Motion by Curvature of Planar Networks II

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Abstract

We prove that the curvature flow of an embedded planar network of three curves connected through a triple junction, with fixed endpoints on the boundary of a given strictly convex domain, exists smooth until the lengths of the three curves stay far from zero. If this is the case for all times, then the evolution exists for all times and the network converges to the Steiner minimal connection between the three endpoints.

1 Introduction

We are interested in the long time behavior of the evolution by curvature of a *triod*, that is, a network of three planar curves meeting at a common point (called *triple junction*) with equal angles (the so-called *Herring condition*) and with fixed endpoints on the boundary of a given convex domain in the Euclidean plane.

As for the mean curvature flow, this evolution can be regarded as the gradient flow of the *Length* functional. In this respect, the Herring condition naturally arises from the variational interpretation of the flow and corresponds to the local stability of the triple junction.

An important motivation for this study is due to the appearance of this evolution in several models of materials science for the motion of grain boundaries in a polycrystalline material or, more generally, of two–dimensional multiple phase systems (see [7, 14, 15] and references therein). Another more theoretical motivation comes from the fact that this is possibly the simplest evolution by curvature of a nonsmooth set. Indeed, while the mean curvature flow of a smooth submanifold is deeply, even if not completely, understood, the evolution of generalized submanifolds admitting singularities, for instance a *varifold*, has not been studied too much in detail after the seminal work by K. Brakke [8] (see also [10, 11] for an alternative approach based on the implicit variational scheme introduced in [2]), we mention anyway the works of T. Ilmanen [19] and K. Kasai and Y. Tonegawa [22].

The mathematical analysis of this flow started in [9] (see also [23]), where short time existence and uniqueness of a smooth flow has been established, and continued in [26] where the authors proved that, at the first singular time, either the curvature blows–up or the length

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of one of the three curves goes to zero on a sequence of times. Extending the analysis performed by G. Huisken for the mean curvature flow (see [17] and references therein), they could also rule out certain kinds of singularities, namely the so–called *Type I* singularities, corresponding to a specific blow–up rate of the curvature at the singular time. A significant difficulty of this analysis is the lack of maximum principle, due to the presence of the triple junction, which requires new arguments in order to estimate geometric quantities such as the curvature and its derivatives.

In this paper we complete the program started in [26] and we prove that no singularity can arise during the evolution of a triod, independently of the type of singularity. More precisely, our main result is the following.

Theorem 1.1. *For any smooth, embedded, initial triod $\mathbb{T}_0$ in a strictly convex set $\Omega \subset \mathbb{R}^2$, with fixed endpoints $P^1, P^2, P^3 \in \partial\Omega$, there exists a unique smooth evolution by curvature of $\mathbb{T}_0$ which at every time is a nondegenerate smooth embedded triod in Ω, in a maximal time interval $[0, T)$. Then, either the inferior limit of the length of one of three curves of the triod $\mathbb{T}_t$ goes to zero as $t \to T$, or $T = +\infty$ and $\mathbb{T}_t$ tends as $t \to +\infty$ to the unique Steiner triod connecting the three fixed endpoints.*

Our strategy is based on the analysis of the blow–up of the flow at a given point, independently of the behavior of the curvature. Using some ideas presented in [20] (see also [25]), which are based on Huisken's *monotonicity formula* (see [17]), we are able to classify all the possible blow–up limits. It turns out that the only admissible configurations are a straight line, a halfline or a flat unbounded triod (see Proposition 2.19). As none of them arises from a singular point of the flow, we obtain our main result. A fundamental ingredient in our analysis are the interior regularity estimates of K. Ecker and G. Huisken (see [12]), which we combine with the estimates on the curvature and its derivatives obtained in [26].

One difficulty in this classification is to show that the possible limits necessarily have multiplicity one. This follows from a geometric argument proposed in [16] (see also [18]) and extended in [26] to the case of a triod, consisting in estimating from below a kind of "embeddedness measure", which is strictly positive when no self–intersections are present and showing that it is monotonically increasing for an evolving triod. We underline that it is not clear to us how to obtain a similar bound for a general network (with multiple triple junctions), since the analogous quantity is no longer monotone if there are more than two triple junctions.

Recently, T. Ilmanen, A. Neves and F. Schulze announced a comprehensive analysis of the evolution by curvature of a general network with several *multiple* (not only triple) junctions, with any angles between the concurring edges. This would clearly include and greatly generalize our work.

In their preliminary paper [21], which the authors kindly sent us, it is proved a local regularity result stating that if in an open set in space–time the Gaussian densities are bounded away from two and the network is locally tree–like, then the flow is smooth with bounded curvature and its derivatives.

Independently of such a general result, this paper deals with the simpler situation of a single triod with fixed endpoints in a strictly convex domain. Our goal is simply to show that singularities cannot happen in this special case, hence completing the program started in [26]. We point out that our method cannot be directly extended to the case of a network with more than two triple junctions, due to the aforementioned main difficulty in showing that the blow–ups at the singular points have multiplicity one.

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2 Definitions and Preliminary Results

Definition 2.1. Let $\Omega \in \mathbb{R}^2$ be a smooth open set and $\mathbb{T} = \cup_{i=1}^3 \sigma^i$ the union of three embedded (at least C^2), regular (i.e. $\sigma_x \neq 0$ for all $x \in [0,1]$) curves $\sigma^i : [0,1] \to \overline{\Omega}$. Let $P^i \in \partial\Omega$, for $i \in \{1,2,3\}$, three distinct points. We say that $\mathbb{T}$ is a *triod* in Ω if

- $\sigma^i(x) \in \partial\Omega$ if and only if $x = 1$, for all $i \in \{1,2,3\}$;
- $O = \sigma^i(0)$ for all $i \in \{1,2,3\}$ and $\sigma^1(x) \neq \sigma^2(y) \neq \sigma^3(z)$ for all $x,y,z \in (0,1]$;
- $\sigma^i(1) = P^i$ for all $i \in \{1,2,3\}$;
- $\sum_{i=1}^3 \frac{\sigma^i_x(0)}{|\sigma^i_x(0)|} = 0$.

Under these conditions, we will call O the *3–point of the triod* $\mathbb{T}$ and P^i the *endpoints of the triod* $\mathbb{T}$.

For a given "initial" triod $\mathbb{T}_0 = \cup_{i=1}^3 \sigma^i$, we consider the following motion by curvature (see [9] and [26]).

Definition 2.2. We say that the one parameter family of triods $\mathbb{T}_t = \cup_{i=1}^3 \gamma(\cdot, t)$ evolve by curvature (staying embedded) in the time interval $[0,T)$ ($T > 0$), if the three family of curves $\gamma^i : [0,1] \times [0,T) \to \overline{\Omega}$ are at least of class C^2 in the first variable and of class C^1 in the second one, and satisfy the following quasilinear parabolic system,

$$
\begin{cases}
\gamma^i_x(x,t) \neq 0 & & \text{regularity} \\
\gamma^i(x,t) \in \partial\Omega & \text{iff } x = 1 & \text{intersection with } \partial\Omega \text{ only at the endpoints} \\
\gamma^i(x,t) \neq \gamma^i(y,t) & \text{if } x \neq y & \text{simplicity} \\
\gamma^i(x,t) = \gamma^j(y,t) \Leftrightarrow x, y = 0 & \text{if } i \neq j & \text{intersection only at the 3–point} \\
\sum_{i=1}^3 \frac{\gamma^i_x(0,t)}{|\gamma^i_x(0,t)|} = 0 & & \text{angles of 120 degrees at the 3–point} \\
\gamma^i(1,t) = P^i & & \text{fixed endpoints condition} \\
\gamma^i(x,0) = \sigma^i(x) & & \text{initial data} \\
\gamma^i_t(x,t) = \frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2} & & \text{motion by curvature}
\end{cases}
\tag{2.1}
$$

for every $x \in [0,1]$, $t \in [0,T)$ and $i,j \in \{1,2,3\}$.

To denote a flow we will often write simply $\mathbb{T}_t$ instead of letting explicit the curves γ^i which compose the triods.
Moreover, it will be also useful to describe a triod as a map $F : \mathbb{T} \to \overline{\Omega}$ from a fixed *standard* triod $\mathbb{T}$ in $\mathbb{R}^2$, composed of three unit segments from the origin in the plane, forming angles of 120 degrees. In this case we will still denote with O the 3–point of $\mathbb{T}$ and with P^i the three endpoints of such standard triod.

The evolution then will be given by a map $F : \mathbb{T} \times [0, T) \to \overline{\Omega}$, constructed naturally from the curves γ^i, so $\mathbb{T}_t = F(\mathbb{T}, t)$.

In [26] the following short time existence and uniqueness theorem has been proven.

Theorem 2.3. *For any smooth initial triod $\mathbb{T}_0$ in a convex set $\Omega \subset \mathbb{R}^2$, there exists a unique smooth solution of Problem (2.1) in a maximal time interval $[0, T)$, with $T > 0$. In particular, the evolving triod does not exit the open set Ω (with exception of the three fixed endpoints P^i).*

The goal of this paper is to show the following result which, with the above theorem, gives Theorem 1.1 in the introduction.

Theorem 2.4. *Given a triod $F : \mathbb{T} \times [0, T) \to \overline{\Omega}$ evolving by curvature, where Ω is a strictly convex subset of $\mathbb{R}^2$, -either the inferior limit of the length of one of three curves of the triod $\mathbb{T}_t$ goes to zero as $t \to T$, or $T = +\infty$ and $\mathbb{T}_t$ tends as $t \to +\infty$ to the unique Steiner triod connecting the three fixed endpoints.*

We remark that the first situation can actually happen, for instance, if the triangle formed by the points P^1, P^2, P^3 has one angle larger than 120 degrees. Notice that the strict convexity of Ω implies that such triangle is nondegenerate.

