ON CRITICAL POINTS OF THE RELATIVE FRACTIONAL PERIMETER

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Abstract. We study the localization of sets with constant nonlocal mean curvature and prescribed small volume in a bounded open set with smooth boundary, proving that they are *sufficiently close* to critical points of a suitable non-local potential. We then consider the fractional perimeter in half-spaces. We prove the existence of a minimizer under fixed volume constraint, showing some of its properties such as smoothness and symmetry, being a graph in the x_N-direction, and characterizing its intersection with the hyperplane $\{x_N = 0\}$.

Contents

1. Introduction 1
2. Notation and preliminary results 4
3. Proof of Theorem 1.1 8
4. Proof of Theorem 1.3 15
5. Appendix 19
References 21

1. Introduction

Isoperimetric problems play a crucial role in several areas such as geometry, linear and nonlinear PDEs, probability, Banach space theory and others. Its classical version consists in studying least-area sets contained in a fixed region (the Euclidean space or any given domain). If the ambient space is an N-dimensional manifold M^N with or without boundary, the goal would be to find, among all the compact hypersurfaces $\Sigma \subset M$ which bound a region Ω of given volume $V(\Omega) = m$ (for $0 < m < V(M)$), those of minimal area $A(\Sigma)$. Such a region Ω is called an *isoperimetric region* and its boundary Σ is called an *isoperimetric hypersurface.*

A first general existence and regularity result can be obtained for example combining the results in [2] with those in [22, 26]. In particular we have that if $N \leq 7$, Σ is smooth. We also refer the reader to the interesting survey [35].

Beyond the existence and the regularity problem, it is also interesting to study the geometry and the topology of the solutions, and to give a qualitative description of the isoperimetric regions. Concerning these issues, we recall that in [31] it was proved that a region of small prescribed volume in a smooth and compact Riemannian manifold has asymptotically (as the volume tends to zero) at least as much perimeter as a round ball.

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Afterwards, regarding critical points of the perimeter relative to a given set, in [18] the existence of surfaces with the shape of half spheres was shown, surrounding a small volume near nondegenerate critical points of the mean curvature of the boundary of an open smooth set in $\mathbb{R}^3$. It was proved that the boundary mean curvature determines the main terms, studying the problem via a Lyapunov-Schmidt reduction. In [17], the same author showed that isoperimetric regions with small volume in a bounded smooth domain Ω are near global maxima of the mean curvature of Ω.

Results of this type were proven in [13] and [39]. The authors considered closed manifolds and proved that isoperimetric regions with small volume locate near the maxima of the scalar curvature. In [39] a viceversa was also shown: for every non-degenerate critical point p of the scalar curvature there exists a neighborhood of p foliated by constant mean curvature hypersurfaces. Moreover, in [38] the boundary regularity question for the capillarity problem was studied.

In recent years fractional operators have received considerable attention for both in pure and applied motivations. In particular, regarding perimeter questions, in [5] the link between the fractional perimeter and the classical De Giorgi's perimeter was analyzed, showing the equi-coercivity and the Γ-convergence of the fractional perimeter, up to a scaling factor $\omega_{N-1}^{-1}(1-2s)$, to the classical perimeter in the sense of De Giorgi and a local convergence result for minimizers was deduced.

Another relevant result about fractional perimeter was obtained in [20], generalizing a quantitative isoperimetric inequality to the fractional setting. Indeed, in the Euclidean space, it is known that among all sets of prescribed measure, balls have the least perimeter, i.e. for any Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure, one has

$$N|B_1|^{\frac{1}{N}}|E|^{\frac{N-1}{N}} \leq P(E) \tag{1.1}$$

with B_1 denoting the unit ball of $\mathbb{R}^N$ with center at the origin and $P(E)$ is the distributional perimeter of E. The equality in (1.1) holds if and only if E is a ball.

In [21] a similar result for the fractional perimeter P_s (defined as in (2.3)) was obtained, improved then in [20] showing the following fact: for every $N \geq 2$ and any $s_0 \in (0,1)$ there exists $C(N, s_0) > 0$ such that

$$P_s(E) \geq \frac{P_s(B_1)}{|B_1|^{\frac{N-s}{N}}}|E|^{\frac{N-s}{N}}\left\{1 + \frac{A(E)^2}{C(N,s)}\right\} \tag{1.2}$$

whenever $s \in [s_0, 1]$ and $0 < |E| < \infty$. Here

$$A(E) := \inf\left\{\frac{|E\triangle(B_{r_E}(x))|}{|E|} : x \in \mathbb{R}^N\right\}$$

stands for the *Fraenkel asymmetry* of E, measuring the L^1-distance of E from the set of balls of volume $|E|$ and $r_E = (|E|/|B_1|)^{1/N}$ so that $|E| = |B_{r_E}|$.

In the same spirit of extension of classical results to the fractional setting, we also mention [28]. Here the authors modify the classical Gauss free energy functional used in capillarity theory by considering surface tension energies of nonlocal type. They analyzed a family of problems including a nonlocal isoperimetric problem of geometric interest. In particular, given $N \geq 2$, $s \in (0,1)$, $\lambda \geq 1$ and $\varepsilon \in [0,\infty]$ they considered the family of interaction kernels $\mathbf{K}(N,s,\lambda,\varepsilon)$, i.e. even functions $K:\mathbb{R}^N\setminus\{0\}\to[0,+\infty)$ such that

$$\frac{\chi_{B_\varepsilon}(z)}{\lambda|z|^{N+s}}\le K(z)\le\frac{\lambda}{|z|^{N+s}}\quad\forall\, z\in\mathbb{R}^N\setminus\{0\}$$

where $B_\varepsilon(x)$ is the ball of center x and radius ε. Taking $\Omega\subset\mathbb{R}^N$ and $\sigma\in(-1,1)$ the authors studied the nonlocal capillarity energy of $E\subset\Omega$ defined as

$$\mathcal{E}(E)=\int_E\int_{E^C\cap\Omega}K(x,y)\,\mathrm{d}x\,\mathrm{d}y+\sigma\int_E\int_{\Omega^C}K(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

with $K\in\mathbf{K}(N,s,\lambda,\varepsilon)$, giving existence and regularity results, density estimates and new equilibrium conditions with respect to those of the classical Gauss free energy.

As it concerns constant nonlocal mean curvature, we mention the paper [10], where it was proved the existence of Delaunay type surfaces, i.e. a smooth branch of periodic topological cylinders with the same constant nonlocal mean curvature, and [30], where the author constructs two families of hypersurfaces with constant nonlocal mean curvature.

Moreover we notice that, recently, in [29] the axial symmetry of *smooth* critical points of the fractional perimeter in a half-space was shown, using a variant of the moving plane method.

Motivated by these results, in the first part of this paper our aim is to study the localization of sets with constant nonlocal mean curvature and small prescribed volume relative to an open bounded domain. The notions of relative fractional perimeter $P_S(E,\Omega)$ and of relative fractional mean curvature H_s^Ω we are going to use are given by formulas (2.3) and (2.5) in the next section.

Theorem 1.1. *Let $s\in(0,1/2)$ and $\Omega\subseteq\mathbb{R}^N$ be a bounded open set with smooth boundary. For x in a given compact set Θ of Ω, set*

$$V_\Omega(x):=\int_{\Omega^C}\frac{1}{|x-y|^{N+2s}}\,\mathrm{d}y.$$

Then for every strict local extremal or non-degenerate critical point x_0 of V_Ω in Ω, there exists $\overline{\varepsilon}>0$ such that for every $0<\varepsilon<\overline{\varepsilon}$ there exist spherical-shaped surfaces S_ε with constant $H^\Omega_{s,\partial S_\varepsilon}$ curvature and enclosing volume identically equal to ε, approaching x_0 as $\varepsilon\to0$.

Notice that in (2.3) (as well as in the above formula) we are using the exponent $2s$ in the denominator, and hence in our notation the range $(0,1/2)$ for s is natural. One of the main tools for proving this result relies on the non-degeneracy of spheres with respect to the linearized non-local mean curvature equation, which follows from a result in [9]. After non-degeneracy is established, we can use a Lyapunov-Schmidt reduction to study a finite-dimensional problem, which is treated by carefully expanding the relative fractional perimeter of balls with small volume. Thanks to classical results in min-max theory, we obtain as a corollary a multiplicity result. Here and in the following, $\mathrm{cat}(\Omega)$ denotes the Lusternik-Schnirelman category of the set Ω (see [27] and Section 2 below for more details).

