Euler's elastica functional as a large mass limit of a two-dimensional non-local isoperimetric problem

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Abstract

We consider a large mass limit of the non-local isoperimetric problem with a repulsive Yukawa potential in two space dimensions. In this limit, the non-local term concentrates on the boundary, resulting in the existence of a critical regime in which the perimeter and the non-local terms cancel each other out to leading order. We show that under appropriate scaling assumptions the next-order Γ-limit of the energy with respect to the L^1 convergence of the rescaled sets is given by a weighted sum of the perimeter and Euler's elastica functional, where the latter is understood via the lower-semicontinuous relaxation and is evaluated on the system of boundary curves. As a consequence, we prove that in the considered regime the energy minimizers always exist and converge to either disks or annuli, depending on the relative strength of the elastica term.

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1 Introduction

In many physical systems the onset of spatial pattern formation is driven by a competition of short-range attractive and long-range repulsive foces [20, 39, 44]. In binary systems, this is often captured by a prototypical model in which the short-range attractive interactions between the two phases are modeled by an interfacial energy term, while the long-range repulsion is due to a two-body interaction through a positive kernel:

$$E(\Omega) = P(\Omega) + \frac{1}{2}\int_\Omega \int_\Omega K(x-y)\,\mathrm{d}^n y\,\mathrm{d}^n x. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is the spatial domain occupied by the minority phase whose "mass" $|\Omega| = m > 0$ is fixed, $P(\Omega)$ is the perimeter [1] of Ω, and $K : \mathbb{R}^n \to [0, \infty)$ is a suitable kernel. A typical example is given by the Coulombic kernel $K(x) = \frac{1}{4\pi|x|}$ in three space dimensions, giving rise to the celebrated Gamow's model of the atomic nucleus [11, 18]. However, many other kernels may be considered, notably the regularized dipolar kernel $K(x) \sim \frac{1}{|x|^3}$ in two dimensions that arises in the context of magnetic domains, ferrofluids and Langmuir monolayers [4, 5, 32, 33, 44]. Yet another variant is obtained by considering a Yukawa potential $K = K_\alpha$ in the plane:

$$K_\alpha(x) = \frac{e^{-\alpha|x|}}{2\pi|x|} \qquad x \in \mathbb{R}^2, \tag{1.2}$$

where $\alpha > 0$ is a screening parameter, which naturally arises in the studies of Langmuir monolayers in the presence of weak ionic solutions (see Appendix A).

The behavior of the minimizers of the non-local isoperimetric problem governed by (1.1) depends rather crucially on the rate of decay of the kernel $K(x)$ as $|x| \to \infty$. In particular, there is a notable difference for large masses: In the case of the three-dimensional Coulombic kernel and two- or three-dimensional domains Ω, minimizers fail to exist beyond a certain critical mass [16, 25, 26, 30], while for a *screened* three-dimensional Coulombic kernel represented by the Yukawa potential the minimizers do exist for all $\alpha > \alpha_c$ for some explicit $\alpha_c = \alpha_c(n) > 0$, provided that $m \geq m_c$ for some $m_c = m_c(\alpha, n) > 0$, see Pegon [41]. Moreover, in two dimensions the minimizers for sufficiently large values of α and all large enough masses are known to be disks [35], something that in the absence of screening ($\alpha = 0$) is known to occur at small masses $m \ll 1$ instead [15, 24, 25, 26]. Thus, one can imagine that for a given $m \gg 1$ a transition occurs at some threshold value of $\alpha > 0$ that may lead to the onset of minimizers which are no longer necessarily disks as the value of α is lowered.

Our work attempts to look into the transition that bridges the gap between the two regimes described above in two space dimensions. We focus on the parameters for which the non-local term at large masses cancels the interfacial energy term of the energy (1.1) to the leading order. It turns out that to next order in the asymptotic expansion of the energy as $m \to \infty$, this yields Euler's elastica functional plus a term proportional to the perimeter:

$$E_0(\Omega) = \int_{\partial\Omega} \left(\sigma + \frac{\pi}{2}\kappa^2\right) \mathrm{d}\mathcal{H}^1. \tag{1.3}$$

Here, κ is the curvature of $\partial\Omega$ and $\sigma > 0$. More precisely, we will show that a relaxed version of the energy in (1.3) can be obtained as the Γ-limit of a suitably rescaled energy in (1.1) with the kernel from (1.2), as the value of α approaches the critical value $\alpha_c = \frac{1}{\sqrt{2\pi}}$ with the right rate (see the following section for the precise statement). As a consequence, we can conclude that the minimizers of the energy in (1.1) in the considered limit change from disks to annuli as the parameter of the asymptotic expansion is varied. Note that annular domains are frequently observed in the experiments on lipid monolayers [34].

Similar regimes may be studied when instead of screened Coulombic repulsion one considers regularized and renormalized dipolar repulsion for $n = 2$. In this setting, Muratov and Simon proved that in regimes where perimeter asymptotically still carries a cost, minimizers are disks even for finite regularization lengths [40]. They also identified the next-order limit in the case of vanishing cost of the perimeter, which by a result of Cesaroni and Novaga [10] coincides with the second-order expansion of the fractional perimeters close to the local one. Muratov and Simon also proved existence of non-spherical minimizers for a modified, yet still isotropic kernel [40]. Closely related results were obtained for a class of general kernels in the regime of large mass by Pegon [41], Merlet and Pegon [35], and Goldman, Merlet and Pegon [21], as well as by Knüpfer and Shi [28] in the case of a torus.

We note that Euler's elastica energy is a classical problem in the calculus of variations, which was first analyzed by Euler in 1744 for $\sigma = 0$, after Daniel Bernoulli proposed the energy to him in a letter [14]. While the original motivation was to study thin elastic rods, it has since also appeared in image segmentation problems, see for example Mumford [38]. Its higher-dimensional analog, the Willmore energy, which asks to minimize the L^2-norm of the mean curvature of a hypersurface and, more generally, the Helfrich energy, appear in a variety of fields from differential geometry to the modeling of cell membranes in biology, see for example Willmore [46] and Helfrich [23]. We will require the elastica energy in its relaxed form (with respect to the L^1 topology of the enclosed sets). It has been characterized by Bellettini and Mugnai [8, 9], see also Bellettini, Dal Maso, and Paolini [6]. Its minimizers have been identified by Goldman, Novaga, and Röger [22], even after augmentation by a non-local term as in (1.1). To the best of our knowledge, they were also the first to include curvature-depending terms in the context of non-local isoperimetric problems.

Finally, we remark on results regarding the passage from first-order variational problems to second-order problems. The most prominent body of literature certainly pertains to the rigorous derivation of bending energies from non-linear elasticity with its many contributions being thoroughly outside the scope of this introduction. We thus only mention the seminal paper by Friesecke, James, and Müller [17], which serves as the foundation for virtually all contributions following it. Indeed, it is also where our argument takes part of its inspiration. On the other hand, the question of this type was posed by De Giorgi in the context of phase field models of phase transitions [12]. While the original conjecture from [12] was shown not to lead to an energy of the form of (1.3) [7] (compare with [43]), a natural alternative would be provided by the diffuse interface version of the energy in (1.1) in two space dimensions:

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 + \frac{9}{32}(1-u^2)^2 \right) \mathrm{d}^2 x + \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_\alpha(x-y)u(x)u(y)\,\mathrm{d}^2x\,\mathrm{d}^2y, \quad (1.4)$$

with the kernel K_α from (1.2) and the mass constraint

$$\int_{\mathbb{R}^2} u\,\mathrm{d}^2 x = m. \quad (1.5)$$

Here the choice of the double-well potential ensures that the surface energy associated with the optimal transition layer connecting $u = 0$ and $u = 1$ is equal to unity, hence, yielding the perimeter functional as the Γ-limit of the first term in (1.4) in the limit $m \to \infty$ after rescaling lengths by $m^{1/2}$ [37]. We thus would expect that the limit behavior of the energy in (1.4) would be the same as that of (1.1), yielding an example of a second-order variational problem arising from phase field models of phase transitions.

This paper is organized as follows. In Sec. 2, we give the precise formulation of the problem under consideration and its limit, and present the precise statement of the obtained results, followed by an outline of the proof. In Sec. 3, we prove existence of minimizers in the considered regime and derive the representation of the non-local energy term used throughout the rest of the paper. Then, in Sec. 4 we establish compactness of boundary curves in the considered limit and in Sec. 5 we prove Γ-convergence. We also provide the details of model derivation in the appendix.

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2 Main results

For the screening parameter $\alpha > 0$ and mass $m > 0$, we study the non-local isoperimetric problem whose kernel is given by the Yukawa potential (1.2). Up to a mass-dependent additive constant, the energy (1.1) is then given by

$$E_\alpha(\Omega) := P(\Omega) - \frac{1}{4\pi} \int_\Omega \int_{\Omega^c} \frac{e^{-\alpha|x-y|}}{|x-y|} \, \mathrm{d}^2 y \, \mathrm{d}^2 x, \tag{2.1}$$

on the admissible class

$$\mathcal{A}_m := \{\Omega \subset \mathbb{R}^2 : \ |\Omega| = m, \ P(\Omega) < \infty\}. \tag{2.2}$$

A direct calculation shows that for $\lambda > 0$ we have

$$E_\alpha(\lambda\Omega) = \lambda \left(P(\Omega) - \frac{\lambda^2}{4\pi} \int_\Omega \int_{\Omega^c} \frac{e^{-\lambda\alpha|x-y|}}{|x-y|} \, \mathrm{d}^2 y \, \mathrm{d}^2 x \right).$$

We may therefore instead analyze the energy

$$F_{\lambda,\alpha}(\Omega) := P(\Omega) - \frac{\lambda^2}{4\pi} \int_\Omega \int_{\Omega^c} \frac{e^{-\lambda\alpha|x-y|}}{|x-y|} \, \mathrm{d}^2 y \, \mathrm{d}^2 x \tag{2.3}$$

on the admissible class $\mathcal{A}_\pi$, where $m = \lambda^2 \pi$ gives the relation between λ and m. In particular, studying the $\lambda \to \infty$ limit of the energy in (2.3) over the admissible class $\mathcal{A}_\pi$ is equivalent to studying the limit of $m \to \infty$ of the energy in (2.1) over the admissible class $\mathcal{A}_m$.

The existence of a subcritical regime of screening parameters for this energy is already established by the general results of Pégon and collaborators [21, 35, 41]. Indeed, their result gives the following:

Theorem 2.1 (Merlet, Pégon [35])**.** *For $\alpha > \frac{1}{\sqrt{2\pi}}$, the L^1 Γ-limit of $F_{\lambda_n,\alpha}$ as $\lambda_n \to \infty$ is given by*

$$G_\alpha(\Omega) = \left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega). \tag{2.4}$$

Furthermore, there exists $\lambda_\alpha > 0$ such that for all $\lambda > \lambda_\alpha$ all minimizers of $F_{\lambda,\alpha}$ are, up to translation, given by the disk $B_1(0)$.

By an L^1 Γ-limit, we mean the limit with respect to convergence of the characteristic functions $\chi_{\Omega_n} \to \chi_\Omega$ in $L^1(\mathbb{R}^2)$ of measurable sets $\Omega_n \subset \mathbb{R}^2$ to that of the limiting set $\Omega \subset \mathbb{R}^2$ as $n \to \infty$. Similar results have been obtained by Muratov and Simon for a non-local isoperimetric problem with dipolar repulsion [40].

In this paper, we will instead investigate the large mass behaviour near the *critical* screening length $\alpha = \frac{1}{\sqrt{2\pi}}$ via a Γ-convergence analysis. First, we note that in this regime minimizers always exist and are sufficiently regular.

Proposition 2.2. *Let $\lambda > 0$ and $\alpha > \frac{1}{\sqrt{2\pi}}$. Then a minimizer of $F_{\lambda,\alpha}$ over $\mathcal{A}_\pi$ exists. Furthermore, all minimizers are bounded, connected, open sets with boundary of class $C^{2,\alpha}$ for any $\alpha \in (0,1)$ and have finitely many holes.*

As the next step, we observe that for fixed and sufficiently regular sets the energy has an expansion in terms of the perimeter and the squared L^2-norm of the curvature of the boundary, i.e., the elastica energy. Throughout the rest of the paper, we call a set *regular*, if it is a bounded open set with the boundary of class C^∞. As the energy $F_{\lambda,\alpha}(\Omega)$ of any admissible set $\Omega \in \mathcal{A}_\pi$ may be approximated by that of a regular set, restricting our attention to regular sets will suffice for our purposes.

Proposition 2.3. *Let Ω be a regular set. Then as $\lambda \to \infty$ we have*

$$F_{\lambda,\alpha}(\Omega) = \left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) + \frac{1}{8\pi\alpha^4\lambda^2} \int_{\partial\Omega} \kappa^2 \, \mathrm{d}\mathcal{H}^1 + o\left(\lambda^{-2}\right).$$

We can thus indeed hope to obtain the combination of the perimeter and the elastica energy as a large-mass Γ-limit of the functionals (2.3) in the critical regime $\alpha = \frac{1}{\sqrt{2\pi}}$. However, note that the integral of the curvature squared is ill-behaved on its own, since

$$\int_{\partial\left(B_{\sqrt{1+r^2}}(0)\setminus B_r(0)\right)} \kappa^2 \, \mathrm{d}\mathcal{H}^1 \sim \frac{1}{r} \to 0 \tag{2.5}$$

as $r \to \infty$. Therefore, we will need to retain control over the perimeter in order to obtain a reasonable Γ-limit. To this end, we will consider sequences of screening parameters which approach the critical parameter from above as $\lambda \to \infty$ with an appropriate rate.

Theorem 2.4. *Let $\lambda_n \to \infty$ and $\alpha_n > \frac{1}{\sqrt{2\pi}}$ be sequences such that $\sigma_n := \lambda_n^2 \left(1 - \frac{1}{2\pi\alpha_n^2}\right)$ satisfies*

$$\lim_{n\to\infty} \sigma_n = \sigma > 0. \tag{2.6}$$

Then, the L^1 Γ-limit of $\lambda_n^2 F_{\lambda_n,\alpha_n}$ as $n \to \infty$ is given by

$$F_{\infty,\sigma} := \operatorname{rel} \tilde{F}_{\infty,\sigma}, \tag{2.7}$$

where for a regular set $\Omega \in \mathcal{A}_\pi$ we define

$$\tilde{F}_{\infty,\sigma}(\Omega) := \sigma P(\Omega) + \frac{\pi}{2} \int_{\partial\Omega} \kappa^2 \, \mathrm{d}\mathcal{H}^1, \tag{2.8}$$

and the relaxation is with respect to the L^1-convergence of the characteristic functions.

Since finite energy sequences might break up into multiple pieces which drift infinitely far apart, we have not included a compactness statement here. As minimizers must be connected due to the non-local kernel being repulsive, we do get convergence of minimizers up to translations. The characterization of the minimizers used here is due to Goldman, Novaga, and Röger [22], to which we also refer for more precise descriptions of the minimizers.

Corollary 2.5. *Under the assumptions of Theorem 2.4, minimizers $\Omega_n \subset \mathcal{A}_\pi$ of F_{λ_n,α_n} exist and after suitable translations and along a subsequence converge in the L^1-topology to a minimizer Ω_∞ of $F_{\infty,\sigma}$. In particular, there exists $\bar{\sigma} > 0$ such that for $\sigma > \bar{\sigma}$ we have $\Omega_\infty = B_1(0)$, while for $\sigma < \bar{\sigma}$ there exists $r_\sigma > 0$ such that $\Omega_\infty = B_{\sqrt{1+r_\sigma^2}}(0) \setminus B_{r_\sigma}(x)$ with $x \in \mathbb{R}^2$ such that $|x| \le \sqrt{1+r_\sigma^2} - r_\sigma$. For $\sigma = \bar{\sigma}$, both cases may occur, with $r_{\bar{\sigma}} > 0$.*

The values of $\bar{\sigma}$ and $r_{\bar{\sigma}}$ may be found explicitly as solutions of an algebraic system of equations. Numerically, we have $\bar{\sigma} \approx 0.112736$ and $r_{\bar{\sigma}} \approx 3.66882$.

The inner ball of Ω_∞ in the case $\sigma \le \bar{\sigma}$ in Corollary 2.5 need not be concentric with the outer ball due to locality of $F_{\infty,\sigma}$. We conjecture that this will not actually occur in the limits of Ω_n as $n \to \infty$. Indeed, the following is expected to hold:

Conjecture 2.6. *Under the assumptions of Theorem 2.4, there exists $\bar{n} > 0$ such that for $n > \bar{n}$ minimizers Ω_n of F_{λ_n,σ_n} are given, up to translation, by $B_1(0)$ if $\sigma > \bar{\sigma}$ and by $B_{\sqrt{1+r_{\sigma,n}^2}}(0) \setminus B_{r_{\sigma,n}}(0)$ for some $r_{\sigma,n} > 0$ converging to r_σ as $n \to \infty$ if $\sigma < \bar{\sigma}$.*

This conjecture is supported by the fact that the minimum of $F_{\lambda,\alpha}$ among all sets of the form $\Omega_x := B_{\sqrt{1+r^2}}(0) \backslash B_r(x) \in \mathcal{A}_\pi$ with $r > 0$ and $x \in \overline{B_{\sqrt{1+r^2}-r}(0)}$ is uniquely attained for $x = 0$ for any $\lambda > 0$ and $\alpha > 0$, see Proposition 2.7 below. Proving this conjecture, however, would require to go to higher orders in the expansion of the energy and, in particular, to keep track of the exponentially small terms arising from the non-local interactions between the inner and the outer boundaries of the minimizers, as well as understanding the asymptotic rigidity of concentric annuli with respect to the energy $F_{\lambda,\alpha}$. Such an analysis goes well beyond the scope of the present paper.

Proposition 2.7. *Let $K : (0,\infty) \to (0,\infty)$ be monotone decreasing such that $r \mapsto rK(r)$ is integrable. For $r > 0$ and $x \in \overline{B_{\sqrt{1+r^2}-r}(0)}$, let $\Omega_x := B_{\sqrt{1+r^2}}(0) \backslash B_r(x)$. Then Ω_0 minimizes*

$$\begin{aligned} f(x) &:= \int_{\Omega_x} \int_{\Omega_x} K(|y-z|) \, \mathrm{d}^2 y \, \mathrm{d}^2 z \\ &= -\int_{\Omega_x} \int_{\Omega_x^c} K(|y-z|) \, \mathrm{d}^2 y \, \mathrm{d}^2 z + 2\pi^2 \int_0^\infty rK(r) \, \mathrm{d}r. \end{aligned} \tag{2.9}$$

among all points $x \in \overline{B_{\sqrt{1+r^2}-r}(0)}$. Additionally, if K is strictly monotone decreasing, then Ω_0 is the unique minimizer.

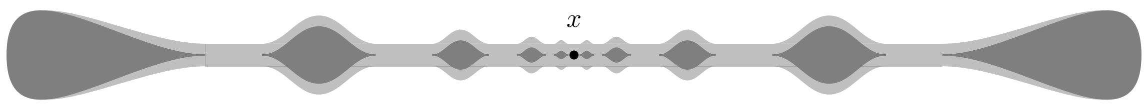

Figure 1: Sketch of a sequence of regular sets (light gray) with finite elastica energy and its limit (dark gray). The singular part of the boundary of the limit has an accumulation point at x.

We finally comment on the issue of the relaxation contained in $F_{\infty,\sigma}$, which is surprisingly subtle and technically challenging. This is due to the L^1-topology controlling the set while the elastica energy controls the boundary. Passing from one to the other is particularly tricky when parts of the boundaries of sequences of sets collapse, in the limit resulting in singular points of the boundary. The singular set may have accumulation points, see Figure 1, and may even have positive $\mathcal{H}^1$ measure, see [6, Example 1].

For curves in two dimensions, the relaxation has been identified by Bellettini and Mugnai [9] building on ideas by Bellettini, Dal Maso, and Paolini [6], which we present below after introducing the necessary notation. For the also physically relevant case of surfaces in three dimensions, the relaxation of the Willmore (or Helfrich) energy is not yet known. Therefore, a three-dimensional analysis of the energy (2.3) currently seems to be out of reach.

Before we describe how to pass from sets to boundaries in a suitable manner, we start with collecting standard notions for single curves.

Definition 2.8. *We will consider regular closed curves $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ to be parametrized by $t \in [0,1]$. The corresponding Sobolev space is*

$$H^2(\mathbb{S}^1;\mathbb{R}^2) := \left\{\gamma \in H^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2) : \gamma(t+1) = \gamma(t) \quad \forall t \in \mathbb{R}\right\}, \tag{2.10}$$

and throughout the paper we will only refer to continuous representatives. A curve $\gamma \in H^2(\mathbb{S}^1;\mathbb{R}^2)$ is called regular if $\gamma'(t) \neq 0$ for any $t \in [0,1]$. The length of such a curve is

$$L(\gamma) := \int_0^1 |\gamma'| \,\mathrm{d}t. \tag{2.11}$$

We abbreviate the image of the curve as $\Gamma := \gamma([0,1])$. For $x \in \mathbb{R}^2 \setminus \Gamma$, we define the winding number of γ around x as

$$\mathcal{I}(\gamma, x) := \frac{1}{2\pi} \int_0^1 \frac{(\gamma(t) - x)^\perp \cdot \gamma'(t)}{|\gamma(t) - x|^2} \,\mathrm{d}t, \tag{2.12}$$

where $y^\perp := (-y_2, y_1)$ for every $y = (y_1, y_2) \in \mathbb{R}^2$. We will say that a regular curve is parametrized by constant speed if for all $t \in [0,1]$ we have

$$|\gamma'(t)| = L(\gamma). \tag{2.13}$$

One can readily check that $\mathcal{I}(\gamma, 0) = 1$ for $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$, which is a constant speed parametrization for $t \in [0,1]$.