Along the paper we will make extensive use of the following notation,

$\tau^i = \tau^i(x,t) = \frac{\gamma^i_x}{\lvert\gamma^i_x\rvert}$	unit tangent vector to γ^i ,
$\nu^i = \nu^i(x,t) = \mathrm{R}\tau^i(x,t) = \mathrm{R}\frac{\gamma^i_x}{\lvert\gamma^i_x\rvert}$	unit normal vector to γ^i ,
$O = O(t) = \gamma^i(0,t)$	3–point of $\mathbb{T}_t$,
$\underline{v}^i = \underline{v}^i(x,t) = \frac{\gamma^i_{xx}}{\lvert\gamma^i_x\rvert^2}$	velocity of the point $\gamma^i(x,t)$,
$\lambda^i = \lambda^i(x,t) = \frac{\langle \gamma^i_{xx} \,\vert\, \tau^i \rangle}{\lvert\gamma^i_x\rvert^2} = \frac{\langle \gamma^i_{xx} \,\vert\, \gamma^i_x \rangle}{\lvert\gamma^i_x\rvert^3}$	tangential velocity of the point $\gamma^i(x,t)$,
$k^i = k^i(x,t) = \frac{\langle \gamma^i_{xx} \,\vert\, \nu^i \rangle}{\lvert\gamma^i_x\rvert^2} = \langle \partial_s \tau^i \,\vert\, \nu^i \rangle = -\langle \partial_s \nu^i \,\vert\, \tau^i \rangle$	curvature at the point $\gamma^i(x,t)$,

where with s we have denoted the arclength parameter on any of the curves and with $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation centered in the origin of $\mathbb{R}^2$ of angle $\pi/2$. Furthermore, we set $\underline{\lambda}^i = \lambda^i \tau^i$ and $\underline{k}^i = k^i \nu^i$, from which it follows that $\underline{v}^i = \underline{\lambda}^i + \underline{k}^i$ and $\lvert\underline{v}^i\rvert^2 = (\lambda^i)^2 + (k^i)^2$. We will also denote by L^i the length of the i–th curve of the triod and by $L = L^1 + L^2 + L^3$ its global length.

We now state some results which have been proven in [26].

Lemma 2.5. *If γ is a curve of a triod moving by curvature, which means that*

$$\gamma_t = \frac{\gamma_{xx}}{\lvert\gamma_x\rvert^2} = \lambda\tau + k\nu \,,$$

then the following commutation rule holds,

$$\partial_t \partial_s = \partial_s \partial_t + (k^2 - \lambda_s)\partial_s \,.$$

With the help of Lemma 2.5 one gets the following formulas.

Lemma 2.6. *For any curve evolving by curvature, there holds*

$$
\begin{aligned}
\partial_t\tau &= \partial_t\partial_s\gamma = \partial_s\partial_t\gamma + (k^2-\lambda_s)\partial_s\gamma = \partial_s(\lambda\tau + k\nu) + (k^2-\lambda_s)\tau = (k_s + k\lambda)\nu\\
\partial_t\nu &= \partial_t(\mathrm{R}\tau) = \mathrm{R}\,\partial_t\tau = -(k_s+k\lambda)\tau\\
\partial_t k &= \partial_t\langle\partial_s\tau,\nu\rangle = \langle\partial_t\partial_s\tau,\nu\rangle = \langle\partial_s\partial_t\tau,\nu\rangle + (k^2-\lambda_s)\langle\partial_s\tau,\nu\rangle\\
&= \partial_s\langle\partial_t\tau,\nu\rangle + k^3 - k\lambda_s = \partial_s(k_s+k\lambda) + k^3 - k\lambda_s\\
&= k_{ss} + k_s\lambda + k^3\\
\partial_t\lambda &= -\,\partial_t\partial_x\frac{1}{|\gamma_x|} = \partial_x\frac{\langle\gamma_x,\gamma_{tx}\rangle}{|\gamma_x|^3} = \partial_x\frac{\langle\tau,\partial_s(\lambda\tau+k\nu)\rangle}{|\gamma_x|} = \partial_x\frac{(\lambda_s-k^2)}{|\gamma_x|}\\
&= \partial_s(\lambda_s-k^2) - \lambda(\lambda_s-k^2) = \lambda_{ss} - \lambda\lambda_s - 2kk_s + \lambda k^2\,.
\end{aligned}
$$

Taking into account the compatibility conditions at the 3–point, we have the following lemma.

Lemma 2.7. *At the 3–point of a triod $\mathbb{T}_t$ evolving as in Problem (2.1) hold*

$$
\begin{aligned}
\lambda^i &= \frac{k^{i-1}-k^{i+1}}{\sqrt{3}}\\
k^i &= \frac{\lambda^{i+1}-\lambda^{i-1}}{\sqrt{3}}
\end{aligned}
$$

where the indices are understood modulo three. Moreover

$$
\sum_{i=1}^{3} k^i = \sum_{i=1}^{3}\lambda^i = 0
$$

$$
k_s^i + \lambda^i k^i = k_s^j + \lambda^j k^j
$$

for every pair $i,j\in\{1,2,3\}$.

The key theorem for the analysis of the singularities is the following result [26, Proposition 3.13].

Proposition 2.8. *If $[0,T)$ is the maximal time interval of existence of a smooth solution $\mathbb{T}_t$ with $T<+\infty$ of Problem (2.1), then one of the following possibilities holds:*

- *the inferior limit of the length of one curve of $\mathbb{T}_t$ tends to zero as $t\to T$,*
- $\limsup_{t\to T}\int_{\mathbb{T}_t} k^2\,ds = +\infty$.

Moreover, if the lengths of the three curves are uniformly bounded away from zero, then the superior limit is actually a limit.

In the next section we will show that if the lengths of the three curves are uniformly bounded away from zero, no singularity can develop. We now introduce the tools and the estimates which we will need.

Let $F:\mathbb{T}\times[0,T)\to\mathbb{R}^2$ a curvature flow for a triod in its maximal time interval of existence, then a modified form of Huisken's monotonicity formula holds.
Let $x_0\in\Omega$ and $\rho_{x_0}:\mathbb{R}^2\times[0,T)$ be the *backward heat kernel* of $\mathbb{R}^2$ relative to (x_0,T), that is

$$
\rho_{x_0}(x,t) = \frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{\sqrt{4\pi(T-t)}}\,.
$$

Proposition 2.9 (Monotonicity Formula – Proposition 6.4 in [26]). *For every $x_0 \in \mathbb{R}^2$ and $t \in [0,T)$ the following identity holds*

$$\frac{d}{dt}\int_{\mathbb{T}_t}\rho_{x_0}(x,t)\,ds = -\int_{\mathbb{T}_t}\left|\underline{k}+\frac{(x-x_0)^\perp}{2(T-t)}\right|^2\rho_{x_0}(x,t)\,ds + \sum_{i=1}^{3}\left\langle \frac{P^i-x_0}{2(T-t)},\tau^i(1,t)\right\rangle\rho_{x_0}(P^i,t)\,. \tag{2.2}$$

Integrating between t_1 and t_2 with $0 \le t_1 \le t_2 < T$ we get

$$\int_{t_1}^{t_2}\int_{\mathbb{T}_t}\left|\underline{k}+\frac{(x-x_0)^\perp}{2(T-t)}\right|^2\rho_{x_0}(x,t)\,ds\,dt = \int_{\mathbb{T}_{t_1}}\rho_{x_0}(x,t_1)\,ds - \int_{\mathbb{T}_{t_2}}\rho_{x_0}(x,t_2)\,ds + \sum_{i=1}^{3}\int_{t_1}^{t_2}\left\langle \frac{P^i-x_0}{2(T-t)},\tau^i(1,t)\right\rangle\rho_{x_0}(P^i,t)\,dt\,.$$

Remark 2.10. Notice that the monotonicity formula for a triod differs from the standard one because of a boundary term. Thanks to the next lemma, this extra term will not change to a big extent the blow–up analysis for the curvature motion of triods.

Lemma 2.11 (Lemma 6.3 in [26]). *Setting $|P^i - x_0| = d^i$, for every index $i \in \{1,2,3\}$ the following estimate holds*

$$\left|\int_t^T\left\langle \frac{P^i-x_0}{2(T-\xi)},\tau^i(1,\xi)\right\rangle\rho_{x_0}(P^i,\xi)\,d\xi\right| \le \frac{1}{\sqrt{2\pi}}\int\limits_{d^i/\sqrt{2(T-t)}}^{+\infty} e^{-y^2/2}\,dy \le 1/2\,.$$

As a consequence, for every point $x_0 \in \mathbb{R}^2$, we have

$$\lim_{t\to T}\sum_{i=1}^{3}\int_t^T\left\langle \frac{P^i-x_0}{2(T-\xi)},\tau^i(1,\xi)\right\rangle\rho_{x_0}(P^i,\xi)\,d\xi = 0\,.$$

Proposition 2.12. *If for every $x_0 \in \mathbb{R}^2$ we define the functions $\Theta : \mathbb{T}\times[0,T)\to\mathbb{R}$ as*

$$\Theta(x_0,t) = \int_{\mathbb{T}_t}\rho_{x_0}(x,t)\,ds\,,$$

then, the limit

$$\widehat{\Theta}(x_0) = \lim_{t\to T}\Theta(x_0,t) = \lim_{t\to T}\int_{\mathbb{T}_t}\rho_{x_0}(x,t)\,ds$$

exists and it is finite.
Moreover, the map $\widehat{\Theta} : \mathbb{R}^2 \to \mathbb{R}$ is upper semicontinuous.

Proof. We consider the function $b : \mathbb{R}^2 \times [0,T) \to \mathbb{R}$ given by

$$b(x_0,t) = \int_t^T\sum_{i=1}^{3}\left\langle \frac{P^i-x_0}{2(T-\xi)}\,\middle|\,\tau^i(1,\xi)\right\rangle\rho_{x_0}(P^i,\xi)\,d\xi\,.$$

Lemma 2.11 says that b is uniformly bounded and for every $x_0 \in \mathbb{R}^2$ we have $\lim_{t\to T} b(x_0, t) = 0$. Hence, the monotonicity formula (2.2) can be rewritten as

$$\frac{d}{dt}\big(\Theta(x_0,t) + b(x_0,t)\big) = -\int_{\mathbb{T}_t} \left| \underline{k} + \frac{(x-x_0)^{\perp}}{2(T-t)} \right|^2 \rho_{x_0}(x,t)\, ds \leq 0\,,$$

hence, being nonincreasing and bounded from below, the functions $\big(\Theta(\cdot,t) + b(\cdot,t)\big)$ pointwise converge on all $\mathbb{R}^2$ when $t \to T$. Since we have seen that $b(\cdot,t)$ pointwise converge to zero everywhere, the limit $\widehat{\Theta}(x_0)$ exists for every $x_0 \in \mathbb{R}^2$.
As the convergence of the continuous functions $\big(\Theta(\cdot,t) + b(\cdot,t)\big)$ to $\widehat{\Theta} : \mathbb{R}^2 \to \mathbb{R}$ is monotone nonincreasing, this latter is upper semicontinuous. $\square$