Corollary 1.2. *Let $s\in(0,1/2)$ and $\Omega\subseteq\mathbb{R}^N$ be a bounded open set with smooth boundary. Then there exists $\overline{\varepsilon}>0$ such that for every $0<\varepsilon<\overline{\varepsilon}$ there exist at least $\mathrm{cat}(\Omega)$ spherical-shaped surfaces S_ε with constant $H^\Omega_{s,\partial S_\varepsilon}$ curvature and enclosing volume identically equal to ε.*

In the last part of this work we aim to study the existence and some properties of sets minimizing the fractional perimeter in a particular domain, namely a half-space:

Theorem 1.3. *There exists a minimizer E for the problem*

$$\inf\Big\{P_s(A,\mathbb{R}^N_+),\ |A|=m\Big\},\quad m\in(0,+\infty), \tag{1.3}$$

where $\mathbb{R}^N_+ := \{x\in\mathbb{R}^N : x_N>0\}$. Moreover ∂E is a radially-decreasing symmetric graph of class C^∞ in the interior, intersecting orthogonally the hyperplane $\{x_N=0\}$.

This result is proved by showing first the existence of a properly rearranged minimizing sequence which is axially symmetric and graphical over the boundary hyperplane. After this is done, we employ some results from [6], [11], [28] to prove a diameter bound and smoothness of the minimizing limit.

The paper is organized as follows: In Section 2 we introduce some notation on fractional perimeter and mean curvature, and we show some preliminary results, especially on the linearized fractional mean curvature. We prove in particular the minimal degeneracy for spheres, also relative to suitably large domains. In Section 3 we prove Theorem 1.1 via a Lyapunov-Schmidt reduction and Corollary 1.2 through a well known result about the Lusternik-Schnirelman category. Finally, in Section 4 we prove Theorem 1.3 in two steps: the existence of minimizers in a bounded domain is a rather standard consequence of the direct method of Calculus of Variations. We then show the symmetry of minimizers and, using the density estimates holding for the fractional perimeter, we prove also the connectedness and hence the free minimality.

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2. Notation and preliminary results

In this section we introduce the notation that will be used throughout the paper. We first define fractional perimeter spaces and fractional mean curvature, listing some of their properties.

For $0<s<1/2$ the *fractional perimeter* (or s-perimeter) of a measurable set $E\subset\mathbb{R}^N$ is defined as

$$P_s(E) := \int_E\int_{E^C}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}, \tag{2.1}$$

where E^C is the complement of E. It has also a simple representation in terms of the usual seminorm in the fractional Sobolev space $H^s(\mathbb{R}^N)$, that is

$$P_s(E) = [\chi_E]^2_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y,$$

where χ_E denotes the characteristic function of E. We say that a set $E \subset \mathbb{R}^N$ has *finite s-perimeter* if (2.1) is finite. If E is an open set and ∂E is a smooth bounded surface, we have from [5, Theorem 2] that as $s \to 1/2$

$$(1-2s)P_s(E) \to \omega_{N-1}P(E), \tag{2.2}$$

where ω_{N-1} denote the volume of the unit ball in $\mathbb{R}^{N-1}$ for $N \geq 2$ and $P(E)$ is the perimeter in the sense of De Giorgi.

This nonlocal notion of perimeter can be considered also relative to a bounded open set Ω by the formula

$$P_s(E,\Omega) := \int_E \int_{\Omega\setminus E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}. \tag{2.3}$$

Definition 2.1. We say that a set $E \subset \mathbb{R}^N$ is a *minimizer* for the fractional perimeter relative to Ω if

$$P_s(E,\Omega) \leq P_s(F,\Omega) \tag{2.4}$$

for any measurable set F that coincides with E outside Ω, i.e. $F \setminus \Omega = E \setminus \Omega$.

Let $s \in (0,1/2)$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. We recall that the nonlocal mean curvature of a set E at a point $x \in \partial E$ is defined as follows

$$H^{\Omega}_{s,\partial E}(x) := \int_{\Omega} \frac{\chi_{E^c\cap\Omega}(y) - \chi_E(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \tag{2.5}$$

(see [28, Theorem 1.3 and Proposition 3.2 with $\sigma = 0$ and $g = 0$]) where χ_E denotes the characteristic function of E, E^C is the complement of E, and the integral has to be understood in the principal value sense.

If E is smooth and compactly contained in Ω, let w be a smooth function defined on on ∂E, with small L^∞ norm. We call E_w the set whose boundary ∂E_w is parametrized by

$$\partial E_w = \{x + w(x)\nu_E(x) | x \in \partial E\} \tag{2.6}$$

where ν_E is a normal vector field to ∂E exterior to E.

The first variation of the s-perimeter (2.3) along these normal perturbations is given by

$$\mathrm{d}_t P_s(E_{tw},\Omega)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} P_s(E_{tw},\Omega) = \int_{\partial E} H^{\Omega}_{s,\partial E} w\,\mathrm{d}\sigma, \tag{2.7}$$

see [14].

In the following, we take $B_1(\xi)$ a ball with center $\xi \in \mathbb{R}^N$ and unit radius, $w \in C^1(\partial B_1(\xi))$, and we denote by $\mathbb{B}(\xi,w)$ the set such that

$$\partial\mathbb{B}(\xi,w) := \{y \in \mathbb{R}^N : y = x + w(x)\nu_{\partial B_1(\xi)}(x), x \in \partial B_1(\xi)\}, \tag{2.8}$$

where $\nu_{\partial B_1(\xi)}$ is the outer unit normal to $\partial B_1(\xi)$.

Then we set

$$S_\xi := \partial B_1(\xi) \qquad \text{and} \qquad P^{\Omega}_{s,\xi}(w) := P^{\Omega}_s(\mathbb{B}(\xi,w),\Omega). \tag{2.9}$$

Moreover, for $\beta \in (2s,1)$ and $\varphi \in C^{1,\beta}(\partial\mathbb{B}(\xi,w))$, we set

$$\left(P^{\Omega}_{s,\xi}\right)'(w)[\varphi] := \int_{\partial\mathbb{B}(\xi,w)} H^{\Omega}_{s,\partial\mathbb{B}(\xi,w)}\varphi\,\mathrm{d}\sigma_w$$

where $d\sigma_w$ stands for the area element of $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$.

Consider next the *spherical fractional Laplacian*

$$L_s\varphi(\theta) := P.V. \int_S \frac{\varphi(\theta)-\varphi(\sigma)}{|\theta-\sigma|^{N+2s}}\,\mathrm{d}\sigma,$$

where $S = \partial B_1$ and the above integral is understood in the principal value sense.

It turns out that (see e.g. [9])

$$L_s : C^{1,\beta}(S) \to C^{\beta-s}(S). \tag{2.10}$$

The operator L_s has an increasing sequence of eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ whose explicit expression is given by

$$\lambda_k := \frac{\pi^{(N-1)/2}\Gamma((1-2s)/2)}{(1+2s)2^{2s}\Gamma((N+2s)/2)}\left(\frac{\Gamma\Big(\dfrac{2k+N+2s}{2}\Big)}{\Gamma\Big(\dfrac{2k+N-2s-2}{2}\Big)} - \frac{\Gamma\Big(\dfrac{N+2s}{2}\Big)}{\Gamma\Big(\dfrac{N-2s-2}{2}\Big)}\right), \tag{2.11}$$

see [36, Lemma 6.26], where Γ is the Euler Gamma function. The eigenfunctions are the usual spherical harmonics, i.e. one has

$$L_s\psi = \lambda_k\psi \qquad \text{for every } k \in \mathbb{N} \text{ and } \psi \in \mathcal{E}_k,$$

where $\mathcal{E}_k$ is the space of spherical harmonics of degree k and dimension $n_k = N_k - N_{k-2}$, with

$$N_k = \frac{(n+k-1)!}{(n-1)!k!}, \quad k \geq 0, \qquad\qquad N_k = 0 \quad k < 0.$$

We recall that $n_0 = 1$ and that $\mathcal{E}_0$ consists of constant functions, whereas $n_1 = N$ and $\mathcal{E}_1$ is spanned by the restrictions of the coordinate functions in $\mathbb{R}^N$ to the unit sphere S.