It is instructive to consider the passage from a bounded set $\Omega \subset \mathbb{R}^2$ with smooth boundary to its boundary curves in detail. Of course, the boundary $\partial\Omega$ of such a set can always be decomposed into the union of the images of a finite collection of smooth, disjoint Jordan curves $\{\gamma_i\}_{i=1}^N \subset C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ for some $N \in \mathbb{N}$, i.e., smooth, closed curves γ_i parametrized by $t \in [0,1]$ without self-intersections. Such a decomposition in the case of regular sets is classical and can be found, e.g., in the appendix of Milnor's book on differential topology [36]. See also Ambrosio *et al.* for the corresponding and much deeper result on sets of finite perimeter in the plane [2]. Throughout the paper, we always order such curves by decreasing length. With this notation, we have $\partial\Omega = \bigcup_{i=1}^N \Gamma_i$, where $\Gamma_i = \gamma_i([0,1])$. Furthermore, we always orient the curves such that at every point of the boundary, the *outward* normal ν_i is given by

$$\nu_i := -\tau_i^\perp, \tag{2.14}$$

where

$$\tau_i := \frac{\gamma_i'}{|\gamma_i'|} \tag{2.15}$$

is the unit tangent along the curve. This way, the curvature κ of $\partial\Omega$ (positive if Ω is convex) and the curvature $\kappa_i : \mathbb{S}^1 \to \mathbb{R}$ of the boundary curve γ_i coincide in the sense that $\kappa(\gamma_i(t)) = \kappa_i(t)$ for all $t \in [0,1]$. For constant speed curves we have the identities

$$\nu_i' = \kappa_i L(\gamma_i)\tau_i, \tag{2.16}$$

$$\gamma_i'' = -\kappa_i L^2(\gamma_i)\nu_i. \tag{2.17}$$

For $\sigma > 0$, the limit energy of Ω in (2.8) can then be written in terms of the family of constant speed boundary curves $\{\gamma_i\}_{i=1}^N$ as

$$\tilde{F}_{\infty,\sigma}(\Omega) = \sum_{i=1}^N \left(\sigma L(\gamma_i) + \frac{\pi}{2} L(\gamma_i) \int_0^1 \kappa_i^2(t)\,\mathrm{d}t \right). \tag{2.18}$$

It remains to recover Ω by means of its boundary curves. By the Jordan decomposition theorem, for every $i = 1,\dots,N$, we can always decompose $\mathbb{R}^2$ into an interior of the curve γ_i and an exterior as

$$\mathrm{int}(\gamma_i) := \left\{ x \in \mathbb{R}^2 \setminus \Gamma_i : |\mathcal{I}(\gamma_i, x)| = 1 \right\}, \tag{2.19}$$

$$\mathrm{ext}(\gamma_i) := \left\{ x \in \mathbb{R}^2 \setminus \Gamma_i : \mathcal{I}(\gamma_i, x) = 0 \right\}. \tag{2.20}$$

Here, the absolute value in (2.19) accounts for both counter-clockwise and clockwise oriented curves. Indeed, if γ_i is oriented counter-clockwise, we have $\mathcal{I}(\gamma_i, x) = 1$ for all $x \in \mathrm{int}(\gamma_i)$, while for a curve γ_i oriented clockwise we have $\mathcal{I}(\gamma_i, x) = -1$ for all $x \in \mathrm{int}(\gamma_i)$. Via elementary combinatorics, one can then recover the original bounded set Ω as

$$\Omega = \left\{ x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^N \Gamma_i : \sum_{i=1}^N \mathcal{I}(\gamma_i, x) \equiv 1 \,(\mathrm{mod}\, 2) \right\}. \tag{2.21}$$

In fact, with the orientations of the boundary curves chosen for identity (2.14) to hold, we even have $\sum_{i=1}^N \mathcal{I}(\gamma_i, x) = \chi_\Omega(x)$ for $x \in \mathbb{R}^2 \setminus \bigcup_{i=1}^N \Gamma_i$. However, to streamline the arguments in this paper, not fixing the orientation and taking the sum modulo 2 instead is more convenient.

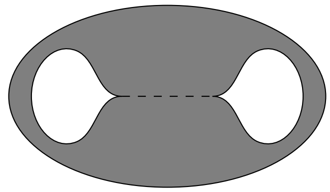

Figure 2: An example of a set with a "collapsed" interior boundary shown as dashed.

We now extend these notions to collections of regular, closed, but not necessarily simple curves in the Sobolev space $H^2(\mathbb{S}^1;\mathbb{R}^2)$ and thus having square integrable curvature. We note from the start that the formulas in (2.14)–(2.17) clearly remain valid a.e. for such curves parametrized with constant speed.

Definition 2.9. *Let $\sigma > 0$. Let $I \subset \mathbb{N}$ be finite, let $\{\gamma_i\}_{i\in I} \subset H^2(\mathbb{S}^1;\mathbb{R}^2)$ be a collection of regular closed curves, and for each $i \in I$ let*

$$\kappa_i := \frac{(\gamma_i')^\perp \cdot \gamma_i''}{|\gamma_i'|^3} \in L^2(\mathbb{S}^1) \tag{2.22}$$

be the curvature of γ_i. We abbreviate $\gamma := \{\gamma_i\}_{i\in I}$ and $\Gamma := \bigcup_{i\in I}\Gamma_i$, where $\Gamma_i := \gamma_i([0,1])$. We then define

$$\hat{F}_{\infty,\sigma}(\gamma) := \sum_{i\in I}\left(\sigma L(\gamma_i) + \frac{\pi}{2}\int_0^1 \kappa_i^2(t)|\gamma_i'(t)|\,\mathrm{d}t\right) \tag{2.23}$$

and, for $x \in \mathbb{R}^2 \setminus \Gamma$,

$$\mathcal{I}(\gamma, x) := \sum_{i\in I}\mathcal{I}(\gamma_i, x). \tag{2.24}$$

Finally, we define

$$A_\gamma^o := \{x \in \mathbb{R}^2 \setminus \Gamma : \mathcal{I}(\gamma,x) \equiv 1 \,(\mathrm{mod}\, 2)\}. \tag{2.25}$$

Notice that while $\Omega = A_\gamma^o$ when every curve γ_i is simple, this need not hold in the relaxation process: If two *interior* boundaries collapse, as in the example in Figure 2, then A_γ^o excludes a one-dimensional segment. This exceptional set of course has measure zero, but needs to be taken care of in topological statements, motivating the following definition.

Definition 2.10 (Bellettini, Mugnai [9])**.** *Given a set of finite perimeter $\Omega \subset \mathbb{R}^2$, we define the open set*

$$\Omega^* := \left\{x \in \mathbb{R}^2 : \exists r > 0 : |B_r(x) \setminus \Omega| = 0\right\}, \tag{2.26}$$

while $\partial^*\Omega$ denotes the reduced boundary of Ω. The set of its system of H^2-boundary curves is then defined as

$$\begin{aligned} G(\Omega) := \Big\{&\{\gamma_i\}_{i\in I} \subset H^2(\mathbb{S}^1;\mathbb{R}^2) : I \subset \mathbb{N},\ |I| < \infty,\ \partial^*\Omega \subset \Gamma, \\ &\Omega^* = \mathrm{int}\left(A_\gamma^o \cup \Gamma\right),\ |\gamma_i'| \equiv \mathit{const}\ \forall i \in I\Big\}, \end{aligned} \tag{2.27}$$

with the convention that $G(\Omega) := \emptyset$ if such a system of curves does not exist.

In the example of Figure 2 the system of boundary curves will consist of an outer circle and a single interior curve which traverses the collapsed interior boundary interval twice. Here the set A^o_γ is shown in gray, while the set Ω^* is obtained from A^o_γ by adding back the white interval without the cusp points (resulting in a disk with two holes).

The following representation of the relaxed elastica functional was established by Bellettini and Mugnai.

Theorem 2.11 (Bellettini, Mugnai [9]). *For a bounded set $\Omega \subset \mathbb{R}^2$ of finite perimeter, we have*

$$F_{\infty,\sigma}(\Omega) = \inf_{\gamma \in G(\Omega)} \hat{F}_{\infty,\sigma}(\gamma). \tag{2.28}$$

2.1 Outline of the proof

Theorem 2.4 being a Γ-convergence statement, its proof is roughly split into a compactness part, a lower bound, and an upper bound. However, here we take "compactness" to mean that limit sets essentially have an H^2-regular boundary rather than showing that all finite energy sequences have an L^1-convergent subsequence, which is wrong, as noted below Theorem 2.4.

The first step is to rewrite the energy in the form, following the ideas of Muratov and Simon [40]:

$$\begin{aligned} F_{\lambda,\alpha}(\Omega) = &\left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) \\ &+ \frac{1}{4\pi\alpha} \int_{\partial^*\Omega} \int_{H^0_-(\nu(y))\Delta\lambda(\Omega-y)} \left|\nu(y) \cdot \frac{z}{|z|}\right| \frac{e^{-\alpha|z|}}{|z|} \,\mathrm{d}^2 z \,\mathrm{d}\mathcal{H}^1(y), \end{aligned} \tag{2.29}$$

where $H^0_-(\nu(y))$ denotes the half-plane through 0 sharing the outward normal $\nu(y)$ with Ω at $y \in \partial^*\Omega$. See Figure 3 for an illustration and Lemma 3.1 for the precise statement.

The strategy for proving compactness loosely follows ideas of the derivation of plate theory by Friesecke, James, and Müller [17] in that we provide an L^2-bound for difference quotients along the sequence. However, our situation is much simpler as the representation (2.29) directly provides a quantitative, non-local comparison of the set with its tangent half-planes without having to first establish further rigidity properties. Therefore, two tangent half-planes at two close boundary points cannot deviate too much without increasing the energy. We can also only have finitely many boundary curves as the elastica energy of short, closed curves blows up.

For the upper and lower bounds, we introduce an anisotropic version of the blowup used in the identity (2.29), so that we can expect the blowup to approach the subgraph of a parabola with curvature at the vertex determined by the curvature of Ω. In the upper bound, we will be able to work with a fixed and regular set to make this intuition rigorous and to compute the resulting energy contribution. We will argue similarly for the lower bound, but even if we can restrict ourselves to only considering sequences of regular sets by a density argument, we will have to deal with quite a few measure-theoretic details to handle the geometric consequences of weak H^2-convergence.

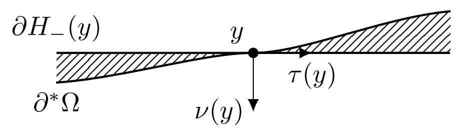

Figure 3: Sketch indicating the domain of integration in z (hatched) around $y \in \partial^*\Omega$ in the representations (2.29) and (3.7). The domain Ω is located above the solid curve and the half-plane $H_-(y)$ is located above the solid line, respectively.

3 Preliminaries and existence of minimizers

Before we turn to the individual steps, we present a rewriting via integration by parts of the non-local term in $F_{\lambda,\alpha}$ in terms of a mixed boundary/bulk integral. A similar computation was already crucial in identifying the critical Γ-limit in the case of dipolar repulsion [40].

To this end we solve the equation $\Delta\Phi_\alpha(|z|) = \frac{e^{-\alpha|z|}}{|z|}$ in $\mathbb{R}^2\backslash\{0\}$ with sufficient decay at infinity. This gives

$$\Phi_\alpha(r) = \frac{1}{\alpha}\,\mathrm{E}_1(\alpha r), \tag{3.1}$$

$$\Phi'_\alpha(r) = -\frac{e^{-\alpha r}}{\alpha r}, \tag{3.2}$$

where $E_1(z) := \int_z^\infty \frac{e^{-t}}{t}\,\mathrm{d}t$ for $z > 0$ is the exponential integral. For $\nu \in \mathbb{S}^1$ and $y \in \partial^*\Omega$ for a set Ω of finite perimeter we also define

$$H^0_-(\nu) := \left\{x \in \mathbb{R}^2 : \nu \cdot x < 0\right\}, \tag{3.3}$$

$$H_-(y) := \left\{x \in \mathbb{R}^2 : \nu(y) \cdot (x - y) < 0\right\}, \tag{3.4}$$

where $\nu(y)$ denotes the outward unit normal of Ω at y. Let furthermore

$$R_\nu := e_2 \otimes \nu - e_1 \otimes \nu^\perp, \tag{3.5}$$

$$A_\lambda := \lambda e_1 \otimes e_1 + \lambda^2 e_2 \otimes e_2, \tag{3.6}$$

where $\nu^\perp = (-\nu_2, \nu_1)$ is the 90-degree counter-clockwise rotation of $\nu = (\nu_1, \nu_2)$, i.e., $R_\nu \in SO(2)$ is the unique rotation such that $R_\nu \nu = e_2$, and A_λ is a matrix of anisotropic dilations along the first and the second coordinate directions. Here and everywhere below $z = (z_1, z_2)$.

Lemma 3.1. *Let $\Omega \in \mathcal{A}_\pi$. Then we have the representations*

$$\begin{aligned} F_{\lambda,\alpha}(\Omega) = &\left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) \\ &+ \frac{1}{4\pi\alpha} \int_{\partial^*\Omega} \int_{H^0_-(\nu(y))\Delta\lambda(\Omega-y)} \left|\nu(y) \cdot \frac{z}{|z|}\right| \frac{e^{-\alpha|z|}}{|z|}\,\mathrm{d}^2 z\,\mathrm{d}\mathcal{H}^1(y) \end{aligned} \tag{3.7}$$

$$\begin{aligned} = &\left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) \\ &+ \frac{1}{4\pi\alpha\lambda^2} \int_{\partial^*\Omega} \int_{H^0_-(e_2)\Delta A_\lambda R_{\nu(y)}(\Omega-y)} |z_2|\, \frac{e^{-\alpha\sqrt{z_1^2 + \frac{z_2^2}{\lambda^2}}}}{z_1^2 + \frac{z_2^2}{\lambda^2}}\,\mathrm{d}^2 z\ \mathrm{d}\mathcal{H}^1(y). \end{aligned} \tag{3.8}$$

In particular, we have

$$F_{\lambda,\alpha}(\Omega) \geq \left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega). \tag{3.9}$$

These representations have the advantage that the non-local term penalizes the deviation of Ω from its tangent half-plane at each $y \in \partial^*\Omega$, see Figure 3. Furthermore, the perimeter term already exhibits the correct leading order behaviour. In particular, together with Proposition 2.3 we immediately get the already-known Γ-convergence statement of Theorem 2.1 for sufficiently strong screening as a corollary.

The proof of Lemma 3.1 relies on integrating the kernel by parts, as already mentioned, and at each point of the boundary moving the expected non-local contribution of the tangent half-plane to the perimeter.

With this representation, the proof of existence of minimizers is a surprisingly simple computation.

Proof of Proposition 2.2. As is common in the field, the full proof operates by the concentration-compactness dichotomy. We refer the reader to, for example, the proof of [40, Lemma 4.4] (see also [27]) for the details. In the following we only prove that for all $\beta \in (0,1)$ we have

$$\inf_{\mathcal{A}_\pi} F_{\lambda,\alpha} < \inf_{\mathcal{A}_{\beta\pi}} F_{\lambda,\alpha} + \inf_{\mathcal{A}_{(1-\beta)\pi}} F_{\lambda,\alpha}, \tag{3.10}$$

which can then be used to rule out the splitting case in the concentration-compactness principle.

For every $\Omega \in \mathcal{A}_\pi$, we compute, using the representation (3.7) and the condition $\alpha > \frac{1}{\sqrt{2\pi}}$, that

$$\begin{aligned} F_{\lambda,\alpha}\left(\beta^{\frac12}\Omega\right) &= \beta^{\frac12}\left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) \\ &\quad + \frac{\beta}{4\pi\alpha} \int_{\partial^*\Omega} \int_{H_-(\nu(y))\Delta\lambda(\Omega - y)} \left|\nu(y) \cdot \frac{z}{|z|}\right| \frac{e^{-\beta^{\frac12}\alpha|z|}}{|z|} \,\mathrm{d}^2 z \,\mathrm{d}\mathcal{H}^1(y) \\ &\geq \beta\left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) + \beta^{\frac12}(1 - \beta^{\frac12})\left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) \\ &\quad + \frac{\beta}{4\pi\alpha} \int_{\partial^*\Omega} \int_{H_-(\nu(y))\Delta\lambda(\Omega - y)} \left|\nu(y) \cdot \frac{z}{|z|}\right| \frac{e^{-\alpha|z|}}{|z|} \,\mathrm{d}^2 z \,\mathrm{d}\mathcal{H}^1(y) \\ &\geq \beta \inf_{\mathcal{A}_\pi} F_{\lambda,\alpha} + \beta^{\frac12}(1 - \beta^{\frac12})\left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega), \end{aligned} \tag{3.11}$$

and thus $\inf_{\mathcal{A}_{\beta\pi}} F_{\lambda,\alpha} > \beta \inf_{\mathcal{A}_\pi} F_{\lambda,\alpha}$. Similarly, we have $\inf_{\mathcal{A}_{(1-\beta)\pi}} F_{\lambda,\alpha} > (1-\beta) \inf_{\mathcal{A}_\pi} F_{\lambda,\alpha}$, so adding the two inequalities gives the claim (3.10).

The rest of the statement can be proved as in [25, Proposition 2.1]. The regularity theory for quasi-minimizers of the perimeter implies $C^{1,\beta}$-regularity of Ω for any $\beta \in (0, \frac12)$, see for example [31, Theorem 21.8] or [42, Theorem 1.4.9]. By [19, Theorem 5.2], the potential

$$v(x) := \frac{1}{2\pi} \int_\Omega \frac{e^{-\alpha|x-y|}}{|x-y|} \,\mathrm{d}x, \qquad x \in \mathbb{R}^2, \tag{3.12}$$

is of class $C^{0,\alpha}(\mathbb{R}^2)$ for any $\alpha \in (0,1)$. Therefore, the Euler-Lagrange equation

$$\kappa(x) + v(x) = \mu, \qquad x \in \partial\Omega, \tag{3.13}$$

where $\mu \in \mathbb{R}$ is the Lagrange multiplier for the mass constraint, holds in the weak sense in a local Cartesian frame in which $\partial\Omega$ is a $C^{1,\beta}$ graph. Consequently, $\partial\Omega$ is of class $C^{2,\alpha}$ for any $\alpha \in (0,1)$. In particular, Ω is a bounded open set with finitely many holes. Finally, as the kernel is repulsive, any minimizer Ω must be connected, since otherwise moving different connected components far apart lowers the energy. □

Proof of Lemma 3.1. As in [40], the proof relies on the application of the Gauss-Green theorem to the double integral in (2.3). However, due to a mild singularity of the kernel absent in [40] we need an additional approximation argument to express the non-local term as an integral over the interior and the reduced boundary of the set of finite perimeter Ω. To that end, let $\eta \in C^\infty(\mathbb{R})$ be a cutoff function with $\eta' \leq 0$ such that $\eta(t) = 1$ for all $t \leq \frac{1}{2}$ and $\eta(t) = 0$ for all $t \geq 1$. For $\varepsilon > 0$ and $R > 0$ we define a short-range cutoff $\eta_\varepsilon(t) = \eta(t/\varepsilon)$ and a long-range cutoff $\eta_R(t) = \eta(t/R)$, respectively, and observe that by the monotone convergence theorem we have

$$\int_\Omega \int_{\Omega^c} \frac{e^{-\lambda\alpha|x-y|}}{|x-y|} \,\mathrm{d}^2y\,\mathrm{d}^2x = \lim_{\varepsilon\to 0,\, R\to\infty} \int_\Omega \int_{\Omega^c} (1 - \eta_\varepsilon(|y-x|))\eta_R(|y-x|) \frac{e^{-\lambda\alpha|y-x|}}{|y-x|} \,\mathrm{d}^2y\,\mathrm{d}^2x. \tag{3.14}$$

Then recalling the definition of Φ_α and integrating by parts in y, which is now justified [1], with the help of Fubini's theorem we obtain

$$\begin{aligned}
&\int_\Omega \int_{\Omega^c} (1 - \eta_\varepsilon(|y-x|))\eta_R(|y-x|) \frac{e^{-\lambda\alpha|y-x|}}{|y-x|} \,\mathrm{d}^2y\,\mathrm{d}^2x \\
&= \int_\Omega \int_{\Omega^c} (1 - \eta_\varepsilon(|y-x|))\eta_R(|y-x|) \Delta_y \Phi_{\lambda\alpha}(|y-x|) \,\mathrm{d}^2y\,\mathrm{d}^2x \\
&= -\int_\Omega \int_{\partial^*\Omega} (1 - \eta_\varepsilon(|y-x|))\eta_R(|y-x|)\, \nu(y) \cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|) \,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}^2x \\
&\quad + \int_\Omega \int_{\Omega^c} \eta_R(|y-x|) \nabla_y \eta_\varepsilon(|y-x|) \cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|) \,\mathrm{d}^2y\,\mathrm{d}^2x \\
&\quad - \int_\Omega \int_{\Omega^c} (1 - \eta_\varepsilon(|y-x|)) \nabla_y \eta_R(|y-x|) \cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|) \,\mathrm{d}^2y\,\mathrm{d}^2x.
\end{aligned} \tag{3.15}$$

Notice that from (3.2) we have

$$|\nu(y) \cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|)| \leq \frac{e^{-\alpha\lambda|x-y|}}{\alpha\lambda|x-y|}, \tag{3.16}$$

which is integrable over $(x,y) \in \Omega \times \partial^*\Omega$ by Fubini's theorem. Hence applying the dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0,\, R\to\infty} &\int_\Omega \int_{\partial^*\Omega} (1 - \eta_\varepsilon(|y-x|))\eta_R(|y-x|)\, \nu(y) \cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|) \,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}^2x \\
&= \int_\Omega \int_{\partial^*\Omega} \nu(y) \cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|) \,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}^2x \\
&= -\int_{\partial^*\Omega} \int_\Omega \nu(y) \cdot \nabla_x \Phi_{\lambda\alpha}(|y-x|) \,\mathrm{d}^2x\,\mathrm{d}\mathcal{H}^1(y).
\end{aligned} \tag{3.17}$$

Similarly, the last term in the right-hand side of (3.15) vanishes in the limit.