We now introduce the rescaling procedure of Huisken [17].
For a fixed $x_0 \in \mathbb{R}^2$, let $\widetilde{F}_{x_0} : \mathbb{T} \times [-1/2 \log T, +\infty) \to \mathbb{R}^2$ be the map

$$\widetilde{F}_{x_0}(p,\mathfrak{t}) = \frac{F(p,t(\mathfrak{t})) - x_0}{\sqrt{2(T - t(\mathfrak{t}))}} \qquad \mathfrak{t}(t) = -\frac{1}{2}\log\left(T-t\right).$$

Then the rescaled triods are given by

$$\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}} = \frac{\mathbb{T}_{t(\mathfrak{t})} - x_0}{\sqrt{2(T - t(\mathfrak{t}))}}$$

and they evolve according to the equation

$$\frac{\partial}{\partial \mathfrak{t}} \widetilde{F}_{x_0}(p,\mathfrak{t}) = \underline{\widetilde{v}}(p,\mathfrak{t}) + \widetilde{F}_{x_0}(p,\mathfrak{t})\,,$$

where

$$\underline{\widetilde{v}}(p,\mathfrak{t}) = \frac{\underline{v}(p,t(\mathfrak{t}))}{\sqrt{2(T-t(\mathfrak{t}))}} = \underline{\widetilde{k}} + \underline{\widetilde{\lambda}} = \widetilde{k}\nu + \widetilde{\lambda}\tau \qquad \text{and} \qquad t(\mathfrak{t}) = T - e^{-2\mathfrak{t}}\,.$$

Notice that we did not put the "tilde" over the unit tangent and normal, since they do not change under rescaling.
We will often write $\widetilde{O}(\mathfrak{t}) = \widetilde{F}_{x_0}(0,\mathfrak{t})$ for the 3–point of the rescaled triod $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}$, when there is no ambiguity on the point x_0.
The rescaled curvature evolves according to the following equation

$$\partial_{\mathfrak{t}}\widetilde{k} = \widetilde{k}_{\sigma\sigma} + \widetilde{k}_\sigma\widetilde{\lambda} + \widetilde{k}^3 - \widetilde{k}\,,$$

which can be obtained by means of the commutation law

$$\partial_{\mathfrak{t}}\partial_\sigma = \partial_\sigma\partial_{\mathfrak{t}} + (\widetilde{k}^2 - \widetilde{\lambda}_\sigma - 1)\partial_\sigma\,,$$

where we denoted with σ the arclength parameter for $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}$.

By a straightforward computation (see [17] and [26, Lemma 6.7]) we have the following rescaled version of the monotonicity formula.

Proposition 2.13 (Rescaled Monotonicity Formula). *Let $x_0 \in \mathbb{R}^2$ and set*

$$\widetilde{\rho}(x) = e^{-\frac{|x|^2}{2}}$$

For every $\mathfrak{t} \in [-1/2 \log T, +\infty)$ the following identity holds

$$\frac{d}{d\mathfrak{t}} \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}} \widetilde{\rho}(x)\, d\sigma = -\int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}} |\,\widetilde{\underline{k}} + x^{\perp}|^2 \widetilde{\rho}(x)\, d\sigma + \sum_{i=1}^{3} \left\langle\, \widetilde{P}^i_{x_0,\mathfrak{t}} \,\middle|\, \tau^i(1, t(\mathfrak{t})) \right\rangle \widetilde{\rho}(\widetilde{P}^i_{x_0,\mathfrak{t}})$$

where $\widetilde{P}^i_{x_0,\mathfrak{t}} = \frac{P^i - x_0}{\sqrt{2(T - t(\mathfrak{t}))}}$.
Integrating between $\mathfrak{t}_1$ and $\mathfrak{t}_2$ with $-1/2 \log T \leq \mathfrak{t}_1 \leq \mathfrak{t}_2 < +\infty$ we get

$$\begin{aligned}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}} |\,\widetilde{\underline{k}} + x^{\perp}|^2 \widetilde{\rho}(x)\, d\sigma\, d\mathfrak{t} = &\int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_1}} \widetilde{\rho}(x)\, d\sigma - \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_2}} \widetilde{\rho}(x)\, d\sigma \\ &+ \sum_{i=1}^{3} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \left\langle\, \widetilde{P}^i_{x_0,\mathfrak{t}} \,\middle|\, \tau^i(1, t(\mathfrak{t})) \right\rangle \widetilde{\rho}(\widetilde{P}^i_{x_0,\mathfrak{t}})\, d\mathfrak{t}\,.\end{aligned} \tag{2.3}$$

Then, we have the analog of Lemma 2.11.

Lemma 2.14 (Lemma 6.8 in [26]). *For every index $i \in \{1, 2, 3\}$ the following estimate holds*

$$\left| \int_{\mathfrak{t}}^{+\infty} \left\langle\, \widetilde{P}^i_{x_0,\xi} \,\middle|\, \tau^i(1, t(\xi)) \right\rangle \widetilde{\rho}(\widetilde{P}^i_{x_0,\xi})\, d\xi \right| \leq \sqrt{\pi/2}\,.$$

Then, for every $x_0 \in \mathbb{R}^2$,

$$\lim_{\mathfrak{t} \to +\infty} \sum_{i=1}^{3} \int_{\mathfrak{t}}^{+\infty} \left\langle\, \widetilde{P}^i_{x_0,\xi} \,\middle|\, \tau^i(1, t(\xi)) \right\rangle \widetilde{\rho}(\widetilde{P}^i_{x_0,\xi})\, d\xi = 0\,.$$

Before showing the key proposition about the blow-up limits of the flow at a singularity, we need some technical lemmas.

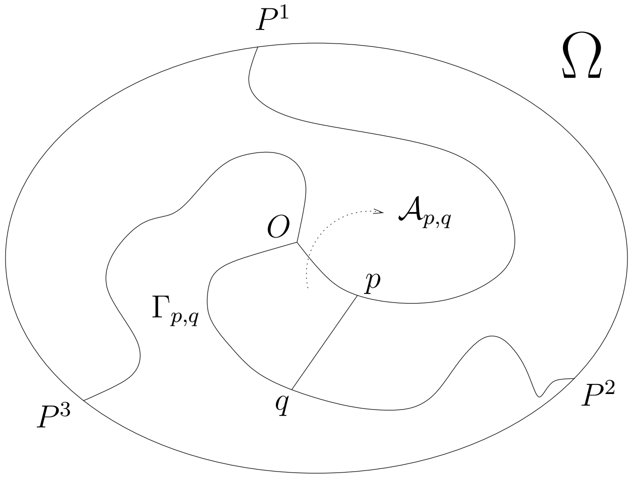
Lemma 2.15 (Second statement in Lemma 6.10 in [26]). *For every ball B_R centered at the origin of $\mathbb{R}^2$, we have the following estimates with a constant C_R independent of $x_0 \in \mathbb{R}^2$ and $\mathfrak{t} \in [-1/2 \log T, +\infty)$*

$$\mathcal{H}^1(\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}} \cap B_R) \leq C_R\,.$$

Definition 2.16. We say that a sequence of triods converges in the C^r_{loc} topology if, after reparametrizing all their curves with the arclength, they converge in C^r on every compact set of $\mathbb{R}^2$.
The definition of convergence in $W^{n,p}_{\mathrm{loc}}$ is analogous.

Given the smooth flow $\mathbb{T}_t = F(\mathbb{T}, t)$, we consider two points $p = F(x, t)$ and $q = F(y, t)$ belonging to $\mathbb{T}_t$ and we define $\Gamma_{p,q}$ to be the *geodesic* curve contained in $\mathbb{T}_t$ connecting p and q. Then we let $A_{p,q}$ to be the area of the open region $\mathcal{A}_{p,q}$ in $\mathbb{R}^2$ enclosed by the segment $[p, q]$ and the curve $\Gamma_{p,q}$, as in the figure.

If the region $\mathcal{A}_{p,q}$ is not connected, we let $A_{p,q}$ to be the sum of the areas of its connected components.



We consider the function $\Phi_t : \mathbb{T} \times \mathbb{T} \to \mathbb{R} \cup \{+\infty\}$ as

$$\Phi_t(x,y) = \begin{cases} \frac{|p-q|^2}{A_{p,q}} & \text{if } x \neq y, \\ 4\sqrt{3} & \text{if } x \text{ and } y \text{ coincide with the 3–point } O \text{ of } \mathbb{T}, \\ +\infty & \text{if } x = y \neq O \end{cases}$$

Since $\mathbb{T}_t$ is smooth and the 120 degrees condition holds, it is easy to check that Φ_t is a lower semicontinuous function. Hence, by the compactness of $\mathbb{T}$, the following infimum is actually a minimum

$$E(t) = \inf_{x,y \in \mathbb{T}} \Phi_t(x,y)$$

for every $t \in [0,T)$.

If the triod $\mathbb{T}_t$ has no self–intersections we have $E(t) > 0$, the converse is clearly also true.
Moreover, $E(t) \leq \Phi_t(0,0) = 4\sqrt{3}$ always holds, thus when $E(t) > 0$ the two points (p,q) of a minimizing pair (x,y) can coincide if and only if $p = q = O$.
Finally, since the evolution is smooth it is easy to see that the function $E : [0,T) \to \mathbb{R}$ is continuous.

Proposition 2.17 (Theorem 4.6 in [26]). *If Ω is bounded and* strictly *convex, there exists a constant $C > 0$ depending only on $\mathbb{T}_0$ such that $E(t) > C > 0$ for every $t \in [0,T)$.*
Hence, the triods $\mathbb{T}_t$ remain embedded in all the maximal interval of existence of the flow.

Lemma 2.18. *The function*

$$E(\mathbb{T}) = \inf_{\substack{p,q \in \mathbb{T} \\ p \neq q}} \frac{|p-q|^2}{A_{p,q}},$$

defined on the class of C^1 triods without self–intersections (bounded or unbounded and with or without end points or 3–points), is upper semicontinuous with respect to the C^1_{loc} convergence.
Moreover, E is dilation and translation invariant.
Consequently, every C^1_{loc} limit $\mathbb{T}_\infty$ of a sequence of rescaled triods $\widetilde{\mathbb{T}}_t$ has no self–intersections

and has multiplicity *one (outside the endpoints and the 3–point if present), it actually satisfies* $E(\mathbb{T}_\infty) > C > 0$ *where the uniform positive constant* C *is given by Proposition 2.17.*

Proof. The dilation–translation invariance and the upper semicontinuity of the function E are straightforward, by the C^1_{loc} convergence. This latter obviously implies the final statement of the theorem once we show the embeddedness and the multiplicity one properties.

