm 1\}\big),$$

while the Γ-limit is $+\infty$ if $u \notin BV\big(\Omega; \{\pm 1\}\big)$. For $\alpha > 0$ this "splitting" is not expected to hold anymore since the frequency of $\nabla v(x/\varepsilon^\alpha)$ is strong enough to compete with the gradient and double-well potential in the resulting free energy. In order to get a consistent physical model, the free energy G_ε must satisfy the following two requirements:

(a) The functional G_ε must have precisely two global minimizers $u_\varepsilon^\pm$, in order to describe the two pure phases of the material. These minimizers will play the role of the constants ± 1 in functional (1.2);

(b) In order to derive a macroscopic description of the system, the energy of these pure phase must converge in the limit $\varepsilon \to 0$ to the minimizers of an asymptotic functional that will be interpreted as the "surface tension" energy.

The first requirement can be satisfied by mainly assuming the L^∞-norm of the vector field ∇v to be small enough. Indeed the Euler-Lagrange equation associated to G_ε is

$$2\varepsilon \Delta u - \frac{W'(u)}{\varepsilon} + \frac{1}{\varepsilon^\alpha} \Delta v \left(\frac{x}{\varepsilon^\alpha}\right) = 0 \;\text{ on } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \;\text{ on } \partial\Omega. \tag{1.5}$$

Thus the function Δv appears as a forcing term, which may be discontinuous. If the norm of v in $W^{1,\infty}(\Omega)$ is small enough, Problem (1.5) admits two solutions u_ε^+ and u_ε^- that are close respectively to 1 and -1. Conditions on (W, v) will ensure that these critical points have same energy and are actually global minimizers (non-constant if $v \not\equiv 0$) of the energy functional G_ε (see Proposition 3.5).

But these minimizers cannot satisfy the second requirement stated above. Indeed their energy can be estimated and one can see that $G_\varepsilon(u_\varepsilon^\pm)$ is strictly negative and of order $|\Omega|\varepsilon^{-1}$. To overcome the fact that $\lim_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon^\pm) = -\infty$ we need to introduce an additive renormalization to the free energy modeling the physical system. The appearance of such a renormalization is in fact quite natural for phase transitions problems. Indeed the energy associated with an interface is the *excess* free energy due to the fact that more than one phase is present, so it is actually a difference of energies, determined only up to adding constants. Hence we are led to replace G_ε by the following functional (see Section 2 for the precise definition):

$$F_\varepsilon(u) := G_\varepsilon(u) - \inf_{\varphi\in H^1(\Omega)} \{G_\varepsilon(\varphi)\}.$$

So defined the functional F_ε can be a good candidate for connecting meso and macroscopic scale. This renormalization corresponds to the first term in the Γ-expansion of G_ε in the sense of [5]. A different model for the competition of inhomogeneities and concentration is studied in [1], but in many respects our setting requires new techniques to handle, among others, the fact that the global minimizers $u_\varepsilon^\pm$ of G_ε are non-constant.

The behavior of the renormalized functional depends on the range of the parameter α. The case $0 < \alpha < 1$ was already considered in [8], where we proved that the global minimizers converge to ± 1 strongly in L^1. We were able then to show that the renormalized functionals F_ε Γ-converge to an anisotropic surface energy in the L^1-topology. In this paper we focus on the case $\alpha = 1$, i.e. oscillations and phase transition have the same scale, and all terms in the functional (1.1) compete on the same scale. We shall prove that there is an anisotropic surface energy also for $\alpha = 1$, although no scale separation between oscillations and formation of a transition layer appears.

The case $\alpha = 1$ differs in several ways from the case considered in [8]. A main difference is that the minimizers $u_\varepsilon^\pm$ of G_ε oscillate with frequency ε^{-1} and an amplitude which is small but finite, hence they converge only weakly in L^2. So we are naturally led to consider the Γ-convergence of the family F_ε with respect to the L^2-weak topology. This allows much less control over the behavior of sequences $u_\varepsilon \rightharpoonup u$ than in the case of L^1-convergence. In particular, such sequences have relatively large gradients everywhere, not only near the limiting interfaces. However, if the sequences have uniformly bounded renormalized energy, then they are forced to be close to the minimizers $u_\varepsilon^\pm$ in most of the space, i.e. their oscillations are not arbitrary, but close to those of the minimizers. This fact allows to overcome most of the difficulties created by the lack of strong convergence.

Using these facts, we will first prove that any sequence with bounded energy converges weakly (up to a subsequence) to some function in $BV\big(\Omega; \{a^-, a^+\}\big)$, but in general $a^\pm \neq \pm 1$ (unlike the case $0 < \alpha < 1$). The second main step will be to derive the so-called "fundamental estimate", which is crucial in order to ensure the Γ-limit to be a Borel-measure (see Thm. 5.3). In this part a main difficulty is due to the fact that we cannot interpolate between two sequences having same weak limit without increasing the energy. Based then on a general representation result for local functionals together with a precise characterization of finite energy sequences, we will finally prove that the Γ-limit of the family F_ε with respect to the weak-L^2 convergence has the form

$$\int_{\partial^* E\cap\Omega} \varphi(\nu_E)d\mathcal{H}^{N-1}, \quad E := \{u > 0\},\, u \in BV\big(\Omega; \{a^-, a^+\}\big) \tag{1.6}$$

where E is a finite perimeter set and ν_E is the unit normal to $\partial^* E$.

The paper is organized as follows. In Section 2 we collect our basic assumptions on the functionals (1.1) and state our main result of Γ-convergence. Section 3 provides several crucial estimates on the minimizers of the functional G_ε and shows the existence of precisely two minimizers under some smallness assumptions on the field v. Using these estimates we are able in Section 4 to prove an equi-coerciveness result on the family of functionals F_ε. Section 5 is mainly dedicated to the proof of the so-called "fundamental estimate". Based then on a usual procedure we can in Section 6 derive our main result of Γ-convergence and a representation formula. In Section 7.1 we explain that the Γ-limit can be of the form (1.6), even if the wells of the double-well potential W do not have the same depth. The minimizer close to a deep but narrow well cannot follow the oscillations of the field as much as the minimizer close to a higher but flat well, hence both may still have the same energy.

2 Notation and main results

Henceforth we fix an open bounded set with Lipschitz boundary $\mathcal{O} \subset \mathbb{R}^N$, $N \geq 2$, and denote by $\mathcal{A}$ the class of all open subsets of $\mathcal{O}$. The unit cube in $\mathbb{R}^N$ centered at the origin and its scalings are denoted by

$$Q := (-1/2, 1/2)^N, \qquad Q_\varepsilon(z) := \varepsilon(z + Q) \quad (\varepsilon > 0,\, z \in \mathbb{Z}^N).$$

The space of *functions of bounded variations* $BV(\Omega)$ is the set of $u \in L^1(\Omega)$ whose distributional derivatives $Du = (D_1 u, \dots, D_n u)$ is a vector-valued Radon measure. If $u \in BV(\Omega)$, by the Radon-Nykodym Theorem we can decompose $Du = D^a u + D^s u$ where $D^a u$ is absolutely continuous with respect to the Lebesgue measure called the *absolutely continuous part* of Du. Given $E \subset \mathbb{R}$ we also define

$$BV(\Omega; E) := \{u \in BV(\Omega) : \ u(x) \in E \ \text{ for a.e. } x \in \Omega\}\,.$$

Moreover we denote by $\partial^* E$ the reduced boundary of E whenever $\chi_E \in BV(\Omega)$, where χ_E stands for the usual characteristic function of the set E.

Given a metric space X and a family $F_\varepsilon : X \to \mathbb{R} \cup \{\infty\}$, we recall that $F' = \Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon$ and $F'' = \Gamma-\limsup_{\varepsilon\to 0} F_\varepsilon$ are defined as follows:

$$\begin{aligned} F'(x) &= \inf\left\{\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \text{ sequence, } x_\varepsilon \to x \text{ in } X\right\}, \\ F''(x) &= \inf\left\{\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \text{ sequence } x_\varepsilon \to x \text{ in } X\right\}. \end{aligned}$$

If $F' = F''$ then this common function is denoted $\Gamma-\lim_{\varepsilon\to 0} F_\varepsilon$.

Given $\Omega \in \mathcal{A}$ and $\varepsilon > 0$, we consider the functional

$$G_\varepsilon(u,\Omega) := \begin{cases} \displaystyle\int_\Omega \left[\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon} + \left\langle \nabla v\left(\frac{x}{\varepsilon}\right), \nabla u\right\rangle\right] \mathrm{d}x & \text{if } u \in H^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

We require that v and W satisfy the following assumptions:

(H1) $W \in C^2(\mathbb{R})$, $W \geq 0$, $W^{-1}(0) = \{-1,1\}$, W is even, $W'(s) < 0$ for $s \in (0,1)$, there exist $\delta_0 \in (0,1)$ and $\omega_0 > 0$, such that W is strictly convex on $[1-\delta_0, +\infty)$ and

$$\begin{aligned} \omega_0^{-1}(s-1)^2 \leq W(s) \leq \omega_0(s-1)^2 \quad & \forall s \in (1-\delta_0, +\infty), \\ W(-1+s) = W(1+s) \quad & \forall s \in (-\delta_0, \delta_0); \end{aligned}$$

(H2) the function v belongs to $W^{1,\infty}(\mathbb{R}^N)$;

(H3) v is Q-periodic and satisfies:

$$v(x_1, \ldots x_i, \ldots x_n) = -v(x_1, \ldots - x_i, \ldots x_n) \qquad \text{for all } 1 \leq i \leq n. \tag{2.2}$$

Under these only assumptions the infimum of the functional $G_\varepsilon(\cdot,\Omega)$ may tend to $-\infty$ as $\varepsilon \to 0$. This is a difficulty to find a "limit functional" in the sense of the Γ-convergence. To overcome this obstacle we introduce a renormalization by following a procedure already used in [8]. Given $\Omega \subset\subset \mathbb{R}^N$, we consider the following class of "polyrectangles" contained in Ω:

$$\mathcal{R}_\varepsilon(\Omega) := \left\{P \subset \Omega : \ P = \mathrm{int}\left(\cup_{z \in I}\overline{Q_\varepsilon(z)}\right) \ \text{ for some } I \subset \mathbb{Z}^N\right\},$$

where $\mathrm{int}(E)$ stands for the interior of a set E. The functional G_ε is then "renormalized" as follows:

$$F_\varepsilon(u,\Omega) := \begin{cases} \displaystyle\sup_{R \in \mathcal{R}_\varepsilon(\Omega)} \left\{G_\varepsilon(u,R) - \inf_{H^1(R)} G_\varepsilon(\cdot,R)\right\} & \text{if } \mathcal{R}_\varepsilon(\Omega) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

For each $u \in H^1(B)$ fixed, $F_\varepsilon(u,\cdot)$ defines a set function $\mathcal{A} \to [0,\infty)$ that enjoys of the following properties:

(a) For each $\Omega \in \mathcal{A}$ the set $\mathcal{R}_\varepsilon(\Omega)$ is finite. Hence the supremum in the definition of (2.3) is a maximum achieved by some polyrectangle R_ε.

(b) $F_\varepsilon(u,\emptyset) = 0$ and $F_\varepsilon(u,\cdot)$ is *increasing*:

$$A_1, A_2 \in \mathcal{A}, \ A_1 \subseteq A_2 \quad \Longrightarrow \quad F_\varepsilon(\cdot, A_1) \leq F_\varepsilon(\cdot, A_2).$$

(c) The set function $F_\varepsilon(u,\cdot)$ is *additive* in $\mathcal{A}$:

$$A_1, A_2 \in \mathcal{A}, \ A_1 \cap A_2 = \emptyset \quad \Longrightarrow \quad F_\varepsilon(u, A_1 \cup A_2) = F_\varepsilon(u, A_1) + F_\varepsilon(u, A_2). \tag{2.4}$$

This property follows by observing that for any $R \in \mathcal{R}_\varepsilon(A_1 \cup A_2)$ we have $R \cap A_i \in \mathcal{R}_\varepsilon(A_i)$ $(i = 1,2)$ and also

$$\inf_{H^1(R)} G_\varepsilon(\cdot,R) = \inf_{H^1(R\cap A_1)} G_\varepsilon(\cdot, R \cap A_1) + \inf_{H^1(R \cap A_2)} G_\varepsilon(\cdot, R \cap A_2).$$

(d) The functional F_ε is *local.* I.e. given $A \in \mathcal{A}$ and $u, v \in H^1(\mathcal{O})$ it holds:

$$u|_A \equiv v|_A \qquad \Longrightarrow \qquad F_\varepsilon(u,A) = F_\varepsilon(v,A). \tag{2.5}$$

(e) The set function $F_\varepsilon(u,\cdot)$ is *not subadditive* in $\mathcal{A}$, i.e. it does not satisfy the property:

$$A \subset A_1 \cup A_2, \quad A, A_1, A_2 \in \mathcal{A} \quad \Longrightarrow \quad F_\varepsilon(u,A) \leq F_\varepsilon(u,A_1) + F_\varepsilon(u,A_2). \tag{2.6}$$

Indeed as soon as $\mathcal{R}_\varepsilon(\mathcal{O}) \neq \emptyset$ we can find $A_1, A_2 \in \mathcal{A}$ containing none of the cubes $Q_\varepsilon(z)$ for all $z \in \mathbb{Z}^N$, but such that $Q_\varepsilon(z_0) \subset A_1 \cup A_2$ for some $z_0 \in \mathbb{Z}^N$. For such sets we have:

$$F_\varepsilon(u,A_1) = F_\varepsilon(u,A_2) = 0 \quad \text{and} \quad F_\varepsilon(u, A_1 \cup A_2) > 0,$$

and so (2.6) cannot hold. For similar reasons the set function $F_\varepsilon(u,\cdot)$ is *not inner regular.* In particular $F_\varepsilon(u,\cdot)$ is not the restriction of a Borel measure.