For sets that are suitable graphs over the unit sphere S of $\mathbb{R}^N$, we have the following result concerning fractional mean curvature relative to the whole space, see [9, Theorem 2.1, Lemma 5.1 and Theorem 5.2 (see also formula (1.3) in the latter paper)].

Proposition 2.2. *Given $\beta \in (2s, 1)$, consider the family of functions*

$$\Upsilon := \left\{\varphi \in C^{1,\beta}(S) \ : \ \|\varphi\|_{L^\infty(S)} < \frac{1}{2}\right\}.$$

Then the map $\varphi \mapsto H^{\mathbb{R}^N}_{s,\partial\mathbb{B}(0,\varphi)}$ is a C^∞ function from Υ into $C^{\beta-2s}(S)$. Moreover, its linearization at $\varphi \equiv 0$ is given by

$$\varphi \longmapsto 2d_{N,s}(L_s - \lambda_1)\varphi, \tag{2.12}$$

where λ_1 is defined in (2.11) *and $d_{N,s} := \frac{1-2s}{(N-1)|B_1^{N-1}|}$ where B_1^{N-1} is the unit ball in $\mathbb{R}^{N-1}$.*

As a consequence of the latter result we have than every function in the kernel of the above linearized nonlocal mean curvature is a linear combination of first-order spherical harmonics, i.e. if $w \in \mathrm{Ker}\ (L_s - \lambda_1)$, we have

$$w = \sum_{i=1}^{N} \lambda_i Y_i, \tag{2.13}$$

where $\{Y_i\}_{i=1,\cdots,N} \in \mathcal{E}_1$ and $\lambda_i \in \mathbb{R}$. Therefore, defining

$$W := \left\{ w \in C^{1,\beta}(S_\xi) : \int_{S_\xi} w\, Y_i = 0 \text{ for } i = 1, \dots, N, \right\}, \tag{2.14}$$

it follows by Fredholm's theory that $L_s - \lambda_1$ is invertible on W.

As a consequence of the above proposition, using a perturbation argument, we deduce also the following result, for which we need to introduce some notation. Let Ω be a bounded set in $\mathbb{R}^N$, for $\varepsilon > 0$ let $\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega$. Fix a compact set Θ in Ω, and let $\xi \in \frac{1}{\varepsilon}\Theta$. Consider then the operator $L^{\Omega_\varepsilon}_{s,\xi}$ corresponding to the linearization of the s-mean curvature at $B_1(\xi)$ relative to Ω_ε, namely the non-local operator such that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,t\varphi)} = (L^{\Omega_\varepsilon}_{s,\xi}\varphi).$$

We have then the following result.

Proposition 2.3. *Let Ω, Θ, ξ and $L^{\Omega_\varepsilon}_{s,\xi}$ be as above, and let $\beta \in (2s, 1)$. Consider the family of functions*

$$\Upsilon := \left\{ \varphi \in C^{1,\beta}(S_\xi) \ : \ \|\varphi\|_{L^\infty(S_\xi)} < \frac{1}{2} \right\}.$$

Then the map $\varphi \mapsto H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,\varphi)}$ is a C^∞ function from Υ into $C^{\beta-2s}(S_\xi)$. Moreover, if W is as in (2.14)*, $L^{\Omega_\varepsilon}_{s,\xi}$ is invertible with uniformly bounded inverse on W.*

Given a topological space M and a subset $A \subseteq M$, we recall next the definition and some properties of the Lusternik-Schnirelman category.

Definition 2.4. [3, Definition 9.2] The category of A with respect to M, denoted by $\mathrm{cat}_M(A)$, is the least integer k such that $A \subseteq A_1 \cup \cdots \cup A_k$ with A_i closed and contractible in M for every $i = 1, \cdots, k$.

We set $\mathrm{cat}(\varnothing) = 0$ and $\mathrm{cat}_M(A) = +\infty$ if there are no integers with the above property. We will use the notation $\mathrm{cat}(M)$ for $\mathrm{cat}_M(M)$.

Remark 2.5. From Definition 2.4, it is easy to see that $\mathrm{cat}_M(A) = \mathrm{cat}_M(\bar{A})$. Moreover, if $A \subset B \subset M$, we have that $\mathrm{cat}_M(A) \le \mathrm{cat}_M(B)$, see [3, Lemma 9.6].

Then assuming that

$$M = F^{-1}(0), \text{ where } F \in C^{1,1}(E, \mathbb{R}) \text{ with } E \supset M \text{ and } F'(u) \neq 0 \ \forall\, u \in M, \tag{2.15}$$

we set

$$\mathrm{cat}_k(M) = \sup\{\mathrm{cat}_M(A) : A \subset M \text{ and } A \text{ is compact}\}.$$

Note that if M is compact, $\mathrm{cat}_k(M)$ =$\mathrm{cat}(M)$. At this point we can state a useful result about the Lusternik-Schnirelman category (see e.g. [3] for the definition of Palais-Smale ((PS)-condition).

Theorem 2.6. [3, Theorem 9.10] *Let M be a Hilbert space or a complete Banach manifolds. Let* (2.15) *hold, let $J \in C^{1,1}(M, \mathbb{R})$ be bounded from below on M and let J satisfy (PS)-condition. Then J has at least $\mathrm{cat}_k(M)$ critical points.*

Remark 2.7. If M has boundary, under the same assumptions of Theorem 2.6 one can still find at least $\mathrm{cat}_k(M)$ critical points for J provided ∇J is non zero on ∂M and points in the outward direction.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 via a finite-dimensional reduction. This will determine the location of critical points of the relative s-perimeter depending on s and the geometry of the domain. One of the main tools is the following asymptotic expansion of the relative s-perimeter. From now on, for every $\varepsilon > 0$, we set $\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega$, and we aim to prove that the nonlocal mean curvature H_s^Ω is *sufficiently close* to $H_s^{\mathbb{R}^N}$. Hereafter we will write simply H_s to denote $H_s^{\mathbb{R}^N}$.

Lemma 3.1. *Let $\Theta \subseteq \Omega$ be a fixed compact set. For all $\varepsilon > 0$ we consider $B_1(\bar{x})$ a ball of center $\bar{x} \in \Theta_\varepsilon := \frac{1}{\varepsilon}\Theta$ and with unit radius. Then, for the fractional perimeter, the following expansion holds*

$$P_s(B_1(\bar{x}), \Omega_\varepsilon) = P_s(B_1(\bar{x})) - \omega_N \varepsilon^{2s} V_\Omega(\varepsilon\bar{x}) + O(\varepsilon^{1+2s}) \qquad \text{as } \varepsilon \to 0, \tag{3.1}$$

where ω_N is the volume of the N-dimensional unit ball and

$$V_\Omega(\varepsilon\bar{x}) := \int_{\Omega^C} \frac{1}{|\bar{x}-y|^{N+2s}}\,\mathrm{d}y. \tag{3.2}$$

Moreover one has that

$$\nabla_{\bar{x}} P_s(B_1(\bar{x}), \Omega_\varepsilon) = -\omega_N \varepsilon^{2s+1} \nabla_{\bar{x}} V_\Omega(\varepsilon\bar{x}) + O(\varepsilon^{2+2s}). \tag{3.3}$$

Proof. Taking ε small enough, we can assume $B_1(\bar{x}) \subset \Omega_\varepsilon$. From (2.3) we have

$$P_s(B_1(\bar{x}), \Omega_\varepsilon) - P_s(B_1(\bar{x})) = -\int_{B_1(\bar{x})}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \frac{1}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.4}$$

If we replace x with $\bar{x}$ in the last integrand, we obtain

$$\frac{1}{|x-y|^{N+2s}} = \frac{1}{|\bar{x}-y|^{N+2s}} + O\left(\frac{1}{|\bar{x}-y|^{N+2s+1}}\right); \qquad x \in B_1(\bar{x}), \quad y \in \mathbb{R}^N \setminus \Omega_\varepsilon.$$

Therefore

$$\int_{B_1(\bar{x})}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \frac{1}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \omega_N \int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \frac{1}{|\bar{x}-y|^{N+2s}}\,\mathrm{d}y + \int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \frac{O(1)}{|\bar{x}-y|^{N+2s+1}}\,\mathrm{d}y.$$