Thus, it remains to evaluate the second integral on the right-hand side of (3.15), which for $R > 2\varepsilon$ can be written as

$$\begin{aligned} &\int_\Omega \int_{\Omega^c} \eta_R(|y-x|)\nabla_y \eta_\varepsilon(|y-x|)\cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|)\,\mathrm{d}^2y\,\mathrm{d}^2x \\ &\qquad = \frac{1}{\varepsilon}\int_\Omega \int_{B_\varepsilon(x)} |\eta'(\varepsilon^{-1}|y-x|)\Phi'_{\lambda\alpha}(|y-x|)|\,\mathrm{d}^2y\,\mathrm{d}^2x. \end{aligned} \tag{3.18}$$

Therefore, by (3.2) and defining $\phi_\varepsilon(x) := \frac{1}{2\pi\varepsilon|x|}|\eta'(|x|/\varepsilon)|$ for $x \in \mathbb{R}^2$ we have

$$\begin{aligned} &\frac{1}{2\pi}\left|\int_\Omega \int_{\Omega^c} \eta_R(|y-x|)\nabla_y \eta_\varepsilon(|y-x|)\cdot \nabla_y \Phi_{\lambda\alpha}(|y-x|)\,\mathrm{d}^2y\,\mathrm{d}^2x\right| \\ &\le \frac{1}{\lambda\alpha}\int_\Omega \int_{B_\varepsilon(x)} \frac{|\eta'(\varepsilon^{-1}|y-x|)|}{2\pi\varepsilon|y-x|}\,\mathrm{d}^2y\,\mathrm{d}^2x \\ &= \frac{1}{\lambda\alpha}\int_\Omega \int_{\Omega^c} \phi_\varepsilon(x-y)\,\mathrm{d}^2y\,\mathrm{d}^2x. \end{aligned} \tag{3.19}$$

The function ϕ_ε is non-negative with $\phi_\varepsilon(x) = 0$ for all $x \in \mathbb{R}^2$ such that $|x| > \varepsilon$, and $\int_{\mathbb{R}^2} \phi_\varepsilon(x)\,\mathrm{d}^2x = 1$, so that it is an approximation of a Dirac delta as $\varepsilon \to 0$. Thus by the standard approximation argument for the characteristic function of Ω in $L^1(\mathbb{R}^2)$ by uniformly bounded smooth functions with compact support we have

$$\lim_{\varepsilon\to 0}\int_\Omega \int_{\Omega^c} \phi_\varepsilon(|x-y|)\,\mathrm{d}^2y\,\mathrm{d}^2x = 0. \tag{3.20}$$

Putting together equations (3.14), (3.15), and (3.17)–(3.20) we have

$$\int_\Omega \int_{\Omega^c} \frac{e^{-\lambda\alpha|x-y|}}{|x-y|}\,\mathrm{d}^2y\,\mathrm{d}^2x = \int_{\partial^*\Omega}\int_\Omega \nu(y)\cdot\nabla_x \Phi_{\lambda\alpha}(|y-x|)\,\mathrm{d}^2x\,\mathrm{d}\mathcal{H}^1(y). \tag{3.21}$$

Therefore, we can write

$$\begin{aligned} &-\int_\Omega \int_{\Omega^c} \frac{e^{-\lambda\alpha|x-y|}}{|x-y|}\,\mathrm{d}^2y\,\mathrm{d}^2x \\ &= \int_{\partial^*\Omega}\int_\Omega \nu(y)\cdot\frac{x-y}{|x-y|}|\Phi'_{\lambda\alpha}(|y-x|)|\,\mathrm{d}^2x\,\mathrm{d}\mathcal{H}^1(y) \\ &= \int_{\partial^*\Omega}\int_{H_-(y)} \nu(y)\cdot\frac{x-y}{|x-y|}|\Phi'_{\lambda\alpha}(|y-x|)|\,\mathrm{d}^2x\,\mathrm{d}\mathcal{H}^1(y) \\ &\quad + \int_{\partial^*\Omega}\int_{\Omega\setminus H_-(y)} \nu(y)\cdot\frac{x-y}{|x-y|}|\Phi'_{\lambda\alpha}(|y-x|)|\,\mathrm{d}^2x\,\mathrm{d}\mathcal{H}^1(y) \\ &\quad - \int_{\partial^*\Omega}\int_{H_-(y)\setminus\Omega} \nu(y)\cdot\frac{x-y}{|x-y|}|\Phi'_{\lambda\alpha}(|y-x|)|\,\mathrm{d}^2x\,\mathrm{d}\mathcal{H}^1(y). \end{aligned} \tag{3.22}$$

For every $y \in \partial^*\Omega$, we compute

$$\begin{aligned} \int_{H_-(y)} \nu(y)\cdot\frac{x-y}{|x-y|}|\Phi'_{\lambda\alpha}(|y-x|)|\,\mathrm{d}^2x &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^\infty \frac{e^{-\lambda\alpha r}\cos\theta}{\lambda\alpha}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= -\frac{2}{\lambda^2\alpha^2} \end{aligned} \tag{3.23}$$

Together with the combinatorics of the sign of $\nu(y) \cdot \frac{x-y}{|x-y|}$ for $x \in H_-(y)$ and $x \notin H_-(y)$, (3.22) and (3.23) thus result in

$$\begin{aligned} &- \int_\Omega \int_{\Omega^c} \frac{e^{-\lambda\alpha|x-y|}}{|x-y|} \, \mathrm{d}^2 y \, \mathrm{d}^2 x \\ &= -\frac{2}{\lambda^2\alpha^2} P(\Omega) + \int_{\partial^*\Omega} \int_{H_-(y)\Delta\Omega} \left| \nu(y) \cdot \frac{x-y}{|x-y|} \right| |\Phi'_{\lambda\alpha}(|y-x|)| \, \mathrm{d}^2 x \, \mathrm{d}\mathcal{H}^1(y) \\ &= -\frac{2}{\lambda^2\alpha^2} P(\Omega) + \frac{1}{\lambda^2\alpha} \int_{\partial^*\Omega} \int_{H^0_-(\nu(y))\Delta\lambda(\Omega-y)} \left| \nu(y) \cdot \frac{z}{|z|} \right| \frac{e^{-\alpha|z|}}{|z|} \, \mathrm{d}^2 z \, \mathrm{d}\mathcal{H}^1(y), \end{aligned} \tag{3.24}$$

which proves equation (3.7).

Finally, we calculate

$$\begin{aligned} &\int_{H^0_-(\nu(y))\Delta\lambda(\Omega-y)} \left| \nu(s) \cdot \frac{z}{|z|} \right| \frac{e^{-\alpha|z|}}{|z|} \, \mathrm{d}^2 z \\ &= \frac{1}{\lambda^2} \int_{H^0_-(e_2)\Delta A_\lambda R_{\nu(y)}(\Omega-y)} |z_2| \frac{e^{-\alpha\sqrt{z_1^2 + \frac{z_2^2}{\lambda^2}}}}{z_1^2 + \frac{z_2^2}{\lambda^2}} \, \mathrm{d}^2 z, \end{aligned} \tag{3.25}$$

giving equation (3.8). □

4 Compactness

4.1 Single boundary curves

We start out by proving compactness for a single sequence of boundary curves of a finite energy sequence. By density of regular sets in the sets of finite perimeter, we may as well assume that the sequence consists of regular sets. The main point here is to prove that the limit is sufficiently regular to have curvature in L^2.

To this end, the first step is to obtain a discrete H^1 estimate for the normals along the sequence, that is, for fixed λ and α. In order to control the geometry of the curves in the lower bound, we also need an estimate for how often two boundary points (be they from the same boundary curve doubling up on itself or from two different boundary curves) with wildly different tangents can be close to each other. Hence we also record a consequence of the arguments pertaining to two mismatched, close-by normals regardless of which boundary curve they belong to.

Lemma 4.1. *Let $\alpha_0 > 0$ and $K > 0$. Then there exist $C, C', C'' > 0$ with the following property: If $\lambda > 0$, $\Omega \in \mathcal{A}_\pi$ is regular and $\gamma : [0,1] \to \mathbb{R}^2$ is a smooth Jordan boundary curve of Ω parametrized by constant speed, then for all $\alpha \in (0, \alpha_0)$ and $s \in [-K, K]$ we have the estimate*

$$\begin{aligned} &L(\gamma) \int_0^1 \left| \nu\left(t + (L(\gamma)\lambda)^{-1} s\right) - \nu(t) \right|^2 \, \mathrm{d}t \\ &\le C \int_\Gamma \int_{H^0_-(\nu(y))\Delta\lambda(\Omega-y)} \left| \nu(y) \cdot \frac{z}{|z|} \right| \frac{e^{-\alpha|z|}}{|z|} \, \mathrm{d}^2 z \, \mathrm{d}\mathcal{H}^1(y) \\ &\le C' F_{\lambda,\alpha}(\Omega). \end{aligned} \tag{4.1}$$

Furthermore, for

$$Z_{K,\lambda} := \big\{(y_1,y_2) \in \partial\Omega \times \partial\Omega : \\ |y_1 - y_2| \le K\lambda^{-1},\ \nu(y_1)\cdot\nu(y_2) \le 0\big\}. \tag{4.2}$$

we have

$$\mathcal{H}^2(Z_{K,\lambda}) \le C'' P(\Omega) F_{\lambda,\alpha}(\Omega). \tag{4.3}$$

Using this information, we can prove the compactness statement for sequences of single boundary curves. As the L^1-topology disregards sets with vanishing mass, we only have to consider sequences of curves whose length does not converge to zero in the limit. We exclude the natural lack of compactness due to the translational symmetry of the problem by pinning one point on each curve.

Lemma 4.2. *Let $\lambda_n > 0$ and $\alpha_n > \frac{1}{\sqrt{2\pi}}$ be such that $\lambda_n \to \infty$ as $n \to \infty$ and such that $\sigma_n := \lambda_n^2\left(1 - \frac{1}{2\pi\alpha_n^2}\right)$ satisfies*

$$\lim_{n\to\infty} \sigma_n = \sigma > 0. \tag{4.4}$$

Let $(\Omega_n) \subset \mathcal{A}_\pi$ be a sequence of regular sets such that

$$\limsup_{n\to\infty} \lambda_n^2 F_{\lambda_n,\alpha_n}(\Omega_n) < \infty. \tag{4.5}$$

Let $\gamma_n : [0,1] \to \mathbb{R}^2$ be a smooth Jordan boundary curve of Ω_n, and assume that all γ_n are parametrized with constant speed, with

$$\liminf_{n\to\infty} L(\gamma_n) > 0. \tag{4.6}$$

Then there exists a subsequence (γ_{n_k}) of (γ_n) and $\gamma_\infty \in H^2(\mathbb{S}^1;\mathbb{R}^2)$ such that

$$\gamma_{n_k} - \gamma_{n_k}(0) \to \gamma_\infty \qquad \text{in} \quad H^1(\mathbb{S}^1;\mathbb{R}^2), \tag{4.7}$$

as $k \to \infty$. In particular, we have $\lim_{k\to\infty} L(\gamma_{n_k}) = L(\gamma_\infty) > 0$. Furthermore, there exists a universal constant $C > 0$ such that along a further subsequence (not relabeled) we have

$$\begin{aligned} &\hat{F}_{\infty,\sigma}(\gamma_\infty) \\ &\le C \liminf_{k\to\infty} \Bigg(\sigma_{n_k} L(\gamma_{n_k}) \\ &\quad + \lambda_{n_k}^2 \int_{\Gamma_{n_k}} \int_{H^0_-(\nu_{n_k}(y))\Delta\lambda_{n_k}(\Omega_{n_k}-y)} \left|\nu_{n_k}(y)\cdot\frac{z}{|z|}\right| \frac{e^{-\alpha_{n_k}|z|}}{|z|}\,\mathrm{d}^2 z\,\mathrm{d}\mathcal{H}^1(y)\Bigg). \end{aligned} \tag{4.8}$$

Proof of Lemma 4.1. For $x \in \mathbb{R}^2$ and $i = 1, 2$, we abbreviate

$$g_i(x) := \nu(y_i)\cdot(x - \lambda y_i), \tag{4.9}$$

$$\mu_i(x) := |g_i(x)| \frac{e^{-\alpha|x-\lambda y_i|}}{|x - \lambda y_i|^2}. \tag{4.10}$$

Step 1: We claim that for all $y_1, y_2 \in \partial\Omega$ with $\lambda|y_1 - y_2| \le K$, we have

$$
\begin{aligned}
|\nu(y_1) - \nu(y_2)|^2 \le C \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_1)} - \chi_{\lambda\Omega}\right| \mu_1 \, \mathrm{d}^2 x \\
+ C \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_2)} - \chi_{\lambda\Omega}\right| \mu_2 \, \mathrm{d}^2 x,
\end{aligned}
\tag{4.11}
$$

for some $C > 0$ depending only on K and α_0.

With the goal of comparing tangent spaces at y_1 and y_2, we have by the triangle inequality

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_1)} - \chi_{\lambda\Omega}\right| \mu_1 \, \mathrm{d}^2 x + \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_2)} - \chi_{\lambda\Omega}\right| \mu_2 \, \mathrm{d}^2 x \\
&\ge \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_1)} - \chi_{\lambda H_-(y_2)}\right| \min\left(\mu_1(x), \mu_2(x)\right) \, \mathrm{d}^2 x.
\end{aligned}
\tag{4.12}
$$

Let $\bar{y} := \frac{\lambda(y_1+y_2)}{2}$. By the assumption $\lambda|y_1 - y_2| \le K$, for $i = 1, 2$ we have

$$
|\lambda y_i - \bar{y}| \le \frac{K}{2}. \tag{4.13}
$$

Therefore, for all $x \in B_K(\bar{y})$ we obtain

$$
\max\left\{|x - \lambda y_1|, |x - \lambda y_2|\right\} \le \frac{3}{2} K \tag{4.14}
$$

so that we have

$$
\min\left(\frac{e^{-\alpha|x-\lambda y_1|}}{|x - \lambda y_1|^2}, \frac{e^{-\alpha|x-\lambda y_2|}}{|x - \lambda y_2|^2}\right) \ge \frac{4e^{-\frac{3}{2}\alpha_0 K}}{9K^2}. \tag{4.15}
$$

Together with (4.12), we arrive at

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_1)} - \chi_{\lambda\Omega}\right| \mu_1 \, \mathrm{d}^2 x + \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_2)} - \chi_{\lambda\Omega}\right| \mu_2 \, \mathrm{d}^2 x \\
&\ge C^{-1} \int_{(\lambda H_-(y_1) \Delta \lambda H_-(y_2)) \cap B_K(\bar{y})} \min\left\{|g_1(x)|, |g_2(x)|\right\} \, \mathrm{d}^2 x,
\end{aligned}
\tag{4.16}
$$

for some $C > 0$ depending only on K and α_0.

We now interpret the integral on the right-hand side if (4.16) as the volume of a three-dimensional body, aiming to estimate $|\nu(y_1) - \nu(y_2)|^2$ from above. Therefore, we may assume that $\nu(y_1) \ne \nu(y_2)$ and, without loss of generality, that $\partial H_-(y_1)$ and $\partial H_-(y_2)$ intersect at the origin. Thus there exists a closed cone $\mathcal{C}^+ \subset \mathbb{R}^2$ with vertex at 0 and half-angle $\theta \in [0, \frac{\pi}{2})$, see Figure 4, such that

$$
\mathcal{C}^+ \cup \mathcal{C}^- = \overline{\lambda H_-(y_1) \, \Delta \, \lambda H_-(y_2)}, \tag{4.17}
$$

where $\mathcal{C}^- := -\mathcal{C}^+$. Now recall that by its definition $|\lambda^{-1} g_i(x)|$ is the distance from the point $\lambda^{-1} x$ to $\partial H_-(y_i)$. Hence the set

$$
L := \left\{x \in \mathcal{C}^+ \cup \mathcal{C}^- : |g_1(x)| = |g_2(x)|\right\} \tag{4.18}
$$

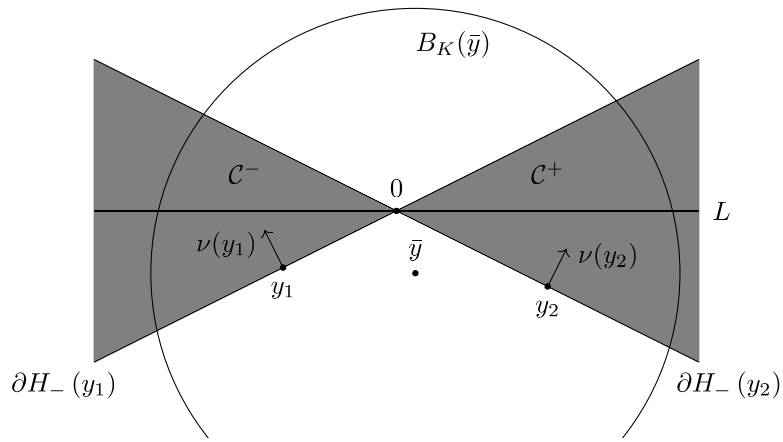

Figure 4: Sketch of the cones $\mathcal{C}^{\pm}$, the line L, the points y_1, y_2, $\bar{y}$, the normals ν_1 and ν_2, and the ball $B_K(\bar{y})$.

defines a line that bisects $\mathcal{C}^{\pm}$, that is, it separates $\mathcal{C}^+$ and $\mathcal{C}^-$ into two respective sub-cones with apertures $\theta \in [0, \frac{\pi}{2})$.

Let $x \in L$. Then for $i = 1, 2$, we have $x - g_i(x)\nu(y_i) \in \partial H_-(y_i)$ and

$$|g_i(x)| = |x| \sin\theta \tag{4.19}$$

Let $\tau \in L$ be such that $|\tau| = 1$ and $\rho\tau \in \mathcal{C}^+ \subset \mathbb{R}^2$ for all $\rho > 0$. We define the three-dimensional sets

$$\tilde{L}^{\pm} := \left\{(\pm\rho\tau, \rho\sin\theta) \in \mathbb{R}^3 : \rho \in (0, \infty)\right\}, \tag{4.20}$$

$$\tilde{\mathcal{C}}^{\pm} := \operatorname{conv}\left(\left(\mathcal{C}^{\pm} \times \{0\}\right) \cup \tilde{L}^{\pm}\right). \tag{4.21}$$

In particular, $\tilde{\mathcal{C}}^{\pm}$ are two three-dimensional cones, see an illustration in Figure 5.

By linearity of g_i for $i = 1, 2$, the definition of L, and the identity (4.19), we can interpret the integral on the right hand side of (4.16) as

$$\begin{aligned} &\int_{((\lambda H_-(y_1))\Delta(\lambda H_-(y_2)))\cap B_K(\bar{y})} \min\{|g_1(x)|, |g_2(x)|\}\, \mathrm{d}^2x \\ &= \left|\left(\tilde{\mathcal{C}}^+ \cup \tilde{\mathcal{C}}^-\right) \cap (B_K(\bar{y}) \times \mathbb{R})\right|. \end{aligned} \tag{4.22}$$

As a result of estimate (4.13), we have $B_{\frac{K}{2}}(\lambda y_i) \subset B_K(\bar{y})$ for $i = 1, 2$, so that

$$\left|\left(\tilde{\mathcal{C}}^+ \cup \tilde{\mathcal{C}}^-\right) \cap (B_K(\bar{y}) \times \mathbb{R})\right| \geq \left|\left(\tilde{\mathcal{C}}^+ \cup \tilde{\mathcal{C}}^-\right) \cap \left(B_{\frac{K}{2}}(\lambda y_i) \times \mathbb{R}\right)\right|. \tag{4.23}$$

Without loss of generality, we may assume $y_2 \in \mathcal{C}^+$, as in Figure 4, so that

$$\left|\left(\tilde{\mathcal{C}}^+ \cup \tilde{\mathcal{C}}^-\right) \cap (B_K(\bar{y}) \times \mathbb{R})\right| \geq \left|\tilde{\mathcal{C}}^+ \cap \left(B_{\frac{K}{2}}(\lambda y_2) \times \mathbb{R}\right)\right|. \tag{4.24}$$

In turn, by monotonicity of the right-hand side of (4.24) with respect to sliding the ball center along $\partial H_-(y_2)$, we have

$$\left|\tilde{\mathcal{C}}^+ \cap \left(B_{\frac{K}{2}}(\lambda y_2) \times \mathbb{R}\right)\right| \geq \left|\tilde{\mathcal{C}}^+ \cap \left(B_{\frac{K}{2}}(0) \times \mathbb{R}\right)\right|. \tag{4.25}$$

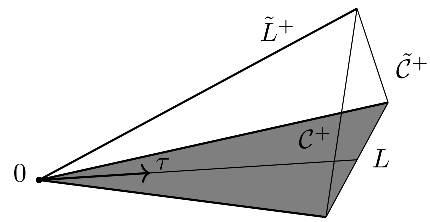

Figure 5: Sketch of the three-dimensional cone $\tilde{\mathcal{C}}^+$.