Though the functionals F_ε are not a measure (in the set variable), we will see that they Γ-converge in an appropriate metric space to a measure. To reach this conclusion we need first more information on the minimizers of the functional $G_\varepsilon\big(\cdot, Q_\varepsilon(z)\big)$. We introduce for each $\Omega \subset\subset \mathbb{R}^N$ the functionals:

$$\widetilde{G}(u,\Omega) := \int_\Omega \big[|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\,\big]\,\mathrm{d}x, \qquad u \in H^1(\Omega), \tag{2.7}$$

$$\overline{G}(u,\Omega) := \int_\Omega \left[|\nabla u|^2 + W\left(u - \frac{v}{2}\right)\right]\mathrm{d}x\,. \tag{2.8}$$

The functional $\widetilde{G}$ is the rescaled version of G_ε, i.e. by the change of variables $y = \varepsilon^{-1}x$ and setting $\tilde{u}(y) = u(\varepsilon y)$, we obtain the identity

$$G_\varepsilon(u,\Omega) = \varepsilon^{N-1}\widetilde{G}\left(\tilde{u}, \varepsilon^{-1}\Omega\right). \tag{2.9}$$

Whereas $\overline{G}$ is a way of rewriting the functional G_ε as a functional with nonnegative integrands, but with an x-dependent potential. We have the relation

$$\widetilde{G}(u,\Omega) = \overline{G}\left(u + \frac{v}{2}, \Omega\right) - \frac{1}{4}\int_\Omega |\nabla v|^2 \mathrm{d}x\,,$$

from which we easily get the following

Proposition 2.1. *Let $w(u,t) := \operatorname{sign}(u)(|u| \wedge (1+t))$. If $2t \geq \|v\|_{L^\infty}$, then*

$$\overline{G}(u,\Omega) \geq \overline{G}(w(u,t),\Omega),$$

and the inequality is strict unless $u = w(u,t)$ a.e. in Ω.

Hence for any $(u,\Omega) \in H^1(\Omega) \times \mathcal{A}$ and for any $2t \geq \|v\|_\infty$, Proposition 2.1 implies that

$$\widetilde{G}\big(u,\Omega\big) \;\geq\; \widetilde{G}\big(-(1+t) \vee u \wedge (1+t), \Omega\big) \tag{2.10}$$

with strict inequality unless $\|u\|_\infty \leq 1+t$. Thus any minimizers u of $\widetilde{G}(\cdot,\Omega)$ is bounded and satisfy the a priori estimate

$$\|u\|_\infty \leq 1 + \|v\|_\infty/2. \tag{2.11}$$

A rearangement argument will further show that in the unit cube Q any minimizer is axis-symmetric (see Proposition 3.1). Therefore if u_Q is a minimizer of $\widetilde{G}(\cdot, Q)$, and extending it by periodicity, we deduce that on a polyrectangle $R \in \mathcal{R}_\varepsilon(\Omega)$ the function $u(\frac{x}{\varepsilon})$ is a minimizer of $G_\varepsilon(\cdot, R)$. In particular we get

$$\inf_{H^1(\mathrm{int}(\overline{Q_1 \cup Q_2}))} G_\varepsilon\big(u, \mathrm{int}(\overline{Q_1 \cup Q_2})\big) = \inf_{H^1(Q_1)} G_\varepsilon\big(u, Q_1\big) + \inf_{H^1(Q_2)} G_\varepsilon\big(u, Q_2\big), \tag{2.12}$$

for any $Q_i = Q_\varepsilon(z_i)$ $i = 1, 2$ with $z_i \in \mathbb{Z}^N$, $z_1 \neq z_2$. As a consequence of (2.12), we can in the definition of $F_\varepsilon(\cdot, \Omega)$ just consider the "maximal polyrectangle" contained in Ω, i.e.

$$F_\varepsilon(u, \Omega) = F_\varepsilon(u, \Omega_\varepsilon) \quad \text{with } \Omega_\varepsilon := \bigcup_{R \in \mathcal{R}_\varepsilon(\Omega)} R. \tag{2.13}$$

This preliminary discussion shows that the renormalization in (2.3) amounts to adding a positive constant, uniformly bounded by $C|\Omega|/\varepsilon$ for some constant $C > 0$ depending only on (W, v). Indeed denoting by u_Q any minimizer of $\widetilde{G}(\cdot, Q)$, for each $R \in \mathcal{R}_\varepsilon(\Omega)$ we have

$$0 \geq \inf_{H^1(R)} G_\varepsilon(\cdot, R) = \frac{|R|}{\varepsilon} \widetilde{G}(u_Q, Q) \geq -\frac{|\Omega|}{\varepsilon} \big|\widetilde{G}(u_Q, Q)\big|.$$

In order to discuss the uniqueness of minimizers for $\widetilde{G}(\cdot, Q)$, we introduce for each $\Omega \subset\subset \mathbb{R}^N$ the following closed convex subsets of $H^1(\Omega)$:

$$H^1_\pm(\Omega) := \{u \in H^1(\Omega) : \ \pm u \geq 1 - \delta_0 \text{ a.e. in } \Omega\}, \tag{2.14}$$

where the constant δ_0 is defined in (H1). The functional $\widetilde{G}(\cdot, Q)$ restricted to $H^1_\pm(\Omega)$ admits a unique minimizer $u^\pm$. When $\|v\|_{W^{1,\infty}}$ is small we can prove that these two functions $u^\pm$ are actually the unique global minimizers for the functional $\widetilde{G}(\cdot, Q)$. More precisely

Theorem 2.2. *Under the assumptions (H1) to (H3) there exists a constant $c_0 := c_0(W) > 0$ such that if $\|v\|_{W^{1,\infty}} \leq c_0$, then the functional $\widetilde{G}(\cdot, Q)$ admits precisely two global minimizers u^+, u^-. Furthermore*

$$\begin{cases} u^\pm \in H^1_\pm(Q), \quad \widetilde{G}(u^+, Q) = \widetilde{G}(u^-, Q), \quad u^+ - u^- = 2, \\ u(x_1, \ldots x_i, \ldots x_n) = u(x_1, \ldots - x_i, \ldots x_n) \qquad \text{for all } 1 \leq i \leq n, \quad x \in Q. \end{cases} \tag{2.15}$$

The two minimizers given by Theorem 2.2 substitute the role played by the constant minimizers ± 1 in the classical Modica-Mortola functional. By extending periodically the function $u^\pm$ to $\mathbb{R}^N$, we easily see that $u^\pm_\varepsilon(x) := u^\pm(\frac{x}{\varepsilon})$ restricted to a polyrectangle $R \in \mathcal{R}_\varepsilon(\Omega)$ (with $\Omega \in \mathcal{A}$) give the two global minimizers of the functionals $G_\varepsilon(\cdot, R)$ and $F_\varepsilon(\cdot, R)$. In order to identify a limit problem we need to choose a topology that provides convergence of these minimizers as $\varepsilon \to 0$. Since $u^\pm_\varepsilon$ satisfy the a priori estimate (2.11) we deduce that

$$u^\pm_\varepsilon \rightharpoonup \int_Q u^\pm \quad \text{weakly in } L^2(Q).$$

Therefore it is natural to work in the weak $L^2(\mathcal{O})$-topology and in order to set the problem in a metric space we note that (2.10) implies

$$F_\varepsilon\big(u, \Omega\big) \geq F_\varepsilon\big(-(1+t) \vee u \wedge (1+t), \Omega\big),$$

for any $\Omega \in \mathcal{A}$ and $2t \geq \|v\|_\infty$. Hence, after cutting, we may work with sequences $u_\varepsilon \in L^2(\mathcal{O})$ which are such that $\|u_\varepsilon\|_\infty \leq 1 + \|v\|_\infty$. which in turn implies that there exists a constant $M > 0$ (depending only on W, v and R) such that $\|u\|_{L^2(\Omega)} \leq M$ for all $\Omega \in \mathcal{A}$. From now on we shall restrict our analysis to the metrizable space

$$X := \Big\{u \in L^2(\mathcal{O}) : \ \|u\|_2 \leq M\Big\}, \text{ endowed with the weak } L^2 - \text{topology}.$$

Our goal is to characterize the Γ-limit of the functionals F_ε restricted to the metric space X (which we still denote by F_ε). Let us emphasize the following properties of F':

(a) $0 \leq F' \leq F''$ and $F'(u, \cdot)$ is increasing;

(b) Since $F_\varepsilon(u, \cdot)$ is superadditive in the class $\mathcal{A}$, we know that F' is superadditive in $\mathcal{A}$ (see [Prop. 16.12, [6]]).

The main difficulty in the present paper will be to prove the subadditivity of F'. To state our main result of Γ-convergence we need to consider the values:

$$a^\pm := \int_Q u^\pm(x)\mathrm{d}x,$$

where $u^\pm$ are the minimizers given by Theorem 2.2 and for $E \subseteq \mathbb{R}^N$ we introduce the following notation:

$$\chi_{E,a^\pm}(x) := a^+\chi_E + a^-(1-\chi_E) = \begin{cases} a^+ & \text{if } x \in E, \\ a^- & \text{otherwise.} \end{cases} \tag{2.16}$$

Our main theorem on Γ-convergence of the family F_ε with respect to the weak L^2-topology reads then as follows:

Theorem 2.3. *Assume that (H1), (H2) and (H3) are satisfied. Then there exists a constant $c_0 = c_0(W, \Omega)$ such that, if $\|v\|_{W^{1,\infty}} \leq c_0$, there exists a function $\varphi : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ which is strictly positive, homogeneous and convex such that*

$$\Gamma - \lim_{\varepsilon \to 0} F_\varepsilon(u, \Omega) = \begin{cases} \displaystyle\int_{\partial^* E \cap \Omega} \varphi(\nu_E) d\mathcal{H}^{N-1} & \text{if } u = \chi_{E,a^\pm} \in BV(\Omega; \{a^-, a^+\}), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.17}$$

for any set $\Omega \in \mathcal{A}$ with Lipschitz boundary.

Unlike in the case of the usual Modica-Mortola functional, the assumption that W is strictly decreasing in $[0, 1]$, see **(H1)**, is important in order to ensure that $\widetilde{G}$ admits only two global minimizers. Indeed the example in Section 7.1 shows that there may be a global minimizer which oscillates around a local minimizer of W.
Further note that if we strengthen **(H1)** by assuming $W(s) = \omega_0(s-1)^2$ for $s \in (1-\delta_0, 1-\delta_0)$, instead of just requiring bounds by quadratic functions, then W' is a linear function and we can easily show that $(a^+, a^-) = (1, -1)$. Finally let us mentioned that the periodicity of v is used in order to derive that the Γ-limit does not depend on x and on the particular subsequence $\varepsilon_j \to 0$ (see Proposition 6.4 and Theorem 6.5).

3 Estimates for the minimizers

Under our assumptions classical arguments show that the functional $\widetilde{G}(\cdot,\Omega)$ defined by (2.7) admits at least one minimizer. In this section we discuss the symmetry of such minimizers, present some a priori bound and prove that there are precisely two minimizers when $\|v\|_{W^{1,\infty}}$ is small enough.

Proposition 3.1. *Assume (H1), (H2) and let $\Omega \subset\subset \mathbb{R}^N$. Then for any minimizer $u \in H^1(\Omega)$ of $\widetilde{G}(\cdot,\Omega)$ it holds*

$$\|u\|_\infty \le 1 + \|v\|_\infty \qquad \text{and} \qquad u \in W^{1,\infty}_{loc}(\Omega). \tag{3.1}$$

If Ω is the unit cube $Q = (-1/2, 1/2)^N$, then with the additional assumption (H3) any minimizers u is axis-symmetric:

$$u(x_1,\dots x_i,\dots x_n) = u(x_1,\dots -x_i,\dots x_n) \qquad \text{for all } 1 \le i \le n, \quad x \in Q, \tag{3.2}$$

and in particular its Q-periodic extension is in $H^1_{loc}(\mathbb{R}^N)$.