From the latter formulas and a change of variables one then finds

$$P_s(B_1(\bar{x}), \Omega_\varepsilon) - P_s(B_1(\bar{x})) = -\varepsilon^{2s}\omega_N \int_{\Omega^C} \frac{1}{|\bar{x}-y|^{N+2s}}\,\mathrm{d}y + O(\varepsilon^{1+2s}),$$

which concludes the proof of (3.1). Formula (3.3) follows in a similar manner. ☐

Now we want to evaluate the deviation of the nonlocal mean curvature from a constant, when it is computed relatively to a large domain. To do that, we define

$$\begin{aligned}\tilde{H}_{s,\xi} &: S^{N-1} \to \mathbb{R} \\ \tilde{H}_{s,\xi}(x) &:= H^{\Omega_\varepsilon}_{s,S_\xi}(x+\xi).\end{aligned} \tag{3.5}$$

Lemma 3.2. *Let $\beta \in (2s,1)$. For the (relative) fractional mean curvature defined in* (2.5)*, the following expansion holds:*

$$\tilde{H}_{s,\xi} = c_{N,s} + O(\varepsilon^{2s}) \qquad \text{in } C^{\beta-2s}(S^{N-1}), \tag{3.6}$$

where $c_{N,s} := H_{s,S_\xi}$ and we recall that $S_\xi = \partial B_1(\xi)$ with $B_1(\xi)$ denoting the ball of center at ξ and unit radius. Moreover, one has that for all $i = 1, \dots, N$,

$$\frac{\partial}{\partial \xi_i}\tilde{H}_{s,\xi} = O(\varepsilon^{2s+1}) \qquad \text{in } C^{\beta-2s}(S^{N-1}). \tag{3.7}$$

Proof. Using the definition of (relative) fractional mean curvature (see (2.5)) and [37, Lemma 2], for $x \in \partial B_1$, we can write

$$\tilde{H}_{s,\xi}(x) = c_{N,s} + \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s}}. \tag{3.8}$$

where $c_{N,s} := H_{s,\xi}(\cdot + \xi)$.

Therefore we get that, for $x \in \partial B_1$,

$$\tilde{H}_{s,\xi}(x) = c_{N,s} + O(\varepsilon^{2s}). \tag{3.9}$$

Then, using (3.8) and differentiating with respect to ξ_i, we find that, for all $i = 1, \dots, N$,

$$\begin{aligned}\frac{\partial}{\partial \xi_i}\tilde{H}_{s,\xi} &= \frac{\partial}{\partial \xi_i}\left(c_{N,s} + \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s}}\right) \\ &= O\left(\int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s+1}}\right) = O(\varepsilon^{2s+1}).\end{aligned} \tag{3.10}$$

Thus, we proved (3.6) and (3.7) in a pointwise sense. It is easy however to see that they also hold in the C^1 sense on the unit sphere S_ξ, and therefore also in $C^{\beta-2s}(S^{N-1})$. □

We turn next to a finite-dimensional reduction of the problem, which is possible by the smallness of volume in the statement of Theorem 1.1. We refer to [4] for a general treatment of the subject.

Proposition 3.3. *Suppose that Ω is a smooth bounded set of $\mathbb{R}^N$, Θ a set compactly contained in Ω, and let $\beta \in (2s,1)$. For $\varepsilon > 0$ small, let $\xi \in \Theta_\varepsilon$. Then there exist $w_\varepsilon : S_\xi \to \mathbb{R}$ in W and $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{R}^N$ such that*

$$Vol(\mathbb{B}(\xi, w_\varepsilon)) = \omega_N; \qquad \int_{S_\xi} w_\varepsilon Y_i \,\mathrm{d}\sigma = 0; \qquad H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)} = c + \sum_{i=1}^{N} \lambda_i Y_i,$$

where $c \in \mathbb{R}$ is close to $c_{N,s}$ and where $\{Y_i\}_{i=1,\cdots,N} \in \mathcal{E}_1$ (extended as zero-homogeneous function in a neighborhood of the unit sphere). Moreover, there exists $C > 0$ (depending on Θ, Ω, N and s) such that $\|w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \le C\varepsilon^{2s}$ and such that $\|\partial_\xi w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \le C\varepsilon^{2s+1}$.

To make the above formula for $H_s^{\Omega_\varepsilon}$ more precise, we mean that

$$H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}(\xi + x(1+w_\varepsilon(x))) = c + \sum_{i=1}^{N} \lambda_i Y_i(x) \qquad \text{for every } x \in S_\xi.$$

Proof. Let us denote by $\overline{W}$ the family of functions in $C^{\beta-2s}(S_\xi)$ that are L^2-orthogonal, with respect to the <u>standard</u> volume element of S_ξ, to constants and to the first-order spherical harmonics. Notice that $\overline{W} \subseteq W$, see (2.14). Let us consider the two-component function $F_{\overline{W}} : \Theta_\varepsilon \times C^{1,\beta}(S_\xi) \to C^{\beta-2s}(S_\xi) \times \mathbb{R}$ defined by

$$F_{\overline{W}}(\xi, w) := \Big(P_{\overline{W}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w)}), Vol(\mathbb{B}(\xi, w)) - \omega_N\Big)\,; \qquad w \in W,$$

where $\omega_N := Vol(B_1(\xi))$ and $P_{\overline{W}} : C^{\beta-2s}(S_\xi) \mapsto \overline{W}$ the orthogonal L^2-projection onto the space $\overline{W}$, with respect to the <u>standard</u> volume element of S_ξ. With this notation, we want to find $w \in W$ such that $F_{\overline{W}}(\xi, w) = (0, 0)$.

By Lemma 3.2 we have that

$$F_{\overline{W}}(\xi, 0) = (O(\varepsilon^{2s}), 0), \tag{3.11}$$

where the latter quantity is intended to be bounded by $C\varepsilon^{2s}$ in the $C^{\beta-2s}(S_\xi)$ sense. In our notation, the constant C is allowed to vary from one formula to the other.

By Proposition 2.3 and by the fact that

$$\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} Vol(\mathbb{B}(\xi, w))[\varphi] = \int_{S_\xi} \varphi \, d\sigma,$$

we have that $L_\xi := \nabla_w F_{\overline{W}}(\xi, 0) \in Inv(W, \overline{W} \times \mathbb{R})$ with $\|L_\xi^{-1}\|_{L(\overline{W}\times\mathbb{R},W)} \leq C$. Hence $F_{\overline{W}}(\xi, w) = (0, 0)$ if and only if $F_{\overline{W}}(\xi, 0) + L_\xi[w] - L_\xi[w] + F_{\overline{W}}(\xi, w) - F_{\overline{W}}(\xi, 0) = (0, 0)$, which can be written as

$$w = T_\xi(w) := -L_\xi^{-1}[F_{\overline{W}}(\xi, 0) - L_\xi[w] + F_{\overline{W}}(\xi, w) - F_{\overline{W}}(\xi, 0)].$$

Therefore $F_{\overline{W}}(\xi, w) = (0, 0)$ if and only if w is a fixed point for T_ξ.

Let us show that T_ξ is a contraction in $B_{\overline{C}\varepsilon^{2s}}(\xi)$ for $\overline{C}$ sufficiently large. From the definition of T_ξ, the above estimate (3.11) and the fact that

$$\|L_\xi^{-1}\|_{L(\overline{W}\times\mathbb{R},W)} \leq C,$$

we have

$$\|T_\xi(0)\|_{C^{1,\beta}(S_\xi)} = \|L_\xi^{-1}[F_{\overline{W}}(\xi, 0)]\|_{C^{1,\beta}(S_\xi)} \leq C^2\varepsilon^{2s}. \tag{3.12}$$

Then, taking w_1 and $w_2 \in B_{\bar{C}\varepsilon^{2s}}(\xi) \subseteq W$ it follows that

$$\|T_\xi(w_1) - T_\xi(w_2)\|_{C^{1,\beta}(S_\xi)} \leq C\|F_{\overline{W}}(\xi, w_1) - F_{\overline{W}}(\xi, w_2) - L_\xi[w_1 - w_2]\|_{C^{1,\beta}(S_\xi)}. \tag{3.13}$$