As $\tilde{\mathcal{C}}^+$ is a cone with opening half-angle θ in the horizontal direction and aperture θ in the vertical direction, we have

$$\left|\tilde{\mathcal{C}}^+ \cap \left(B_{\frac{K}{2}}(0) \times \mathbb{R}\right)\right| \geq CK^3 \sin^2\theta, \tag{4.26}$$

for some $C > 0$ universal. Finally, by the observation that

$$|\nu(y_1) - \nu(y_2)| = 2\sin\theta, \tag{4.27}$$

and estimates (4.16) and (4.22)–(4.26), we obtain

$$\begin{aligned} |\nu(y_1) - \nu(y_2)|^2 \leq C &\int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_1)} - \chi_{\lambda\Omega}\right| \mu_1 \, \mathrm{d}^2 x \\ &+ C \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_2)} - \chi_{\lambda\Omega}\right| \mu_2 \, \mathrm{d}^2 x, \end{aligned} \tag{4.28}$$

for some $C > 0$ depending only on K and α_0, proving the claim.

Step 2: Estimate the normals along a curve. As γ is parametrized by constant speed, for all $t \in [0,1]$ and $s \in [-K, K]$ we have

$$\lambda \left|\gamma(t) - \gamma\left(t + (L(\gamma)\lambda)^{-1} s\right)\right| \leq K. \tag{4.29}$$

The first estimate in (4.1) then follows by taking $y_1 = \gamma(t)$ and $y_2 = \gamma\left(t + (L(\gamma)\lambda)^{-1}s\right)$ in Step 1 and integrating in t, while the second one is obtained with the help of Lemma 3.1.

Step 3: Estimate mismatched normals. For $(y_1, y_2) \in Z_{K,\lambda}$, Step 1 implies

$$\begin{aligned} 2 &\leq |\nu(y_1) - \nu(y_2)|^2 \\ &\leq C \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_1)} - \chi_{\lambda\Omega}\right| \mu_1 \, \mathrm{d}^2 x \\ &\quad + C \int_{\mathbb{R}^2} \left|\chi_{\lambda H_-(y_2)} - \chi_{\lambda\Omega}\right| \mu_2 \, \mathrm{d}^2 x. \end{aligned} \tag{4.30}$$

Integrating jointly in y_1 and y_2 over $Z_{K,\lambda} \subset \Gamma \times \Gamma$, as in Step 2 we obtain

$$\mathcal{H}^2(Z_{K,\lambda}) \leq C' P(\Omega) F_{\lambda,\alpha}(\Omega), \tag{4.31}$$

for some $C' > 0$ depending only on K and α_0. This concludes the proof. □

Proof of Lemma 4.2. Throughout the following, we will never relabel the sequences after passing to subsequences. Without loss of generality, we may also assume that $\gamma_n(0) = 0$ for all $n \in \mathbb{N}$. We recall that (4.4) implies that $\alpha_n \to \frac{1}{\sqrt{2\pi}}$ as $n \to \infty$.

Step 1: Establish the limit behavior of the normals. We begin by observing that by (4.5), (4.6) and Lemma 3.1 the limit $L_\infty := \lim_{n\to\infty} L(\gamma_n) \in (0,\infty)$ exists along a subsequence.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be the standard mollifier and let $\phi_\delta = \frac{1}{\delta}\phi\left(\frac{\cdot}{\delta}\right)$ be an approximation for the Dirac delta in one dimension with support in $(-\delta,\delta)$ for $\delta \to 0$. Consider the maps $\tilde{\nu}_n := \phi_{(L(\gamma_n)\lambda_n)^{-1}} * \nu_n$ as 1-periodic convolutions, where ν_n is the outward normal to γ_n defined in (2.14). In particular, we have $|\tilde{\nu}_n| \le 1$. Note that we are mimicking convolution in arc-length coordinates on the fixed domain $[0,1]$. For $L(\gamma_n)\lambda_n > 2$, which holds for n large enough due to $L_\infty > 0$, we may apply the Cauchy-Schwarz inequality to split off ϕ and get

$$\begin{aligned}\int_0^1 |\tilde{\nu}_n - \nu_n|^2 \,\mathrm{d}t &= \int_0^1 \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_{(L(\gamma_n)\lambda_n)^{-1}}(\tau)(\nu_n(t+\tau) - \nu_n(t))\,\mathrm{d}\tau \right|^2 \mathrm{d}t \\ &\le \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_{(L(\gamma_n)\lambda_n)^{-1}}(\tau) \int_0^1 |\nu_{\lambda_n}(t+\tau) - \nu_{\lambda_n}(t)|^2 \,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_{-1}^1 \phi(s) \int_0^1 \left|\nu_{\lambda_n}(t + (L(\gamma_n)\lambda_n)^{-1}s) - \nu_{\lambda_n}(t)\right|^2 \,\mathrm{d}t\,\mathrm{d}s.\end{aligned} \tag{4.32}$$

Applying the L^∞-type estimate (4.1) to the last term in (4.32) for $K = 1$, we get

$$\begin{aligned}&L(\gamma_n)\int_0^1 |\tilde{\nu}_n - \nu_n|^2\,\mathrm{d}t \\ &\le C \int_{\Gamma_n} \int_{H^0_-(\nu_n(y))\Delta\lambda_n(\Omega - y)} \left|\nu_n(y)\cdot\frac{z}{|z|}\right| \frac{e^{-\alpha|z|}}{|z|}\,\mathrm{d}^2 z\,\mathrm{d}\mathcal{H}^1(y),\end{aligned} \tag{4.33}$$

for some $C > 0$ universal and all n large enough. In particular, by Lemma 3.1 this ensures tha

$$\tilde{\nu}_n - \nu_n \to 0 \tag{4.34}$$

in $L^2(\mathbb{S}^1;\mathbb{R}^2)$ as $n \to \infty$.

Similarly, due to $\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi'_{(L(\gamma_n)\lambda_n)^{-1}}\,\mathrm{d}t = 0$ for $n \in \mathbb{N}$ sufficiently large and the Cauchy–Schwarz inequality to split off $|\phi'|$, we obtain

$$\begin{aligned}\int_0^1 |\tilde{\nu}_n'|^2\,\mathrm{d}t &= \int_0^1 \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi'_{(L(\gamma_n)\lambda_n)^{-1}}(\tau)(\nu_n(t+\tau) - \nu_n(t))\,\mathrm{d}\tau\right|^2 \mathrm{d}t \\ &\le \|\phi'_{(L(\gamma_n)\lambda_n)^{-1}}\|_{L^1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\phi'_{(L(\gamma_n)\lambda_n)^{-1}}(\tau)\right| \int_0^1 |\nu_n(t+\tau) - \nu_n(t)|^2\,\mathrm{d}t\,\mathrm{d}\tau.\end{aligned} \tag{4.35}$$

Combining the fact that $\|\phi'_{(L(\gamma_n)\lambda_n)^{-1}}\|_{L^1} \le CL(\gamma_n)\lambda_n$ for some $C > 0$ universal with the L^∞-type estimate (4.1) for $K = 1$, we furthermore get

$$\begin{aligned}&\frac{1}{L(\gamma_n)}\int_0^1 |\tilde{\nu}_n'|^2\,\mathrm{d}t \\ &\le C\lambda_n^2 \int_{\Gamma_n}\int_{H^0_-(\nu_n(y))\Delta\lambda(\Omega_n - y)} \left|\nu_n(y)\cdot\frac{z}{|z|}\right| \frac{e^{-\alpha|z|}}{|z|}\,\mathrm{d}^2 z\,\mathrm{d}\mathcal{H}^1(y),\end{aligned} \tag{4.36}$$

again, for some $C > 0$ universal and all n large enough. In particular, by (4.1), (4.5) and Lemma 3.1 we have that $\tilde{\nu}_n'$ is uniformly bounded in $L^2(\mathbb{S}^1;\mathbb{R}^2)$, and so by uniform boundedness of $|\tilde{\nu}_n|$ there exists $\tilde{\nu}_\infty \in H^1(\mathbb{S}^1;\mathbb{R}^2)$ such that upon extraction of a subsequence $\tilde{\nu}_n \rightharpoonup \tilde{\nu}_\infty$ in $H^1(\mathbb{S}^1;\mathbb{R}^2)$ as $n \to \infty$. In turn, by the Rellich-Kondrachov theorem we get $\tilde{\nu}_n \to \tilde{\nu}_\infty$ strongly in $L^2(\mathbb{S}^1;\mathbb{R}^2)$ as $n \to \infty$. Hence, due to convergence (4.34), we also get

$$\nu_n \to \tilde{\nu}_\infty \qquad \text{strongly in} \quad L^2(\mathbb{S}^1;\mathbb{R}^2). \tag{4.37}$$

Furthermore, since $|\nu_n| = 1$, we obtain that $|\tilde{\nu}_\infty| = 1$ as well (recall that $\tilde{\nu}_\infty \in C^{\frac{1}{2}}(\mathbb{S}^1;\mathbb{R}^2)$ by the corresponding Sobolev embedding).

Step 2: Construct a limit curve $\gamma_\infty \in H^2(\mathbb{S}^1;\mathbb{R}^2)$. By (4.37), (2.14) and the fact that $|\gamma_n'| = L(\gamma_n)$ we have

$$\gamma_n' = L(\gamma_n)\nu_n^\perp \to L_\infty \tilde{\nu}_\infty^\perp \qquad \text{strongly in} \quad L^2(\mathbb{S}^1;\mathbb{R}^2). \tag{4.38}$$

At the same time, since we assumed without loss of generality that $\gamma_n(0) = 0$, upon extraction of a subsequence we get $\gamma_n \rightharpoonup \gamma_\infty$ weakly in $H^1(\mathbb{S}^1;\mathbb{R}^2)$ for some $\gamma_\infty \in H^1(\mathbb{S}^1;\mathbb{R}^2)$ with $\gamma_\infty(0) = 0$ as $n \to \infty$. In particular, by (4.38) we have

$$\gamma_\infty' = L_\infty \tilde{\nu}_\infty^\perp, \tag{4.39}$$

and $\gamma_n \to \gamma_\infty$ strongly in $H^1(\mathbb{S}^1;\mathbb{R}^2)$. From the strong convergence, it follows that $L_\infty = \lim_{n\to\infty} L(\gamma_n) = L(\gamma_\infty)$. From (4.39) and the fact that $|\tilde{\nu}_\infty| = 1$ we thus obtain that $|\gamma_\infty'| = L(\gamma_\infty)$, i.e., that γ_∞ is a closed curve parametrized with constant speed. Finally, we conclude that $\gamma_\infty \in H^2(\mathbb{S}^2;\mathbb{R}^2)$ from (4.39) and the fact that $\tilde{\nu}_\infty \in H^1(\mathbb{S}^1;\mathbb{R}^2)$.

Step 3: Estimate the elastica energy up to constants. Observe that by the identities (2.22) and (4.39), together with the constant speed parametrization, we have

$$|\kappa_\infty| = \frac{|\gamma_\infty''|}{L^2(\gamma_\infty)} = \frac{|\tilde{\nu}_\infty'|}{L(\gamma_\infty)} \in L^2(\mathbb{S}^2). \tag{4.40}$$

By $\lim_{n\to\infty} L(\gamma_n) = L(\gamma_\infty)$, weak convergence of $\tilde{\nu}_n'$, and lower semi-continuity of the norms, we therefore obtain (recall Definition 2.9)

$$\hat{F}_{\infty,\sigma}(\gamma_\infty) = L(\gamma_\infty)\left(\sigma + \frac{\pi}{2}\int_0^1 \kappa_\infty^2 \,\mathrm{d}t\right) \le \liminf_{n\to\infty}\left(\sigma_n L(\gamma_n) + \frac{\pi}{2L(\gamma_n)}\int_0^1 |\tilde{\nu}_n'|^2 \,\mathrm{d}t\right). \tag{4.41}$$

Estimate (4.36) then yields (4.8), concluding the proof. □

4.2 Identifying the asymptotic system of boundary curves

The main issues in compactness for all boundary curves are, first, proving that there may only exist finitely many limit curves and, second, that their limits are a system of boundary curves to the limiting set.

For the first part, we use the estimate (4.8) together with the Gauss-Bonnet theorem for closed curves to show that the limit energy of curves blows up as their length approaches zero. The isoperimetric inequality ensures that curves whose lengths vanish in the limit $\lambda_n \to \infty$ do not contribute to the L^1 Γ-limit. Combined, these two facts also ensure that not all boundary curves would vanish in the limit.

Lemma 4.3. *Let $\lambda_n > 0$ and $\alpha_n > \frac{1}{\sqrt{2\pi}}$ be such that $\lambda_n \to \infty$ as $n \to \infty$ and such that $\sigma_n = \lambda_n^2 \left(1 - \frac{1}{2\pi\alpha_n^2}\right)$ satisfies*

$$\lim_{n\to\infty} \sigma_n = \sigma > 0. \tag{4.42}$$

Let $(\Omega_n) \subset \mathcal{A}_\pi$ be a sequence of regular sets such that

$$M := \limsup_{n\to\infty} \lambda_n^2 F_{\lambda_n,\alpha_n}(\Omega_n) < \infty, \tag{4.43}$$

and for $n \in \mathbb{N}$, let $N_n \in \mathbb{N}$ be the number of constant speed boundary curves $\{\gamma_{n,i}\}_{i=1}^{N_n}$ of the set Ω_n, enumerated by decreasing length. Finally, let $N \geq 0$ be the number of non-vanishing boundary curves as $n \to \infty$:

$$N := \sup \left(\{0\} \cup \left\{ i \in \mathbb{N} : \limsup_{n\to\infty} L(\gamma_{n,i}) > 0 \right\} \right), \tag{4.44}$$

with the convention that $L(\gamma_{n,i}) = 0$ if $i > N_n$. Then the following holds:

i) For the non-vanishing curves, we have

$$1 \leq N \leq CM\sigma^{-\frac{1}{2}}, \tag{4.45}$$

for some $C > 0$ universal.

ii) For the vanishing curves, we have

$$\lim_{n\to\infty} \sup_{i>N} L(\gamma_{n,i}) = 0 \tag{4.46}$$

and

$$\lim_{n\to\infty} \sum_{i>N} |\operatorname{int}(\gamma_{n,i})| = 0. \tag{4.47}$$

We remark that for sequences of sets whose energy is comparable (within a universal constant) to that of the minimizers we have

$$2\pi\sigma \leq M \leq C(\sigma + \sqrt{\sigma}), \tag{4.48}$$

for some $C > 0$ universal: The lower bound is due to the isoperimetric inequality and the upper bound is obtained by testing with either a disk for $\sigma \geq 1$ or an annulus for $\sigma < 1$, see estimate (2.5). Therefore, for such sets the upper estimate in (4.45) translates into

$$N \leq C\left(1 + \sqrt{\sigma}\right), \tag{4.49}$$

for some $C > 0$ universal. In particular, counterintuitively, this estimate shows that the number of non-vanishing boundary curves in a sequence of sets under consideration remains uniformly bounded for $\sigma \lesssim 1$ as $n \to \infty$. At the same time, for $\sigma \gg 1$ the number of non-vanishing boundary curves could be large as $n \to \infty$, as can be seen from an example of a configuration consisting of one disk of $O(1)$ radius and $N = O(\sigma^{1/2})$ small disks of radius $r = O(\sigma^{-1/2})$ far apart. Thus, for such sequences of sets the estimate in (4.45) is sharp.

It remains to prove that the non-vanishing curves asymptotically provide a system of boundary curves for an admissible limiting set. In order to handle the technical issue that even relatively long boundary curves may escape to infinity, we only take those curves that stay close to the origin, resulting in a further restriction on the set of indices I to consider in the system of boundary curves.

Proposition 4.4. *Under the assumptions of Lemma 4.3, let $\chi_{\Omega_n} \to \chi_{\Omega_\infty}$ in $L^1(\mathbb{R}^2)$ for some $\Omega_\infty \in \mathcal{A}_\pi$ as $n \to \infty$. Then there exist a subsequence and a family $\{\gamma_{\infty,i}\}_{i\in I} \in G(\Omega_\infty)$ of H^2-regular, constant speed curves such that for all $i \in I$ we have*

$$\gamma_{n,i} \to \gamma_{\infty,i} \qquad \text{in} \quad H^1(\mathbb{S}^1;\mathbb{R}^2), \tag{4.50}$$

where $\gamma_{n,i}$ for $i \in I$ is some sub-collection of curves from the decomposition of Ω_n into its boundary curves (modulo re-indexing).

Proof of Lemma 4.3. We begin by observing that by (3.9) and (4.43) we have that $P(\Omega_n)$ is uniformly bounded and, in particular, so are $L(\gamma_{n,i})$. By the fact that our enumeration of $\gamma_{n,i}$ in i has decreasing lengths for each n, for any subsequence n_m with $m \in \mathbb{N}$ and any $j \leq k$, $j, k \in \mathbb{N}$ we have

$$\limsup_{m\to\infty} L(\gamma_{n_m,j}) \geq \limsup_{m\to\infty} L(\gamma_{n_m,k}). \tag{4.51}$$

Step 1: Bound on the number of long curves. If $N = 0$, there is nothing to prove. Therefore, we may assume that $N \geq 1$ and let $i \in \mathbb{N}$ be such that $i \leq N \leq \infty$. Going to a subsequence n_m, $m \in \mathbb{N}$, that depends on i, we may assume that

$$\limsup_{n\to\infty} L(\gamma_{n,i}) = \lim_{m\to\infty} L(\gamma_{n_m,i}) > 0 \tag{4.52}$$

exists. By the inequality (4.51), we may repeatedly apply Lemma 4.2 to get a further, non-relabeled subsequence obeying (4.52) and closed limit curves $\gamma_{\infty,j} \in H^2(\mathbb{S}^1;\mathbb{R}^2)$ with constant speed and $L(\gamma_{\infty,j}) = \lim_{m\to\infty} L(\gamma_{n_m,j})$ for all $j = 1,\dots,i$ with the following property: We have

$$\gamma_{n_m,j} - \gamma_{n_m,j}(0) \to \gamma_{\infty,j} \qquad \text{in} \quad H^1(\mathbb{S}^1;\mathbb{R}^2), \tag{4.53}$$

as $m \to \infty$, and

$$\begin{aligned} &\sum_{j=1}^{i} L(\gamma_{\infty,j}) \left(\sigma + \frac{\pi}{2}\int_0^1 \kappa_{\infty,j}^2 \, \mathrm{d}t\right) \\ &\leq C \sum_{j=1}^{i} \liminf_{m\to\infty} \Bigg(\sigma_{n_m} L(\gamma_{n_m,j}) \\ &\qquad + \lambda_{n,m}^2 \int_{\Gamma_{n_m,j}} \int_{H_-(\nu(y))\Delta\lambda_{n_m}(\Omega_{n_m}-y)} \left|\nu(y)\cdot\frac{z}{|z|}\right| \frac{e^{-|z|}}{|z|} \, \mathrm{d}^2 z \, \mathrm{d}\mathcal{H}^1(y)\Bigg), \end{aligned} \tag{4.54}$$

for some $C > 0$ universal, where $\kappa_{\infty,j}$ is the curvature of $\gamma_{\infty,j}$. Consequently, we have

$$\sum_{j=1}^{i} L(\gamma_{\infty,j}) \left(\sigma + \frac{\pi}{2} \int_0^1 \kappa_{\infty,j}^2 \, \mathrm{d}t \right) \leq C \limsup_{n\to\infty} \lambda_n^2 F_{\lambda_n,\alpha_n}(\Omega_n) = CM, \tag{4.55}$$

for some $C > 0$ universal.

By Fenchel's theorem [13, Theorem 3 and Remark 5, Section 5.7], together with a straightforward approximation argument to remove the regularity assumption therein, for each $j = 1, \dots, i$ we have

$$\int_{\Gamma_{\infty,j}} |\kappa_{\infty,j}| \, \mathrm{d}\mathcal{H}^1 \geq 2\pi. \tag{4.56}$$

As a result of Jensen's inequality, we therefore obtain

$$L(\gamma_{\infty,j}) \int_0^1 \kappa_{\infty,j}^2 \, \mathrm{d}t \geq \frac{1}{L(\gamma_{\infty,j})} \left(\int_{\Gamma_{\infty,j}} |\kappa_{\infty,j}| \, \mathrm{d}\mathcal{H}^1 \right)^2 = \frac{4\pi^2}{L(\gamma_{\infty,j})}. \tag{4.57}$$

Combining this with estimate (4.55) and Young's inequality, we obtain

$$\left(8\pi^3 \sigma\right)^{\frac{1}{2}} i \leq \sum_{j=1}^{i} \left(\sigma L(\gamma_{\infty,j}) + \frac{2\pi^3}{L(\gamma_{\infty,j})} \right) \leq CM, \tag{4.58}$$

for some $C > 0$ universal. In particular, in view of the arbitrariness of i we have $N < \infty$, and the upper bound in (4.45) holds.