Proof. Once we know that $u \in L^\infty(\Omega)$, using the Euler-Lagrange satisfied by u we deduce that $\Delta(u-v) \in L^\infty(\Omega)$. Standard elliptic regularity [11] yield then $u - v \in C^1(\Omega)$. Since $v \in W^{1,\infty}(\mathbb{R}^N)$ we conclude $u \in W^{1,\infty}_{loc}(\Omega)$.
We now prove (3.2). Given $1 \le i \le n$, let $u^\sigma \in H^1(Q)$ be defined as

$$u^\sigma(x_1,\dots x_i,\dots x_n) := u(x_1,\dots -x_i,\dots x_n),$$

and set

$$\overline{u} := \sup\{u, u^\sigma\} \qquad \text{and} \qquad \underline{u} := \inf\{u, u^\sigma\}.$$

We claim that

$$(i) \quad \widetilde{G}(\overline{u}, Q) = \widetilde{G}(\underline{u}, Q) = \widetilde{G}(u, Q) \qquad \text{and} \qquad (ii) \quad u \equiv u^\sigma. \tag{3.3}$$

Assume first

$$\int_{\{u-u^\sigma>0\}} \left\{|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\right\} \le \int_{\{u-u^\sigma<0\}} \left\{|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\right\}. \tag{3.4}$$

We note that

$$\int_{\{u-u^\sigma<0\}} |\nabla u^\sigma|^2 = \int_{\{u-u^\sigma>0\}} |\nabla u|^2 \qquad \int_{\{u-u^\sigma<0\}} W(u^\sigma) = \int_{\{u-u^\sigma>0\}} W(u) \tag{3.5}$$

and thanks to assumption (H3) we also get

$$\int_{\{u-u^\sigma<0\}} \langle \nabla v, \nabla u^\sigma\rangle = \int_{\{u-u^\sigma>0\}} \langle \nabla v, \nabla u\rangle. \tag{3.6}$$

Hence using successively the definition of $\overline{u}$, (3.5) together with (3.6), and (3.4), we obtain

$$\begin{aligned}
\widetilde{G}(\overline{u}, Q) &= \int_{\{u-u^\sigma>0\}} \left\{|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\right\} + \int_{\{u-u^\sigma\le 0\}} \left\{|\nabla u^\sigma|^2 + W(u^\sigma) + \langle \nabla v, \nabla u^\sigma\rangle\right\} \\
&= \int_{\{u-u^\sigma>0\}} \left\{|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\right\} + \int_{\{u-u^\sigma\ge 0\}} \left\{|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\right\} \\
&\le \int_{\{u-u^\sigma<0\}} \left\{|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\right\} + \int_{\{u-u^\sigma\ge 0\}} \left\{|\nabla u|^2 + W(u) + \langle \nabla v, \nabla u\rangle\right\} \\
&= \widetilde{G}(u, Q).
\end{aligned} \tag{3.7}$$

Since u is a minimizer we deduce that $\widetilde{G}(\overline{u}, Q) = \widetilde{G}(u, Q)$. From the equalities

$$\widetilde{G}(\underline{u}, Q) + \widetilde{G}(\overline{u}, Q) \;=\; \widetilde{G}(u, Q) + \widetilde{G}(u^\sigma, Q) \;=\; 2\widetilde{G}(u, Q),$$

we also deduce $\widetilde{G}(\underline{u}, Q) = \widetilde{G}(u, Q)$. Hence statement (i) in (3.3) follows if (3.4) holds. If in (3.4) we assume the reverse inequality, same arguments with $\underline{u}$ instead of $\overline{u}$ allow also to conclude.

To show statement (ii) in (3.3) we consider the Euler-Lagrange equation satisfied by $\overline{u}, \underline{u}$ and deduce that they solve

$$-2\Delta(\overline{u} - \underline{u}) + c(x)(\overline{u} - \underline{u}) = 0, \qquad \overline{u} - \underline{u} \geq 0,$$

where

$$c(x) := \begin{cases} \frac{W'(\overline{u}) - W'(\underline{u})}{\overline{u} - \underline{u}}(x) & \text{if } \overline{u}(x) \neq \underline{u}(x), \\ 0 & \text{if } \overline{u}(x) = \underline{u}(x). \end{cases}$$

Since $\overline{u}, \underline{u} \in L^\infty(Q)$ by (3.1), we deduce that $c \in L^\infty(Q)$. Hence by the Strong Maximum Principle the following alternative holds:

$$\overline{u} - \underline{u} > 0 \text{ in } Q \qquad \text{or} \qquad \overline{u} \equiv \underline{u} \text{ in } Q. \tag{3.8}$$

Using the continuity of $\overline{u}, \underline{u}$ (by (3.1)) and noting that $\overline{u} - \underline{u} = 0$ on the set $\{x_i = 0\} \cap Q$, we deduce that the first alternative in (3.8) cannot hold. Hence $u^\sigma \equiv u$ in Q which concludes the proof of (3.2). □

As a consequence our Proposition 3.1, for any disjoint pair of cube $Q_i \in \mathcal{R}_\varepsilon(\Omega)$ $(i = 1, 2)$ the following additivity property hold

$$\inf_{H^1(\operatorname{int}(\overline{Q_1 \cup Q_2}))} G_\varepsilon\big(u, \operatorname{int}(\overline{Q_1 \cup Q_2})\big) = \inf_{H^1(Q_1)} G_\varepsilon\big(u, Q_1\big) + \inf_{H^1(Q_2)} G_\varepsilon\big(u, Q_2\big), \tag{3.9}$$

$$F_\varepsilon\big(u, \operatorname{int}(\overline{Q_1 \cup Q_2})\big) = F_\varepsilon\big(u, Q_1\big) + F_\varepsilon\big(u, Q_2\big). \tag{3.10}$$

The following definition introduces a cutting and reflection procedure, which gives a function u^t, which assumes values only in one of the convex regions of the potential W. First we distinguish whether u has large or small oscillations on the set where it is in the "minority well", then we perform a reflection-and-cutting procedure.

Definition 3.2. *Let $t, K > 0$ and $v \in W^{1,\infty}(\Omega)$. For each $u \in H^1(\Omega)$ we define*

$$T_1(u) \;:=\; \begin{cases} u, & \text{if } \displaystyle\fint_{\left\{u < -t + \frac{\|v\|_\infty}{2}\right\}} |\nabla u|^2 < K; \\ u \vee (-t + \|v\|_\infty/2), & \text{if } \displaystyle\fint_{\left\{u < -t + \frac{\|v\|_\infty}{2}\right\}} |\nabla u|^2 \geq K; \end{cases}$$

$$T_2(u) \;:=\; |u| \vee (t - \frac{v}{2}).$$

We finally set

$$u^t := \begin{cases} (T_2 \circ T_1)(u) & \text{if } |\{u \leq 0\}| \leq \frac{1}{2}|\Omega|, \\ -(T_2 \circ T_1)(-u) & \text{if } |\{u \leq 0\}| > \frac{1}{2}|\Omega|. \end{cases} \tag{3.11}$$

Lemma 3.3. *Assume (H1)-(H2) hold and let T_1 be as in Def. 3.2. Then $\widetilde{G}(u,\Omega) \ge \widetilde{G}(T_1(u),\Omega)$ whenever*

$$K \ge \fint_\Omega |\nabla v|^2 + 2W(-t + \|v\|_\infty/2). \tag{3.12}$$

Proof. Consider the sublevel set $\mathcal{T} := \{u < -t + \|v\|_\infty/2\}$ and assume $T_1(u) \not\equiv u$. Then by using $\langle \nabla v, \nabla u\rangle \ge -\frac{1}{2}(|\nabla v|^2 + |\nabla u|^2)$ and $W(u) \ge 0$ we get

$$\begin{aligned}
&\widetilde{G}(u,\Omega) - \widetilde{G}(T_1(u),\Omega)\\
&= \int_{\mathcal{T}} \Big\{ \big[|\nabla u|^2 - 0\big] + \big[W(u) - W(-t + \|v\|_\infty/2)\big] + \big[\langle \nabla v, \nabla u\rangle - 0\big] \Big\}\\
&\ge |\mathcal{T}| \left(\frac{1}{2} \fint_{\mathcal{T}} |\nabla u|^2 - \frac{1}{2} \fint_{\mathcal{T}} |\nabla v|^2 - W(-t + \|v\|_\infty/2) \right)\\
&\ge |\mathcal{T}| \left(\frac{K}{2} - \frac{1}{2} \fint_\Omega |\nabla v|^2 - W(-t + \|v\|_\infty/2) \right).
\end{aligned}$$

This last expression is clearly non-negative whenever the constant K in the definition of T_1 is chosen as in (3.12). □

We will need the following well known result.

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain (open and connected) with Lipschitz boundary.*

(a) (**Relative Isoperimetric Inequality**). *There exists a constant $\mathbf{i}_\Omega > 0$ such that*

$$\mathcal{P}(E,\Omega) \ge \mathbf{i}_\Omega \big(\min\{|E|, |\Omega \setminus E|\} \big)^{\frac{N-1}{N}} \qquad \text{for any } E \subseteq \Omega.$$

(b) Let $u \in BV(\Omega)$ with $|\{u \le 0\}| \le \frac{|\Omega|}{2}$. Then for any $t_0 \ge 0$ we have

$$t_0 \big|\{u < -t_0\}\big| \le \frac{|\Omega|^{\frac{1}{N}}}{\mathbf{i}_\Omega} \int_{-t_0}^0 \mathcal{P}\big(\{u < s\},\Omega\big)\, ds. \tag{3.13}$$

Proof. For the first statement we refer to [10, Section 5.6]. Concerning (3.13) we note that

$$\begin{aligned}
t_0\big|\{u < -t_0\}\big| &\le t_0 |\Omega|^{\frac{1}{N}} \big|\{u < -t_0\}\big|^{\frac{N-1}{N}} \le |\Omega|^{\frac{1}{N}} \int_{-t_0}^0 \big|\{u < s\}\big|^{\frac{N-1}{N}} ds\\
&\le \frac{|\Omega|^{\frac{1}{N}}}{\mathbf{i}_\Omega} \int_{-t_0}^0 \mathcal{P}\big(\{u < s\},\Omega\big)\, ds.
\end{aligned}$$

□

The following estimate will play a crucial role in our arguments.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lispchitz domain, and assume (H1), (H2) hold. There exist two positive constants $C(W,\Omega)$, $\widetilde{W} := \widetilde{W}(W)$ such that if*

$$\|v\|_{L^\infty(\Omega)} < \rho \text{ for some } 0 < \rho < \frac{\delta_0}{3}, \qquad \|\nabla v\|_{L^\infty(\Omega)} \le C(W,\Omega), \tag{3.14}$$

then for each $t \in (1 - \delta_0 + \rho, 1 - \frac{\rho}{2})$ and each $u \in H^1(\Omega)$, we have

$$\widetilde{G}(u,\Omega) - \widetilde{G}(u^t,\Omega) \ge \widetilde{W} \int_{-t+\rho}^{t-\rho} \mathcal{P}\big(\{u < s\},\Omega\big)\, ds + \widetilde{W}^2 \big|\{|u| < t - \rho\}\big|. \tag{3.15}$$

Proof. We consider the case $\{u \le 0\} \le |\Omega|/2$.
Let the constant K in Def. 3.2 be chosen as in Lemma 3.3. Recalling that $(T_2 \circ T_1)(u) = u^t$ (see Def. 3.2), Lemma 3.3 implies

$$\begin{aligned}\widetilde{G}(u,\Omega) - \widetilde{G}(u^t,\Omega) &= \widetilde{G}(u,\Omega) - \widetilde{G}(T_1(u),\Omega) + \widetilde{G}(T_1(u),\Omega) - \widetilde{G}(u^t,\Omega) \\ &\ge \widetilde{G}(T_1(u),\Omega) - \widetilde{G}(u^t,\Omega).\end{aligned}$$

Hence since $T_1 \circ T_1 = T_1$ and $\{T_1(u) \ge 0\} = \{u \ge 0\}$ (because $-t + \|v\|_\infty/2 < 0$), we may without loss of generality assume that $T_1(u) \equiv u$, i.e.

$$|\{u \le 0\}| \le |\Omega|/2, \qquad ⨍_{\left\{u < -t + \frac{\|v\|_\infty}{2}\right\}} |\nabla u|^2 < K, \tag{3.16}$$

with

$$K = ⨍_{\left\{u < -t + \frac{\|v\|_\infty}{2}\right\}} |\nabla v|^2 dx + 2W\left(-t + \frac{\|v\|_\infty}{2}\right) \le \|\nabla v\|_\infty^2 + 2W(0).$$