We notice that the function $w \mapsto Vol(\mathbb{B}(\xi, w))$ is a smooth function from the metric ball of radius $\frac{1}{2}$ in $C^{1,\beta}(S_\xi)$ into $\mathbb{R}$. Thanks also to the smoothness statement in Proposition 2.3, the right hand side in the latter formula can be bounded by

$$\begin{aligned} F_{\overline{W}}(\xi, w_1) - F_{\overline{W}}(\xi, w_2) - L_\xi[w_1 - w_2] = \int_0^1 \Big(&\nabla_w F_{\overline{W}}(\xi, w_2 + s(w_1 - w_2)) \\ &- \nabla_w F_{\overline{W}}(\xi, 0)[w_1 - w_2]\Big) \, \mathrm{d}s \leq C\|w_1 - w_2\|^2_{C^{1,\beta}(S_\xi)}. \end{aligned} \tag{3.14}$$

Hence, in $B_{\bar{C}\varepsilon^{2s}}(\xi) \subseteq W$ the Lipschitz constant of T_ξ is $C\bar{C}\varepsilon^{2s}$. So choosing first any $\bar{C} \geq 2C$, and then $\varepsilon > 0$ small enough, we find therefore that T_ξ is a contraction in $B_{\bar{C}\varepsilon^{2s}}(\xi)$. As a consequence, there exists $w_\varepsilon : S_\xi \to \mathbb{R}$ in W such that $\|w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \leq \bar{C}\varepsilon^{2s}$ and such that $F_{\overline{W}}(\xi, w_\varepsilon) = (0,0)$.

We also recall that the fixed point w can be proved to be continuous and differentiable with respect to the parameter ξ, (see e.g. [7], Section 2.6). Recall that $w_\varepsilon = w_\varepsilon(\xi)$ solves

$$Vol(\mathbb{B}(\xi, w_\varepsilon)) = \omega_N \quad \text{and} \quad P_{\overline{W}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}) = 0 \qquad \text{for all } \xi \in \mathbb{R}^N.$$

We want next to differentiate the above relations with respect to ξ. For this purpose, it is convenient to fix an index i, and to consider the one-parameter family of centers

$$\xi(t) = (\xi_1, \dots, \xi_i + t, \dots, \xi_N) = \xi + t\mathbf{e}_i. \tag{3.15}$$

Our aim is to understand the variation of $\partial\mathbb{B}(\xi_t, w_\varepsilon(\xi_t))$ normal to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$. The above variation is characterized by a translation in the i-th component and by a variation of w_ε, which is in the radial direction with respect to the center ξ. Therefore, letting ν_{w_ε} denote the unit outer normal vector to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$, the normal variation of $\partial\mathbb{B}(\xi(t), w_\varepsilon(\xi(t)))$ with respect to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ (computed at $t = 0$) is the scalar product between the pointwise shift $\mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i}$ and the unit outer normal vector to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ that is ν_{w_ε}, i.e.

$$\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i}(x - \xi) \cdot \nu_{w_\varepsilon}, \qquad x \in S_\xi. \tag{3.16}$$

Hence we have that

$$\frac{\partial}{\partial \xi_i} Vol(\mathbb{B}(\xi, w_\varepsilon)) = 0 \quad \text{and} \quad P_{\overline{W}}\left(\frac{\partial}{\partial \xi_i} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon(\xi))}\right)\left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i}(x - \xi) \cdot \nu_{w_\varepsilon}\right] = 0.$$

Using (3.7) and Proposition 2.3 one finds from the second equation in the latter formula that $\|v_{i,\varepsilon}\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s+1}$, where $v_{i,\varepsilon} = P_{\overline{W}}\partial_{\xi_i} w_\varepsilon$. Since $\dfrac{\partial w_\varepsilon}{\partial \xi_i} \in W$, it remains to control then the component of $\partial_{\xi_i} w_\varepsilon$ in the orthogonal complement of $\bar{W}$, namely its average.

Let us write

$$\partial_{\xi_i} w_\varepsilon = v_{i,\varepsilon} + c_{i,\varepsilon} \qquad \text{with } c_{i,\varepsilon} \in \mathbb{R}.$$

From a direct computation we have that

$$0 = \frac{\partial}{\partial \xi_i} Vol(\mathbb{B}(\xi, w_\varepsilon)) = \int_{S_\xi} (1 + w_\varepsilon)^{N-1} (v_{i,\varepsilon} + c_{i,\varepsilon})\, d\sigma.$$

Since we know that $\|v_{i,\varepsilon}\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s+1}$, it follows from the latter formula that also $|c_{i,\varepsilon}| \leq C\varepsilon^{2s+1}$. Therefore one deduces

$$\|\partial_{\xi_i} w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s+1}, \tag{3.17}$$

which is the desired conclusion, possibly relabelling the constant C. □

We next show how to find ξ's so that the Lagrange multipliers λ_i in the statement of Proposition 3.3 vanish, thus obtaining surfaces with constant relative fractional mean curvature.

Proposition 3.4. *Let $w_\varepsilon : S_\xi \to \mathbb{R}$ given by Proposition 3.3. Recalling* (2.9)*, for $\xi \in \Theta_\varepsilon$, we define $\Phi_\xi := P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon))$. Then, for $\varepsilon > 0$ sufficiently small, if $\nabla_\xi \Phi_{\xi_{|_{\xi=\bar{\xi}}}} = 0$ for some $\bar{\xi} \in \Theta_\varepsilon$, one has*

$$H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\bar{\xi},w_\varepsilon)} \equiv c,$$

where $c = c(\varepsilon, \bar{\xi})$.

Proof. Recall that $w_\varepsilon = w_\varepsilon(\xi)$ solves

$$Vol(\mathbb{B}(\xi, w_\varepsilon)) = \omega_N \quad \text{and} \quad P_{\overline{W}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}) = 0 \qquad \text{for all } \xi \in \mathbb{R}^N.$$

Since $Vol(\mathbb{B}(\xi, w_\varepsilon)) = \omega_N$ for any choice of ξ, it follows that the integral over $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ of the normal variation vanishes, i.e., recalling (3.16), we have for $\xi = \bar{\xi}$

$$\int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon} = 0, \tag{3.18}$$

where $d\sigma_{w_\varepsilon}$stands for the area element of $\partial\mathbb{B}(\bar{\xi}, w_\varepsilon(\bar{\xi}))$.

For the same reason, recalling (2.7) and (3.15), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi(t), w_\varepsilon(\xi(t)))) = \int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\bar{\xi},w_\varepsilon)} \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon}.$$

By our choice of $\bar{\xi}$ we have that, for all $i = 1, \dots, N$

$$\frac{\partial}{\partial \xi_i}|_{\xi=\bar{\xi}} \Phi_\xi = 0.$$

Recalling also that by Proposition 3.3, $H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)} = c + \sum_{i=1}^N \lambda_i Y_i$ (see Section 2 for the definition of the first-order sphereical harmonics Y_i) , from (3.18) we have that for all $i = 1, \dots, N$

$$0 = \int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} \left(\sum_{j=1}^N \lambda_j Y_j\right) \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon}. \tag{3.19}$$

Notice that by the estimates on w_ε and $\partial_\xi w_\varepsilon$ in Proposition 3.3 one has

$$\int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} Y_j \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon} = \delta_{ij} + o_\varepsilon(1); \qquad i, j = 1, \dots, N.$$

Therefore the system (3.19) implies the vanishing of all λ_j's, which gives the desired conclusion. □

The next step is to show that fractional perimeter of $B_1(\xi)$ is sufficiently close to fractional perimeter of the deformed ball $\mathbb{B}(\xi, w_\varepsilon)$, also when differentiating with respect to ξ.

Proposition 3.5. *Let w_ε be as Proposition 3.4. The following Taylor expansion holds:*

$$P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{4s}). \tag{3.20}$$

Moreover one has

$$\frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = \frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{1+4s}). \tag{3.21}$$

Proof. Thanks to the first statement of Lemma 3.2, following the notation in Section 2, we get that

$$\begin{aligned} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi,w_\varepsilon)) &= P_s^{\Omega_\varepsilon}(B_1(\xi)) + (P_{s,\xi}^{\Omega_\varepsilon})'(0)[w_\varepsilon] + P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi,w_\varepsilon)) - (P_{s,\xi}^{\Omega_\varepsilon})'(0)[w_\varepsilon] - P_s^{\Omega_\varepsilon}(B_1(\xi)) \\ &= P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{4s}) + \int_0^1 \Big((P_{s,\xi}^{\Omega_\varepsilon})'(t\,w_\varepsilon) - (P_{s,\xi}^{\Omega_\varepsilon})'(0)\Big)[w_\varepsilon]\,\mathrm{d}t, \end{aligned} \tag{3.22}$$

where $(P_s^{\Omega_\varepsilon})'$ is defined as in the formula after (2.7).