Suppose that a sequence of rescaled triods $\widetilde{\mathbb{T}}_{\mathfrak{t}_j}$ converges to some limit $\mathbb{T}_\infty$. If this latter has a transversal self–intersection, the triods of the approximating sequence must definitively have self–intersections too, but this contradicts Proposition 2.17. By the same reason, a self–intersection is also impossible at the 3–point. Since the limit can contain an endpoint P^i_∞ only if we rescale the evolving triods around one of their endpoints P^i, the convexity of the set Ω implies that the limit set $\mathbb{T}_\infty$ must lie in a halfspace of $\mathbb{R}^2$ and that the only way a self–intersection can happen is that the limit curve starting at P^i_∞ is tangent at such endpoint to another limit piece of a curve of $\mathbb{T}_\infty$. As we assumed that the lengths of the curves are uniformly bounded below away from zero, then either $\mathbb{T}_\infty$ contains a loop (as it is connected), or a piece of the limit triod close to P^i_∞ must have multiplicity two, coming from the "collapsing" of two pieces of curve in the sequence of rescaled triods. The first case is excluded again by Proposition 2.17, the second one is managed by means the argument below, which deals with the multiplicity of the limit set.
The only other possible self–intersections of the limit set can happen at self–tangency points (i.e. there exist "internal" points of $\mathbb{T}_\infty$ with integer multiplicity larger than one).
By the C^1_{loc} convergence, in a sufficiently small ball of radius R around any of such points $x \in \mathbb{R}^2$, definitively, for every rescaled triod $\widetilde{\mathbb{T}}_{\mathfrak{t}_j}$, there must be some number of curves which are "pieces" of $\widetilde{\mathbb{T}}_{\mathfrak{t}_j}$, such that they are all disjoint, all graphs on the tangent line L to $\mathbb{T}_\infty$ at x and all converging to the same limit C^1 graph $\mathbb{T}_\infty \cap B_R$. Considering two of such pieces of curves, say σ^j_1 and σ^j_2, we take the point p_j and q_j which are the intersections of the orthogonal line to L at x and the two curves. By hypotheses, the distance d_j between p_j and q_j goes to zero. Moreover, as every rescaled triod is connected there must be a geodesic curve (in the entire rescaled triod) connecting such two points. This means that the open region $\mathcal{A}_{p_j,q_j}$ is well defined and its area A_{p_j,q_j} is larger than the area S_j contained between the two curves–graphs σ^j_1 and σ^j_2 in B_R. Hence we get $E(\widetilde{\mathbb{T}}_{\mathfrak{t}_j}) \leq d^2_j/S_j$. If we now rescale the ball B_R by a factor $1/d_j$, the line L and the the two curves inside σ^j_1 and σ^j_2 converge to three parallel unbounded straight lines, as $j \to \infty$, while at the same time the distance between the rescaled of the points p_j, q_j is one (hence, also between the two lines coming from the two curves in the limit). As the function E is dilation invariant and the rescaling of the region between the two curves in B_R converges to a half–strip in $\mathbb{R}^2$, we conclude that $\lim_{j\to\infty} S_j/d^2_j = +\infty$, hence $E(\widetilde{\mathbb{T}}_{\mathfrak{t}_j}) \to 0$ which is a contradiction, by Proposition 2.17. □

We can now show the following results which is analogous to the (stronger) one for Type I singularities proved in [26, Proposition 6.16].

Proposition 2.19. *Assume that the lengths of the three curves of the triods* $\mathbb{T}_t$ *are uniformly bounded away from zero during the evolution.*

For every $x_0 \in \mathbb{R}^2$ *and every subset* $\mathcal{I}$ *of* $[-1/2 \log T, +\infty)$ *with infinite Lebesgue measure, there exists a sequence of rescaled times* $\mathfrak{t}_j \to +\infty$, *with* $\mathfrak{t}_j \in \mathcal{I}$, *such that the sequence of rescaled triods* $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ *converges in the* C^1_{loc} *topology to a limit set* $\mathbb{T}_\infty$ *which,* if not empty, *is one of the following:*

- *a straight line through the origin with multiplicity one (in this case $\widehat{\Theta}(x_0) = 1$);*

- *an infinite flat triod centered at the origin with multiplicity one, except its 3–point (in this case $\widehat{\Theta}(x_0) = 3/2$);*

- *a halfline from the origin of multiplicity one, except the origin (in this case $\widehat{\Theta}(x_0) = 1/2$).*

Moreover, the L^2 norm of the curvature in every ball $B_R \in \mathbb{R}^2$ along such sequence goes to zero, as $j \to \infty$.

For every sequence of rescaled triods $\mathbb{T}_{x_0,\mathfrak{t}_j}$ converging at least in the C^1_{loc} topology to a limit $\widetilde{\mathbb{T}}_\infty$, as $\mathfrak{t}_j \to +\infty$, we have

$$\lim_{j\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}} \widetilde{\rho}\, d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}_\infty} \widetilde{\rho}\, d\sigma = \widehat{\Theta}(x_0)\,. \tag{2.4}$$

Proof. Assume that we have a sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ converging in the C^1_{loc} topology to a limit $\mathbb{T}_\infty$, as $\mathfrak{t}_j \to +\infty$. Since by Lemma 2.18 the limit must be embedded with multiplicity one, the convergence on every compact subset of $\mathbb{R}^2$ implies that the Radon measures $\mathcal{H}^1 \llcorner \widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_{j_l}}$ weakly* converge in $\mathbb{R}^2$ to the Radon measure $\mathcal{H}^1 \llcorner \mathbb{T}_\infty$. Moreover, as in the proof of Proposition 6.20 in [26], we can pass to the limit in the following Gaussian integral:

$$\lim_{j\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}} \widetilde{\rho}\, d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}_\infty} \widetilde{\rho}\, d\sigma\,.$$

Consequently,

$$\frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}} \widetilde{\rho}\, d\sigma = \int_{\mathbb{T}_{t(\mathfrak{t}_j)}} \rho_{x_0}(x, t(\mathfrak{t}_j))\, ds = \Theta(x_0, t(\mathfrak{t}_j)) \to \widehat{\Theta}(x_0)\,,$$

as $j \to \infty$ and equality (2.4) follows.

We now show the first statement.

Setting $\mathfrak{t}_1 = -1/2 \log T$ and letting $\mathfrak{t}_2$ go to $+\infty$ in the rescaled monotonicity formula 2.3, by Lemma 2.14 we get

$$\int\limits_{-1/2\log T}^{+\infty} \int\limits_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, d\mathfrak{t} < +\infty\,,$$

then, *a fortiori*,

$$\int\limits_{\mathcal{I}} \int\limits_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, d\mathfrak{t} < +\infty\,.$$

Being the last integral finite and being the integrand a nonnegative function on a set of infinite Lebesgue measure, we can extract within $\mathcal{I}$ a sequence of times $\mathfrak{t}_j \to +\infty$, such that $\int\limits_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}} |\,\underline{\widetilde{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma$ converges to zero.

It thus follows that, for every ball of radius R in $\mathbb{R}^2$, the triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ have curvatures uniformly bounded in $L^2(B_R)$. Then, reparametrizing all the triods with arclength, we obtain curves with uniformly bounded first derivatives (from above and below away from

zero by the assumption on the lengths) and with second derivatives in L^2_{loc}. Moreover, by Lemma 2.15, for every ball B_R centered at the origin of $\mathbb{R}^2$ we have a uniform bound $\mathcal{H}^1(\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j} \cap B_R) \leq C_R$, for some constants C_R independent of $j \in \mathbb{N}$.
By standard compactness arguments (see [17, 24]), the sequence $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ of reparametrized triods admits a subsequence $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_{j_l}}$ which converges weakly in $W^{2,2}_{\mathrm{loc}}$ and also in the C^1_{loc} topology, to a (possibly empty) set $\mathbb{T}_\infty$. If the point $x_0 \in \mathbb{R}^2$ is distinct from all the endpoints P^i, then $\mathbb{T}_\infty$ has no endpoints, since they go to infinity along the rescaled flow. If $x_0 = P^i$, the set $\mathbb{T}_\infty$ has a single endpoint at the origin of $\mathbb{R}^2$.
As we have already pointed out, by Lemma 2.18, the limit set (if not empty) has no self–intersections and multiplicity one, moreover, if a 3–point is present then the angles are of 120 degrees by the convergence of the curves in C^1_{loc}.
Since the integral functional

$$\widetilde{\mathbb{T}} \mapsto \int_{\widetilde{\mathbb{T}}} |\,\widetilde{\underline{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma$$

is lower semicontinuous with respect to this convergence (see [27]), the limit $\mathbb{T}_\infty$ distributionally satisfies $\underline{k}_\infty + x^\perp = 0$. In principle, the limit set is composed by curves in $W^{2,2}_{\mathrm{loc}}$, but from the relation $\underline{k}_\infty + x^\perp = 0$, it follows that $\underline{k}_\infty$ is continuous, since the curves are C^1_{loc}. By a bootstrap argument, it is then easy to see that the $\mathbb{T}_\infty$ is actually smooth.
Such a limit set is an unbounded triod or curve with at most one endpoint (depending on the choice of the point x_0), moreover, by Lemma 2.18, (if not empty) it has no self–intersections. As the relation above implies $k_\infty = -\langle x \,|\, \nu \rangle$ at every point $x \in \mathbb{T}_\infty$, repeating the argument of Lemma 5.2 in [26], if a triod is present, it must be centered at the origin of $\mathbb{R}^2$ and this excludes the presence of an endpoint at the same time. Indeed, in such case, it must be $x_0 = P^1$ (for instance) and any blow–up must be contained in a half space (since the triod does not "escape" the convex set Ω during the evolution) which is clearly impossible for a triod.
Thus, by the same relation, the classification Lemmas 5.2, 5.3, 5.4 and Proposition 5.5 in [26], we can conclude that in any case the curvature of the limit set is zero everywhere and that $\mathbb{T}_\infty$ is among the sets in the statement.