Using the definition of $\widetilde{G}$, the left hand-side of (3.15) is given by

$$\begin{aligned}&\widetilde{G}(u,\Omega) - \widetilde{G}(u^t,\Omega) = \overline{G}\left(u + \frac{v}{2}, \Omega\right) - \overline{G}\left(u^t + \frac{v}{2}, \Omega\right) \\ &= \int_{\{u + \frac{v}{2} < t\}} \left\{ \left|\nabla\left(u + \frac{v}{2}\right)\right|^2 - \left|\nabla\left(u^t + \frac{v}{2}\right)\right|^2 + W(u) - W(u^t) \right\} \\ &= \underbrace{\int_{\{|u| < t - \frac{v}{2}\}} \left\{ \left|\nabla\left(u + \frac{v}{2}\right)\right|^2 + \frac{1}{2}\left(W(u) - W\left(t - \frac{v}{2}\right)\right)\right\}}_{I_1} + \underbrace{2\int_{\{u < -t + \frac{v}{2}\}} \langle \nabla v, \nabla u\rangle}_{I_2} \\ &\quad + \underbrace{\frac{1}{2}\int_{\{|u| < t - \frac{v}{2}\}} \left(W(u) - W\left(t - \frac{v}{2}\right)\right)}_{I_3}.\end{aligned} \tag{3.17}$$

Let us estimate I_1. Since $t - v/2 \le t - \rho/2 < 1$ and W is decreasing in the interval $(0,1)$, $W(u) - W(t - v/2) > 0$ on $\{|u| < t - v/2\}$, and the inequality $a^2 + b^2 \ge 2ab$ yields

$$\begin{aligned} I_1 &= \int_{\{|u| < t - \frac{v}{2}\}} \left\{ |\nabla u|^2 + W(u) - W\left(t - \frac{v}{2}\right) + \frac{|\nabla v|^2}{4} + \langle \nabla v, \nabla u\rangle \right\} \\ &\ge \int_{\{|u| < t - \frac{v}{2}\}} |\nabla u| \left\{ \sqrt{2\left[W(u) - W\left(t - \frac{v}{2}\right)\right] + |\nabla v|^2} - |\nabla v| \right\} \\ &\ge \int_{\{|u| < t - \frac{\rho}{2}\}} |\nabla u| \left\{ \sqrt{2\left[W(u) - W\left(t - \frac{\rho}{2}\right)\right] + |\nabla v|^2} - |\nabla v| \right\} \\ &\ge \int_{\{|u| < t - \rho\}} |\nabla u| \left\{ \sqrt{2\left[W\left(t - \rho\right) - W\left(t - \frac{\rho}{2}\right)\right] + |\nabla v|^2} - |\nabla v| \right\}.\end{aligned} \tag{3.18}$$

Since W is convex in the interval $(1 - \delta_0, \infty)$, the function $t \mapsto W(t - \rho) - W(t - \frac{\rho}{2})$ is decreasing and so we infer

$$W(t - \rho) - W(t - \frac{\rho}{2}) \ \ge \ \widetilde{W}^2 := W(1 - \frac{3\rho}{2}) - W(1 - \rho). \tag{3.19}$$

Using (3.19) and the fact that the function $s \mapsto \sqrt{2\widetilde{W}^2 + s^2} - s$ is decreasing, we can estimate (3.18) as follows:

$$\begin{aligned} I_1 &\geq \left(\sqrt{2\,\widetilde{W}^2 + \|\nabla v\|_\infty^2} - \|\nabla v\|_\infty\right) \int_{\{|u|<t-\rho\}} |\nabla u| \\ &= \left(\sqrt{2\,\widetilde{W}^2 + \|\nabla v\|_\infty^2} - \|\nabla v\|_\infty\right) \int_{-t+\rho}^{t-\rho} \mathcal{P}\big(\{u < s\}, \Omega\big)\, \mathrm{d}s, \end{aligned} \tag{3.20}$$

the latter equality being a consequence of the co-area formula.
Similar arguments yield

$$I_3 \geq \int_{\{|u|<t-\rho\}} \frac{1}{2}\Big[W\Big(t-\rho\Big) - W\Big(t - \frac{\rho}{2}\Big)\Big] \geq \widetilde{W}^2 |\{|u| < t-\rho\}|.$$

Let us estimate the integral I_2 in (3.17). By setting $\mathcal{T} = \{u < -t + \|v\|_\infty/2\}$, using successively the Cauchy-Schwarz inequality, (3.16) and then (3.13) we get

$$\begin{aligned} |I_2| &\leq \left(\fint_{\mathcal{T}} |\nabla v|^2\right)^{1/2} \left(\fint_{\mathcal{T}} |\nabla u|^2\right)^{1/2} |\mathcal{T}| \leq \|\nabla v\|_\infty K^{1/2} \left|\left\{u < -t + \frac{\rho}{2}\right\}\right| \\ &\leq \|\nabla v\|_\infty K^{1/2} \left|\{u < -t+\rho\}\right| \leq \|\nabla v\|_\infty K^{1/2} \frac{|\Omega|^{\frac{1}{N}} \mathbf{i}_\Omega^{-1}}{t-\rho} \int_{-t+\rho}^{0} \mathcal{P}\big(\{u < s\}, \Omega\big)\, ds \\ &\leq \|\nabla v\|_\infty K^{1/2} \frac{|\Omega|^{\frac{1}{N}} \mathbf{i}_\Omega^{-1}}{1-\delta_0} \int_{-t+\rho}^{t-\rho} \mathcal{P}\big(\{u < s\}, \Omega\big)\, ds. \end{aligned} \tag{3.21}$$

Therefore (3.20) and (3.21) plugged in (3.17) yields

$$\widetilde{G}(u,\Omega) - \widetilde{G}(u^t,\Omega) \geq \Big(2\widetilde{W} + O(\|\nabla v\|_\infty)\Big) \int_{-t+\rho}^{t-\rho} \mathcal{P}\big(\{u < s\}, \Omega\big).$$

By choosing $\|\nabla v\|_\infty$ small enough we can have $2\widetilde{W} + O(\|\nabla v\|_\infty) \geq \widetilde{W}$. I.e., we can find a constant $C(W,\Omega)$ such that

$$\widetilde{G}(u,\Omega) - \widetilde{G}(u^t,\Omega) \geq \widetilde{W} \int_{-t+\rho}^{t-\rho} \mathcal{P}\big(\{u < s\}, \Omega\big) ds,$$

provided $\|\nabla v\|_{L^\infty(\Omega)} \leq C(W,\Omega)$. This concludes the proof of the proposition. □

Proposition 3.5 implies as in [8] the existence of exactly two global minimizers $u^\pm$, whose range are contained in the convex regions of the potential W. Recalling the definition of the sets $H^1_\pm(\Omega)$ (see (2.14)), we get the following result which is a more precise formulation of Theorem 2.2:

Proposition 3.6. *Let Ω be a bounded Lipschitz domain, and let (H1), (H2), (3.14) be satisfied. Then the functional* (2.7) *admits precisely two global minimizers u^+, u^- and there holds*

$$u^\pm \in H^1_\pm(\Omega), \quad \widetilde{G}(u^+,\Omega) = \widetilde{G}(u^-,\Omega), \quad u^+ - u^- = 2, \tag{3.22}$$

$$\widetilde{G}(u,\Omega) - \widetilde{G}(u^\pm,\Omega) \geq \widetilde{W} \int_{-1+\delta_0}^{1-\delta_0} \mathcal{P}\big(\{u < s\}, \Omega\big)\, ds \quad \text{for all } u \in H^1(\Omega). \tag{3.23}$$

When Ω is the unit cube Q, the minimizers $u^\pm$ satisfy the symmetry property (3.2).

4 A priori bounds and a compactness result

The results of the previous section will be applied to the case where the domain is the unit cube Q. In particular we emphasize that under the assumptions (H1)-(H2), Proposition 3.5 provides a constant $C(W,Q)$ such that when

$$\|v\|_{L^\infty(Q)} < \rho \text{ for some } 0<\rho<\frac{\delta_0}{3}, \qquad \|\nabla v\|_{L^\infty(Q)} \le C(W,Q), \tag{4.1}$$

then inequality (3.15) holds (with $\Omega := Q$). In order to formulate a similar inequality on a cube $Q_\varepsilon(z)$ we introduce the following definition.

Definition 4.1. *Let $t>0$ and $u \in H^1(\Omega)$. For each $z \in \mathbb{Z}^N$ with $Q_\varepsilon(z) \subseteq \Omega$ we consider in the unit cube Q the rescaled function*

$$\widetilde{u} : Q \to \mathbb{R}, \quad x \mapsto u(\varepsilon[x+z]),$$

and let $\widetilde{u}^t$ be as in Definition 3.2 (with the constant K given by Lemma 3.3). Now we define

$$C^t_\varepsilon[u] : \Omega_\varepsilon \to \mathbb{R}, \quad C^t_\varepsilon[u](x) := \widetilde{u}^t\left(\frac{x-\varepsilon z}{\varepsilon}\right) \quad \text{if } x \in Q_\varepsilon(z).$$

For t as in Proposition 3.5, the restriction of $C^t_\varepsilon[u]$ to a cube $Q_\varepsilon(z) \subseteq \Omega$ has a range contained in one of the convex regions of W. Furthermore if v satisfies (4.1), by using (2.9), Proposition 3.5 (in the unit cube) and then by scaling we derive:

$$\begin{aligned} G_\varepsilon\big(u, Q_\varepsilon(z)\big) - G_\varepsilon\big(C^t_\varepsilon[u], Q_\varepsilon(z)\big) &= \varepsilon^{N-1}\left[\widetilde{G}(\widetilde{u}, Q) - \widetilde{G}(\widetilde{u}^t, Q)\right] \\ &\ge \widetilde{W}\int_{-t+\rho}^{t-\rho} \mathcal{P}\big(\{\widetilde{u}<s\}, Q\big)ds + \widetilde{W}^2\left|\{|\widetilde{u}|<t-\rho\}\right| \\ &= \widetilde{W}\int_{-t+\rho}^{t-\rho} \mathcal{P}\big(\{u<s\}, Q_\varepsilon(z)\big)ds + \frac{\widetilde{W}^2}{\epsilon}\left|\{|u|<t-\rho\}\right|. \end{aligned} \tag{4.2}$$

The main result of this section is Theorem 4.8, whose proof relies on the idea that a sequence u_ε with a uniform energy bound is "close" to a minimizer $u^\pm$. To make this precise we are led to introduce the following definition:

Definition 4.2. *For any Lebesgue-measurable function $u : \Omega \to \mathbb{R}$ and $z \in \mathbb{Z}^N$ we set:*

$$\operatorname{sign}_\varepsilon[u,z] := \begin{cases} 1 & \text{if } \left|\{u(x)>0\} \cap Q_\varepsilon(z)\right| > \dfrac{|Q_\varepsilon(z)|}{2}, \\ -1 & \text{if } \left|\{u(x)>0\} \cap Q_\varepsilon(z)\right| \le \dfrac{|Q_\varepsilon(z)|}{2}. \end{cases}$$

By considering the two global minimizers $u^\pm$ of $\widetilde{G}(\cdot, Q)$ given by Proposition 3.6 we define:

$$\begin{aligned} M_\varepsilon[u,z] &:= \left[u^+\left(\frac{x}{\varepsilon}\right)\frac{1+\operatorname{sign}_\varepsilon[u,z]}{2} + u^-\left(\frac{x}{\varepsilon}\right)\frac{1-\operatorname{sign}_\varepsilon[u,z]}{2}\right]\chi_{Q_\varepsilon(z)}, \\ H_\varepsilon[u,z] &:= \left[a^+\frac{1+\operatorname{sign}_\varepsilon[u,z]}{2} + a^-\frac{1-\operatorname{sign}_\varepsilon[u,z]}{2}\right]\chi_{Q_\varepsilon(z)}, \end{aligned}$$

where $a^\pm := \int_Q u^\pm(x)\mathrm{d}x$. Finally we define

$$M_\varepsilon[u] := \sum_{z\in\mathbb{Z}^N} M_\varepsilon[u,z] \quad \text{and} \quad H_\varepsilon[u] := \sum_{z\in\mathbb{Z}^N} H_\varepsilon[u,z].$$

Note that in the definition of $M_\varepsilon[u], H_\varepsilon[u]$ the sum is actually finite, and the restriction of these functions to Ω is a BV-function with possible jumps only on the boundary of a cube $Q_\varepsilon(z)$. We have the following lemma.