Using the fact that the nonlocal mean curvature is smooth, we deduce then that

$$\int_0^1 \Big((P_{s,\xi}^{\Omega_\varepsilon})'(t\,w_\varepsilon) - (P_{s,\xi}^{\Omega_\varepsilon})'(0)\Big)[w_\varepsilon]\,\mathrm{d}t = O(\varepsilon^{4s}),$$

so the last two formulas imply (3.20).

To prove (3.21), we use the estimate $\|\partial_\xi w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \le C\varepsilon^{2s+1}$ from Proposition 3.3. Calling τ_i the quantity in (3.16) and recalling the notation from Section 2, we write that

$$\frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi,w_\varepsilon)) = (P_{s,\xi}^{\Omega_\varepsilon})'(w_\varepsilon)[\tau_i].$$

Taylor-expanding the latter quantity we can write that

$$\begin{aligned} \frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi,w_\varepsilon)) &= (P_{s,\xi}^{\Omega_\varepsilon})'(0)[\tau_i] + \frac{1}{2}(P_{s,\xi}^{\Omega_\varepsilon})''(0)[\tau_i] + o(\varepsilon^{1+4s}) \\ &= \frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{1+4s}). \end{aligned} \tag{3.23}$$

This concludes the proof. ☐

Proof of Theorem 1.1. Suppose x_0 is a strict local extremal of V_Ω, without loss of generality a minimum. Then there exists an open set $\Upsilon \subset\subset \Omega$ such that $V_\Omega(x_0) < \inf_{\partial\Upsilon} V_\Omega - \delta$ for some $\delta > 0$. Let Φ_ξ be defined as in Proposition 3.4: by the estimates (3.1) and (3.20) it follows that for every $\bar{x} \in \frac{1}{\varepsilon}\Upsilon$

$$\Phi_{\bar{x}} = P_s^{\mathbb{R}^N}(B_1(\bar{x})) - \omega_N \varepsilon^{2s} V_\Omega(\varepsilon\bar{x}) + O(\varepsilon^{1+2s}). \tag{3.24}$$

Since $P_s^{\mathbb{R}^N}(B_1(\bar{x})) = P_s^{\mathbb{R}^N}(B_1(\frac{x_0}{\varepsilon})$, we get

$$\begin{aligned} \Phi_{\frac{x_0}{\varepsilon}} - \Phi_{\bar{x}} &= \omega_N \varepsilon^{2s}(V_\Omega(\varepsilon\bar{x}) - V_\Omega(x_0)) + O(\varepsilon^{1+2s}) \\ &\ge \omega_N \varepsilon^{2s}(\inf_{\partial\Upsilon} V_\Omega(\varepsilon\bar{x}) - V_\Omega(x_0)) + O(\varepsilon^{1+2s}) \\ &> \delta\omega_N\varepsilon^{2s} + O(\varepsilon^{1+2s}) \ge \delta\omega_N\varepsilon^{2s} + C\varepsilon^{1+2s} > 0 \end{aligned} \tag{3.25}$$

for $\varepsilon < \frac{\delta\omega_N}{C}$ where $C > 0$ is a constant.

Hence, for ε sufficiently small,

$$\Phi_{\frac{x_0}{\varepsilon}} > \sup_{\frac{1}{\varepsilon}\Upsilon} \Phi_\cdot .$$

As a consequence $\Phi_\cdot$ attains a maximum in the dilated domain $\frac{1}{\varepsilon}\Upsilon$, and the conclusion follows from Proposition 3.4.

Suppose now that x_0 is a non-degenerate critical point of V_Ω. From (3.3) and (3.21) one can find an open set $\Upsilon \subset\subset \Omega$ such that

$$\deg\left(\nabla\Phi, \frac{1}{\varepsilon}\Upsilon, 0\right) \neq 0.$$

This implies that Φ_ξ has a critical point in $\frac{1}{\varepsilon}\Upsilon$, and the conclusion again follows from Proposition 3.4.

Since in both cases the set Υ containing x_0 can be taken arbitrarily small, the localization statement in the theorem is also proved. □

Remark 3.6. From [4, Theorem 2.24] one has a relation between the Morse index of a critical point as found in Proposition 3.4 and the Morse index of the corresponding critical point of Φ. In our case, since round spheres are global minimizers for the s-perimeter relative to $\mathbb{R}^N$, these two indices coincide.

To prove Corollary 1.2, we need the following Lemma.

Lemma 3.7. *For all $x \in \partial\Omega$ one has*

$$\lim_{y \to x} V_\Omega(y) = +\infty,$$

and

$$\lim_{\Omega \ni y \to x} \nabla V_\Omega(y) \cdot \nu_{\partial\Omega}(x) = +\infty,$$

where $\nu_{\partial\Omega}$ denotes the outer unit normal to $\partial\Omega$.

Proof. Letting $d := \mathrm{dist}(x, \partial\Omega)$ for all $x \in \Omega$, thanks to the change of variables $x' = \frac{x}{d}$, we get that

$$V_\Omega(x) = \int_{\Omega^C} \frac{1}{|x-y|^{N+2s}}\,\mathrm{d}y = \int_{(\Omega/d)^C} \frac{1}{|dx'-y'|^{N+2s}}\,\mathrm{d}y' \tag{3.26}$$

from which, if $d \to 0$, setting $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x > 0\}$, we have

$$\int_{(\Omega/d)^C} \frac{1}{|dx'-y'|^{N+2s}}\,\mathrm{d}y' \to \int_{(\mathbb{R}^N_+)^C} \frac{1}{|y'|^{N+2s}}\,\mathrm{d}y' < +\infty,$$

i.e. V_Ω behaves asymptotically as d^{-N-2s} when $d \to 0$. With a similar proof, one finds that the component of ∇V_Ω normal to $\partial\Omega$ behaves as $d^{-N-2s-1}$. □

Proof of Corollary 1.2. Given $\delta > 0$ small enough, let us define the set $\Omega^\delta \subseteq \Omega$ by

$$\Omega^\delta = \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}.$$

From Remark 3.7 we have

$$\nabla V_\Omega \cdot \nu_{\partial\Omega^\delta} > 0 \quad \text{on } \partial\Omega^\delta.$$

As in the proof of Theorem 1.1, it turns out that

$$\nabla\Phi. \cdot \nu_{\partial\frac{1}{\varepsilon}\Omega^\delta} > 0 \quad \text{on } \partial\frac{1}{\varepsilon}\Omega^\delta.$$

Clearly, since $\bar{\Omega}$ is compact, the (PS)-condition holds. So the conclusion follows from Theorem 2.6 and Remark 2.7. □

Remark 3.8. It is interesting to see how the geometry of the domain (and not just the topology, as in Corollary 1.2) plays a role in order to obtain either uniqueness of multiplicity of solutions.

In the Appendix we will prove uniqueness for the unit ball B_1, i.e. we will show that V_{B_1} has a unique critical point at the origin which is a non-degenarate minimum.

Secondly, we will give an example of dumble-bell domain, topologically equivalent to a ball, such that the reduced functional Φ_ξ (defined as in Proposition 3.4) has at least three critical points, while Corollary 1.2 would give us only one solution.

4. Proof of Theorem 1.3

Let us consider a bounded open set with smooth boundary $\Omega \subseteq \mathbb{R}^N$, and $s \in (0, 1/2)$.

First of all we point out that, using the direct method of Calculus of Variations and the Sobolev embeddings (which hold for fractional spaces too, see [15]), it is easy to show that there exist minimizers for

$$\{P_s(E, \Omega), |E| = m\} \quad m \in (0, +\infty). \tag{4.1}$$

Our goal is to show that minimizers exist also relatively to half-spaces, and to characterize them to some extent.

Definition 4.1. Let $s \in (0, 1/2)$ and $E \subset \mathbb{R}^N$ be a measurable set. We denote with

$$P_s(E, \mathbb{R}^N_+) := \int_E \int_{\mathbb{R}^N_+ \setminus E} \frac{\mathrm{d}x\, \mathrm{d}y}{|x-y|^{N+2s}}, \tag{4.2}$$

where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$ is the half-space.