Since on every ball B_R the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ can converge (in the C^1 topology) only to a limit set with zero curvature, satisfying $x^\perp = 0$ and

$$\lim_{j\to\infty} \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j} \cap B_R} |\,\widetilde{\underline{k}} + x^\perp|^2\, d\sigma = 0\,,$$

as the term $x^\perp$ is continuous in the C^1_{loc} convergence, we actually have that

$$\lim_{j\to\infty} \int_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j} \cap B_R} \widetilde{\underline{k}}^2\, d\sigma = 0\,.$$

Finally, the values of $\widehat{\Theta}(x_0)$ in the statement are obtained through a computation by means of formula (2.4). $\square$

Lemma 2.20. *The existence of the limit $\widehat{O} = \lim_{t\to T} O(t)$ together with $\widehat{\Theta}(x_0) = 3/2$ are equivalent to $x_0 = \widehat{O}$.*

Proof. We first show that $\widehat{\Theta}(x_0)$ can be equal to $3/2$ at only one point in Ω. Assuming that $\widehat{\Theta}(x_0) = \widehat{\Theta}(y_0) = 3/2$, we define the two sets of rescaled times

$$\begin{aligned}\mathcal{I}_{x_0} &= \left\{\, \mathfrak{t} \in [-1/2 \log T, +\infty) \text{ such that } |\,O(t(\mathfrak{t})) - x_0| \geq \sqrt{2(T - t(\mathfrak{t}))}\,\right\},\\ \mathcal{I}_{y_0} &= \left\{\, \mathfrak{t} \in [-1/2 \log T, +\infty) \text{ such that } |\,O(t(\mathfrak{t})) - y_0| \geq \sqrt{2(T - t(\mathfrak{t}))}\,\right\}\end{aligned}$$

and we claim that both have finite Lebesgue measure. Indeed, if the Lebesgue measure of $\mathcal{I}_{x_0}$, for instance, is not finite, we have

$$\int\limits_{\mathcal{I}_{x_0}} \int\limits_{\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}}} |\,\widetilde{\underline{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, d\mathfrak{t} < +\infty\,.$$

Hence, since we assumed $\widehat{\Theta}(x_0) = 3/2$, we can extract a sequence of times $\mathfrak{t}_j \in \mathcal{I}_{x_0}$ such that the rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ converge in the C^1_{loc} topology to an infinite flat triod centered at the origin of $\mathbb{R}^2$. This is clearly in contradiction with the fact that, by construction, every set $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j} \cap B_{1/2}$ does not contain the 3–point of the rescaled triod $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$.
If the points x_0 and y_0 are distinct, we have a contradiction, as $[\mathfrak{t}_0, +\infty) \setminus \mathcal{I}_{y_0} \subset \mathcal{I}_{x_0}$, if $\mathfrak{t}_0$ is large enough and the set $[\mathfrak{t}_0, +\infty) \setminus \mathcal{I}_{y_0}$ would have finite Lebesgue measure as well, which is clearly not possible.

We now see that $\widehat{\Theta}(x_0) = 3/2$ holds for every point $x_0 \in \overline{\Omega}$ such that there exists a sequence $t_j \to T$ with $\lim_{j\to\infty} O(t_j) = x_0$. This fact, by the compactness of $\overline{\Omega}$ and the uniqueness of point x_0, implies the statement of the lemma.

Fixing any $r \in [0, T)$, we definitely have $t_j > r$, hence if $O(t_j) \to x_0$, we get

$$\begin{aligned}\Theta(x_0, r) + b(x_0, r) &= \int_{\mathbb{T}_r} \frac{e^{-\frac{|x-x_0|^2}{4(T-r)}}}{\sqrt{4\pi(T-r)}}\, ds + \int_r^T \sum_{i=1}^3 \left\langle \frac{P^i - x_0}{2(T-t)}, \tau^i(1,t) \right\rangle \frac{e^{-\frac{|P^i-x_0|^2}{4(T-t)}}}{\sqrt{4\pi(T-t)}}\, dt\\ &= \lim_{j\to\infty} \left\{ \int_{\mathbb{T}_r} \frac{e^{-\frac{|x-O(t_j)|^2}{4(t_j-r)}}}{\sqrt{4\pi(t_j-r)}}\, ds + \int_r^{t_j} \sum_{i=1}^3 \left\langle \frac{P^i - O(t_j)}{2(t_j-t)}, \tau^i(1,t) \right\rangle \frac{e^{-\frac{|P^i-O(t_j)|^2}{4(t_j-t)}}}{\sqrt{4\pi(t_j-t)}}\, dt \right\}\\ &\geq \lim_{j\to\infty} \lim_{r\to t_j^-} \left\{ \int_{\mathbb{T}_r} \frac{e^{-\frac{|x-O(t_j)|^2}{4(t_j-r)}}}{\sqrt{4\pi(t_j-r)}}\, ds + \int_r^{t_j} \sum_{i=1}^3 \left\langle \frac{P^i - O(t_j)}{2(t_j-t)}, \tau^i(1,t) \right\rangle \frac{e^{-\frac{|P^i-O(t_j)|^2}{4(t_j-t)}}}{\sqrt{4\pi(t_j-t)}}\, dt \right\}\\ &= \lim_{j\to\infty} \lim_{r\to t_j^-} \int_{\mathbb{T}_r} \frac{e^{-\frac{|x-O(t_j)|^2}{4(t_j-r)}}}{\sqrt{4\pi(t_j-r)}}\, ds\,,\end{aligned}$$

where in the last passage we applied the analogue of Lemma 2.11 with t_j in place of T. Indeed, the monotonicity formula (actually all the previous strategy) holds also if T is not the maximal time of smooth existence. Repeating all the argument in the Proposition 2.19 at time t_j, we then see that the last integral inside the limit must be equal to $3/2$ (as we are rescaling exactly around the 3–point $O(t_j)$) and thus the only possible limit of rescaled triods is an unbounded triod in $\mathbb{R}^2$ centered at the origin.
Hence, we can conclude that for every $r \in [0, T)$ it holds $\Theta(x_0, r) + b(x_0, r) \geq 3/2$, which, when $r \to T$, implies that $\widehat{\Theta}(x_0) = 3/2$. $\square$

In the following, given $\overline{x} \in \mathbb{R}^2$ and $R > 0$, we denote by $Q_R(\overline{x})$ the square

$$Q_R(\overline{x}) := \left\{ x \in \mathbb{R}^2 : \ |x_1 - \overline{x}_1| \le R, \, |x_2 - \overline{x}_2| \le R \right\}.$$

Proposition 2.21. *Suppose that the curve γ_0 is a graph over $\langle e_1 \rangle$ in the square $Q_{2R}(x_0)$, and assume that the curve $\gamma_t \cap Q_{2R}(x_0)$ is contained in the horizontal strip $\{|x_2| \le \delta\}$ for any $t \in [0, \tau)$, with $\tau > 0$ and $0 < \delta < R$.*
Then $\gamma_t \cap Q_{2R}(x_0)$ is a graph over $\langle e_1 \rangle$ for all $t \in [0, \tau)$.

Proof. We claim that the number of intersections of γ_t with any vertical segment of the form $\ell_x := \{x + se_2 : x \in Q_{2R}(x_0), s \in \mathbb{R}\} \cap Q_{2R}(x_0)$ is nonincreasing in time, hence it is constantly equal to 1 as γ_0 is a graph in $Q_{2R}(x_0)$ over $\langle e_1 \rangle$. It then follows that $\gamma_t \cap Q_{2R}(x_0)$ is a graph over $\langle e_1 \rangle$ for all $t \in [0, \tau)$ and the thesis is proven.

In order to prove the claim, let us assume by contradiction that there exist a vertical segment ℓ and a time $\overline{t} \ge 0$ such that the set $\gamma_t \cap \ell$ is a single point for $t \in [0, \overline{t})$, and has cardinality strictly greater than 1 for a sequence $t_n \downarrow \overline{t}$. In particular, there exist a point $\overline{x} \subset \gamma_{\overline{t}} \cap \ell$ and two sequences x_n, y_n such that

$$x_n, y_n \in \gamma_{t_n} \cap \ell, \, x_n \ne y_n \text{ and } \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \overline{x}. \tag{2.5}$$

It follows that $\gamma_{\overline{t}}$ has a vertical tangent line at $\overline{x}$, so that we can write $\gamma_{\overline{t}} \cap Q_\delta(\overline{x})$ as a smooth graph over ℓ for a suitably small $\delta > 0$.

By [13, Theorem 2.1] there exists $\varepsilon > 0$ such that $\gamma_t \cap Q_{\frac{\delta}{2}}(\overline{x})$ is also a graph over ℓ for all $t \in [\overline{t}, \overline{t} + \varepsilon]$ and, by continuity, at the intersection with $\partial Q_{\frac{\delta}{2}}(\overline{x})$ the curve γ_t does not intersect ℓ. Then, by the Sturmian theorem of Angenent in [5, Proposition 1.2] and [4, Section 2] (see [3] for the proof), we have that the cardinality of $\gamma_t \cap \ell$ in $Q_{\frac{\delta}{2}}(\overline{x})$ is nonincreasing in time on $[\overline{t}, \overline{t} + \varepsilon]$, thus contradicting property (2.5). □

Corollary 2.22. *Assume that $\gamma_0 \cap B_{7R}(x_0)$ is a graph over $\langle e_1 \rangle$, contained in the horizontal strip $\{|x_2| \le R\}$. Then $\gamma_t \cap B_{2R}(x_0)$ is a graph over $\langle e_1 \rangle$ for all $t \in [0, \tau)$, with $\tau = R^2/2$. Moreover, letting $v = \langle \nu \,|\, e_2 \rangle^{-1}$, we have*

$$\sup_{t \in [0,\tau)} \ \sup_{\gamma_t \cap B_R(x_0)} v \le C \sup_{\gamma_0 \cap B_{2R}(x_0)} v$$

for some $C > 0$ independent of R.

Proof. Letting $x_\pm = x_0 \pm 4Re_2$, by assumption we have that γ_0 is contained in the complementary of the set $B_{3R}(x_+) \cup B_{3R}(x_-) \subset B_{7R}(x_0)$.

By comparison principle, it follows that γ_t does not intersect the set $B_{R(t)}(x_+) \cup B_{R(t)}(x_-)$, with $R(t) = \sqrt{9R^2 - 2t}$, for all $t \in [0, 9R^2/2)$. In particular, $\gamma_t \cap Q_{2R}(x_0)$ does not intersect the upper and lower edge of the square $Q_{2R}(x_0)$ if $t \in [0, \tau)$, with $\tau = R^2/2$. Therefore, from Proposition 2.21 it follows that $\gamma_t \cap Q_{2R}(x_0)$, hence also $\gamma_t \cap B_{2R}(x_0)$, is a graph over $\langle e_1 \rangle$ for all $t \in [0, R^2/2)$.
The last assertion of the corollary then follows from Theorem 2.3 in [13], noticing that if γ_t is the graph of the function $u(\cdot\,, t)$, then $v = \sqrt{1 + |u'|^2}$. □

We recall the following result [13, Corollary 3.2 and Corollary 3.5].