Step 2: Estimates for vanishingly short curves. To handle the boundary curves $\gamma_{\lambda,i}$ for $i > N$, assume towards a contradiction that there exists a sequence of $i_n \in \mathbb{N}$ with $i_n > N$ such that

$$\limsup_{n\to\infty} L(\gamma_{n,i_n}) > 0. \tag{4.59}$$

By discreteness of $\mathbb{N}$, we have $i' := \min_{n>0} i_n \in \mathbb{N}$ and $i' > N$. As the curves are ordered by decreasing length, we therefore also get

$$\limsup_{n\to\infty} L(\gamma_{n,i'}) \geq \limsup_{n\to\infty} L(\gamma_{n,i_n}) > 0, \tag{4.60}$$

which by way of definition (4.44) would imply a contradiction. This yields (4.46).

As a result of the isoperimetric inequality and (3.9) we have

$$\sum_{i>N} |\operatorname{int}(\gamma_{n,i})| \leq \frac{1}{4\pi} \sum_{i>N} L^2(\gamma_{n,i}) \leq \frac{\sup_{i>N} L(\gamma_{n,i})}{4\pi} \sum_{i>N} L(\gamma_{n,i}) \leq \frac{P(\Omega_n)}{4\pi} \sup_{i>N} L(\gamma_{n,i}) \to 0 \tag{4.61}$$

as $n \to \infty$, proving (4.47).

Lastly, if $N = 0$ then (4.47) would imply $|\Omega_n| \to 0$ as $n \to \infty$, contradicting the fact that $\Omega_n \in \mathcal{A}_\pi$ for all n large enough. This concludes the proof. □

Proof of Proposition 4.4. Step 1: Construct the limiting curves.

Let $N \in \mathbb{N}$ be as in Lemma 4.3. We take a non-relabeled subsequence such that for all $i = 1, \ldots, N$ we have

$$\limsup_{n\to\infty} L(\gamma_{n,i}) = \lim_{n\to\infty} L(\gamma_{n,i}) \tag{4.62}$$

and such that we either have $\limsup_{n\to\infty} \|\gamma_{n,i}\|_\infty < \infty$ or $\lim_{n\to\infty} \|\gamma_{n,i}\|_\infty = \infty$. Let $A \subset \{1, \ldots, N\}$ be the set of indices such that the former alternative holds, i.e., such that $\gamma_{n,i}$ is uniformly bounded in n for all $i \in A$. This set is non-empty, as otherwise by Lemma 4.3 and uniform boundedness of $L(\gamma_{n,i})$ this would contradict the L^1-convergence of the characteristic functions of Ω_n to a set of positive Lebesgue measure. Thus we may apply Lemma 4.2 to get another subsequence such that for all $i \in A$ there exist closed limit curves $\gamma_{\infty,i} \in H^2(\mathbb{S}^1; \mathbb{R}^2)$ with constant speed and $L(\gamma_{\infty,i}) > 0$ such that $\gamma_{n,i} \to \gamma_{\infty,i}$ in $H^1(\mathbb{S}^1; \mathbb{R}^2)$ as $n \to \infty$.

In the following, we use the abbreviations $\gamma_\infty = \{\gamma_{\infty,i}\}_{i\in A}$ and $\Gamma_\infty = \bigcup_{i\in A} \gamma_{\infty,i}([0,1])$.

Step 2: Prove $\partial^*\Omega_\infty \subset \Gamma_\infty$.

Let $x \in \partial^*\Omega_\infty$. Then there exists $r_0 \in (0,1)$ such that for all $r \in (0, r_0)$ we have [1]

$$\frac{1}{3} < \frac{|\Omega_\infty \cap B_r(x)|}{\pi r^2} < \frac{2}{3}. \tag{4.63}$$

As $\chi_{\Omega_n} \to \chi_{\Omega_\infty}$ in L^1, we have for n sufficiently big depending on r that also

$$\frac{1}{3} < \frac{|\Omega_n \cap B_r(x)|}{\pi r^2} < \frac{2}{3}, \tag{4.64}$$

so that the relative isoperimetric inequality implies

$$\frac{\mathcal{H}^1(\partial\Omega_n \cap B_r(x))}{r} > \frac{1}{C}, \tag{4.65}$$

for some $C > 0$ universal. Therefore, from (4.65) and Lemma 4.3, we get for n sufficiently large that

$$\frac{\mathcal{H}^1\left(\left(\bigcup_{i\in A} \Gamma_{n,i}\right) \cap B_r(x)\right)}{r} > \frac{1}{C}. \tag{4.66}$$

Consequently, there exists a subsequence n_k for $k \in \mathbb{N}$ with $\frac{1}{k} < r_0$, $i \in A$, and $t_k, t_\infty \in [0,1]$ such that $|\gamma_{n_k,i}(t_k) - x| < \frac{1}{k}$ and $t_k \to t_\infty$ as $k \to \infty$. Because the curves $\gamma_{n_k,i}$ converge in $C^0(\mathbb{S}^1; \mathbb{R}^2)$, we get $\gamma_{\infty,i}(t_\infty) = x$.

Step 3: For almost all $x \in \mathbb{R}^2$ we have $x \notin \Gamma_\infty$ and

$$\mathcal{I}(\gamma_\infty, x) \equiv \chi_{\Omega_\infty}(x) \pmod 2. \tag{4.67}$$

As Γ_n for $n \in \mathbb{N} \cup \{\infty\}$ has Hausdorff dimension 1, we have $\left|\bigcup_{n\in\mathbb{N}\cup\{\infty\}} \Gamma_n\right| = 0$. Let $x \in \mathbb{R}^2 \setminus \Gamma_\infty$. Since the curves $\gamma_{n,i}$ for $i \in A$ converge in $H^1(\mathbb{S}^1; \mathbb{R}^2)$ and $C^0(\mathbb{S}^1; \mathbb{R}^2)$ and since the set $\mathbb{R}^2 \setminus \Gamma_\infty$ is open, we have

$$\mathcal{I}(\gamma_\infty, x) = \lim_{n\to\infty} \sum_{i\in A} \mathcal{I}(\gamma_{n,i}, x). \tag{4.68}$$

As for $i \in \mathbb{N}$ with $i \leq N$ and $i \notin A$ the curves $\gamma_{n,i}$ run off to infinity, we must have $\mathcal{I}(\gamma_{n,i}, x) = 0$ for sufficiently large n and those values of i. We thus get

$$\mathcal{I}(\gamma_\infty, x) = \lim_{n\to\infty} \sum_{i=1}^{N} \mathcal{I}(\gamma_{n,i}, x) \tag{4.69}$$

Due to the convergence (4.47), we can choose a non-relabeled subsequence such that

$$\sum_{n=1}^{\infty} \sum_{i>N} \left|\overline{\mathrm{int}(\gamma_{n,i})}\right| < \infty. \tag{4.70}$$

The Borel-Cantelli Lemma [45, p. 42] therefore implies that for almost all $x \in \mathbb{R}^2$ and $n_0(x)$ sufficiently big we have $x \notin \overline{\mathrm{int}(\gamma_{n,i})}$ for all $i > N$ and all $n > n_0(x)$. In particular, for almost all $x \in \mathbb{R}^2$ we get

$$\mathcal{I}(\gamma_\infty, x) = \lim_{n\to\infty} \sum_{i=0}^{N_n} \mathcal{I}(\gamma_{n,i}, x), \tag{4.71}$$

since eventually, the sum only has at most N non-zero terms.

According to the representation (2.21), for almost all $x \in \mathbb{R}^2$ we have

$$\sum_{i=1}^{N_n} \mathcal{I}(\gamma_{n,i}, x) \equiv \chi_{\Omega_n}(x) \pmod 2. \tag{4.72}$$

By $|\Omega_\infty \Delta \Omega_n| \to 0$ as $n \to \infty$, we may choose another, non-relabeled subsequence such that $\chi_{\Omega_n} \to \chi_{\Omega_\infty}$ pointwise almost everywhere. Combining these insights with the convergence (4.71), for almost all $x \in \mathbb{R}^2$ we get

$$\mathcal{I}(\gamma_\infty, x) \equiv \chi_{\Omega_\infty}(x) \pmod 2. \tag{4.73}$$

Step 4: Prove $\Omega^*_\infty = \mathrm{int}\left(A^o_{\gamma_\infty} \cup \Gamma_\infty\right)$.

We recall that $A^o_{\gamma_\infty}$ and Ω^*_∞ are defined via (2.25) and (2.26), respectively. We first prove the inclusion $\Omega^*_\infty \subset \mathrm{int}\left(A^o_{\gamma_\infty} \cup \Gamma_\infty\right)$. Notice that since the set Ω^*_∞ is open, it is enough to show that $\Omega^*_\infty \subset A^o_{\gamma_\infty} \cup \Gamma_\infty$. So, let $x \in \Omega^*_\infty$. If $x \in \Gamma_\infty$, there is nothing to prove. Otherwise, there exists $r \in (0,1)$ such that $|B_r(x) \setminus \Omega_\infty| = 0$ and $B_r(x) \cap \Gamma_\infty = \emptyset$. By Step 3, for almost all $\tilde{x} \in B_r(x)$ we have

$$\mathcal{I}(\gamma_\infty, \tilde{x}) \equiv 1 \pmod 2. \tag{4.74}$$

Continuity of the index in $B_r(x)$ then gives

$$\mathcal{I}(\gamma_\infty, x) \equiv 1 \pmod 2. \tag{4.75}$$

Consequently, we get $x \in A^o_{\gamma_\infty}$.

Let now $x \notin \Omega^*_\infty$. Then for every $r \in (0,1)$ we have

$$|B_r(x) \setminus (\Omega_\infty \cup \Gamma_\infty)| > 0. \tag{4.76}$$

Again by Step 3, for all $r \in (0,1)$ and almost all $\tilde{x} \in B_r(x) \setminus (\Omega_\infty \cup \Gamma_\infty)$ we get that

$$\mathcal{I}(\gamma_\infty, \tilde{x}) \equiv 0 \pmod 2, \tag{4.77}$$

so that there exists a sequence of points $\tilde{x}_k \in B_{k^{-1}}(x) \setminus \left(A^o_{\gamma_\infty} \cup \Gamma_\infty\right)$ for all $k \in \mathbb{N}$. Therefore, we have $x \notin \mathrm{int}\left(A^o_{\gamma_\infty} \cup \Gamma_\infty\right)$, proving the claim of this step.

Finally, combining the statements of Steps 2 and 4 we obtain that $\{\gamma_{\infty,i}\}_{i\in A} \in G(\Omega_\infty)$, concluding the proof. □

5 Γ-convergence

5.1 Upper bound

Both the upper and the lower bounds crucially depend on the representation (3.8). In the upper bound presented in Lemma 5.1, we observe that the anisotropic blowup $A_\lambda R_{\nu(y)}(\Omega - y)$ of a sufficiently regular set $\Omega \subset \mathbb{R}^2$ converges to the subgraph of the parabola $z_1 \mapsto -\frac{1}{2}\kappa(y)z_1^2$. The integral in the non-local term in representation (3.8) can then be explicitly calculated in the limit using Fubini's theorem, the integral over z_2 giving the dependence of the energy on $\kappa^2(y)$.

Lemma 5.1. *Let $\alpha > 0$ and let Ω be a regular set. Then, as $\lambda \to \infty$ we have*

$$F_{\lambda,\alpha}(\Omega) \leq \left(1 - \frac{1}{2\pi\alpha^2}\right) P(\Omega) + \frac{1}{8\pi\alpha^4\lambda^2} \int_{\partial\Omega} \kappa^2 \, \mathrm{d}\mathcal{H}^1 + o\left(\lambda^{-2}\right).$$

Proof. All constants in this proof may depend on Ω and α, in contrast to the rest of the paper.

By the identity (3.8), we only need to compute the non-local term therein. Let $y \in \partial\Omega$. For the sake of convenience, in this proof we parametrize $\mathbb{S}^1$ on $[-\frac{1}{2}, \frac{1}{2}]$ instead of the usual parametrization on the unit interval. Let $\gamma : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}^2$ be a constant speed parametrization of the connected component of $\partial\Omega$ containing y and such that $\gamma(0) = y$.

For $\lambda > 1$ and $s \in \mathbb{R}$, let $T_\lambda s := (L(\gamma)\lambda)^{-1}s$, so that s plays the role of an arc length parameter after blowup by $T_\lambda^{-1} = L(\gamma)\lambda$. Let $\tau(t) := \gamma'(t)/L(\gamma)$ be the unit tangent vector to γ at point $\gamma(t)$, and let

$$g_\lambda(s) := \lambda\tau(0) \cdot (\gamma(T_\lambda s) - \gamma(0)), \tag{5.1}$$
$$h_\lambda(s) := \lambda^2\nu(0) \cdot (\gamma(T_\lambda s) - \gamma(0)). \tag{5.2}$$

be the local Cartesian coordinates of $\gamma(s)$ with respect to the orthonormal basis $\{\tau(0), \nu(0)\}$ after anisotropic blowup, see Figure 6. By Taylor expansion and identity (2.17), for $s \in \mathbb{R}$ we have

$$g_\lambda(s) = s + O(\lambda^{-2}s^3), \tag{5.3}$$
$$g_\lambda'(s) = 1 + O(\lambda^{-2}s^2), \tag{5.4}$$
$$h_\lambda(s) = -\frac{\kappa(y)}{2}s^2 + O(\lambda^{-1}s^3), \tag{5.5}$$

as well as

$$|g_\lambda'(s)| \leq 1 \tag{5.6}$$

for all s. In particular, if $\lambda^{-1}s$ is sufficiently small, the map g_λ is monotone increasing with $g_\lambda'(s) \geq \frac{1}{2}$ and is therefore invertible. Thus for all $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} &A_\lambda\left(R_{\nu(y)}(\Omega - y) \cap (-\varepsilon, \varepsilon)^2\right) \\ &= \left\{(z_1, z_2) : z_1 \in (-\lambda\varepsilon, \lambda\varepsilon), z_2 \in \left(-\lambda^2\varepsilon, h_\lambda(g_\lambda^{-1}(z_1))\right)\right\}. \end{aligned} \tag{5.7}$$

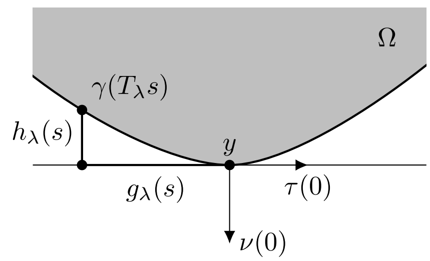

Figure 6: Sketch of the anisotropic blowup of $\gamma(T_\lambda s)$ in the basis $\{\tau(0), \nu(0)\}$ with components $g_\lambda(s)$ and $h_\lambda(s)$.

An explicit calculation in polar coordinates for $z = (z_1, z_2)$ gives

$$\int_{\mathbb{R}^2 \setminus A_\lambda(-\varepsilon,\varepsilon)^2} |z_2| \, \frac{e^{-\alpha\sqrt{z_1^2+\frac{z_2^2}{\lambda^2}}}}{z_1^2 + \frac{z_2^2}{\lambda^2}} \, \mathrm{d}^2 z \ \le \ \lambda^2 \int_{B^c_{\lambda\varepsilon}(0)} \frac{e^{-\alpha|z|}}{|z|} \, \mathrm{d}^2 z = O\left(\frac{\lambda^2}{\alpha} e^{-\varepsilon\alpha\lambda}\right). \tag{5.8}$$

By Fubini's theorem, we therefore have

$$\begin{aligned} &\int_{H^0_-(e_2)\Delta A_\lambda R_{\nu(y)}(\Omega - y)} |z_2| \, \frac{e^{-\alpha\sqrt{z_1^2+\frac{z_2^2}{\lambda^2}}}}{z_1^2 + \frac{z_2^2}{\lambda^2}} \, \mathrm{d}^2 z \\ &= \int_{-\lambda\varepsilon}^{\lambda\varepsilon} \int_0^{|h_\lambda(g_\lambda^{-1}(z_1))|} |z_2| \, \frac{e^{-\alpha\sqrt{z_1^2+\frac{z_2^2}{\lambda^2}}}}{z_1^2 + \frac{z_2^2}{\lambda^2}} \, \mathrm{d}z_2 \, \mathrm{d}z_1 + O\left(\frac{\lambda^2}{\alpha} e^{-\varepsilon\alpha\lambda}\right). \end{aligned} \tag{5.9}$$

By monotonicity of the exponential function and inverse powers, as well as estimate (5.6), we get

$$\begin{aligned} &\int_{-\lambda\varepsilon}^{\lambda\varepsilon} \int_0^{|h_\lambda(g_\lambda^{-1}(z_1))|} |z_2| \, \frac{e^{-\alpha\sqrt{z_1^2+\frac{z_2^2}{\lambda^2}}}}{z_1^2 + \frac{z_2^2}{\lambda^2}} \, \mathrm{d}z_2 \, \mathrm{d}z_1 \\ &\le \int_{-\lambda\varepsilon}^{\lambda\varepsilon} \int_0^{|h_\lambda(g_\lambda^{-1}(z_1))|} |z_2| \, \frac{e^{-\alpha|z_1|}}{z_1^2} \, \mathrm{d}z_2 \, \mathrm{d}z_1 \\ &\le \frac{1}{2} \int_{g_\lambda^{-1}(-\lambda\varepsilon)}^{g_\lambda^{-1}(\lambda\varepsilon)} h_\lambda^2(s) \frac{e^{-\alpha|g_\lambda(s)|}}{g_\lambda^2(s)} \, \mathrm{d}s. \end{aligned} \tag{5.10}$$

Using the expansions (5.3) and (5.5), we get

$$\begin{aligned} &\frac{1}{2} \int_{g_\lambda^{-1}(-\lambda\varepsilon)}^{g_\lambda^{-1}(\lambda\varepsilon)} h_\lambda^2(s) \frac{e^{-\alpha|g_\lambda(s)|}}{g_\lambda^2(s)} \, \mathrm{d}s \\ &= \frac{1}{8} \int_{g_\lambda^{-1}(-\lambda\varepsilon)}^{g_\lambda^{-1}(\lambda\varepsilon)} s^2 \left(\kappa^2(y) + O(\lambda^{-1}|s|)\right) e^{-\alpha|s|(1+O(\lambda^{-2}s^2))} \, \mathrm{d}s. \end{aligned} \tag{5.11}$$

As g_λ' is strictly positive on $(-\lambda\varepsilon, \lambda\varepsilon)$, we have $\lambda^{-1}|s| \leq C\varepsilon$ for all $s \in \big(g_\lambda^{-1}(-\lambda\varepsilon), g_\lambda^{-1}(\lambda\varepsilon)\big)$. For $\varepsilon > 0$ sufficiently small depending only on $\partial\Omega$, the first error term in identity (5.11) is estimated by

$$\int_{g_\lambda^{-1}(-\lambda\varepsilon)}^{g_\lambda^{-1}(\lambda\varepsilon)} \lambda^{-1}|s|^3 e^{-\frac{\alpha}{C}|s|}\,\mathrm{d}s \leq \lambda^{-1}\int_{\mathbb{R}} |s|^3 e^{-\frac{\alpha}{C}|s|}\,\mathrm{d}s = O(\alpha^{-4}\lambda^{-1}). \tag{5.12}$$

Similarly, for all $s \in \big(g_\lambda^{-1}(-\lambda\varepsilon), g_\lambda^{-1}(\lambda\varepsilon)\big)$ we have

$$e^{-\alpha|s|\left(1+O(\lambda^{-2}s^2)\right)} \leq e^{-(1-C\varepsilon)\alpha|s|}, \tag{5.13}$$

so that the identity (5.11) can be estimated from above as

$$\begin{aligned} &\frac{1}{2}\int_{g_\lambda^{-1}(-\lambda\varepsilon)}^{g_\lambda^{-1}(\lambda\varepsilon)} h_\lambda^2(s)\frac{e^{-\alpha|g_\lambda(s)|}}{g_\lambda^2(s)}\,\mathrm{d}s \\ &\leq \frac{\kappa^2(y)}{8}\int_{g_\lambda^{-1}(-\lambda\varepsilon)}^{g_\lambda^{-1}(\lambda\varepsilon)} s^2 e^{-(1-C\varepsilon)\alpha|s|}\,\mathrm{d}s + O(\alpha^{-4}\lambda^{-1}). \end{aligned} \tag{5.14}$$

Finally, explicit integration gives

$$\begin{aligned} &\frac{\kappa^2(y)}{8}\int_{g_\lambda^{-1}(-\lambda\varepsilon)}^{g_\lambda^{-1}(\lambda\varepsilon)} s^2 e^{-(1-C\varepsilon)\alpha)|s|}\,\mathrm{d}s \\ &\leq \frac{\kappa^2(y)}{8}\int_{\mathbb{R}} s^2 e^{-(1-C\varepsilon)\alpha|s|}\,\mathrm{d}s \\ &= \frac{\kappa^2(y)}{4\alpha^3(1-C\varepsilon)^3}\int_0^\infty s^2 e^{-s}\,\mathrm{d}s \\ &= \frac{\kappa^2(y)}{2\alpha^3(1-C\varepsilon)^3}. \end{aligned} \tag{5.15}$$

Combining the estimates (5.9), (5.10), (5.14), and (5.15), for all $\varepsilon > 0$ sufficiently small depending only on $\partial\Omega$ we get

$$\begin{aligned} &\int_{H_-^0(e_2)\Delta A_\lambda R_{\nu(y)}(\Omega-y)} |z_2|\,\frac{e^{-\alpha\sqrt{z_1^2+\frac{z_2^2}{\lambda^2}}}}{z_1^2+\frac{z_2^2}{\lambda^2}}\,\mathrm{d}^2 z \\ &\leq \frac{\kappa^2(y)}{2\alpha^3(1-C\varepsilon)^3} + O\left(\alpha^{-4}\lambda^{-1} + \frac{\lambda^2}{\alpha}e^{-\varepsilon\alpha\lambda}\right) \end{aligned} \tag{5.16}$$

Choosing $\varepsilon = \lambda^{-\frac{1}{2}}$ and integrating over $y \in \partial\Omega$, we obtain the statement. □

5.2 Lower bound

The argument for the lower bound follows much the same strategy as the upper bound. However, we of course have to ensure that the computation is valid along a sequence of only weakly convergent objects. We first point out that the microscopic difference quotients considered in the proof of Lemma 4.2 in fact converge to the curvature in a weak sense.