We begin by studying minimizers of

$$\{P_s(E, \mathbb{R}^N_+) : \ E \subseteq B_R^+, \ |E| = m\} \quad m \in (0, +\infty), \tag{4.3}$$

with $B_R^+ := B_R \cap \mathbb{R}^N_+$ denoting the half ball of radius $R > 0$ centred at the origin. Without loss of generality we can assume that $m = 1$ and, since we look for minimizers in a half-ball, we can assume that E is closed. With completely similar arguments, one can also prove the following result.

Proposition 4.2. *Problem* (4.3) *admits a minimizer* $E \subseteq B_R^+$.

We have next the following lemma.

Lemma 4.3. *If E is a minimizer for* (4.3)*, then dist* $(E, \{z_N = 0\}) = 0$.

Proof. By contradiction suppose that the minimizer $E \subseteq B_R^+$ does not intersect the plane $\{z_N = 0\}$. Then, if $e := (e_1, \cdots, e_N)$ is the canonical basis of $\mathbb{R}^N$ and $\lambda :=$

$\operatorname{dist}(E,\{z_N = 0\}) > 0$, we consider the shifted set $E-\lambda e_N$. Using the following change of variables (i.e. translating downwards the set E by $\lambda\vec{e}_N$)

$$E \ni x \longmapsto x' = x-\lambda e_N \in E-\lambda e_N,$$
$$\mathbb{R}^N_+ \setminus E \ni y \longmapsto y' = y-\lambda e_N \in \mathbb{R}^N_+ \setminus (E-\lambda e_N),$$

we have

$$\begin{aligned} P_s(E,\mathbb{R}^N_+) &= \int_E \int_{\mathbb{R}^N_+\setminus E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-\lambda e_N - y + \lambda e_N|^{N+2s}} \\ &> \int_{E-\lambda e_N} \int_{\mathbb{R}^N_+\setminus(E-\lambda e_N)} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}} = P_s(E-\lambda e_N,\mathbb{R}^N_+). \end{aligned} \tag{4.4}$$

This is in contradiction to the minimality of E. □

Now we want to show other basic properties of minimizers for (4.3). To see these, we premise a useful

Definition 4.4. Given a function $u:\mathbb{R}^N\to\mathbb{R}^+$, we define $u^*:\mathbb{R}^N\to\mathbb{R}^+$ the radially symmetric rearrangement of u with respect to x_N so that, given $x_N>0$, $t>0$, the superlevel set $\{u^*(\cdot,x_N)>t\}$ is a ball B in $\mathbb{R}^{N-1}$ centered at the origin and

$$|\{u^*(\cdot,x_N)>t\}| = |\{u(\cdot,x_N)>t\}|,$$

see Figure 1.

If $u=\chi_E$, we call E^* the ball such that $\chi_{E^*}=(\chi_E)^*$.

(a) Level set of u. **(b)** Level set of u^*.

Figure 1. The radially symmetric rearrangement of u.

Definition 4.5. Given a function $u:\mathbb{R}^N\to\mathbb{R}^+$, we define $\hat{u}:\mathbb{R}^N\to\mathbb{R}^+$ to be the decreasing rearrangement of u with respect to x_N: given $x'>0$, $t>0$, $\{x_N : \hat{u}(x',x_N)>t\}\subseteq\mathbb{R}^+$ is a segment of the form $[0,\alpha)$ with $\alpha := |\{x_N : \hat{u}(x',x_N)>t\}|$, as in Figure 2.

If $u=\chi_E$, we call $\hat{E}$ the set such that $\chi_{\hat{E}} = (\widehat{\chi_E})$. Notice that $\partial\hat{E}$ is a graph in the direction $\vec{e}_N$.

With these definitions at hand, we can show a first property of minimizers of (4.3):

Lemma 4.6. *If E is a minimizer of* (4.3)*, we have that*

$$P_s(E^*,\mathbb{R}^N_+) \le P_s(E,\mathbb{R}^N_+)$$

and the equality holds if and only if $E=E^*$.

(a) Level set of u. **(b)** Level set of $\hat{u}$.

Figure 2. The decreasing rearrangement of u.

Proof. Proceeding as in [34], we define

$$\mathcal{H}^s(\mathbb{R}^N_+) := \{u \in L^2(\mathbb{R}^N_+) : [u]_{\mathcal{H}^s(\mathbb{R}^N_+)} < +\infty\},$$

where

$$[u]^2_{\mathcal{H}^s(\mathbb{R}^N_+)} := \inf\left\{ \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla v|^2 + |\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y : v \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+ \times \mathbb{R}^+), v(\cdot,0) = u(\cdot)\right\}. \tag{4.5}$$

The space $\mathcal{H}^s(\mathbb{R}^N_+)$ is endowed with the Hilbert norm

$$\|u\|^2_{\mathcal{H}^s(\mathbb{R}^N_+)} = \|u\|^2_{L^2(\mathbb{R}^N_+)} + [u]^2_{\mathcal{H}^s(\mathbb{R}^N_+)}.$$

According to (4.5) we get

$$P_s(E,\mathbb{R}^N_+) = \frac{1}{2}\inf\left\{ \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v|^2+|\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y : v \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+\times\mathbb{R}^+), v(\cdot,0) = \chi_E(\cdot)\right\}, \tag{4.6}$$

and we define

$$H^1(\mathbb{R}^N_+\times\mathbb{R}^+, y^{1-2s}\,\mathrm{d}y) := \left\{v \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+\times\mathbb{R}^+) : \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|v|^2+|\nabla_x v|^2+|\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y < \infty\right\}.$$

For all $v \in H^1(\mathbb{R}^N_+ \times \mathbb{R}^+, y^{1-2s}\,\mathrm{d}y)$, we set $v^*(\cdot, y) = [v(\cdot, y)]^*$. Then

a) since the symmetrization preserves characteristic functions, we have that

$$(\chi_E(\cdot))^* = \chi_{E^*}(\cdot); \tag{4.7}$$

b) from [8, Theorem 1] we get that

$$\int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v^*|^2 + |\partial_y v^*|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v|^2 + |\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.8}$$

Hence combining (4.6), (4.7) and (4.8) we deduce the desired conclusion. □

In a similar way we obtain the following

Lemma 4.7. *Let E be a minimizer of* (4.3). *Then*

$$P_s(\hat{E}, \mathbb{R}^N_+) \le P_s(E, \mathbb{R}^N_+)$$

and the equality holds if and only if $E = \hat{E}$.

Proof. Proceeding as in Lemma 4.6 and denoting with $\hat{v}(\cdot, y) = [v(\hat{\cdot}, y)]$, we have that

$$(\hat{\chi_E(\cdot)}) = \chi_{\hat{E}}(\cdot), \tag{4.9}$$

and from [8, Theorem 1] we get

$$\int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x \hat{v}|^2 + |\partial_y \hat{v}|^2) y^{1-2s}\, \mathrm{d}x\, \mathrm{d}y \le \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v|^2 + |\partial_y v|^2) y^{1-2s}\, \mathrm{d}x\, \mathrm{d}y. \tag{4.10}$$

Recalling (4.6) and using (4.9) and (4.10) we conclude the proof. □

Remark 4.8. Note that from these two symmetrizations we obtain a connected minimizer for (1.3).

We next prove an estimate on the diameter of a set minimizing (4.3):

Theorem 4.9. *If E is a minimizer of* (4.3)*, then*

$$|diam\ E| \le \frac{2\sqrt{2}C_0}{r_0^{N-1}}, \tag{4.11}$$

where $diam\ E$ denotes the diameter of the set E and both $C_0 > 0$ and $r_0 > 0$ come from [28, Theorem 1.7].