Proposition 2.23. *Suppose that γ_t is a graph over $\langle e_1\rangle$ in $B_R(x_0)$ for all $t\in[0,\tau)$. Then letting $\theta\in(0,1)$ and $m\geq 0$, we have*

$$\sup_{\gamma_t\cap B_{\sqrt{\theta R^2-2t}}(x_0)} t^{m+1}|\partial_s^m k|^2\leq C_{m,v}$$

for all $t\in[0,\tau)$, where the constant $C_{m,v}$ depends only on m, θ and $\sup_{t\in[0,\tau)}\sup_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)} v$.

Proposition 2.24. *Let γ_t be as in Proposition 2.23. For all $\theta\in(0,1)$ we have*

$$\sup_{\gamma_t\cap B_{\sqrt{\theta R^2-2t}}(x_0)}|k|^2\leq\frac{C_v}{(1-\theta)^2}\left(\frac{1}{R^2}+\sup_{\gamma_0\cap B_R(x_0)}|k|^2\right)\tag{2.6}$$

for all $t\in[0,\tau)$, where the constant C_v depends only on $\sup_{t\in[0,\tau)}\sup_{\gamma_t\cap B_R(x_0)} v$.

Proof. Let $g=k^2\varphi(v^2)$, with $\varphi(s):=s(1-cs)^{-1}$ and $c>0$ to be chosen later, and let $\eta=(R^2-|x|^2-2t)^2$. By a direct computation as in the proof of Theorem 3.1 in [13], we obtain

$$(\partial_t-\Delta)\,g\eta\leq-2cg^2\eta-2\langle\varphi v^{-3}\nabla v+\eta^{-1}\nabla\eta,\nabla(g\eta)\rangle+C(n)\left(\left(1+\frac{1}{cv^2}\right)\left(|x|^2+2t\right)+R^2\right)g\,.\tag{2.7}$$

At a point where $m(t):=\max_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)}(g\eta)$ is attained in space, multiplying inequality (2.7) by $\frac{\eta}{2c}$ we get

$$m'(t)\frac{\eta}{2c}\leq-m(t)^2+\frac{C(n)}{2c}\left(\left(1+\frac{1}{cv^2}\right)\left(|x|^2+2t\right)+R^2\right)m(t)\leq 0\,,$$

as soon as

$$m(t)\geq\frac{C(n)}{2c}\left(\left(1+\frac{1}{cv^2}\right)\left(|x|^2+2t\right)+R^2\right)\,.$$

Choosing then the constant $c=\frac{1}{2}\min_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)} v^{-2}$, we obtain that

$$cv^2\leq\frac{1}{2}\min_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)} v^{-2}\cdot\max_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)} v^2=1/2$$

in $\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)$, hence the function ηg is well defined, moreover we have $m'(t)\leq 0$ whenever

$$m(t)\geq 4C(n)\,R^2\max_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)} v^2.$$

As $\varphi(v)\geq 1$ and $\eta\geq(1-\theta)^2R^4$ in $B_{\sqrt{\theta R^2-2t}}(x_0)$, it follows

$$\begin{aligned}\sup_{\gamma_t\cap B_{\sqrt{\theta R^2-2t}}(x_0)}|k|^2&\leq(1-\theta)^{-2}R^{-4}\max_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)}(g\eta)\\&\leq(1-\theta)^{-2}\max\left\{\frac{m(0)}{R^4},\frac{4C(n)}{R^2}\max_{\gamma_t\cap B_{\sqrt{R^2-2t}}(x_0)} v^2\right\}\end{aligned}$$

which gives estimate (2.6) as $m(0)\leq R^2k^2\max_{\gamma_0\cap B_R(x_0)} v^2(\cdot,0)$. ☐

3 The Proof of Theorem 2.4

The fact that if $T = +\infty$ and the lengths of the three curves of a triod moving by curvature are bounded below away from zero uniformly in time, then the evolving triod $\mathbb{T}_t$ tends as $t \to +\infty$ to the unique Steiner triod connecting the three fixed endpoints is shown in Section 8 of [26].

This section is devote to exclude finite time singularities (i.e. $T < +\infty$) for a triod moving by curvature, whose curves have lengths bounded away from zero from below, uniformly in time. From this fact, Theorem 2.4 follows.

To this aim, we will proceed with an argument by contradiction relying on the C^1_{loc} convergence (with the L^2 norm of the curvature going to zero in every compact subset of $\mathbb{R}^2$) of a sequence of rescaled triods to any of the three singularity models in Proposition 2.19. The argument is similar in spirit to the one in [25], adapted to the case of an evolving triod.

To set the notation, let $F : \mathbb{T} \times [0, T) \to \mathbb{R}^2$, with $T < \infty$, be a triod moving by curvature in its maximal time interval of smooth existence. We assume that the lengths of the three curves of the triod $\mathbb{T}_t$ are uniformly bounded below away from zero and that $T < +\infty$. We are going to show that the full L^2 norm of the curvature of the evolving triod stays uniformly bounded up to time T, hence contradicting Proposition 2.8.

We define the set of *reachable points* of the flow as

$$\mathcal{R} = \left\{ x \in \mathbb{R}^2 \,\middle|\, \text{there exist } p_i \in \mathbb{T} \text{ and } t_i \nearrow T \text{ such that } \lim_{i\to\infty} F(p_i, t_i) = x \right\}.$$

Such a set is easily seen to be closed, contained in $\overline{\Omega}$, hence compact, and the following lemma holds.

Lemma 3.1. *A point $x \in \mathbb{R}^2$ belongs to $\mathcal{R}$ if and only if for every time $t \in [0, T)$ the closed ball with center x and radius $\sqrt{2(T-t)}$ intersects $\mathbb{T}_t$.*

Proof. One of the two implications is trivial. We have to prove that if $x \in \mathcal{R}$, then $F(\mathbb{T}, t) \cap \overline{B}_{\sqrt{2(T-t)}}(x) \neq \emptyset$. To this aim, we define the function $d_x(t) = \inf_{p\in\mathbb{T}} |F(p, t) - x|$, where, due to the compactness of $\mathbb{T}$ the infimum is actually a minimum. Since the function $d_x : [0, T) \to \mathbb{R}$ is locally Lipschitz, we can use Hamilton's trick to compute its time derivative and get (for any point q where at the instant t the minimum of $|F(p, t) - q|$ is attained)

$$\begin{aligned}
\partial_t d_x(t) = \partial_t |F(q, t) - x| &\geq \frac{\langle k(q, t)\nu(q, t) + \lambda(q, t)\tau(q, t), F(q, t) - x\rangle}{|F(q, t) - x|} \\
&= \frac{\langle k(q, t)\nu(q, t), F(q, t) - x\rangle}{|F(q, t) - x|} \geq -\frac{1}{d_x(t)},
\end{aligned}$$

since at a point of minimum distance the vector $\frac{F(q,t)-x}{|F(q,t)-x|}$ is parallel to $\nu(q, t)$ (unless x is one of the endpoints, but in this case the Lemma is obvious). Integrating this inequality over time, we get

$$d_x^2(t) - d_x^2(s) \leq 2(s - t).$$

We now use the hypothesis that x is reachable (i.e. $\lim_{t_i\to T} d_x(t_i) = 0$) and we conclude

$$d_x^2(t) = \lim_{i\to\infty} [d_x^2(t) - d_x^2(t_i)] \leq 2 \lim_{i\to\infty} (t_i - t) = 2(T - t).$$

$\square$

As a consequence, when we consider the blow–up of the evolving triods around points of $\overline{\Omega}$, we have a dichotomy among the them. Either the limit of any sequence of rescaled triods is not empty and we are rescaling around a point in $\mathcal{R}$, or the blow–up limit is empty, since the distance of the evolving triod from the point of blow–up is too big. Conversely, if the blow up point belongs to $\mathcal{R}$, the above lemma ensures that any rescaled triod contains at least one point of the closed unit ball of $\mathbb{R}^2$.

Fixing any point $x_0 \in \mathcal{R}$, by Proposition 2.19 there is a sequence $\mathfrak{t}_i \nearrow \infty$ of rescaled triods such that $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ converges in the C^1_{loc} topology to a nonempty limit which must be either a straight line, a halfline or an infinite flat triod. Moreover, in every ball $B_R \in \mathbb{R}^2$, the L^2 norm of the curvature along such sequence goes to zero as $j \to \infty$.

We start considering the case when the blow–up limit is a straight line.

Proposition 3.2. *If the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ converges to a straight line, then the curvature of the evolving triod is uniformly bounded for $t \in [0, T)$ in a ball around the point x_0.*

Proof. Assume that there is a straight line L through the origin of $\mathbb{R}^2$ such that the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ converges to L as $j \to \infty$.
Recalling Lemma 2.20 this implies that the distance $|O(t) - x_0|$ is uniformly bounded from below, so that there exists $i \in \{1, 2, 3\}$ such that the rescaled curves $\frac{\gamma^i_{t_j}}{\sqrt{2(T-t_j)}}$ converge to L as $j \to \infty$. In particular, for all $M > 1$ there exists $j_M \in \mathbb{N}$ such that the curve $\gamma^i_{t_{j_M}} \cap B_{7M\sqrt{2(T-t_{j_M})}}(x_0)$ is a graph over the line $x_0 + L$. By Corollary 2.22 it follows that $\gamma^i_t \cap B_{M\sqrt{2(T-t_{j_M})}}(x_0)$ is also a graph over the line $x_0 + L$ for all $t \in [t_{j_M}, t_{j_M} + M^2(T - t_{j_M})) \supset [t_{j_M}, T)$, and its slope v^i (with respect to the line $x_0 + L$) is uniformly bounded by a constant independent of M and t. Therefore, if $M > 2$, from Proposition 2.23 (applied with $\theta = 1/2$) it follows that the curvature of the curve $\gamma^i_t \cap B_{M\sqrt{2(T-t_{j_M})}}(x_0)$ and all its derivatives are bounded for $t \in [t_{j_M}, T)$ and we are done. □

We then consider the case of a halfline.