Lemma 5.2. *Under the assumptions of Lemma 4.2, let* (γ_{n_k}) *be the subsequence and let* γ_∞ *be its limit from the conclusion of Lemma 4.2. Then, if* κ_∞ *is the curvature of* γ_∞ *we have*

$$\lambda_{n_k}\left[\nu_{n_k}\left(t+\frac{s}{L(\gamma_{n_k})\lambda_{n_k}}\right)-\nu_{n_k}(t))\right] \rightharpoonup s\kappa_\infty(t)\frac{\gamma_\infty'(t)}{L(\gamma_\infty)} \tag{5.17}$$

as $k\to\infty$ *in* $\mathcal{D}'(\mathbb{S}^1;\mathbb{R}^2)$ *for all* $s\in\mathbb{R}$*, as well as in* $\mathcal{D}'(\mathbb{S}^1\times\mathbb{R};\mathbb{R}^2)$*.*

For the proof of the lower bound proper, the convergence (5.17) ensures that the limiting object after the anisotropic blowup is the expected parabola, while the second part of Lemma 4.1 allows us to work at points on which the boundaries along the sequence are not too ill-behaved. However, the proof is somewhat heavy on standard, measure-theoretic details. We recall the definition of $\hat{F}_{\infty,\sigma}$ in (2.23).

Proposition 5.3. *Let* $\lambda_n\to\infty$ *and* $\alpha_n>\frac{1}{\sqrt{2\pi}}$ *be sequences such that* $\sigma_n=\lambda_n^2\left(1-\frac{1}{2\pi\alpha_n^2}\right)$ *satisfies*

$$\lim_{n\to\infty}\sigma_n=\sigma>0. \tag{5.18}$$

Let $\Omega_n\in\mathcal{A}_\pi$ *for* $n\in\mathbb{N}$ *be regular sets such that* $\chi_n\to\chi_{\Omega_\infty}$ *in* $L^1(\mathbb{R}^2)$ *for* $\Omega_\infty\in\mathcal{A}_\pi$ *and such that*

$$\limsup_{n\to\infty}\lambda_n^2F_{\lambda_n,\alpha_n}(\Omega_n)<\infty. \tag{5.19}$$

Furthermore, let there exist $I\subset\mathbb{N}$ *finite, and a family* $\gamma_\infty:=\{\gamma_{\infty,i}\}_{i\in I}\in G(\Omega_\infty)$ *of* H^2*-regular, constant speed curves such that for all* $i\in I$ *we have*

$$\gamma_{n,i}\to\gamma_{\infty,i} \qquad \text{in } H^1(\mathbb{S}^1;\mathbb{R}^2), \tag{5.20}$$

as $n\to\infty$*, where* $\gamma_{n,i}$ *for* $i\in I$ *is some sub-collection of curves from the decomposition of* Ω_n *into its boundary curves. Then*

$$\hat{F}_{\infty,\sigma}(\gamma_\infty)\le\liminf_{n\to\infty}\lambda_n^2F_{\lambda_n,\alpha_n}(\Omega_{\lambda_n}). \tag{5.21}$$

Proof of Lemma 5.2. Let $s\in\mathbb{R}$. Let $\xi\in C^\infty(\mathbb{S}^1)$ be a smooth, periodic test function parametrized by $t\in[0,1]$. By the strong convergence of ν_n to ν_∞ obtained in Lemma 4.2 with the help of identity (2.14), we get

$$\begin{aligned}
&\lim_{n_k\to\infty}\int_0^1\lambda_{n_k}\left[\nu_{n_k}\left(t+\frac{s}{L(\gamma_{n_k})\lambda_{n_k}}\right)-\nu_{n_k}(t)\right]\xi(t)\,\mathrm{d}t\\
&=\lim_{k\to\infty}\int_0^1\nu_{n_k}(t)\lambda_{n_k}\left[\xi\left(t-\frac{s}{L(\gamma_{n_k})\lambda_{n_k}}\right)-\xi(t)\right]\,\mathrm{d}t\\
&=-\int_0^1\nu_\infty(t)\frac{s}{L(\gamma_\infty)}\partial_t\xi(t)\,\mathrm{d}t.
\end{aligned} \tag{5.22}$$

Together with the fact that $\nu_\infty'=\kappa_\infty\gamma_\infty'$ a.e., see identity (2.16), we get the first desired convergence in (5.17). To obtain the second, simply repeat the above argument after testing with a function $\xi\in C^\infty(\mathbb{S}^1\times\mathbb{R})$ with compact support in the second variable. □

Proof of Proposition 5.3. Step 1: Choose appropriate subsequences.

We first choose a subsequence (not relabeled) such that we may apply Lemma 4.2 to the sequences $\gamma_{n,i}$ for all $i \in I$ and such that

$$\sum_{n\in\mathbb{N}} \lambda_n^{-\frac{1}{2}} < \infty, \tag{5.23}$$

the latter being chosen to be able to subsequently apply the Borel-Cantelli lemma in Steps 3 and 4.

Let $i \in I$. Furthermore, let us abbreviate $L(\gamma_{n,i})$ by $L_{n,i}$, and recall that $|\gamma_{n,i}'| = L_{n,i}$ everywhere. For all $K \in \mathbb{N}$ and $t \in [0,1]$ we define

$$Z_{K,n,i,t} := \left\{ y \in \partial\Omega \cap B_{K\lambda_n^{-1}}(\gamma_{n,i}(t)) : \nu(y)\cdot\nu(\gamma_{n,i}(t)) \le 0 \right\}. \tag{5.24}$$

Due to the estimate (4.3), Fubini's theorem, and estimate (3.9), we have for all $K \in \mathbb{N}$ and $i \in I$ that

$$L_{n,i}\int_0^1 \mathcal{H}^1(Z_{K,n,i,t})\,\mathrm{d}t \le C_K P(\Omega_n) F_{\lambda_n,\alpha}(\Omega_n) \le \frac{C_K}{\sigma_n}\lambda_n^2 F_{\lambda_n,\alpha}^2(\Omega_n), \tag{5.25}$$

for some $C_K > 0$ depending only on K and all n sufficiently large. Therefore, after taking yet another subsequence, by (5.19) we have for almost all $t \in [0,1]$, all $K \in \mathbb{N}$, and all $i \in I$ that

$$\lim_{n\to\infty} \lambda_n^{\frac{3}{2}} \mathcal{H}^1(Z_{K,n,i,t}) = 0, \tag{5.26}$$

where the exponent $\frac{3}{2} < 2$ was chosen as sufficient to complement estimate (5.23) when passing from estimate (5.80) to estimate (5.82) in what follows.

Let $n \in \mathbb{N}$ and $i \in I$. For $s \in \mathbb{R}$, let $T_{n,i}s := (L_{n,i}\lambda_n)^{-1}s$, so that, as in the proof of Lemma 5.1, the variable s plays the role of a microscopic arc length parameter. In analogy with the definitions (5.1) and (5.2), for $t \in [0,1]$ and $s \in \mathbb{R}$ we define $\tau_{n,i}(t) := \gamma_{n,i}'(t)/L_{n,i}$ and the functions

$$g_{n,i,t}(s) := \lambda_n \tau_{n,i}(t)\cdot\left(\gamma_{n,i}(t+T_{n,i}s) - \gamma_{n,i}(t)\right), \tag{5.27}$$

$$h_{n,i,t}(s) := \lambda_n^2 \nu_{n,i}(t)\cdot\left(\gamma_{n,i}(t+T_{n,i}s) - \gamma_{n,i}(t)\right), \tag{5.28}$$

giving

$$|g_{n,i,t}'(s)| \le 1. \tag{5.29}$$

Then for $s \in \mathbb{R}$, we have

$$\begin{aligned} &\int_0^1 \left(1 - g_{n,i,t}'(s)\right)\,\mathrm{d}t \\ &= \frac{1}{2L_{n,i}^2}\int_0^1 \left(|\gamma_{n,i}'(t)|^2 + |\gamma_{n,i}'(t+T_{n,i}s)|^2 - 2\gamma_{n,i}'(t)\cdot\gamma_{n,i}'(t+T_{n,i}s)\right)\,\mathrm{d}t \\ &= \frac{1}{2L_{n,i}^2}\int_0^1 |\gamma_{n,i}'(t+T_{n,i}s) - \gamma_{n,i}'(t)|^2\,\mathrm{d}t, \end{aligned} \tag{5.30}$$

so that together with (5.19) the bound (4.1) implies

$$\lim_{n\to\infty} \lambda_n^{\frac{1}{2}} \int_0^1 \left(1 - g'_{n,i,t}(s)\right) \mathrm{d}t = 0 \tag{5.31}$$

locally uniformly in s, where the exponent $\frac{1}{2} < 2$ is chosen to again complement estimate (5.23) to arrive at (5.34) in what follows.

Let $K \in \mathbb{N}$. Using $g_{n,i,t}(0) = 0$, integrating in s and using Fubini's theorem, we obtain

$$\lambda_n^{\frac{1}{2}} \int_0^1 \max_{s\in[-K,K]} |s - g_{n,i,t}(s)| \,\mathrm{d}t \to 0 \tag{5.32}$$

in the limit $n \to \infty$. Passing to a subsequence, for all $i \in I$ and almost all $t \in [0,1]$ we get

$$\lambda_n^{\frac{1}{2}} \max_{s\in[-K,K]} |s - g_{n,i,t}(s)| \to 0, \tag{5.33}$$

and together with the summability (5.23) that

$$\sum_{n\in\mathbb{N}} \max_{s\in[-K,K]} |s - g_{n,i,t}(s)| < \infty. \tag{5.34}$$

Additionally, for $t \in [0,1]$ and all $s \in \mathbb{R}$ we calculate

$$h'_{n,i,t}(s) = \frac{\lambda_n}{L_{n,i}} \nu_{n,i}(t) \cdot \gamma'_{n,i}(t + T_{n,i}s) = -\lambda_n \frac{\gamma'_{n,i}(t)}{L_{n,i}} \cdot \left(\nu_{\lambda_n}(t + T_{n,i}s) - \nu_{n,i}(t)\right) \tag{5.35}$$

Consequently, by (5.19) and Lemma 4.1 we have

$$\sup_{n\in\mathbb{N}} \sup_{s\in[-K,K]} \int_0^1 |h'_{n,i,t}(s)|^2 \,\mathrm{d}t < \infty, \tag{5.36}$$

and we get from convergence (5.17), a weak-times-strong argument, and $|\gamma'_{n,i}| = L_{n,i}$ that

$$h'_{n,i,t}(s) \rightharpoonup -s\kappa_{\infty,i}(t) \tag{5.37}$$

in $L^2_t(0,1)$ for all $s \in \mathbb{R}$, as well as in $L^2_{t,s}((0,1) \times (-K,K))$. Here and in the following, the subscripts in the notation for Lebesgue spaces denote the variables in which the integration is performed.

Again, using $h_{n,i,t}(0) = 0$ we may apply the fundamental theorem of calculus in s and Jensen's inequality to obtain

$$\begin{aligned} \sup_{n\in\mathbb{N}} \int_0^1 \sup_{s\in[-K,K]} \frac{|h_{n,i,t}(s)|^2}{|s|} \,\mathrm{d}t &\le \sup_{n\in\mathbb{N}} \int_0^1 \int_{-K}^K |h'_{n,i,t}(s)|^2 \,\mathrm{d}s \,\mathrm{d}t \\ &\le 2K \sup_{n\in\mathbb{N}} \sup_{s\in[-K,K]} \int_0^1 |h'_{n,i,t}(s)|^2 \,\mathrm{d}t \\ &< \infty. \end{aligned} \tag{5.38}$$

For all $s \in \mathbb{R}$ and $\xi \in L^2(0,1)$, we also get

$$\int_0^1 h_{n,i,t}(s)\xi(t) \,\mathrm{d}t = \int_0^s \int_0^1 h'_{n,i,t}(s')\xi(t) \,\mathrm{d}t \,\mathrm{d}s' \to -\int_0^1 \frac{s^2}{2} \kappa_{\infty,i}(t)\xi(t) \,\mathrm{d}t, \tag{5.39}$$

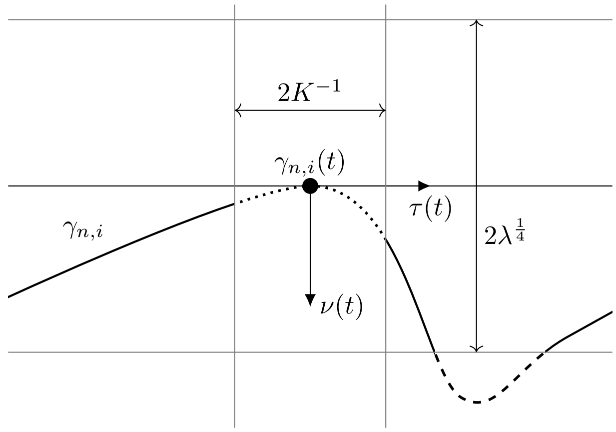

Figure 7: Sketch of $\gamma_{n,i}(t + T_\lambda s)$ on $S^1_{K,n,i,t} \cap S^2_{K,n,i,t}$. The dotted part of $\gamma_{n,i}$ is outside $S^1_{K,n,i,t}$, the dashed part is outside of $S^2_{K,n,i,t}$.

from the weak $L^2_{t,s}((0,1) \times (-K,K))$ convergence (5.37). Therefore, for all $s \in \mathbb{R}$ we have

$$h_{n,i,t}(s) \rightharpoonup -\frac{s^2}{2}\kappa_{\infty,i}(t) \tag{5.40}$$

in $L^2_t(0,1)$. Furthermore, estimate (5.38) implies for all $i \in I$ and $K \in \mathbb{N}$ that

$$\lambda_n^{-\frac{1}{4}} \max_{s\in[-K,K]} |h_{n,i,t}(s)| \to 0 \tag{5.41}$$

in $L^2_t(0,1)$ as $n \to \infty$ and thus also for almost all $t \in [0,1]$ after passage to one more, final subsequence. Here, we chose the exponent $-\frac{1}{4} < 0$, so that we can later deduce the convergence (5.95) from the estimate (5.31).

Step 2: Given $i \in I$, identify sets over which the curve $\gamma_{n,i}$ is locally a graph over its tangent space for sufficiently large $n \in \mathbb{N}$.

Let $i \in I$ and $K \in \mathbb{N}$ be fixed throughout this step of the proof.

For every $n \in \mathbb{N}$ and $t \in [0,1]$, we consider the sets

$$S^1_{K,n,i,t} := \left\{ s \in [-K,K] : |g_{n,i,t}(s)| \geq K^{-1} \right\}, \tag{5.42}$$

$$S^2_{K,n,i,t} := \left\{ s \in [-K,K] : |h_{n,i,t}(s)| \leq \lambda_n^{\frac{1}{4}} \right\}. \tag{5.43}$$

The set $S^1_{K,n,i,t}$ cuts away the origin, while the set $S^2_{K,n,i,t}$ makes sure that $h_{n,i,t}$ is not too large, see Figure 7.

We will want to consider $g_{n,i,t}(s)$ as a parametrization for $\gamma_{n,i}$ around its own tangent line at $\gamma_{n,i}(t)$. However, there is no reason why it should be injective, see Figure 8. Somewhat abusing notation, we therefore define the generalized inverse of $g_{n,i,t}$ for $n \in \mathbb{N}$, $t \in [0,1]$, and $z_1 \in g_{n,i,t}([-K,K])$ as

$$g^{-1}_{n,i,t}(z_1) := \begin{cases} \inf\{s' \in [0,K] : g_{n,i,t}(s') \geq z_1\} & \text{if } z_1 \geq 0, \\ \sup\{s' \in [-K,0] : g_{n,i,t}(s') \leq z_1\} & \text{if } z_1 < 0 \end{cases} \tag{5.44}$$

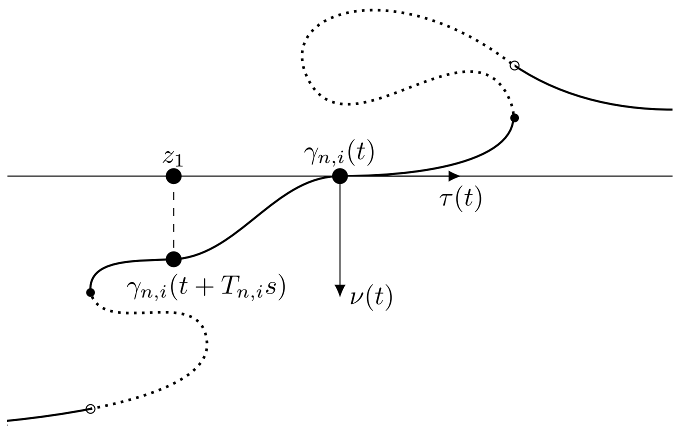

Figure 8: Sketch of $\gamma_{n,i}(t+T_\lambda s)$ on $S^3_{K,n,i,t}$. Parts of $\gamma_{n,i}$ outside of $S^3_{K,n,i,t}$ are shown as dotted. The solid points signify the endpoints contained in $S^3_{K,n,i,t}$. For $z_1 \in \mathbb{R}$, the inverse $g^{-1}_{n,i,t}(z_1)$ is the microscopic arc length parameter $s \in S^3_{K,n,i,t}$ such that $g_{n,i,t}(s) = z_1$.

to be left-continuous for positive s and right-continuous for negative s. By continuity of $g_{n,i,t}$, we indeed have $g_{n,i,t} \circ g^{-1}_{n,i,t}(z_1) = z_1$ for all $n \in \mathbb{N}$, $t \in [0,1]$, and $z_1 \in g_{n,i,t}([-K,K])$. While we may not have $g^{-1}_{n,i,t} \circ g_{n,i,t}(s) = s$ for all $s \in [-K,K]$, this does hold for all $n \in \mathbb{N}$, $t \in [0,1]$ on

$$\begin{aligned} S^3_{K,n,i,t} := &\left\{ s \in [0,K] : g_{n,i,t}(s') < g_{n,i,t}(s)\, \forall s' \in [0,s) \right\} \\ &\cup \left\{ s \in [-K,0] : g_{n,i,t}(s') > g_{n,i,t}(s)\, \forall s' \in (s,0] \right\}, \end{aligned} \tag{5.45}$$

by construction. As a result of $g_{n,i,t}(0) = 0$, we have $\operatorname{sgn}(g_{n,i,t}(s)) = \operatorname{sgn}(s)$ for all $s \in S^3_{K,n,i,t}$.

Finally, when comparing Ω with its tangent half-plane, we have to contend with the possibility of small holes in Ω or small pieces of Ω lying between $\gamma_{n,i}$ and its tangent line at t, see Figure 9. Therefore, we also consider the parametrized line segment

$$l_{n,i,t,s}(r) := g_{n,i,t}(s)e_1 + r h_{n,i,t}(s) e_2 \tag{5.46}$$

for $r \in [0,1]$ and the set

$$\begin{aligned} S^4_{K,n,i,t} := \Big\{ &s \in [-K,K] : \forall r \in (0,1) : \\ &l_{n,i,t,s}(r) \in \left(\overline{H^0_-(e_2)} \setminus A_{\lambda_n} R_{\nu_{n,i}(t)} \left(\Omega_n - \gamma_{n,i}(t) \right) \right) \\ &\qquad \cup \left(A_{\lambda_n} R_{\nu_{n,i}(t)} \left(\overline{\Omega_n} - \gamma_{n,i}(t) \right) \setminus H^0_-(e_2) \right) \Big\}, \end{aligned} \tag{5.47}$$

which rules out this pathological behaviour. Notice that by definition we have $l_{n,i,t,s}(0) \in \partial H^0_-(e_2)$, while $l_{n,i,t,s}(1) \in A_{\lambda_n} R_{\nu_{n,i}(t)}(\partial \Omega_n - \gamma_{n,i}(t))$.