Lemma 4.3. *Let (H1) to (H3) and* (4.1) *be satisfied. Then there exists* $W_0 > 0$ *such that*

$$\varepsilon \int_R \left|D^a(C^t_\varepsilon[u] - M_\varepsilon[u])\right|^2 + \frac{W_0}{\varepsilon} \int_R \left|C^t_\varepsilon[u] - M_\varepsilon[u]\right|^2 \leq F_\varepsilon(u, R),$$

for any $u \in H^1(\Omega)$, $t \in (1 - \delta_0 + \rho, 1 - \frac{\rho}{2})$ *and* $R \in \mathcal{R}_\varepsilon(\Omega)$.

Proof. Let $Q_\varepsilon(z) \subseteq \Omega$ and assume without loss of generality that $\operatorname{sign}_\varepsilon[u, z] = 1$. Note that both functions $C^t_\varepsilon[u]$ and $M_\varepsilon[u]$ restricted to $Q_\varepsilon(z)$ take values in $(1 - \delta_0, 1 + \delta_0)$, an interval on which W is convex. Hence, setting $\xi := C^t_\varepsilon[u] - M_\varepsilon[u]$, we get

$$\begin{aligned} W(M_\varepsilon[u] + \xi) - W(M_\varepsilon[u]) &= W'(M_\varepsilon[u])\xi + \int_{M_\varepsilon[u]}^{M_\varepsilon[u]+\xi} \int_{M_\varepsilon[u]}^{s} W''(\tau)\mathrm{d}\tau\mathrm{d}s \\ &\geq W'(M_\varepsilon[u])\xi + W_0|\xi|^2. \end{aligned} \tag{4.3}$$

with $W_0 := \inf_{\tau \in (1-\delta_0, 1+\delta_0)} \{W''(\tau)\}$ which is positive by (H1). Therefore inequality (4.2) and (4.3) give

$$\begin{aligned} F_\varepsilon\big(u, Q_\varepsilon(z)\big) &\geq G_\varepsilon\big(C^t_\varepsilon[u], Q_\varepsilon(z)\big) - G_\varepsilon\big(M_\varepsilon[u], Q_\varepsilon(z)\big) \\ &\geq \int_{Q_\varepsilon(z)} \varepsilon\big[|\nabla(M_\varepsilon[u] + \xi)|^2 - |\nabla(M_\varepsilon[u])|^2\big]\,\mathrm{d}x \\ &\quad + \int_{Q_\varepsilon(z)} \left[\frac{W'(M_\varepsilon[u])}{\varepsilon}\,\xi + \frac{W_0}{\varepsilon}|\xi|^2 + \left\langle \nabla v\left(\frac{\cdot}{\varepsilon}\right), \nabla\xi\right\rangle\right] \mathrm{d}x. \end{aligned} \tag{4.4}$$

As the restriction of $M_\varepsilon[u]$ on each cube $Q_\varepsilon(z)$ is an absolute minimizer of $G_\varepsilon(\cdot, Q_\varepsilon(z))$, the associated Euler-Lagrange equation implies

$$\int_{Q_\varepsilon(z)} \left[2\varepsilon\nabla(M_\varepsilon[u])\nabla\xi + \frac{W'(M_\varepsilon[u])}{\varepsilon}\,\xi + \left\langle \nabla v\left(\frac{\cdot}{\varepsilon}\right), \nabla\xi\right\rangle\right] \mathrm{d}x = 0.$$

By plugging this equality in (4.4) we obtain

$$F_\varepsilon(u, Q_\varepsilon(z)) \geq \int_{Q_\varepsilon(z)} \left[\varepsilon|\nabla\xi|^2 + \frac{W_0}{\varepsilon}\xi^2\right] \mathrm{d}x.$$

The thesis follows by using the additivity property (3.10) and summing over all the cubes $Q_\varepsilon(z)$ contained in R. □

In order to derive a bound on $\left\|u - C^t_\varepsilon[u]\right\|_{L^2(\Omega_\varepsilon)}$ and $\left\|u - M_\varepsilon[u]\right\|_{L^2(\Omega_\varepsilon)}$ we introduce the following notation:

Definition 4.4. *Let $\varepsilon, t > 0$. For any $u \in H^1(\Omega)$ we define*

$$\begin{aligned} B_\varepsilon^t(u) &:= \{x \in \Omega_\varepsilon : C_\varepsilon^t[u](x) \neq u(x)\} \\ B_\varepsilon^{t,0}(u) &:= \{x \in \Omega_\varepsilon : -t - \|v\|_\infty/2 \leq u(x) \leq t + \|v\|_\infty/2\} \\ B_\varepsilon^{t,+}(u) &:= \{x \in \Omega_\varepsilon : u(x) < -t - \|v\|_\infty/2\} \cap \Big(\bigcup_{\{z\in\mathbb{Z}^N:\ \mathrm{sign}_\varepsilon[u,z]=1\}} Q_\varepsilon(z)\Big) \\ B_\varepsilon^{t,-}(u) &:= \{x \in \Omega_\varepsilon :\ u(x) > t + \|v\|_\infty/2\} \cap \Big(\bigcup_{\{z\in\mathbb{Z}^N:\ \mathrm{sign}_\varepsilon[u,z]=-1\}} Q_\varepsilon(z)\Big). \end{aligned}$$

The set $B_\varepsilon^{t,0}(u)$ contains mainly the points where $u(x)$ is not in the convex regions of W, whereas $B_\varepsilon^{t,\pm}(u)$ are the sets where u is in the "wrong" well of the potential W. We have

$$B_\varepsilon^t(u) \subseteq B_\varepsilon^{t,0}(u) \cup B_\varepsilon^{t,+}(u) \cup B_\varepsilon^{t,-}(u). \tag{4.5}$$

Lemma 4.5. *Assume (H1) to (H3) hold, and v satisfy (4.1) (in particular $\|v\|_\infty < \rho < \delta_0/3$). Given $t \in (1-\delta_0+\rho, 1-(3/2)\rho - \|v\|_\infty)$, there exists a constant $C(t,W,v) > 0$ such that:*

$$|B_\varepsilon^{t,0}(u)| \leq C\varepsilon F_\varepsilon(u,\Omega), \quad |B_\varepsilon^{t,\pm}(u)| \leq C\varepsilon F_\varepsilon(u,\Omega), \quad \textit{for all } u \in H^1(\Omega). \tag{4.6}$$

Proof. Given $z \in \mathbb{Z}^N$ with $Q_\varepsilon(z) \subseteq \Omega$, it is enough to show that

$$\big|B_\varepsilon^{t,0}(u) \cap Q_\varepsilon(z)\big| \leq C\varepsilon F_\varepsilon(u, Q_\varepsilon(z)), \tag{4.7}$$

$$\big|B_\varepsilon^{t,\pm}(u) \cap Q_\varepsilon(z)\big| \leq C\varepsilon F_\varepsilon\big(u, Q_\varepsilon(z)\big). \tag{4.8}$$

Indeed conclusion (4.6) follows then by using the additivity property (3.10) and summing over all cubes $Q_\varepsilon(z)$ contained in Ω_ε.

We give the proof when $Q_\varepsilon(z)$ is such that $\mathrm{sign}_\varepsilon[u,z] = 1$, since the case where $\mathrm{sign}_\varepsilon[u,z] = -1$ can be treated in the same way. Consider the rescaled function $\widetilde{u}(x) = u(\varepsilon[x+z])$ in the unit cube Q. The idea of the proof is to apply Proposition 3.5 with a t' which is *different* from the one defining $B_\varepsilon^{t,0}(u)$, $B_\varepsilon^{t\pm,0}(u)$. Let

$$t' := t + \frac{1}{2}\|v\|_\infty + \rho,$$

and note that this choice satisfies the conditions of Proposition 3.5.

We first estimate $\big|B_\varepsilon^{t,0}(u) \cap Q_\varepsilon(z)\big|$ by applying Proposition 3.5 as follows

$$\widetilde{G}(\widetilde{u}, Q) - \widetilde{G}(u^\pm, Q) \geq \widetilde{G}(\widetilde{u}, Q) - \widetilde{G}(\widetilde{u}^{t'}, Q) \geq \tilde{W}^2\big|\{|\widetilde{u}| < t' - \rho\} \cap Q\big|.$$

Since $t' - \rho = t + (1/2)\|v\|_\infty$, by rescaling (see (2.9)) we deduce

$$\frac{1}{\varepsilon^{N-1}} F_\varepsilon(u, Q_\varepsilon(z)) \geq \frac{\tilde{W}^2}{\varepsilon^N}\big|B_\varepsilon^{t,0}(u) \cap Q_\varepsilon(z)\big|,$$

and consequently there exists $C > 0$ such that (4.7) holds.

Let us now prove the estimate on $|B_\varepsilon^{t,\pm}(u) \cap Q_\varepsilon(z)|$. By applying Proposition 3.5 we get

$$\begin{aligned} \widetilde{G}(\widetilde{u}, Q) - \widetilde{G}(u^\pm, Q) &\geq \widetilde{G}(\widetilde{u}, Q) - \widetilde{G}(\widetilde{u}^{t'}, Q) \geq \widetilde{W} \int_{-t'+\rho}^{t'-\rho} \mathcal{P}\big(\{\widetilde{u} < s\}, Q\big)\, \mathrm{d}s \\ &\geq \widetilde{W} \int_{-t-\frac{\|v\|_\infty}{2}}^{0} \mathcal{P}\big(\{\widetilde{u} < s\}, Q\big)\, \mathrm{d}s. \end{aligned} \tag{4.9}$$

Since $\operatorname{sign}_\varepsilon[u,z]=1$, we have $|\{\widetilde{u}<s\}\cap Q|\le |Q|/2$ for each $s\le 0$. Hence inequality (3.13) implies

$$\int_{-t-\|v\|_\infty/2}^{0} \mathcal{P}\big(\{\widetilde{u}<s\},Q\big)\,\mathrm{d}s \;\ge\; \mathbf{i}_Q(1-\delta_0)\,\big|\{\widetilde{u}<-t-\|v\|_\infty/2\}\cap Q\big|. \tag{4.10}$$

Hence using (4.9), (4.10) and by rescaling we find

$$\frac{1}{\varepsilon^{N-1}}F_\varepsilon\big(u,Q_\varepsilon(z)\big) \;\ge\; \frac{\mathbf{i}_Q(1-\delta_0)}{\varepsilon^N}\big|B_\varepsilon^{t,+}(u)\cap Q_\varepsilon(z)\big|$$

and therefore (4.8) holds. □

From Lemma 4.5 and Lemma 4.3 we obtain the following estimate.

Lemma 4.6. *Assume (H1) to (H3) and* (4.1) *hold. Then there exists* $C>0$ *such that*

$$\big\|u-M_\varepsilon[u]\big\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon F_\varepsilon(u,\Omega), \qquad \textit{for all } u\in H^1(\Omega).$$

Proof. We first not that there exists $s_0,c_0>0$ (depending only on (W,v)) such that

$$\int_{\{|u|\ge s_0\}} \big|u-M_\varepsilon[u]\big|^2 \;\le\; c_0\varepsilon F_\varepsilon\big(u,\Omega\big). \tag{4.11}$$

Indeed, recalling the definition of F_ε we have

$$\begin{aligned} F_\varepsilon\big(u,\Omega\big) &= \int_{\Omega_\varepsilon}\left[\varepsilon\left|\nabla u(x)+\frac{1}{2\epsilon}\nabla v\left(\frac{x}{\varepsilon}\right)\right|^2+\frac{W(u(x))}{\varepsilon}\right]\,\mathrm{d}x\\ &\quad -\inf_{w\in H^1(\Omega_\varepsilon)}\int_{\Omega_\varepsilon}\left[\varepsilon\left|\nabla w(x)+\frac{1}{2\epsilon}\nabla v\left(\frac{x}{\varepsilon}\right)\right|^2+\frac{W(w(x))}{\varepsilon}\right]\,\mathrm{d}x. \end{aligned}$$

By choosing $w(x)=-\frac{1}{2}v(\frac{x}{\varepsilon})$, we obtain the inequality

$$F_\varepsilon(u,\Omega)\ge \int_{\Omega_\varepsilon}\left[\frac{W(u)}{\varepsilon}-\frac{W\left(v/2\right)}{\varepsilon}\right]. \tag{4.12}$$

Moreover the quadratic growth of the function W assumed in (H1) implies the existence of a constant $c:=c(W,v)>0$ and $s_0>1$ such that

$$W(s)-W\left(v/2\right)\;\ge\; C(s^2-1)\;\ge\; c|u-M_\varepsilon(u)|^2 \quad \text{for all } s\ge s_0. \tag{4.13}$$

From (4.12) and (4.13) we obtain (4.11).