Proposition 3.3. *If the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathfrak{t}_j}$ converges to a halfline, then the curvature of the evolving triod is uniformly bounded for $t \in [0, T)$ in a ball around the point x_0.*

Proof. By the C^1_{loc} convergence of the rescaled flow to the halfline, we can see that the point x_0 must be one of the endpoints of the triod, which we will denote with P. We now perform a reflexion with center P of the triod and we consider the motion by curvature of the union of the two (mutually reflected through P) triods which is still a motion by curvature, now of a *network* of curves (see [26] for more details). Since at the endpoint P the curvature vanishes by construction, the point P stay fixed during the motion of the network and the sequence of rescaled networks around $P = x_0$ converges in the C^1_{loc} topology to a straight line. We can now repeat the proof of Proposition 3.2 to conclude. □

If there is no $x_0 \in \mathbb{R}^2$ with $\widehat{\Theta}(x_0) = 3/2$, by Propositions 3.2 and 3.3, there exists a ball around every reachable point in which the curvature of the evolving triod is uniformly bounded for $t \in [0, T)$.
As the set of reachable points $\mathcal{R}$ is compact, it follows that the curvature is uniformly bounded as $t \to T < +\infty$, which is contradiction to Proposition 2.8. Hence, we can assume that at

some (unique) point $x_0 \in \Omega$ we have $\widehat{\Theta}(x_0) = 3/2$ and that the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\mathrm{t}_j}$ converges to an infinite flat triod $\mathbb{T}_\infty$ centered at the origin. Furthermore, the L^2 norm of the curvature of the rescaled triods goes to zero on every compact subset of $\mathbb{R}^2$. By Lemma 2.20 this means that $x_0 = \widehat{O}$ is the limit of the 3–point $O(t)$ as $t \to T$. We write $\mathbb{T}_\infty = L^1 \cup L^2 \cup L^3$ where the L^i's are halflines from the origin of $\mathbb{R}^2$.

In order to analyze the case of a flat triod arising as a blow–up limit, we need some preliminary estimates, based on the following Gagliardo–Nirenberg interpolation inequalities (see [1, 6], for instance).

Proposition 3.4 (Proposition 3.11 in [26]). *Let γ be a smooth regular curve in $\mathbb{R}^2$ with finite length* L. *If u is a smooth function defined on γ and $m \geq 1$, $p \in [2, +\infty]$, we have the estimates*

$$\|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{\mathrm{L}^{m\sigma}} \|u\|_{L^2}$$

for every $n \in \{0, \dots, m-1\}$ where

$$\sigma = \frac{n + 1/2 - 1/p}{m}$$

and the constants $C_{n,m,p}$ and $B_{n,m,p}$ are independent of γ.

Lemma 3.5. *Let $F : \mathbb{T} \times [0, T) \to \mathbb{R}^2$, with $T < \infty$, be a triod moving by curvature* with moving endpoints $Q^i : [0, T) \to \Omega$ *such that the lengths of the three curves are uniformly bounded from below away from zero by $L > 0$.*
Then, for some constants $C_1 > 0$, $C_2 > 0$, independent of the triod, the following estimate holds:

$$\frac{d}{dt} \int_{\mathbb{T}_t} k^2 \, ds \leq C_1 \Big(\int_{\mathbb{T}_t} k^2 \, ds \Big)^3 + \frac{C_2}{L} \Big(\int_{\mathbb{T}_t} k^2 \, ds \Big)^2 + 2 \sum_{i=1}^{3} k^i (k_s^i + \lambda^i k^i) \Big|_{\text{at the point } Q^i(t)} .$$

Proof. Using Lemma 2.6 and integrating by parts (see computations (3.4), (3.5) and (3.6) in [26] for full details), we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{T}_t} k^2 \, ds = & -2 \int_{\mathbb{T}_t} k_s^2 \, ds + \int_{\mathbb{T}_t} k^4 \, ds - \sum_{i=1}^{3} k^i (k_s^i + \lambda^i k^i) \Big|_{\text{at the 3–point}} \\ & + 2 \sum_{i=1}^{3} k^i (k_s^i + \lambda^i k^i) \Big|_{\text{at the point } Q^i(t)} \\ = & -2 \int_{\mathbb{T}_t} k_s^2 \, ds + \int_{\mathbb{T}_t} k^4 \, ds + 2 \sum_{i=1}^{3} k^i (k_s^i + \lambda^i k^i) \Big|_{\text{at the point } Q^i(t)} , \end{aligned}$$

where we applied the "orthogonality" relation (2.10) in [26], saying that the 3–point contribution above is zero.

Letting L to be the minimum of the length of the three curves of the triod, by the above proposition and Peter–Paul inequality, for any $\varepsilon > 0$ we have the interpolation estimate

$$\begin{aligned} \int_{\mathbb{T}_t} k^4 \, ds & \leq \Big[C \Big(\int_{\mathbb{T}_t} k_s^2 \, ds \Big)^{1/8} \Big(\int_{\mathbb{T}_t} k^2 ds \Big)^{3/8} + \frac{C}{L^{1/4}} \Big(\int_{\mathbb{T}_t} k^2 \, ds \Big)^{1/2} \Big]^4 \\ & \leq C \Big(\int_{\mathbb{T}_t} k_s^2 \, ds \Big)^{1/2} \Big(\int_{\mathbb{T}_t} k^2 ds \Big)^{3/2} + \frac{C}{L} \Big(\int_{\mathbb{T}_t} k^2 \, ds \Big)^2 \\ & \leq \varepsilon \int_{\mathbb{T}_t} k_s^2 \, ds + C_1 \Big(\int_{\mathbb{T}_t} k^2 \, ds \Big)^3 + \frac{C_2}{L} \Big(\int_{\mathbb{T}_t} k^2 \, ds \Big)^2 . \end{aligned}$$

Substituting in the last equation above, after taking $\varepsilon < 2$, we get the thesis. $\square$

We are now ready to prove the main theorem of the paper.

Since the subset $\mathcal{I}$ of $[-1/2\log T, +\infty)$ defined by $\mathcal{I} = \cup_{j=1}^{\infty}(\mathfrak{t}_j + \log\sqrt{3/2}, \mathfrak{t}_j + \log\sqrt{3})$ has obviously infinite Lebesgue measure, by Proposition 2.19, we can assume that there exists another sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\widetilde{\mathfrak{t}}_j}$, with $\widetilde{\mathfrak{t}}_j \in (\mathfrak{t}_j + \log\sqrt{3/2}, \mathfrak{t}_j + \log\sqrt{3})$ for every $j \in \mathbb{N}$, which is also C^1_{loc} converging to a flat triod (*a priori* not necessarily the same one) centered at the origin of $\mathbb{R}^2$ as $j \to \infty$. Indeed, even if the two blow–up limits are different, they both must be a flat triod, as equality (2.4) must hold for both of them. Moreover, the L^2 norm of the curvature of the modified sequence of rescaled triods, as well as the one of the original sequence of rescaled triods, converges to zero on every compact subset of $\mathbb{R}^2$.
Finally, passing to a subsequence, we can also assume that $\mathfrak{t}_j$ and $\widetilde{\mathfrak{t}}_j$ (hence, also t_j and $\widetilde{t}_j$) are increasing sequences.
Notice that, by means of the rescaling relation $\mathfrak{t}(t) = -\frac{1}{2}\log(T-t)$, the condition $\widetilde{\mathfrak{t}}_j \in (\mathfrak{t}_j + \log\sqrt{3/2}, \mathfrak{t}_j + \log\sqrt{3})$ reads, for the original time parameter, as $\widetilde{t}_j \in (\frac{2}{3}t_{j_M} + \frac{1}{3}T, \frac{1}{3}t_{j_M} + \frac{2}{3}T)$.

Repeating the argument in the proof of Proposition 3.2, for any M large enough there exists j_M such that for all $i \in \{1,2,3\}$ the curve $\gamma^i_t \cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{M\sqrt{2(T-t_{j_M})}}(x_0)$ is a graph over $x_0 + L^i$ for all $t \in [t_{j_M}, T)$, with slope (with respect to the line $x_0 + L^i$) uniformly bounded by a constant C_v independent of M and $t \in [t_{j_M}, T)$. Moreover, by Lemma 2.20, we can also assume that the 3–point $O(t)$ in this time interval does not get into the annulus $B_{5M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{M\sqrt{2(T-t_{j_M})}}(x_0)$.
By Proposition 2.23, with $\theta < 1/2 < 9/16 + \frac{1}{2M^2}$, it follows that the subsequent evolution of the curves

$$\gamma^i_{t_M} \cap \left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\right),$$

that, with an abuse of notation (we cannot exclude that other parts of $\mathbb{T}_t$ get into the annulus $B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)$), we still denote with

$$\gamma^i_t \cap \left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\right),$$

for $i \in \{1,2,3\}$, are smooth evolutions for all $t \in [t_{j_M}, T)$ and the following estimate holds

$$|k^i_s(t)|^2 \leq \frac{C_v}{(t-t_{j_M})^2} \leq \frac{C_v}{(\widetilde{t}_{j_M} - t_{j_M})^2} \leq \frac{9C_v}{(T-t_{j_M})^2}, \tag{3.1}$$

for all $t \in [\widetilde{t}_{j_M}, T)$, where the constant C_v depends only on the slope with respect to the line $x_0 + L^i$.
Since, by Proposition 2.19, the L^2 norm of the curvature (in the rescaled ball $\widetilde{B}_{5M}(0)$) of the sequence of rescaled triods $\widetilde{\mathbb{T}}_{x_0,\widetilde{\mathfrak{t}}_j}$, which is given by

$$\sqrt{2(T-\widetilde{t}_j)}\int_{\mathbb{T}_{\widetilde{t}_j} \cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0)} k^2\, ds,$$

converges to zero as $j \to \infty$, the above estimate (3.1) on the derivative of the curvature, which for the sequence rescaled triods becomes $|\widetilde{k}^i_s(\mathfrak{t}_j)| \leq 3\sqrt{C}$, implies that also the L^∞ norm of the curvature of the rescaled curves

$$\gamma^i_{\widetilde{t}_j} \cap \left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\right),$$

which is given by

$$\sqrt{2(T-\widetilde{t}_j)}\left(\sup_{\mathbb{T}_{\widetilde{t}_j}\cap\left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\right)}|k|\right),$$

converges to zero as $j\to\infty$.
Since the above argument holds not only for j_M but for every $j\geq j_M$, fixed any $\varepsilon\in(0,1/2)$, first considering an $M>2$ large enough and then choosing a suitably large j_M, we can assume that

- $M>\max\{1/\sqrt{\varepsilon}, C_2/\varepsilon^{1/6}\}$, where the constant C_2 is the one appearing in Lemma 3.5,
- $$\int_{\mathbb{T}_{\widetilde{t}_{j_M}}\cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0)} k^2\,ds\leq\frac{\varepsilon}{\sqrt{2(T-\widetilde{t}_{j_M})}}\leq\frac{\sqrt{3}\varepsilon}{\sqrt{2(T-t_{j_M})}}\,, \tag{3.2}$$
- $$\sup_{\mathbb{T}_{\widetilde{t}_{j_M}}\cap\left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\right)} k^2\leq\frac{\varepsilon}{2(T-\widetilde{t}_{j_M})}\leq\frac{3\varepsilon}{2(T-t_{j_M})}\,. \tag{3.3}$$

Moreover, if C_v is a constant depending only on the slope of the evolving curve, with respect to the line x_0+L^i, we can clearly always increase j_M as we like, without affecting C_v, since, by Proposition 3.2, as $j\to\infty$, the three curves $\gamma^i_{t_j}\cap B_{5M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{M\sqrt{2(T-t_{j_M})}}(x_0)$ converge to a smooth limit.
Hence, for every such constant C_v we can also assume that $C_v\varepsilon^{1/12}<1$ and $2(C_1+C_v+1)\varepsilon^{1/6}<1$.