Let

$$S_{K,n,i,t} := S^1_{K,n,i,t} \cap S^2_{K,n,i,t} \cap S^3_{K,n,i,t} \cap S^4_{K,n,i,t}. \tag{5.48}$$

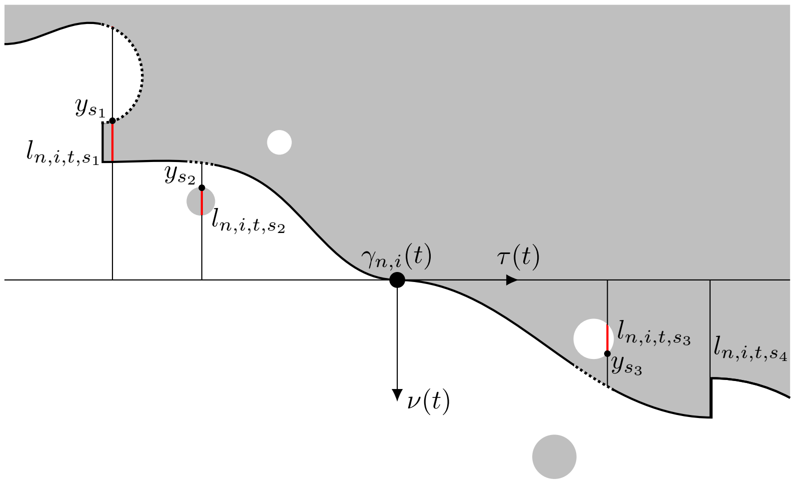

Figure 9: Sketch of $\gamma_{n,i}(t+T_\lambda s)$ on $S^4_{K,n,i,t}$. The parts of $\gamma_{n,i}$ not along $S^4_{K,n,i,t}$ are shown as dotted, the set Ω is shown in gray. We have $0, s_4 \in S^4_{K,n,i,t}$, but $s_1, s_2, s_3 \notin S^4_{K,n,i,t}$. The corresponding sets D_{s_1}, D_{s_2} and D_{s_3} are shown as red lines, while y_{s_1}, y_{s_2} and y_{s_3} are shown as small dots.

Additionally, we define the sets $S_{K,\infty,i,t}$ and $S^j_{K,\infty,i,t}$ for $j = 1,2,3,4$ as

$$S_{K,\infty,i,t} := \bigcup_{n\in\mathbb{N}} \bigcap_{n'\in\mathbb{N}:n'>n} S_{K,n',i,t}, \tag{5.49}$$

$$S^j_{K,\infty,i,t} := \bigcup_{n\in\mathbb{N}} \bigcap_{n'\in\mathbb{N}:n'>n} S^j_{K,n',i,t}, \tag{5.50}$$

being the sets of points that for sufficiently large n lie in all sets $S_{K,n',i,t}$, resp., $S^j_{K,n',i,t}$ for $n' \geq n$. and observe the decomposition

$$S_{K,\infty,i,t} = S^1_{K,\infty,i,t} \cap S^2_{K,\infty,i,t} \cap S^3_{K,\infty,i,t} \cap S^4_{K,\infty,i,t}. \tag{5.51}$$

The convergence (5.33) states that for all $i \in I$, almost all $t \in [0,1]$, and all $s \in [-K,K]$, we have $g_{n,i,t}(s) \to s$ as $n \to \infty$. Therefore, for such $t \in [0,1]$, all $s \in S^1_{K,\infty,i,t}$ satisfy $s \in S^1_{K,n,i,t}$ for $n \in \mathbb{N}$ sufficiently large, giving $|s| = \lim_{n\to\infty} |g_{n,i,t}(s)| \geq K^{-1}$. Similarly, if $|s| > K^{-1}$, then we have $|g_{n,i,t}(s)| > K^{-1}$ for $n \in \mathbb{N}$ sufficiently large, giving $s \in S^1_{K,\infty,i,t}$. Consequently, we have

$$|S^1_{K,\infty,i,t} \Delta \left([-K,K] \setminus [-K^{-1},K^{-1}]\right)| = 0 \tag{5.52}$$

for almost all $t \in [0,1]$. By the convergence (5.41), for almost all $t \in [0,1]$ we have

$$\begin{aligned} &[-K,K] \setminus S^2_{K,\infty,i,t} \\ &= \bigcap_{n>0} \bigcup_{n'\in\mathbb{N}:n'>n} \left\{ s \in [-K,K] : \lambda_{n'}^{-\frac{1}{4}} |h_{n',i,t}|(s) > 1 \right\} \\ &= \emptyset. \end{aligned} \tag{5.53}$$

Step 3: For all $K \in \mathbb{N}$ and $i \in I$, we prove $|[-K,K] \setminus S^3_{K,\infty,i,s}| = 0$, i.e., we may invert $g_{n,i,t}$ almost everywhere for sufficiently large n.

Let $n \in \mathbb{N}$ and $t \in [0,1]$. We decompose $[-K,K] \setminus S^3_{K,n,i,t}$ into its at most countably many connected components in the following way: We claim that there exists a set $J \subset \mathbb{N}$ and $0 < a_j < b_j \leq K$ for $j \in J$ such that the sets $(a_j, b_j]$ are pairwise disjoint and

$$[0,K] \setminus S^3_{K,n,i,t} = \bigcup_{j \in J} (a_j, b_j]. \tag{5.54}$$

Here, the half-open intervals are a result of $g_{n,i,t}(s)$ being left-continuous for $s \geq 0$ and the definition of $S^3_{K,n,i,t}$.

Let $s \in [0,K] \setminus S^3_{K,n,i,t}$. Let

$$a_s := \min \left\{ s' \in [0,s] : g_{n,i,t}(s') = \max_{[0,s]} g_{n,i,t} \right\}, \tag{5.55}$$

which exists by continuity of $g_{n,i,t}$. Furthermore, let

$$b_s := \max \left\{ s' \in [s,K] : g_{n,i,t}(r) \leq \max_{[0,s]} g_{n,i,t} \ \forall r \in [s,s'] \right\}, \tag{5.56}$$

which also exists by continuity of $g_{n,i,t}$. By definition of $S^3_{K,n,i,t}$, we have $a_s < s \leq b_s$ and $a_s \in S^3_{K,n,i,t}$. For all $\tilde{s} \in (a_s, s]$ we have $g_{n,i,t}(\tilde{s}) \leq g_{n,i,t}(a_s)$ by definition of a_s, while for $\tilde{s} \in (s, b_s]$ the same holds by definition of b_s. Thus, for all $\tilde{s} \in (a_s, b_s]$ we have $\tilde{s} \notin S^3_{K,n,i,t}$ and

$$\max_{[0,s]} g_{n,i,t} = g_{n,i,t}(a_s) = \max_{[0,\tilde{s}]} g_{n,i,t}. \tag{5.57}$$

Therefore, we have

$$a_s = \min \left\{ s' \in [0,\tilde{s}] : g_{n,i,t}(s') = \max_{[0,\tilde{s}]} g_{n,i,t} \right\} = a_{\tilde{s}}. \tag{5.58}$$

In particular, applying the identity (5.57) for $\tilde{s} = b_s$ gives $g_{n,i,t}(a_s) = \max_{[0,b_s]} g_{n,i,t}$. Conversely, by definition of b_s for every $\varepsilon > 0$ there exists $r \in (b_s, b_s + \varepsilon)$ such that we have $g_{n,i,t}(r) > \max_{[0,b_s]} g_{n,i,t}$. Consequently, we have

$$b_s = \max \left\{ s' \in [\tilde{s},K] : g_{n,i,t}(r) \leq \max_{[0,\tilde{s}]} g_{n,i,t} \ \forall r \in [\tilde{s},s'] \right\}. \tag{5.59}$$

Thus, the sets $(a_s, b_s]$ for $s \in [0,K] \setminus S^3_{K,n,i,t}$ provide the decomposition of $[0,K] \setminus S^3_{K,n,i,t}$ into connected components. As each half-open interval must contain at least one rational number, there may be at most countably many pairwise disjoint, connected components. This proves the claim, yielding the decomposition in (5.54).

By construction, for all $j \in J$ we have $g_{n,i,t}(b_j) = g_{n,i,t}(a_j)$, unless $b_j = K$, in which case we only have $g_{n,i,t}(b_j) \leq g_{n,i,t}(a_j)$. Therefore, we have

$$|[a_j, b_j]| \leq b_j - a_j + g_{n,i,t}(a_j) - g_{n,i,t}(b_j). \tag{5.60}$$

In total, we consequently get

$$\left|[0,K]\setminus S^3_{K,n,i,t}\right| \le \sum_{j\in J} \left|(b_j - g_{n,i,t}(b_j) - (a_j - g_{n,i,t}(a_j)))\right|. \tag{5.61}$$

By an approximation argument using $\tilde{J}\subset J$ finite, we get

$$\left|[0,K]\setminus S^3_{K,n,i,t}\right| \le TV_{[0,K]}(s - g_{n,i,t}(s)), \tag{5.62}$$

where the latter is the one-dimensional total variation of the function $s\mapsto s-g_{n,i,t}(s)$ for $s\in[0,K]$. Due to (5.29), this map is non-decreasing, and therefore

$$\left|[0,K]\setminus S^3_{K,n,i,t}\right| \le K - g_{n,i,t}(K). \tag{5.63}$$

An analogous argument works for $[-K,0]\setminus S^3_{K,n,i,t}$, giving in total

$$\left|[-K,K]\setminus S^3_{K,n,i,t}\right| \le 2K - g_{n,i,t}(K) + g_{n,i,t}(-K). \tag{5.64}$$

Together with the summability of this expression that follows from (5.34), the Borel-Cantelli lemma implies

$$|[-K,K]\setminus S^3_{K,\infty,i,s}| = 0. \tag{5.65}$$

Step 4: For all $K\in\mathbb{N}$ and $i\in I$, we prove that $S^4_{K,\infty,i,t}$ has full measure.

For $t\in[0,1]$, let $s\in S^2_{K,n,i,t}\setminus S^4_{K,n,i,t}$ for some $n\in\mathbb{N}$. In particular, we have $h_{n,i,t}(s)\neq 0$. As the argument in this step is most naturally done in the original coordinates, we define

$$\begin{aligned}\tilde{l}_{n,i,t,s}(r) &:= \gamma_{n,i}(t) + R^T_{\nu_{n,i}(t)} A^{-1}_{\lambda_n} l_{n,i,t,s}(r)\\ &= \gamma_{n,i}(t) + \lambda_n^{-1} g_{n,i,t}(s)\tau_{n,i}(t) + r\lambda_n^{-2} h_{n,i,t}(s)\nu_{n,i}(t).\end{aligned} \tag{5.66}$$

for $r\in[0,1]$. Recalling the definitions (5.47) and (3.4), the set

$$\begin{aligned}D_s := \Big\{ r\in(0,1) : l_{n,i,t,s}(r) \notin &\Big(\overline{H^0_-(e_2)} \setminus A_{\lambda_n} R_{\nu_{n,i}(t)}\left(\Omega_n - \gamma_{n,i}(t)\right)\Big)\\ &\cup \Big(A_{\lambda_n} R_{\nu_{n,i}(t)}\left(\overline{\Omega_n} - \gamma_{n,i}(t)\right)\setminus H^0_-(e_2)\Big)\Big\},\end{aligned} \tag{5.67}$$

see Figure 9, is non-empty. Therefore, we have

$$r_{\max,s} := \sup D_s \in (0,1]. \tag{5.68}$$

By $\Big(\tilde{l}_{n,i,t,s}(r) - \gamma_{n,i}(t)\Big)\cdot\nu_{n,i}(t) = r\lambda_n^{-2}h_{n,i,t}(s)$, for all $r\in(0,1]$ we have

$$\operatorname{sgn}\Big(\tilde{l}_{n,i,t,s}(r) - \gamma_{n,i}(t)\Big)\cdot\nu_{n,i}(t) = \operatorname{sgn} h_{n,i,t}(s) \neq 0. \tag{5.69}$$

Consequently, for all $r\in(0,1]$ we have either $\tilde{l}_{n,i,t,s}(r)\in H_-(\gamma_{n,i}(t))$ or $\tilde{l}_{n,i,t,s}(r)\in\mathbb{R}^2\setminus\overline{H_-(\gamma_{n,i}(t))}$. Thus if $r_{\max,s}<1$, we have $y_s := \tilde{l}_{n,i,t,s}(r_{\max,s})\in\partial\Omega_n$. If $r_{\max,s}=1$, by the representation (5.66) and the definitions (5.27) and (5.28) we have $\tilde{l}_{n,i,t,s}(r_{\max,s})\in\partial\Omega_n$. At the same time, a direct computation gives

$$\partial_r\Big(\nu(y_s)\cdot\tilde{l}_{n,i,t,s}(r)\Big) = \lambda_n^{-2}\nu(y_s)\cdot\nu_{n,i}(t)h_{n,i,t}(s). \tag{5.70}$$

Let us consider the case $\tilde{l}_{n,i,t,s}(r) \in H_-(\gamma_{n,i}(t))$ for all $r \in (0,1]$. Then by identity (5.69), we have $h_{n,i,t}(s) < 0$, so that

$$\operatorname{sgn} \partial_r \left(\nu(y_s) \cdot \tilde{l}_{n,i,t,s}(r)\right) = -\operatorname{sgn} \nu(y_s) \cdot \nu_{n,i}(t). \tag{5.71}$$

For $r \in D_s$, we furthermore get $\tilde{l}_{n,i,t,s}(r) \in \Omega_n$. As Ω_n is regular and $\nu(y_s)$ is the outer unit normal to Ω_n, there exists $\varepsilon > 0$ such that $\tilde{l}_{n,i,t,s}(r) \in \overline{H_-(y_s)}$ for $r \in (r_{\max,s} - \varepsilon, r_{\max,s})$. Consequently, linearity of $\tilde{l}_{n,i,t,s}(r)$ as a function of r gives

$$\partial_r \left(\nu(y_s) \cdot \tilde{l}_{n,i,t,s}(r)\right) \geq 0. \tag{5.72}$$

so that (5.71) implies $\nu(y_s) \cdot \nu_{n,i}(t) \leq 0$.

In the case $\tilde{l}_{n,i,t,s}(r) \in \mathbb{R}^2 \setminus \overline{H_-(\gamma_{n,i}(t))}$ for all $r \in (0,1]$, we instead have $h_{n,i,t}(s) > 0$ and

$$\operatorname{sgn} \partial_r \left(\nu(y_s) \cdot \tilde{l}_{n,i,t,s}(r)\right) = \operatorname{sgn} \nu(y_s) \cdot \nu_{n,i}(t). \tag{5.73}$$

Additionally, we have $\tilde{l}_{n,i,t,s}(r) \notin \overline{\Omega_n}$ for all $r \in D_s$. Therefore, we have $\tilde{l}_{n,i,t,s}(r) \notin H_-(y_s)$ for all $r \in (r_{\max,s} - \varepsilon, r_{\max,s})$ and $\varepsilon > 0$ small enough. As a result, we also have

$$\partial_r \left(\nu(y_s) \cdot \tilde{l}_{n,i,t,s}(r)\right) \leq 0, \tag{5.74}$$

resulting in $\nu(y_s) \cdot \nu_{n,i}(t) \leq 0$.

Furthermore, as a result of (5.66), (5.29) and $s \in S^2_{K,n,i,t}$ we have

$$|y - \gamma_{n,i}(t)| \leq \lambda_n^{-1}|s| + \lambda_n^{-2}|h_{n,i,t}(s)| \leq K\lambda_n^{-1} + \lambda_n^{-\frac{7}{4}}. \tag{5.75}$$

For $n \in \mathbb{N}$ sufficiently big, we thus have $y \in Z_{2K,n,i,t}$, see the definition in (5.24). Additionally, representation (5.66) implies

$$p(y) = g_{n,i,t}(s), \tag{5.76}$$

where $p(y) := \lambda_n \left(y - \gamma_{n,i}(t)\right) \cdot \tau_{n,i}(t)$ for $y \in \mathbb{R}^2$.

The above can therefore be compiled into the statement

$$g_{n,i,t}\left(S^2_{K,n,i,t} \setminus S^4_{K,n,i,t}\right) \subset p(Z_{2K,n,i,t}), \tag{5.77}$$

so that p being a λ_n-Lipschitz map ensures

$$\left|g_{n,i,t}\left(S^2_{K,n,i,t} \setminus S^4_{K,n,i,t}\right)\right| \leq |p(Z_{2K,n,i,t})| \leq \lambda_n \mathcal{H}^1(Z_{2K,n,i,t}). \tag{5.78}$$

As $g_{K,n,i,t}$ is injective on $S^3_{K,n,i,t}$, the coarea-formula thus gives

$$\begin{aligned}\int_{(S^2_{K,n,i,t} \cap S^3_{K,n,i,t}) \setminus S^4_{K,n,i,t}} \left|g'_{n,i,t}\right|(s)\,\mathrm{d}s &\leq \left|g_{n,i,t}\left(\left(S^2_{K,n,i,t} \cap S^3_{K,n,i,t}\right) \setminus S^4_{K,n,i,t}\right)\right| \\ &\leq \lambda_n \mathcal{H}^1(Z_{2K,n,i,t}).\end{aligned} \tag{5.79}$$

Let $A_{K,n,i,t} := \{s \in [-K,K] : |g'_{n,i,t}(s)| \leq \frac{1}{2K}\}$. Then from (5.79) we have the estimate

$$\begin{aligned} \left|(S^2_{K,n,i,t} \cap S^3_{K,n,i,t}) \setminus S^4_{K,n,i,t}\right| &= \left|\left((S^2_{K,n,i,t} \cap S^3_{K,n,i,t}) \setminus S^4_{K,n,i,t}\right) \setminus A_{K,n,i,t}\right| \\ &\quad + \left|\left((S^2_{K,n,i,t} \cap S^3_{K,n,i,t}) \setminus S^4_{K,n,i,t}\right) \cap A_{K,n,i,t}\right| \\ &\leq 2K \int_{\left((S^2_{K,n,i,t} \cap S^3_{K,n,i,t}) \setminus S^4_{K,n,i,t}\right) \setminus A_{K,n,i,t}} \left|g'_{n,i,t}\right|(s)\,\mathrm{d}s \\ &\quad + \left|\left((S^2_{K,n,i,t} \cap S^3_{K,n,i,t}) \setminus S^4_{K,n,i,t}\right) \cap A_{K,n,i,t}\right| \\ &\leq 2K\lambda_n \mathcal{H}^1(Z_{2K,n,i,t}) + |A_{K,n,i,t}|. \end{aligned} \tag{5.80}$$

On the other hand, using (5.29) we may compute

$$\left(1 - \frac{1}{2K}\right)|A_{K,n,i,t}| \leq \int_{A_{K,n,i,t}} \left(1 - g'_{n,i,t}(s)\right)\,\mathrm{d}s \leq \max_{s \in [-K,K]} |s - g_{n,i,t}(s)|. \tag{5.81}$$

Therefore, combining this with estimates (5.80), (5.23), (5.26), and (5.34) gives

$$\sum_{n \in \mathbb{N}} \left|(S^2_{K,n,i,t} \cap S^3_{K,n,i,t}) \setminus S^4_{K,n,i,t}\right| < \infty. \tag{5.82}$$

Again, we employ the Borel-Cantelli lemma to get

$$\left|\bigcap_{n \in \mathbb{N}} \bigcup_{n' \in \mathbb{N}: n' > n} (S^2_{K,n',i,t} \cap S^3_{K,n',i,t}) \setminus S^4_{K,n',i,t}\right| = 0. \tag{5.83}$$

If $s \in \left(S^2_{K,\infty,i,t} \cap S^3_{K,\infty,i,t}\right) \setminus S^4_{K,\infty,i,t}$, by definition (5.50) we have $s \in S^2_{K,n',i,t} \cap S^3_{K,n',i,t}$ for sufficiently big $n \in \mathbb{N}$ and all $n' \in \mathbb{N}$, $n' \geq n$. Conversely, for all $n \in \mathbb{N}$ there exists $n' \in \mathbb{N}$ with $n' \geq n$ such that $s \notin S^4_{K,n',i,t}$. In particular, for all $n \in \mathbb{N}$ sufficiently big there exists $n' \in \mathbb{N}$ with $n' \geq n$ such that $s \in \left(S^2_{K,n',i,t} \cap S^3_{K,n',i,t}\right) \setminus S^4_{K,n',i,t}$. Therefore, from (5.83) we obtain

$$\left|(S^2_{K,\infty,i,t} \cap S^3_{K,\infty,i,t}) \setminus S^4_{K,\infty,i,t}\right| = 0. \tag{5.84}$$

Finally, computing the set inclusions

$$\begin{aligned} & S_{K,\infty,i,t} \Delta \left([-K,K] \setminus [-K^{-1},K^{-1}]\right) \\ & \subset S^1_{K,\infty,i,t} \setminus \left([-K,K] \setminus [-K^{-1},K^{-1}]\right) \cup \bigcup_{j=1}^{4} \left([-K,K] \setminus [-K^{-1},K^{-1}]\right) \setminus S^j_{K,\infty,i,t} \\ & \subset S^1_{K,\infty,i,t} \Delta \left([-K,K] \setminus [-K^{-1},K^{-1}]\right) \\ & \qquad \cup \left([-K,K] \setminus S^2_{K,\infty,i,t}\right) \cup \left([-K,K] \setminus S^3_{K,\infty,i,t}\right) \\ & \qquad \cup \left(S^2_{K,\infty,i,t} \cap S^3_{K,\infty,i,t}\right) \setminus S^4_{K,\infty,i,t}, \end{aligned} \tag{5.85}$$

we get from the identities (5.51), (5.52), (5.53), (5.65), and (5.84) that

$$\left|S_{K,\infty,i,t} \Delta \left([-K,K] \setminus [-K^{-1},K^{-1}]\right)\right| = 0. \tag{5.86}$$

Step 5: For all $i \in I$, prove

$$\liminf_{n\to\infty} \int_0^1 \int_{H_-^0(e_2)\Delta A_{\lambda_n} R_{\nu_{n,i}(t)}(\Omega_n-\gamma_{n,i}(t))} |z_2| \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{z_2^2}{\lambda_n^2}}}}{z_1^2+\frac{z_2^2}{\lambda_n^2}} \,\mathrm{d}^2 z\,\mathrm{d}t \geq 2^{\frac{1}{2}}\pi^{\frac{3}{2}} \int_0^1 \kappa_{\infty,i}^2(t)\,\mathrm{d}t. \tag{5.87}$$