For u small, we note that the conditions on v allow to find a $t>0$ such that $B_\varepsilon^{t,\pm,0}$ as in Definition 4.4 can be estimated by Lemma 4.5, and Lemma 4.3 may be applied for the same $t>0$. Hence we can estimate

$$\begin{aligned} \int_{\{|u|<s_0\}}\big|u-M_\varepsilon[u]\big|^2 &\le 2\left(\int_{\{|u|<s_0\}}\big|u-C_\varepsilon^t[u]\big|^2+\int_{\{|u|<s_0\}}\big|C_\varepsilon^t[u]-M_\varepsilon[u]\big|^2\right)\\ &\le 2s_0^2\Big(\big|B_\varepsilon^{t,0}(u)\big|+\big|B_\varepsilon^{t,+}(u)\big|+\big|B_\varepsilon^{t,-}(u)\big|\Big)+2\int_{\{|u|<s_0\}}\big|C_\varepsilon^t[u]-M_\varepsilon[u]\big|^2\\ &\le C\varepsilon F_\varepsilon\big(u,\Omega\big) \end{aligned}$$

the last inequality following from Lemma 4.3, property (4.5) and Lemma 4.5. which completes the proof of the lemma. □

Lemma 4.7. *Assume (H1), (H2), (H3) and* (4.1) *hold. Then there exist constants* $\tilde{C}(W)$, $C(W,v)$ *such that if* $\|\nabla v\|_\infty \le \tilde{C}$, *then*

$$\int_{\Omega_\varepsilon} \big|DH_\varepsilon[u]\big| \le CF_\varepsilon(u,\Omega), \qquad \textit{for all } u \in H^1(\Omega). \tag{4.14}$$

Proof. Notice that $a^+ = a^- + 2$. Hence the total variation of the measure $DH_\varepsilon[u]$ can be written as

$$\int_{\Omega_\varepsilon} \big|DH_\varepsilon[u]\big| = \frac{\varepsilon^{(N-1)}}{2} \sum_{|z_i - z_j|=1} (a^+ - a^-)|\mathrm{sign}_\varepsilon[u,z_i] - \mathrm{sign}_\varepsilon[u,z_j]|,$$

which is nothing but the Hamiltonian of a nearest-neighbor Ising model. We use this Hamiltonian to provides the lower bound (4.14) for the energy $F_\varepsilon(\cdot,\Omega)$.
We have to show that each pair of neighboring cubes which have different sign contribute to the energy at least $c\varepsilon^{N-1}$, for some $c > 0$. Let us consider a pair of cubes $Q_\varepsilon(z_i)$, $Q_\varepsilon(z_j)$ such that $|z_i - z_j| = 1$ and $\mathrm{sign}_\varepsilon[u,z_i] \ne \mathrm{sign}_\varepsilon[u,z_j]$.

Let t be as in Lemma 4.5.
Case 1: If there exists $k \in \{i,j\}$ such that either

$$\big|Q_\varepsilon(z_k) \cap \{0 < u < t + \|v\|_\infty/2\}\big| > \varepsilon^N/4 \text{ and } \mathrm{sign}_\varepsilon[u,z_i] = 1,$$

or

$$\big|Q_\varepsilon(z_k) \cap \{0 > u > -t - \|v\|_\infty/2\}\big| > \varepsilon^N/4 \text{ and } \mathrm{sign}_\varepsilon[u,z_i] = -1,$$

then by (4.7) we have

$$\varepsilon^N/4 \le \big|B^0_\varepsilon \cap Q_\varepsilon(z_k)\big| \le \varepsilon F_\varepsilon(u, Q_\varepsilon(z_k)),$$

so that $F_\varepsilon(u,Q_\varepsilon(z_k)) \ge \varepsilon^{N-1}/4$.

Case 2: If not, we have

$$\big|\big(Q_\varepsilon(z_i) \cup Q_\varepsilon(z_j)\big) \cap \{u > t + \|v\|_\infty/2\}\big| \ge \frac{1}{8}\big|Q_\varepsilon(z_i) \cup Q_\varepsilon(z_j)\big|,$$
$$\big|\big(Q_\varepsilon(z_i) \cup Q_\varepsilon(z_j)\big) \cap \{u < -t - \|v\|_\infty/2\}\big| \ge \frac{1}{8}\big|Q_\varepsilon(z_i) \cup Q_\varepsilon(z_j)\big|.$$

Hence for the set which is the union of the *two* cubes we get that on a significant portion of the set the function is in the "wrong" well, whatever the sign of the majority of the set is.
Set $\hat{Q} := \mathrm{int}\big(\overline{Q_\varepsilon(z_i) \cup Q_\varepsilon(z_j)}\big)$. By using Proposition 3.5 (with $\Omega := \hat{Q}$) and the isoperimetric inequality as in the proof of Lemma 4.5, but for a set which is the union of two cubes, we obtain that there exists a constant $c > 0$ such that

$$\begin{aligned} F_\varepsilon(u,\hat{Q}) &= G_\varepsilon\big(u,\hat{Q}\big) - G_\varepsilon\big(u^\pm_\varepsilon,\hat{Q}\big) \\ &= \sum_{k\in\{i,j\}} \big\{G_\varepsilon(u,Q_\varepsilon(z_k)) - G_\varepsilon(u^\pm_\varepsilon, Q_\varepsilon(z_k))\big\} \\ &\ge c\left(\frac{1}{8}\varepsilon^N\right)^{\frac{N-1}{N}}. \end{aligned}$$

The only difference is the fact that now the condition on $\|\nabla v\|_\infty$ depends on the volume and the isoperimetric inequality for a set which is the union of two cubes. Hence two neighboring cubes with different sign contribute to the energy at least $c\varepsilon^{(N-1)}$, for some constant $c > 0$ independent of $u \in H^1(\Omega)$. □

Theorem 4.8. (Equi-coerciveness) *Assume (H1)-(H3) holds and v satisfies the condition of Lemma 4.7. For any sequence $u_\varepsilon \in H^1(\Omega)$ such that*

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) < \infty, \quad u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(\Omega),$$

there holds

$$\int_{\Omega_\varepsilon} \big|DH_\varepsilon[u_\varepsilon]\big| \le C, \quad H_\varepsilon[u_\varepsilon] \to u \text{ in } L^2(\Omega), \quad u \in BV\big(\Omega; \{a^-, a^+\}\big).$$

Proof. The BV estimate on $H_\varepsilon[u_\varepsilon]$ is a direct consequence of Lemma 4.7. As a consequence $H_\varepsilon[u_\varepsilon]$ converges, up to a subsequence, to a function w strongly in $L^1(\Omega)$.
Using Definition 4.2, Corollary 4.6 and the equality $M_\varepsilon[u] = H_\varepsilon[u] + (u^+(x/\varepsilon) - a^+)$ we have

$$H_\varepsilon[u_\varepsilon] \rightharpoonup u. \tag{4.15}$$

By Lemma 4.15 it follows $w = u$, which gives the thesis. □

5 Bounds on F_ε and the Fundamental Estimate

Henceforth we shall always assume that (H1) to (H3) hold and that the function v satisfies besides (4.1) also the conditions of Lemma 4.7.

The following lower bound is a direct consequence of Theorem 4.8 and of Lemma 4.7.

Proposition 5.1. *For each $\Omega \in \mathcal{A}$ we have*

$$(\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon)(u, \Omega) = \infty, \quad \text{for all } u \in X \setminus BV\big(\Omega; \{a^-, a^+\}\big), \tag{5.1}$$

and there exists a constant $C_1 > 0$ such that for any $\chi_{E,a^\pm} \in BV\big(\Omega; \{a^-, a^+\}\big)$ it holds

$$(\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon)\left(\chi_{E,a^\pm}, \Omega\right) \ge C_1 \mathcal{P}(E, \Omega). \tag{5.2}$$

Proof. Consider $u_\varepsilon \xrightarrow{X} u$ such that

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) = (\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon)(u, \Omega).$$

If $u \notin BV\big(\Omega; \{a^-, a^+\}\big)$, Thm. 4.8 implies $\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) = \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) = \infty$, and so (5.1) holds. If $u \in BV\big(\Omega; \{a^-, a^+\}\big)$ we note that Lemma 4.7, Theorem Cor.1.16 together with the lower-semicontinuity property of the total variation yield:

$$(\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon)(u, \Omega) \;=\; \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega) \;\ge\; C_1 \int_\Omega \big|D[H_\varepsilon u_\varepsilon]\big| \;\ge\; C_1 \int_\Omega |Du|,$$

and referring to the definition (2.16) we deduce (5.2). □

The following upper bound can be easily proved by explicit construction as in [8, Prop. 4.9]. The idea is to smooth out $M_\varepsilon[\chi_{E,a^\pm}]$ in a ε-neighborhood of its jump set.

Proposition 5.2. *There exists a constant $C_2 > 0$ such that for any $\Omega \in \mathcal{A}$ with Lipschitz boundary and for any $\chi_{E,a^\pm} \in BV\big(\Omega, \{a^-, a^+\}\big)$, we have*

$$(\Gamma - \limsup_{\varepsilon\to 0} F_\varepsilon)\,\big(\chi_{E,a^\pm}, \Omega\big) \le C_2 \mathcal{P}(E, \Omega) \tag{5.3}$$

It remains to show the so-called fundamental estimate, see e.g. Lemma 3.2 in [1], which implies that $(\Gamma - \liminf F_\varepsilon)(u, \cdot)$ is a subadditive set function.

Lemma 5.3. *Let $U, U', V \in \mathcal{A}$, with $U \Subset U'$, and let $S := \big(U' \setminus \overline{U}\big) \cap V$. Let u_ε and v_ε such that*

$$\limsup_{\varepsilon\to 0} \big(F_\varepsilon(u_\varepsilon, U') + F_\varepsilon(v_\varepsilon, V)\big) < +\infty \tag{5.4}$$

$$\limsup_{\varepsilon\to 0} (\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty) < +\infty \tag{5.5}$$

$$(u_\varepsilon - v_\varepsilon) \rightharpoonup 0 \text{ weakly in } L^2(S). \tag{5.6}$$

Then there exists a function $\varphi \in C^\infty(\mathbb{R}^N, [0,1])$ such that

$$\varphi = 1 \text{ on } \overline{U}, \quad \varphi = 0 \text{ on } \mathbb{R}^N \setminus U', \quad |\nabla\varphi| \le C\varepsilon^{-1},$$

and

$$F_\varepsilon(\varphi u + (1-\varphi)v, U \cup V) \le F_\varepsilon(u, U') + F_\varepsilon(v, V) + \delta_\varepsilon(u, v, U, U', V), \tag{5.7}$$

where

$$\lim_{\varepsilon\to 0} \delta_\varepsilon(u_\varepsilon, v_\varepsilon, U, U', V) = 0.$$

The proof follows closely that of [8, Prop. 4.4]. First we give some definitions.

Definition 5.4. *Let $U, U', V \in \mathcal{A}$, with $U \Subset U'$. For $i \in \mathbb{N}$, we define*

$$U_0 := \operatorname{int}\Big(\textstyle\bigcup_{\{z:\ \operatorname{dist}(Q_\varepsilon(z),U)\le\varepsilon/2,\ Q_\varepsilon(z)\subset U'\}} \overline{Q_\varepsilon(z)}\Big), \qquad V_\varepsilon := \operatorname{int}\Big(\textstyle\bigcup_{\{z:\ Q_\varepsilon(z)\subset V\}} \overline{Q_\varepsilon(z)}\Big),$$

$$U_{i+1} := \operatorname{int}\Big(\textstyle\bigcup_{\{z:\ \operatorname{dist}(Q_\varepsilon(z),U_i)\le\varepsilon/2,\ Q_\varepsilon(z)\subset U'\}} \overline{Q_\varepsilon(z)}\Big), \quad \widetilde{S}_\varepsilon(i) := \big(U_{i+1} - \overline{U}_i\big) \cap V_\varepsilon.$$

The following Lemma lists several condition which are sufficient to obtain (5.7). Hence it remains to verify that these conditions are fulfilled.