Proof. Thanks to Lemma 4.6 and Lemma 4.7, we can suppose that there exists $H > 0$ such that

$$[0, He_N] \subseteq\ E \tag{4.12}$$

and that, for all $t > 0$,

$$E_t = E \cap \{x_N = t\} = B_{R(t)}. \tag{4.13}$$

We consider the interval $[0, He_N]$ and we divide it in M subintervals of length at most $2r_0$, where $r_0 > 0$ comes from [28, Theorem 1.7] and $M = [\frac{H}{2r_0}] + 1$. For every subinterval we take its center x^i where $i = 1, \cdots, M$. From [28, Theorem 1.7], for every x^i, there exists $C_0 > 0$, a ball $B_{r_0}(x^i)$ with center at x^i and radius r_0 such that

$$|E \cap B_{r_0}(x^i)| \ge \frac{r_0^N}{C_0} > 0 \quad \text{for all } i = 1, \cdots, M.$$

Thus

$$1 = |E| \ge \left|\frac{H}{2r_0}\right| \cdot \frac{r_0^N}{C_0}.$$

and hence

$$|H| \le \frac{2C_0}{r_0^{N-1}}. \tag{4.14}$$

We proceed similarly to estimate $R(t)$ for all $t > 0$, obtaining that

$$|R(t)| \le \frac{2C_0}{r_0^{N-1}} \qquad \text{for all } t > 0. \tag{4.15}$$

Combinig (4.14) and (4.15), we deduce the thesis. □

As a corollary we get that a minimizer for (4.3) is a minimizer for (1.3):

Corollary 4.10. *Let E be a minimizer of* (4.3)*. If $R > \frac{2\sqrt{2}C_0}{r_0^{N-1}}$ (where C_0, $r_0 > 0$ comes from* [28, Theorem 1.7]*) it is a free minimizer, i.e.*

$$E \subset B_R.$$

Finally we prove that

Proposition 4.11. *Let E be a minimizer of* (4.3)*. Then ∂E is C^∞.*

Proof. From Lemma 4.7 we know that ∂E is graph along the direction x_N. Then [6, Corollary 3] implies that ∂E is C^∞ outside a closed singular set of Hausdorff dimension $N-8$.

Moreover, since by Lemma 4.6, E is also radially decreasing and symmetric, the singular set has to be its highest point (in the x_N direction of E). Now we consider a blow up of E centered at the singular point and we obtain a singular, symmetrical cone C. By densities estimates (see [28, Theorem 1.7]) which hold for E , we get that $C \neq \varnothing$. Hence C is a lipschitz cone and [19, Theorem 1] tells us that C is a halfspace. As a consequence ∂E is C^∞. □

Proof of Theorem 1.3. From Proposition 4.2 and Corollary 4.10 we have the existence of a minimizer for (1.3). Moreover, thanks to Lemma 4.6, Lemma 4.7, Proposition 4.11 and Lemma 4.3, we deduce the minimizer's properties. □

Remark 4.12. It would be interesting to know whether minimizers, or even critical points, of the functional in (1.3) are unique up to horizontal translations (see for instance [23–25] for similar uniqueness results).

5. Appendix

We prove in this appendix the assertions in Remark 3.8.

Lemma 5.1. *If B_1 is the unit ball of $\mathbb{R}^N$, then $0 \in B_1$ is a non-degenerate global minimum of V_{B_1} and it is the unique critical point.*

Proof. First of all we note that V_{B_1} is a radial function, i.e. $V_{B_1}(x) = v_{B_1}(|x|)$. Hence, since V_{B_1} is smooth in the interior of the ball, it follows that $v'_{B_1}(0) = 0$. It is easily seen that

$$(\Delta V_{B_1})(0) = 2(1+s)(N+2s)\int_{B_1^C} \frac{1}{|y|^{N+2s+2}}\,\mathrm{d}y > 0,$$

where B_1^C denotes the complement of B_1. Therefore, since $v''_{B_1}(0) = \frac{1}{n}\Delta V_{B_1}(0)$, it follows that for fixed $\delta > 0$ one has $v''_{B_1}(t) > 0$ for $t \in [0,\delta]$, which implies the non-degeneracy of the origin as a critical point of V_{B_1}.

It remains to show the monotonicity of v_{B_1} in the whole interval $(0,1)$, but since Lemma 3.7 holds, it is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{B_1}(t\vec{e}_1) \neq 0 \quad \text{for } t \in [\delta, 1-\delta]. \tag{5.1}$$

Recalling the definition (3.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{B_1}(t\vec{e}_1) = \tilde{c}_{N,s}\int_{B_1^C} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}}\,\mathrm{d}y, \tag{5.2}$$

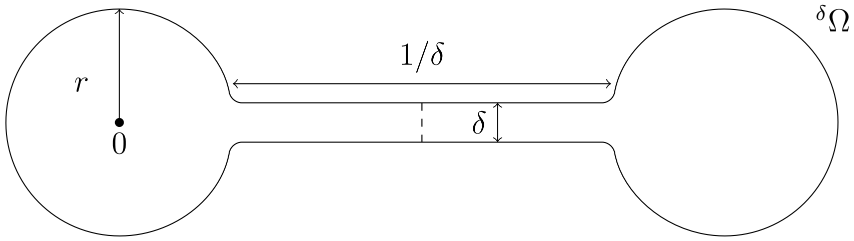

Figure 3. A dumb-bell domain $^\delta\Omega$.

where $\tilde{c}_{N,s}$ is a constant depending only on N and s, $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}$.

By Fubini's Theorem

$$\int_{B_1^C} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \, dy = \int_{\mathbb{R}^{N-1}} dy' \int_{\{y_1 : (y_1, y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \, dy. \tag{5.3}$$

Since $(y_1, y') \in B_1^c \times \mathbb{R}^{N-1}$, we have two cases:

1) if $|y'| \geq 1 \quad \Rightarrow \quad y_1 \in \mathbb{R}$;

2) if $|y'| < 1 \quad \Rightarrow \quad y_1 \leq -\sqrt{1 - |y'|^2} \ \vee \ y_1 \geq \sqrt{1 - |y'|^2}$.

In the first case we obtain by oddness

$$\int_{\{y_1 : (y_1, y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \, dy = \int_{\{y_1 \in \mathbb{R}\}} \frac{y_1 - t}{((y_1 - t)^2 + |y'|^2)^{(N+2s+2)/2}} \, dy = 0. \tag{5.4}$$

In the second case, using the changes of variables $y_1 - t = s$ and $z = t - y_1$, we get

$$\begin{aligned} &\int_{\{y_1 : (y_1, y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \, dy \\ &= \int_{\{y_1 \leq -\sqrt{1 - |y'|^2}\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \, dy + \int_{\{y_1 \geq \sqrt{1 - |y'|^2}\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \, dy \\ &= \int_{\{z \geq t + \sqrt{1 - |y'|^2}\}} \frac{z}{(z^2 + |y'|^2)^{(N+2s+2)/2}} \, dz \\ &+ \int_{\{s \geq \sqrt{1 - |y'|^2} - t\}} \frac{s}{(s^2 + |y'|^2)^{(N+2s+2)/2}} \, dy > 0, \end{aligned} \tag{5.5}$$

since $\{z : z \geq t + \sqrt{1 - |y'|^2}\} \subseteq \{z : z \geq \sqrt{1 - |y'|^2} - t\}$ and since the first integral is negative.

Putting together (5.2), (5.3), (5.4) and (5.5) we obtain (5.1) which concludes the proof. □

Lemma 5.2. *Let Φ_ξ be defined as in Proposition 3.4. There exist a dumble-bell domain (as in Figure 3) with the same topology of the ball, such that Φ_ξ has at least three critical points.*

Sketch of the Proof. We consider a sequence of domains $^\delta\Omega$ as in Figure 3. Fixed $r \in (0, 1)$, it is easy to see that

$$V_{^\delta\Omega} \to V_{B_1} \quad \text{in } C^2(B_r(0)) \quad \text{as } \delta \to 0. \tag{5.6}$$

For δ small, by Lemma 5.1, we get that $V_{^\delta\Omega}$ has a unique non-degenerate minimum x_1 in $B_{r/2}(0)$ and there exists $\gamma > 0$ such that

$$\inf_{\partial B_r(0)} V_{^\delta\Omega} > \sup_{B_{r/2}(0)} V_{^\delta\Omega} + \gamma.$$

By symmetry, we have a non-degenerate minimum point x_2 in the other ball with the same properties. Recall also that from Lemma 3.7 that if $x \in \partial(^\delta\Omega)$, it holds

$$\lim_{^\delta\Omega \ni y \to x} V_{^\delta\Omega}(y) = +\infty.$$

Hence, from (3.24) (with a similar formula for the gradient in ξ) and the above observations, there exists a critical point of Φ other that x_1 and x_2, by Mountain Pass Theorem. □

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