By Proposition 2.24, as $M>2$, at the points

$$\gamma^i_t\cap\left(B_{\frac{7}{2}M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{\frac{5}{2}M\sqrt{2(T-t_{j_M})}}(x_0)\right),$$

we have the estimate

$$|k^i(t)|^2\leq C_v\left(\sup_{\gamma^i_{\widetilde{t}_{j_M}}\cap\left(B_{4M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)\right)}|k^i|^2+\frac{1}{M^2(T-t_{j_M})}\right)$$

for all $t\in[\widetilde{t}_{j_M},T)$, with a constant C_v depending only on the slope of the curve with respect to the line x_0+L^i, which is uniformly bounded. Thus, by the above estimate (3.3) we get

$$|k^i(t)|^2\leq\frac{C_v}{T-t_{j_M}}\left(\varepsilon+\frac{1}{M^2}\right)\leq\frac{2C_v\varepsilon}{T-t_{j_M}} \tag{3.4}$$

as we already chose $M^2>1/\varepsilon$ above, for all the point of the curve $\gamma^i_t\cap\left(B_{\frac{7}{2}M\sqrt{2(T-t_{j_M})}}(x_0)\setminus B_{\frac{5}{2}M\sqrt{2(T-t_{j_M})}}(x_0)\right)$ and times $t\in[\widetilde{t}_{j_M},T)$. We want to underline once more that the constant C depends only on the slope of the curve with respect to the line x_0+L^i.

It follows that for every $t\in[\widetilde{t}_{j_M},T)$, all the triods $\widehat{\mathbb{T}}_t$ determined by "cutting" $\mathbb{T}_t$ at the new (moving in time) endpoints $Q^i(t)=\gamma^i_t\cap\partial B_{3M\sqrt{2(T-t_{j_M})}}(x_0)$ have the lengths of their

three curves uniformly bounded away from zero from below and unit tangent vectors at the endpoints $Q^i(t)$ which form angles with the respective velocity vectors $\partial_t Q^i(t)$ which are also bounded away from zero, uniformly in time, because of the uniform control on the slope of the curves with respect to the line $x_0 + L^i$. This implies that the norm of the curvature $|k^i(Q^i(t))|$ at any endpoint $Q^i(t)$ controls the norm of the tangential velocity $|\lambda^i(Q^i(t))|$, up to a multiplicative constant C_v (depending only on the slope), uniformly bounded in time for $t \in [\widetilde{t}_{j_M}, T)$.
Then, from estimates (3.1), (3.4), we conclude

$$\Big| k^i(Q^i(t)) k^i_s(Q^i(t)) \Big| \leq \frac{C_v \varepsilon^{1/4}}{(T - t_{j_M})^{\frac{3}{2}}} \,,$$

$$\Big| [k^i(Q^i(t))]^2 \lambda^i(Q^i(t)) \Big| \leq C_v \Big| k^i(Q^i(t)) \Big|^3 \leq \frac{C_v \varepsilon^{3/4}}{(T - t_{j_M})^{\frac{3}{2}}} \,,$$

for every $t \in [\widetilde{t}_{j_M}, T)$, where the constant C_v depends only on the slope of the curve with respect to the line $x_0 + L^i$.

Recalling Lemma 3.5 and noticing that the length of every curve of the triod is bounded from below by a uniform factor (depending only on the slope v) times $M\sqrt{T - t_{j_M}}$, being a graph in the annulus $B_{3M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{2M\sqrt{2(T-t_{j_M})}}(x_0)$, we have the following formula for the evolution of the L^2 norm of the curvature of the triods $\widehat{\mathbb{T}}_t$ determined by the three (moving in time) endpoints $Q^i(t)$, for $t \in [\widetilde{t}_{j_M}, T)$, where the constants C_1 and C_2 are "universal" and C_v depends only on the slope of the curve with respect to the line $x_0 + L^i$:

$$\begin{aligned}
\frac{d}{dt} \int_{\widehat{\mathbb{T}}_t} k^2 \, ds &\leq C_1 \Big(\int_{\widehat{\mathbb{T}}_t} k^2 \, ds \Big)^3 + \frac{C_2 C_v}{M\sqrt{T - t_{j_M}}} \Big(\int_{\widehat{\mathbb{T}}_t} k^2 \, ds \Big)^2 + \frac{C_v \varepsilon^{1/4}}{(T - t_{j_M})^{\frac{3}{2}}} \\
&\leq C_1 \Big(\int_{\widehat{\mathbb{T}}_t} k^2 \, ds \Big)^3 + \frac{C_v \varepsilon^{\frac{1}{6}}}{\sqrt{T - t_{j_M}}} \Big(\int_{\widehat{\mathbb{T}}_t} k^2 \, ds \Big)^2 + \frac{C_v \varepsilon^{1/4}}{(T - t_{j_M})^{\frac{3}{2}}} \\
&\leq C_1 \Big(\int_{\widehat{\mathbb{T}}_t} k^2 \, ds \Big)^3 + \frac{\varepsilon^{\frac{1}{12}}}{\sqrt{T - t_{j_M}}} \Big(\int_{\widehat{\mathbb{T}}_t} k^2 \, ds \Big)^2 + \frac{C_v \varepsilon^{1/4}}{(T - t_{j_M})^{\frac{3}{2}}} \,,
\end{aligned}$$

as we chose $M > C_2/\varepsilon^{\frac{1}{6}}$ and $2C_v\varepsilon^{1/12} < 1$.
Then, letting

$$A(t) := \max \left\{ \int_{\widehat{\mathbb{T}}_t} k^2 \, ds \,,\, \frac{\varepsilon^{\frac{1}{12}}}{\sqrt{T - t_{j_M}}} \right\} \,,$$

it follows

$$A'(t) \leq \overline{C}_v A^3(t)$$

for almost every $t \in [\widetilde{t}_{j_M}, T)$, where the constant $\overline{C}_v$ is given by $C_1 + C_v + 1$.
Integrating this differential inequality and recalling estimate (3.2), implying that

$$A(\widetilde{t}_{j_M}) \leq \max \left\{ \frac{\sqrt{3}\varepsilon}{\sqrt{2(T - t_{j_M})}} \,,\, \frac{\varepsilon^{\frac{1}{12}}}{\sqrt{T - t_{j_M}}} \right\} \leq \frac{\varepsilon^{\frac{1}{12}}}{\sqrt{T - t_{j_M}}} \,,$$

as $\varepsilon < 1/2$, we get

$$A(t) \leq \frac{1}{\sqrt{A(\widetilde{t}_{j_M})^{-2} - 2\overline{C}_v (t - \widetilde{t}_{j_M})}} \leq \frac{\varepsilon^{\frac{1}{12}}}{\sqrt{T - t_{j_M} - 2\overline{C}_v \varepsilon^{\frac{1}{6}} (t - \widetilde{t}_{j_M})}} \,,$$

for every $t \in [\widetilde{t}_{j_M}, T)$.
As $(t - \widetilde{t}_{j_M}) \leq (T - t_{j_M})$, it follows that the function $A(t)$ is uniformly bounded on $[\widetilde{t}_{j_M}, T)$ as soon as $2\overline{C}_v \varepsilon^{\frac{1}{6}} < 1$, which is satisfied by our previous assumption on $\varepsilon > 0$.

We now notice that the three curves of the triod $\mathbb{T}_t$, connecting respectively the points P^i and Q^i (determined by $\mathbb{T}_t \setminus \widehat{\mathbb{T}}_t$) cannot get too close to the point $x_0 = \lim_{t \to T} O(t)$ along the flow. Indeed, the parts of these curves in the annulus $B_{5M\sqrt{2(T-t_{j_M})}}(x_0) \setminus B_{3M\sqrt{2(T-t_{j_M})}}(x_0)$ are graphs for every $t \in [\widetilde{t}_{j_M}, T)$, while the remaining pieces "outside" at time $t = \widetilde{t}_{j_M}$, by maximum principle, during their subsequent evolution can never get into the circle of radius $R(t) = \sqrt{16M^2(T - t_{j_M}) - 2(t - t_{j_M})}$ and center x_0, also moving by mean curvature in the time interval $[\widetilde{t}_{j_M}, T)$ and, as $t \to T$, converging to the circle of radius $\sqrt{16M^2(T - t_{j_M}) - 2(T - t_{j_M})} = \sqrt{(16M^2 - 2)(T - t_{j_M})}$, which is clearly positive as $M > 2$, hence far from the point x_0.
Consequently, since the closed subset of the set of reachable points obtained as possible limit points of these three curves as $t \to T$ is contained in a closed set far from x_0, by Propositions 3.2 and 3.3, we can cover such a set by a finite number of balls where the curvature of the evolving triod is uniformly bounded during the flow. Being also the total length of the evolving triods uniformly bounded and being the L^2 norm of the curvature of the "subtriods" $\widehat{\mathbb{T}}_t$, given by the square root of the uniformly bounded function $A(t)$, we conclude that the full L^2 norm of the curvature of the evolving triods $\mathbb{T}_t$ is bounded, in contradiction with Proposition 2.8. This concludes the proof.

Remark 3.6. We point out that the main result of this paper, namely Theorem 1.1, can be extended with a similar proof to a triod evolving by curvature with Neumann boundary conditions. Moreover, whenever the classification given in Proposition 2.19 holds, the same proof also applies to the evolution of a network with multiple triple junctions. For instance, this is true for a network without loops and with at most *two* triple junctions. Indeed, in this case, Proposition 2.17 still holds and all the subsequent arguments can be adapted with minor modifications.

Note. In this respect, we take the occasion to underline a mistake in [26, Remark 4.5] (pointed out to us by T. Ilmanen), where the authors claim that Proposition 2.17 (Theorem 4.6 in [26]) holds for *any* network (without loops), *without any constraint on the number of triple junctions*. Actually, the proof of Proposition 2.17 can be generalized only to networks in the plane with *at most two triple junctions*.

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