Lemma 5.5. *Let $u_\varepsilon, v_\varepsilon \in H^1(\mathcal{O})$, and let U, U', V, $U_i, \widetilde{S}_\varepsilon(i)$ be as in Definition 5.4. Assume that we can find $i_\varepsilon \in \mathbb{N}$ such that the sets $\widetilde{S}_\varepsilon := \widetilde{S}_\varepsilon(i_\varepsilon)$ fulfill*

$$F_\varepsilon(u_\varepsilon, \widetilde{S}_\varepsilon) + F_\varepsilon(v_\varepsilon, \widetilde{S}_\varepsilon) \;\to\; 0 \qquad \text{as } \varepsilon \to 0, \tag{5.8}$$

$$\int_{\widetilde{S}_\varepsilon} \frac{|u_\varepsilon - v_\varepsilon|}{\varepsilon}\, \mathrm{d}x \;\to\; 0 \qquad \text{as } \varepsilon \to 0, \tag{5.9}$$

$$\int_{\widetilde{S}_\varepsilon} \varepsilon |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \, \mathrm{d}x \;\to\; 0 \qquad \text{as } \varepsilon \to 0, \tag{5.10}$$

$$\int_{\widetilde{S}_\varepsilon} \varepsilon \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) \, \mathrm{d}x \;\leq\; C. \tag{5.11}$$

Let $\varphi \in C^1(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on U_{i_ε}, $\varphi \equiv 0$ on $\mathbb{R}^N \setminus U_{i_\varepsilon+1}$ and $|\nabla\varphi| \leq C\varepsilon^{-1}$ for some C independent of ε. Define $z_\varepsilon := \varphi u_\varepsilon + (1-\varphi)v_\varepsilon$, then we have

$$F_\varepsilon(z_\varepsilon, \widetilde{S}_\varepsilon) \to 0.$$

The proof is a minor modification of the proof of Lemma 4.6 in [8] and therefore omitted. We have to verify that the assumptions of Lemma 5.5 hold for any two sequences u_ε and v_ε such that $\limsup_{\varepsilon\to 0} \left(F_\varepsilon(u_\varepsilon, U') + F_\varepsilon(v_\varepsilon, V)\right) < +\infty$, and $(u_\varepsilon - v_\varepsilon) \rightharpoonup 0$ on $S := \left(U' \setminus \overline{U}\right) \cap V$. We will split the proof in several lemmas, the first of them saying that weak convergence to the same limit and bounded energy implies that two sequences are close in L^1.

Lemma 5.6. *If u_ε and v_ε fulfill (5.4)-(5.6), then*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - v_\varepsilon\|_{L^1(S)} = 0.$$

Note that we cannot expect each of the sequences to converge strongly, but we will show that the difference converges strongly.

Proof. First note that a sequence of bounded energy is bounded in $L^2(S)$, which implies, by our assumption, that both sequences (up to passing to a common subsequence) converge weakly in $L^2(S)$ to the same function $w^* \in L^2(S)$. As $H_\varepsilon[u_\varepsilon]$ and $H_\varepsilon[v_\varepsilon]$ are bounded in $BV(S)$ by Corollary 4.8, they have a (common) subsequence which converges strongly in $L^1(S)$ to w^*. From the definition of $H_\varepsilon[u_\varepsilon]$ we see that $H_\varepsilon[u_\varepsilon] - M_\varepsilon[u_\varepsilon] \rightharpoonup 0$ in $L^2(\Omega)$. This implies that $M_\varepsilon[u_\varepsilon]$ also converge to w^* (up to the same subsequence) weakly in $L^2(S)$. We then estimate

$$\|u_\varepsilon - v_\varepsilon\|_{L^1(S)} \;\leq\; \|M_\varepsilon[u_\varepsilon] - u_\varepsilon\|_{L^1(S)} + \|M_\varepsilon[u_\varepsilon] - M_\varepsilon[v_\varepsilon]\|_{L^1(S)} + \|M_\varepsilon[v_\varepsilon] - v_\varepsilon\|_{L^1(S)}.$$

By Corollary 4.6 we know that

$$\|M_\varepsilon[u_\varepsilon] - u_\varepsilon\|_{L^1(S)} + \|M_\varepsilon[v_\varepsilon] - v_\varepsilon\|_{L^1(S)} \to 0 \quad (\text{as } \varepsilon \to 0).$$

So it remains to show the same for $M_\varepsilon[u_\varepsilon] - M_\varepsilon[v_\varepsilon]$. Recalling that $\|H_\varepsilon[u_\varepsilon] - H_\varepsilon[v_\varepsilon]\|_{L^1(S)} \to 0$, by definition of $H_\varepsilon[u]$ we get

$$\left\| \sum_{z\in\mathbb{Z}^N : Q_\varepsilon(z)\subset\Omega} \left(\mathrm{sign}_\varepsilon[u_\varepsilon, z] - \mathrm{sign}_\varepsilon[v_\varepsilon, z]\right) \chi_{Q_\varepsilon(z)} \right\|_{L^1(S)} \to 0.$$

Since, by definition,

$$M_\varepsilon[u] = \frac{1}{2}(u^+ + u^-) + \frac{1}{4}(u^+ - u^-) \sum_{z\in\mathbb{Z}^N: Q_\varepsilon(z)\subset\Omega} \mathrm{sign}_\varepsilon[u,z]\chi_{Q_\varepsilon(z)},$$

this implies that

$$\lim_{\varepsilon\to 0} \|M_\varepsilon[u_\varepsilon] - M_\varepsilon[v_\varepsilon]\|_{L^1(S)} = 0.$$

This concludes the proof. □

The next estimate allows us to bound the gradient term on the bad set by the energy.

Lemma 5.7. *Let t be as in Lemma 4.5. Then there exists a constant $C = C(W, v, t) > 0$ such that*

$$\varepsilon \int_{B^t_\varepsilon(u)} |\nabla u|^2 \le C F_\varepsilon(u, \Omega), \qquad \textit{for all } u \in H^1(\Omega).$$

Proof. Assume w.l.o.g. that $Q_\varepsilon := Q_\varepsilon(z)$ is a positive cube. Now let $\rho = t + \|v\|_\infty/2$ and note that

$$\int_{B^t_\varepsilon(u)\cap Q_\varepsilon} |\nabla u|^2 \le \int_{\{u<\rho\}\cap Q_\varepsilon} |\nabla u|^2.$$

First we estimate the part of the integral where $u \le -\rho$. Let $T_\rho(u) = u$, if $u > -\rho$, and $T_\rho(u) = -\rho$ otherwise. Let $w_\varepsilon(x) := v(\varepsilon^{-1}x)$. We estimate, using $2ab \le a^2 + b^2$,

$$\begin{aligned} F_\varepsilon(u, Q_\varepsilon) \quad &\ge \quad G_\varepsilon(u, Q_\varepsilon) - G_\varepsilon(T_\rho(u), Q_\varepsilon) + \int_{\{u\le -\rho\}} \left(\varepsilon|\nabla u|^2 + \frac{W(u) - W(-\rho)}{\varepsilon} - \varepsilon|\nabla w_\varepsilon||\nabla u| \right) \\ &\ge \quad \int_{\{u\le -\rho\}} \frac{\varepsilon}{2}|\nabla u|^2 - \varepsilon^{-1} \left(\sup_{[-\rho,\rho]} (W(u) - W(\rho)) + \frac{\|v\|^2_{W^{1,\infty}(Q)}}{2} \right) |\{u < -\rho\}|. \end{aligned}$$

Hence

$$\varepsilon \int_{\{u\le -\rho\}\cap Q_\varepsilon(z)} \varepsilon|\nabla u|^2 \le \varepsilon F_\varepsilon(u, Q_\varepsilon) + C(W, t, v)|B^t_\varepsilon \cap Q_\varepsilon|. \tag{5.12}$$

In order to estimate the remaining part of the integral, where $|u| < \rho$, we define $u^\rho := |u| \vee \rho$. As the double-well potential is symmetric and increasing on $[0, 1]$ (see **(H1)** we have $W(u) - W(u^\rho) \ge 0$. Using again $2ab \le a^2 + b^2$ we estimate

$$F_\varepsilon(u, Q_\varepsilon) \quad \ge \quad G_\varepsilon(u, Q_\varepsilon) - G_\varepsilon(u^\rho, Q_\varepsilon) \ge \int_{\{|u|\le\rho\}} \frac{\varepsilon}{2}|\nabla u|^2 - 2\frac{\|v\|^2_{W^{1,\infty}}}{\varepsilon}|\{u < \rho\}| - \int_{\{u<-\rho\}} \frac{\varepsilon}{2}|\nabla u|^2.$$

Using (5.12) we obtain

$$\varepsilon \int_{\{|u|<\rho\}\cap Q_\varepsilon} \varepsilon|\nabla u|^2 \le C'(t, W, v)|B^t_\varepsilon \cap Q_\varepsilon| + \varepsilon F_\varepsilon(u, Q_\varepsilon),$$

and the lemma is shown. □

We let in the following

$$S_\varepsilon := \text{int}\left(\bigcup_{\{z:\ Q_\varepsilon(z)\subset S\}} \overline{Q_\varepsilon(z)}\right).$$

Lemma 5.8. *If u_ε and v_ε fulfill* (5.4)-(5.6)*, on a set $S \in \mathcal{A}$, then*

$$\varepsilon^2 \int_{S_\varepsilon} |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \to 0.$$

Proof. The main idea of the proof is to use Lemma 4.3 on the "good" set, were u_ε is in the correct well of the double-well potential, and then to bound $\int_{B_\varepsilon} \nabla |u_\varepsilon|^2$, where the "bad" set B_ε is as in Definition 4.4.
We have

$$\begin{aligned}
\frac{\varepsilon}{2}\int_{S_\varepsilon} \varepsilon|\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \mathrm{d}x \ &\le \ \int_{S_\varepsilon} \varepsilon^2 \left|D^a(M_\varepsilon[u_\varepsilon]) - D^a(M_\varepsilon[v_\varepsilon])\right|^2 \mathrm{d}x \\
&+ \ \int_{S_\varepsilon} \varepsilon^2 |D^a M_\varepsilon[u_\varepsilon] - D^a C^t_\varepsilon[u_\varepsilon]|^2 \mathrm{d}x + \int_{S_\varepsilon} \varepsilon^2 |D^a M_\varepsilon[v_\varepsilon] - D^a C^t_\varepsilon[v_\varepsilon]|^2 \mathrm{d}x \\
&+ \ \int_{S_\varepsilon} \varepsilon^2 |D^a C^t_\varepsilon[u_\varepsilon] - D^a u_\varepsilon|^2 \mathrm{d}x + \int_{S_\varepsilon} \varepsilon^2 |D^a C^t_\varepsilon[v_\varepsilon] - D^a v_\varepsilon|^2 \mathrm{d}x \\
&= \ I_1 + I_2[u_\varepsilon] + I_2[v_\varepsilon] + I_3[u_\varepsilon] + I_3[v_\varepsilon].
\end{aligned}$$

By Lemma 4.3, we get $I_2[u_\varepsilon] + I_2[v_\varepsilon] \to 0$.
As Proposition 3.6 implies that $\nabla u^+ = \nabla u^-$, we obtain from the definition of $M_\varepsilon[u]$

$$I_1 = |D^a(M_\varepsilon[u_\varepsilon]) - D^a(M_\varepsilon[v_\varepsilon])| = 0.$$

It remains to estimate $I_3[u_\varepsilon] + I_3[v_\varepsilon]$. In order to show that $I_3[u_\varepsilon] \to 0$, first note that for any x with $C^t_\varepsilon[u](x) \neq u(x)$ we have $|D^a C^t_\varepsilon[u](x)| \le \max(\varepsilon^{-1}\|v\|_{W^{1,\infty}}, |\nabla u|)$, hence

$$I_3[u_\varepsilon] \le 2\|v\|^2_{W^{1,\infty}} |B^t_\varepsilon(u_\varepsilon) \cap S_\varepsilon| + 4\varepsilon^2 \int_{\{C^t_\varepsilon[u] \neq u\}} |\nabla u_\varepsilon|^2.$$

Thereofore $I_3 \to 0$ is a consequence of Lemma 4.5 an Lemma 5.7. □

Lemma 5.9. *If u_ε and v_ε are such that*

$$\limsup_{\varepsilon\to 0} \big(F_\varepsilon(u_\varepsilon, U') + F_\varepsilon(v_\varepsilon, V)\big) < +\infty \quad \textit{and} \quad (u_\varepsilon - v_\varepsilon) \rightharpoonup 0 \textit{ on } S,$$

then we can find sets $\widetilde{S}_\varepsilon$ which fulfill the assumptions of Lemma 5.5.

Proof. Property (5.8) follows from an averaging argument (originally due to De Giorgi) as in [8, Lemma 4.7], and from the assumption that both sequences have bounded energy. More precisely, letting k_ε be the largest integer such that $\widetilde{S}_\varepsilon(i) \neq 0$ for $i \le k_\varepsilon$, we have

$$\sum_{i=0}^{k_\varepsilon} \Big(F_\varepsilon(u_\varepsilon, \widetilde{S}_\varepsilon(i)) + F_\varepsilon(v_\varepsilon, \widetilde{S}_\varepsilon(i))\Big) \le F_\varepsilon(u_\varepsilon, S) + F_\varepsilon(v_\varepsilon, S) \le C.$$

Since k_ε is of order $1/\varepsilon$, for at least one half of the indeces i there holds

$$F_\varepsilon(u_\varepsilon, \widetilde{S}_\varepsilon(i)) + F_\varepsilon(v_\varepsilon, \widetilde{S}_\varepsilon(i)) \leq \frac{2C}{k_\varepsilon} \leq \widetilde{C}\varepsilon,$$

which gives (5.8).
Property (5.9) follows as above from an averaging argument, and from Lemma 5.6.
In order to prove (5.10), we use Lemma 5.8 to show that $\varepsilon^2 \int_{S_\varepsilon} |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \to 0$ and apply again an averaging argument. Finally, we can estimate the energy $F_\varepsilon(u_\varepsilon, S)$ as follows

$$\begin{aligned} F_\varepsilon(u_\varepsilon, S) &\geq G_\varepsilon(u_\varepsilon, S_\varepsilon) \geq \int_{S_\varepsilon} \varepsilon\left(|\nabla u_\varepsilon|^2 + \nabla w_\varepsilon \nabla u_\varepsilon\right) \mathrm{d}x \\ &\geq \frac{1}{2}\int_{S_\varepsilon} \varepsilon\left(|\nabla u_\varepsilon|^2 - |\nabla w_\varepsilon|^2\right) \mathrm{d}x \geq \frac{1}{2}\int_{S_\varepsilon} \varepsilon|\nabla u_\varepsilon|^2 - \frac{C}{\varepsilon}|S_\varepsilon|. \end{aligned}$$

Therefore, we obtain

$$\int_{S_\varepsilon} \varepsilon|\nabla u_\varepsilon|^2 \mathrm{d}x \leq \frac{C}{\varepsilon},$$

and (5.11) follows as before from an averaging argument. □

6 Representation and properties of the Γ-limit

Once we have both the fundamental estimate and the estimates from above and below, we can reason as in [4, Th. 10.3, Prop. 11.6]. We get the following result.

Proposition 6.1. *There exist a sequence $\varepsilon_j \to 0$ and $F_0 : L^2(\mathbb{R}^N) \times \mathcal{A} \to [0, +\infty]$, such that*

$$(\Gamma - \lim_{j\to\infty} F_{\varepsilon_j})(\cdot, \Omega) = F_0(\cdot, \Omega),$$

for any $\Omega \in \mathcal{A}$ with Lipschitz boundary (in the weak $L^2(\Omega)$-topology). Moreover, for any $u \in BV(\mathcal{O}; \{a^-, a^+\})$, $F_0(u, \cdot)$ is the restriction to $\mathcal{A}$ of a regular Borel measure, whereas $F_0(u, \cdot) \equiv +\infty$ if $u \notin BV(\mathcal{O}; \{a^-, a^+\})$.

Note that the functional F_0 obtained in previous proposition may depend on the subsequence ε_j. The fact that F_0 is independent of such sequences will be a consequence of the representation formula stated below.

Definition 6.2. *Given $x, \nu \in \mathbb{R}^N$ we introduce the notations:*

$$\mathbb{H}^{\nu,x} := \{y \in \mathbb{R}^N : (y - x) \cdot \nu \leq 0\}, \qquad \chi^{\nu,x}(y) := \begin{cases} a^+ & \text{if } y \in \mathbb{H}^{\nu,x}, \\ a^- & \text{if } y \notin \mathbb{H}^{\nu,x}. \end{cases}$$

Furthermore setting Q^ν to be the unit closed cube centered at the origin with two of its faces orthogonal to ν, we define

$$Q_\rho^{\nu,x} := x + \rho Q^\nu.$$

Recalling [3, Theorem 1.0.3] (see also [1, Theorem 3.5]), we obtain the following representation formula for the functional F_0.

Theorem 6.3. *Consider the constants $C_1, C_2 > 0$ given by Propositions 5.1 and 5.2. Then there exists a Borel function $\varphi : \mathbb{R}^N \times \mathcal{S}^{N-1} \to [C_1, C_2]$ such that*

$$F_0(u,B) \quad = \quad \begin{cases} \displaystyle\int_{\partial^* E\cap B} \varphi(x,\nu_E)\,d\mathcal{H}^{N-1} & \text{if } u = \chi_{E,a^\pm} \in BV\big(\Omega;\{a^-,a^+\}\big), \\ +\infty & \text{otherwise,} \end{cases}$$

for any open set Ω with Lipschitz boundary and any Borel set $B \subseteq \Omega$. In particular, φ satisfies

$$\varphi(x,\nu) = \limsup_{\rho\to 0^+} \frac{1}{\rho^{N-1}}\ \min\Big\{F_0\left(u, Q_\rho^{\nu,x}\right):\ u = \chi^{\nu,x} \text{ in } \mathbb{R}^N \setminus Q_\rho^{\nu,x}\Big\}. \tag{6.1}$$

The representation formula (6.1) allows us to obtain more informations on the function φ. In particular, the following proposition shows that the Γ-limit is homogeneous, i.e. φ does not depend on x. The proof follows exactly as in [8, Prop. 5.6].

Proposition 6.4. *The function φ given by Theorem 6.3 does not depend on x, moreover its one–homogeneous extension*

$$\widetilde{\varphi} : \mathbb{R}^N \to [0,\infty) \quad x \mapsto \begin{cases} |x|\varphi(x/|x|) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is convex.

Given $\nu \in \mathcal{S}^{N-1}$ and $\lambda > 0$, we define

$$[Q_\lambda^\nu] := \bigcup_{z:\ (z+Q^{\nu,0})\subset\lambda Q} \left(z + Q^{\nu,0}\right)\ .$$

We conclude with the following representation result for the function φ, the proof of which follows exactly as in [1, Theorem 4.3].

Theorem 6.5. *We have the following representation for the function φ:*

$$\varphi(\nu) \quad = \quad \lim_{\lambda\to+\infty} \frac{1}{\lambda^{N-1}} \min\Big\{\widetilde{G}(u,[Q_\lambda^\nu]) - \widetilde{G}(u^\pm,[Q_\lambda^\nu])\ : u = \chi^{\nu,0} \text{ on } \mathbb{R}^N \setminus [Q_\lambda^\nu]\Big\}. \tag{6.2}$$

In particular, the Γ–limit does not depend on the subsequence ε_j.

Finally, we note that as in [8, Proposition 5.1] assumption (H3) implies that

$$\varphi(\nu) = \varphi(-\nu) \qquad \forall \nu \in \mathcal{S}^{N-1}\,.$$

7 Final remarks

7.1 The case of two different wells

All the results of this paper can be extended to the more general case of a double-well potential which is not necessarily even. We can consider, for example, a potential W which satisfies, instead of (H1), the assumption

(H1') $W \in C^2(\mathbb{R})$, $W \geq 0$, $W^{-1}(0) = \{1\}$, W is strictly decreasing in $[0,1]$ and strictly increasing in $[-1,0]$. Moreover, there exist positive constants $\delta_0, \delta_1, C_+, C_-$ such that

$$\begin{aligned} W(s) &= C_+(s-1)^2 && \forall s \in (1-\delta_0, +\infty)\,, \\ W(s) &= C_-(s+1)^2 + \delta_1 && \forall s \in (-\infty, -1+\delta_0)\,, \end{aligned}$$

where the constants $C_+ \gg C_- > 0$ and $\delta_1 > 0$ are chosen in such a way that

$$\min_{v \in H^1(Q):\ v \geq 0} \widetilde{G}(v, Q) = \min_{v \in H^1(Q):\ v \leq 0} \widetilde{G}(v, Q).$$

We observe that it is always possible to find such constants for $\|v\|_{W^{1,\infty}}$ sufficiently small. Indeed, since the lower well of the potential (at $s = 1$) is very narrow, while the higher well (at $s = -1$) is very flat, the v-dependent part of the functional is less effective around the higher well than around the lower one. Thus the energy of the positive and negative minimizers can be equal, even though the depth of the wells is different. This is made precise in the following computation. For $\|v\|_{W^{1,\infty}}$ small enough, there exist two solutions $u^\pm \in H^1(Q)$ of the Euler-Lagrange equation of $\widetilde{G}$ such that $\pm u^\pm(x) \geq 1 - \delta_0$ for all $x \in Q$ and they are unique within the class of positive (resp. negative) functions. Therefore the Euler-Lagrange equation is linear, i.e.

$$-2\Delta u^\pm + 2C_\pm \left(u^\pm \mp 1\right) = \Delta v.$$

Let $(\lambda_k)_{k\in\mathbb{N}}$ be the sequence of eigenvalues of the Laplace operator $-\Delta$ on Q with periodic boundary conditions and consider an associated sequence of orthonormal eigenfunctions $(e_k)_{k\in\mathbb{N}}$. If $v_k := \langle v, e_k\rangle_{L^2(Q)}$ are the Fourier coefficients of v, then the Fourier coefficients $u_k^\pm$ of $u^\pm$ are given by

$$u_0^\pm = \pm 1 \quad \text{and} \quad u_k^\pm = -\frac{\lambda_k}{2(\lambda_k + C_\pm)} v_k \quad \text{for } k \geq 1.$$

Consequently,

$$\widetilde{G}(u^+, Q) = -\frac{1}{4}\sum_{k\geq 1} \frac{\lambda_k^2 v_k^2}{\lambda_k + C_+} \quad \text{and} \quad \widetilde{G}(u^-, Q) = \delta_1 - \frac{1}{4}\sum_{k\geq 1} \frac{\lambda_k^2 v_k^2}{\lambda_k + C_-}\,,$$

which implies

$$\widetilde{G}(u^-, Q) - \widetilde{G}(u^+, Q) = \delta_1 - \frac{C_+ - C_-}{4}\sum_{k\geq 1} \frac{\lambda_k^2 v_k^2}{(\lambda_k + C_+)(\lambda_k + C_-)}.$$

For $C_+ > C_-$ there exists $\delta_1 > 0$ such that $\widetilde{G}(u^-, Q) = \widetilde{G}(u^+, Q)$.

7.2 Limit of the parabolic problems

Let us conclude with a brief discussion of the convergence of the parabolic equations corresponding to the gradient flows of the functionals G_ε (properly rescaled in time), i.e.

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta u - \frac{W'(u)}{2\varepsilon^2} + \frac{1}{2\varepsilon^2}\Delta v\left(\frac{x}{\varepsilon}\right) & (x,t) \in \Omega \times (0, +\infty) \\ u(\cdot, 0) = \chi_E - \chi_{\Omega\setminus E} & \text{in } \Omega, \end{cases} \tag{7.1}$$

where $E \subset \Omega$ is a set of finite perimeter.

For v smooth, Problem (7.1) admits a smooth solution $u_\varepsilon : \Omega \times (0, +\infty) \to \mathbb{R}$ (globally defined). Moreover it is well known that, when $v \equiv 0$, the functions u_ε converge as $\varepsilon \to 0$ to the characteristic function of a set $E(t)$, which is the evolution by mean curvature of the set E at time t [2, 12]. Conversely, we expect that the presence of the function v may prevent the motion of the interface ∂E, which remains "trapped" between local minimizers of the approximating functionals. Therefore, for a wide class of forcing terms v, for each $t \in (0, \infty)$ we expect:

$$u_\varepsilon(\cdot, t) \rightharpoonup \chi_{E,a^\pm} \quad \text{(weakly in } L^2(\Omega)) .$$

Namely while there is a relaxation towards the positive and negative minimizers, the set separating the positive and negative region in space does not move significantly. In the simpler one-dimensional case this so-called "pinning" of fronts has been studied in detail in [9].

Acknowledgements. N.D. would like to thank Prof. Enza Orlandi for helpful discussions. The second author was supported by an Alexander von Humboldt fellowship.

References

[1] N. Ansini, A. Braides, V. Chiadò Piat, *Gradient theory of phase transitions in composite media*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), no. 2, 265–296.

[2] G. Barles, H. M. Soner, P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim. **31** (1993), no. 2, 439–469.

[3] G. Bouchitté, I. Fonseca, G. Leoni, L. Mascarenhas, *A global method for relaxation in $W^{1,p}$ and SBV_p*, Arch. Ration. Mech. Anal. **165** (2002), 187–242.

[4] A. Braides, A. Defranceschi, Homogenization of multiple integrals, Oxford University Press, Oxford, 1998.

[5] A. Braides, L. Truskinovsky, *Asymptotic expansions by Γ-convergence*, Preprint (2007) (available at http://cvgmt.sns.it).

[6] G. Dal Maso, An introduction to Γ-convergence, Birkhäuser, Boston, 1993.

[7] E. De Giorgi, *Sulla convergenza di alcune successioni di integrali del tipo dell'area*, Rend. Mat. **8** (1975), 277–294.

[8] N. Dirr, M. Lucia, M. Novaga, *Γ-convergence of the Allen–Cahn energy with an oscillating forcing term*, Interfaces and Free Boundaries, **8** (2006), 47–78.

[9] N. Dirr, N. K. Yip, *Pinning and De-Pinning Phenomena in Front Propagation in Heterogeneous Media*, Interfaces and Free Boundaries, **8** (2006), 79–109.

[10] L.C. Evans, R.L. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[11] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[12] T. Ilmanen, *Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993) 417–461.

[13] L. Modica, S. Mortola, *Un esempio di* Γ^{-}*-convergenza*, Bollettino U.M.I. **14-B** (1977), 285–299.

[14] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1987), 123–142.