Let $t \in [0,1]$, $n \in \mathbb{N}$, and $K \in \mathbb{N}$. By the properties of $S^3_{K,n,i,t}$ and $S^4_{K,n,i,t}$, as well as the fact that $\partial H_-^0(e_2)$ and $\partial\Omega_n$ are sets of two-dimensional Lebesgue measure zero, we have

$$\begin{aligned}&\int_{H_-^0(e_2)\Delta A_{\lambda_n} R_{\nu_{n,i}(t)}(\Omega_n-\gamma_{n,i}(t))} |z_2| \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{z_2^2}{\lambda_n^2}}}}{z_1^2+\frac{z_2^2}{\lambda_n^2}} \,\mathrm{d}^2 z\\ &\geq \int_{g_{n,i,t}\left(S_{K,n,i,t}\right)} \int_0^{|h_{n,i,t}(g_{n,i,t}^{-1}(z_1))|} z_2 \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{z_2^2}{\lambda_n^2}}}}{z_1^2+\frac{z_2^2}{\lambda_n^2}} \,\mathrm{d}z_2\,\mathrm{d}z_1.\end{aligned} \tag{5.88}$$

Due to definition (5.43) and invertibility of $g_{n,i,t}$ on $S^3_{K,n,i,t}$, for all $z_1 \in g_{n,i,t}\left(S_{K,n,i,t}\right)$ we have the bound $|h_{n,i,t}(g_{n,i,t}^{-1}(z_1))| \leq \lambda_n^{\frac{1}{2}}$ for n large enough. Therefore, for all $0 \leq z_2 \leq |h_{n,i,t}(g_{n,i,t}^{-1}(z_1))|$ we have by monotonicity

$$\frac{e^{-\alpha_n\sqrt{z_1^2+\frac{z_2^2}{\lambda_n^2}}}}{z_1^2+\frac{z_2^2}{\lambda_n^2}} \geq \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{1}{\lambda_n}}}}{z_1^2+\frac{1}{\lambda_n}}. \tag{5.89}$$

Furthermore, we calculate using concavity of the square root and convexity of the exponential

$$\begin{aligned}0 &\leq \frac{e^{-\alpha_n|z_1|}}{z_1^2} - \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{1}{\lambda_n}}}}{z_1^2+\frac{1}{\lambda_n}}\\ &= \frac{e^{-\alpha_n|z_1|}}{z_1^2}\left(1-e^{-\alpha_n|z_1|\left(\sqrt{1+\frac{1}{\lambda_n z_1^2}}-1\right)}\right) + \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{1}{\lambda_n}}}}{\lambda_n z_1^2\left(z_1^2+\frac{1}{\lambda_n}\right)}\\ &\leq \frac{e^{-\alpha_n|z_1|}}{z_1^2}\left(1-e^{-\frac{\alpha_n}{2\lambda_n|z_1|}}\right) + \frac{e^{-\frac{\alpha_n}{\sqrt{\lambda_n}}}}{\lambda_n z_1^2\left(z_1^2+\frac{1}{\lambda_n}\right)}\\ &\leq \frac{\alpha_n e^{-\alpha_n|z_1|}}{2\lambda_n|z_1|^3} + \frac{e^{-\frac{\alpha_n}{\sqrt{\lambda_n}}}}{z_1^2}\\ &\leq \frac{\alpha_n}{2\lambda_n|z_1|^3} + \frac{e^{-\frac{\alpha_n}{\sqrt{\lambda_n}}}}{z_1^2}.\end{aligned} \tag{5.90}$$

Similarly, we have

$$\frac{e^{-\frac{|z_1|}{\sqrt{2\pi}}}}{z_1^2} - \frac{e^{-\alpha_n|z_1|}}{z_1^2} = \frac{e^{-\frac{|z_1|}{\sqrt{2\pi}}}}{z_1^2}\left(1-e^{-\left(\alpha_n-\frac{1}{\sqrt{2\pi}}\right)|z_1|}\right) \leq \frac{\alpha_n-\frac{1}{\sqrt{2\pi}}}{|z_1|}. \tag{5.91}$$

Combining (5.90) and (5.91), for all $z_1 \in g_{n,i,t}(S_{K,n,i,t})$ we have by definition (5.42) that

$$\frac{e^{-\frac{|z_1|}{\sqrt{2\pi}}}}{z_1^2} - \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{1}{\lambda_n}}}}{z_1^2+\frac{1}{\lambda_n}} \leq a_n := \frac{\alpha_n K^3}{2\lambda_n} + K^2 e^{-\frac{\alpha_n}{\sqrt{\lambda_n}}} + K\left(\alpha_n - \frac{1}{\sqrt{2\pi}}\right), \tag{5.92}$$

which vanishes in the limit $n \to \infty$. Therefore, explicitly computing the remaining, trivial integral over z_2, we have

$$\begin{aligned} &\int_{g_{n,i,t}(S_{K,n,i,t})} \int_0^{|h_{n,i,t}(g_{n,i,t}^{-1}(z_1))|} z_2 \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{z_2^2}{\lambda_n^2}}}}{z_1^2+\frac{z_2^2}{\lambda_n^2}} \,\mathrm{d}z_2\,\mathrm{d}z_1 \\ &\geq \frac{1}{2}\int_{g_{n,i,t}(S_{K,n,i,t})} h_{n,i,t}^2(g_{n,i,t}^{-1}(z_1)) \left(\frac{e^{-\frac{|z_1|}{\sqrt{2\pi}}}}{z_1^2} - a_n\right) \mathrm{d}z_1 \\ &= \frac{1}{2}\int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \left(\frac{e^{-\frac{|g_{n,i,t}(s)|}{\sqrt{2\pi}}}}{g_{n,i,t}^2(s)} - a_n\right) |g'_{n,i,t}(s)|\,\mathrm{d}s. \end{aligned} \tag{5.93}$$

Now, due to (5.29) and estimate (5.38), we have

$$\limsup_{n\to\infty} a_n \int_0^1 \int_{S_{K,n,i,t}} h_{n,i,t}^2(s)|g'_{n,i,t}(s)|\,\mathrm{d}s\,\mathrm{d}t = 0. \tag{5.94}$$

Combining the bounds $|h_{n,i,t}(s)| \leq \lambda_n^{\frac{1}{4}}$ and $|g_{n,i,t}(s)| > K^{-1}$ for $s \in S_{K,n,i,t}$ with the convergence (5.31), we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{2}\int_0^1 \int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|g_{n,i,t}(s)|}{\sqrt{2\pi}}}}{g_{n,i,t}^2(s)} \left(1 - |g'_{n,i,t}(s)|\right) \mathrm{d}s\,\mathrm{d}t \\ &\leq \limsup_{n\to\infty} \frac{K^2}{2}\int_0^1 \int_{S_{K,n,i,t}} \lambda_n^{\frac{1}{2}} \left(1 - g'_{n,i,t}(s)\right) \mathrm{d}s\,\mathrm{d}t \\ &= 0. \end{aligned} \tag{5.95}$$

Using $|g_{n,i,t}(s)| \leq |s|$ obtained by integrating the bound (5.29) from $g_{n,i,t}(0) = 0$, we get

$$\frac{1}{2}\int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|g_{n,i,t}(s)|}{\sqrt{2\pi}}}}{g_{n,i,t}^2(s)} \,\mathrm{d}s \geq \frac{1}{2}\int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2} \,\mathrm{d}s. \tag{5.96}$$

Let $\bar{n} \in \mathbb{N}$. For $n \in \mathbb{N}$ with $n \geq \bar{n}$, we have

$$\tilde{S}_{K,\bar{n},i,t} := \bigcap_{n'=\bar{n}}^{\infty} S_{K,n',i,t} \subset S_{K,n,i,t} \subset [-K, K]. \tag{5.97}$$

Therefore, using Fubini's theorem we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2}\int_0^1 \int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2}\,\mathrm{d}s\,\mathrm{d}t \\
&\geq \liminf_{n\to\infty} \frac{1}{2}\int_0^1 \int_{\tilde{S}_{K,\bar{n},i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2}\,\mathrm{d}s\,\mathrm{d}t \\
&= \liminf_{n\to\infty} \frac{1}{2}\int_{-K}^K \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2} \int_0^1 \chi_{\tilde{S}_{K,\bar{n},i,t}}(s) h_{n,i,t}^2(s)\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{5.98}$$

As $\chi_{\tilde{S}_{K,\bar{n},i,t}} = \chi^2_{\tilde{S}_{K,\bar{n},i,t}}$, we also get

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2}\int_0^1 \int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2}\,\mathrm{d}s\,\mathrm{d}t \\
&\geq \liminf_{n\to\infty} \frac{1}{2}\int_{-K}^K \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2} \int_0^1 \Big(\chi_{\tilde{S}_{K,\bar{n},i,t}}(s) h_{n,i,t}(s)\Big)^2\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{5.99}$$

For all $s \in \mathbb{R}$, the convergence (5.40) then gives

$$\chi_{\tilde{S}_{K,\bar{n},i,t}}(s) h_{n,i,t}(s) \rightharpoonup -\chi_{\tilde{S}_{K,\bar{n},i,t}}(s)\frac{s^2}{2}\kappa_{\infty,i}(t) \tag{5.100}$$

in $L^2_t(0,1)$. Fatou's Lemma and weak lower-semicontinuity of the L^2-norm imply

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2}\int_{-K}^K \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2} \int_0^1 \Big(\chi_{\tilde{S}_{K,\bar{n},i,t}}(s) h_{n,i,t}(s)\Big)^2\,\mathrm{d}t\,\mathrm{d}s \\
&\geq \frac{1}{2}\int_{-K}^K \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2} \liminf_{n\to\infty} \int_0^1 \Big(\chi_{\tilde{S}_{K,\bar{n},i,t}}(s) h_{n,i,t}(s)\Big)^2\,\mathrm{d}t\,\mathrm{d}s \\
&\geq \frac{1}{8}\int_{-K}^K e^{-\frac{|s|}{\sqrt{2\pi}}} s^2 \int_0^1 \chi_{\tilde{S}_{K,\bar{n},i,t}}(s) \kappa_{\infty,i}^2(t)\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{5.101}$$

Combining this with estimate (5.99), we get

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{2}\int_0^1 \int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2}\,\mathrm{d}s\,\mathrm{d}t \\
&\geq \frac{1}{8}\int_{-K}^K e^{-\frac{|s|}{\sqrt{2\pi}}} s^2 \int_0^1 \chi_{\tilde{S}_{K,\bar{n},i,t}}(s) \kappa_{\infty,i}^2(t)\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned} \tag{5.102}$$

On the other hand, as the sets $\tilde{S}_{K,\bar{n},i,t}$ are increasing in $\bar{n}$, we may take the supremum in $\bar{n}$ on the right hand side of estimate (5.102) and get after an application of Fubini's theorem and the monotone convergence theorem that

$$\liminf_{n\to\infty} \frac{1}{2}\int_0^1 \int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2}\,\mathrm{d}s\,\mathrm{d}t \geq \frac{1}{8}\int_0^1 \kappa_{\infty,i}^2(t) \int_{S_{K,\infty,i,t}} e^{-\frac{|s|}{\sqrt{2\pi}}} s^2\,\mathrm{d}s\,\mathrm{d}t. \tag{5.103}$$

Equation (5.86) then gives

$$\liminf_{n\to\infty} \frac{1}{2}\int_0^1 \int_{S_{K,n,i,t}} h_{n,i,t}^2(s) \frac{e^{-\frac{|s|}{\sqrt{2\pi}}}}{s^2}\,\mathrm{d}s\,\mathrm{d}t \geq \frac{1}{4}\int_0^1 \kappa_{\infty,i}^2(t)\int_{K^{-1}}^{K} e^{-\frac{|s|}{\sqrt{2\pi}}} s^2\,\mathrm{d}s\,\mathrm{d}t. \tag{5.104}$$

Chaining together estimates (5.88), (5.93), (5.94), (5.95), (5.96), and (5.104) and noticing that the estimates apply for all $K \in \mathbb{N}$, we get

$$\begin{aligned} &\liminf_{n\to\infty} \int_0^1 \int_{H^0_-(e_2)\Delta A_{\lambda_n} R_{\nu_{n,i}(t)}(\Omega_n - \gamma_{n,i}(t))} |z_2| \frac{e^{-\alpha_n\sqrt{z_1^2+\frac{z_2^2}{\lambda_n^2}}}}{z_1^2+\frac{z_2^2}{\lambda_n^2}}\,\mathrm{d}^2 z\,\mathrm{d}t \\ &= \frac{1}{4}\sup_{K\in\mathbb{N}} \int_0^1 \kappa_{\infty,i}^2(t)\int_{K^{-1}}^{K} e^{-\frac{|s|}{\sqrt{2\pi}}} s^2\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{4}\int_0^1 \kappa_{\infty,i}^2(t)\int_0^{\infty} e^{-\frac{|s|}{\sqrt{2\pi}}} s^2\,\mathrm{d}s\,\mathrm{d}t \\ &= 2^{\frac{1}{2}}\pi^{\frac{3}{2}} \int_0^1 \kappa_{\infty,i}^2(s)\,\mathrm{d}t, \end{aligned} \tag{5.105}$$

which is what we claimed for this step of the proof.

Step 6: Combine all limit curves. Note that since the boundary curves converge strongly in $H^1(\mathbb{S}^1;\mathbb{R}^2)$, we have $L(\gamma_{n,i}) \to L(\gamma_{\infty,i})$ as $n\to\infty$. Inserting this with $\sigma_n \to \sigma$, $\alpha_n \to \frac{1}{\sqrt{2\pi}}$ as $n\to\infty$, and the estimate (5.87) into the representation (3.8) we get

$$\liminf_{\lambda\to\infty} \lambda_n^2 F_{\lambda,\alpha}(\Omega_\lambda) \geq \sum_{i\in I} L(\gamma_{\infty,i})\left(\sigma + \frac{\pi}{2}\int_0^1 \kappa_{\infty,i}^2(t)\,\mathrm{d}t\right) = \hat{F}_{\infty,\sigma}(\gamma), \tag{5.106}$$

concluding the proof. □

5.3 Concluding arguments

Finally, the proofs of Proposition 2.3, Theorem 2.4 and Corollary 2.5 essentially consist of putting together all the information at hand. Additionally, we provide the proof for Proposition 2.7.

Proof of Proposition 2.3. The upper bound is given by Lemma 5.1, while the lower bound follows by applying Proposition 5.3 to the constant sequence $n \mapsto \Omega$ for $n \in \mathbb{N}$. □

Proof of Theorem 2.4. By the fact that $F_{\infty,\sigma}$ is the L^1-relaxation of the functional in (2.8), the upper bound follows immediately from Proposition 2.3.

To prove the lower bound let $\Omega_n \in \mathcal{A}_\pi$ for $n \in \mathbb{N}$ be sets such that $\chi_{\Omega_n} \to \chi_{\Omega_\infty}$ for $\Omega_\infty \in \mathcal{A}_\pi$ and such that

$$\liminf_{n\to\infty} \lambda_n^2 F_{\lambda_n,\alpha_n}(\Omega_n) < \infty. \tag{5.107}$$

By a standard approximation argument, we may suppose the set Ω_n to be regular for all $n \in \mathbb{N}$. Combining Theorem 2.11 with Propositions 4.4 and 5.3, we get a system of curves $\{\gamma_{\infty,i}\}_{i\in I} \in G(\Omega_\infty)$ such that

$$F_{\infty,\sigma}(\Omega_\infty) \leq \hat{F}_{\infty,\sigma}(\gamma_\infty) \leq \liminf_{n\to\infty} \lambda_n^2 F_{\lambda_n,\alpha_n}(\Omega_n), \tag{5.108}$$

concluding the proof. □

Proof of Corollary 2.5. Existence of minimizers follows from Proposition 2.2. As the minimizers are connected, after a suitable translation and by the bound on the perimeter there exists $\Omega_\infty \in \mathcal{A}_\pi$ and a subsequence (not relabeled) such that $\chi_{\Omega_n} \to \chi_{\Omega_\infty}$ in $L^1(\mathbb{R}^2)$. We then have by the lower bound that

$$\inf_{\mathcal{A}_\pi} F_{\infty,\sigma} \leq F_{\infty,\sigma}(\Omega_\infty) \leq \liminf_{n\to\infty} \lambda_n^2 F_{\lambda_n,\alpha_n}(\Omega_n) = \liminf_{n\to\infty} \inf_{\mathcal{A}_\pi} \lambda_n^2 F_{\lambda_n,\alpha_n}, \tag{5.109}$$

while the upper bound implies

$$\liminf_{n\to\infty} \inf_{\mathcal{A}_\pi} \lambda_n^2 F_{\lambda_n,\alpha_n} \leq \inf_{\mathcal{A}_\pi} F_{\infty,\sigma}. \tag{5.110}$$

Therefore, we have equality everywhere and Ω_∞ is a minimizer of $F_{\infty,\sigma}$. The characterization of minimizers in the limit was carried out by Goldman, Novaga, and Röger [22]. □

Proof of Proposition 2.7. The alternative representation of f immediately follows from

$$\int_{\Omega_x} \int_{\mathbb{R}^2} K(|y-z|)\,\mathrm{d}^2y\,\mathrm{d}^2z = 2\pi^2 \int_0^\infty rK(r)\,\mathrm{d}r. \tag{5.111}$$

Setting $R := \sqrt{1+r^2}$, we calculate

$$\begin{aligned} f(x) = &\int_{B_R(0)} \int_{B_R(0)} K(|y-z|)\,\mathrm{d}^2y\,\mathrm{d}^2z + \int_{B_r(x)} \int_{B_r(x)} K(|y-z|)\,\mathrm{d}^2y\,\mathrm{d}^2z \\ &- 2\int_{B_R(0)} \int_{B_r(x)} K(|y-z|)\,\mathrm{d}^2y\,\mathrm{d}^2z \\ = &\int_{B_R(0)} \int_{B_R(0)} K(|y-z|)\,\mathrm{d}^2y\,\mathrm{d}^2z + \int_{B_r(0)} \int_{B_r(0)} K(|y-z|)\,\mathrm{d}^2y\,\mathrm{d}^2z \\ &- 2\int_{B_R(0)} \int_{B_r(x)} K(|y-z|)\,\mathrm{d}^2y\,\mathrm{d}^2z. \end{aligned} \tag{5.112}$$

By the Riesz rearrangement inequality applied to the last term, see [29, Lemma 3] for a sharp version of the inequality, we get $f(x) \geq f(0)$. If K is additionally strictly monotone decreasing, the sharp version implies that $x = 0$ is the only minimizer. □

A Model derivation

For the sake of the reader's convenience, below we present a first principles derivation of the energy in (2.1) along the lines of Ref. [3], except that we take the sharp interface approach and model the Langmuir layer as an incompressible two-dimensional patch of amphiphilic molecules. For a fixed patch area there is, therefore, no non-trivial local contribution to the energy from the interior of the patch and all the local interactions due to van der Waals forces can be captured by an interfacial energy term representing line tension. We also focus on a regime that takes advantage of the large dielectric constant of water at moderate droplet sizes (see further discussion at the end of this section). For the clarity of the derivation, in this section we adhere to the standard physics notations.

Consider a monolayer of amphiphilic molecules at the air-water interface located at the $z = 0$ plane in $\mathbb{R}^3$, with water occupying the $z < 0$ half-plane. The molecules are restricted to a set $\Omega \subset \mathbb{R}^2$ with fixed area $|\Omega|$ in the xy-plane. The excess energy associated with this monolayer patch may be written as

$$\mathcal{E}(\Omega) = \mathcal{E}_{\text{surf}}(\Omega) + \mathcal{E}_{\text{long-range}}(\Omega), \tag{A.1}$$

where the first term is the surface energy of the patch:

$$\mathcal{E}_{\text{surf}} = \gamma P(\Omega), \tag{A.2}$$

with $P(\Omega)$ being the perimeter of the set Ω, coinciding with the one-dimensional Hausdorff measure $\mathcal{H}^1(\partial\Omega)$ of the boundary of Ω for sufficiently regular sets [1], and $\gamma > 0$ being the line tension. The long-range part $\mathcal{E}_{\text{long-range}}(\Omega)$ of the energy is due to the electrostatic interaction of the charged polar heads of the amphiphilic molecules immersed in water:

$$\mathcal{E}_{\text{long-range}}(\Omega) = \frac{1}{2}\int_\Omega q\rho\bar{U}\,d^2r, \tag{A.3}$$

where $-q$ is the charge taken away from the amphiphilic molecule's polar head by water, ρ is the areal density of the amphiphilic molecules, and $\bar{U}$ is the electrostatic potential at $z = 0$. The latter may be found with the help of the Debye-Hückel theory by solving for the potential U in the whole space (in the SI units) [3]:

$$\Delta U - \kappa^2 U = 0, \qquad z < 0. \tag{A.4}$$

$$\Delta U = 0, \qquad z > 0, \tag{A.5}$$

subject to the conditions at the air-water interface:

$$\lim_{z\to 0^-} U(\cdot, z) = \lim_{z\to 0^+} U(\cdot, z), \tag{A.6}$$

$$\epsilon_d \lim_{z\to 0^-} U_z(\cdot, z) - \lim_{z\to 0^+} U_z(\cdot, z) = \frac{q}{\epsilon_0}\rho\chi_\Omega, \tag{A.7}$$

with U vanishing at infinity. Here κ is the Debye-Hückel screening parameter equal to the inverse of the screening length in water, ϵ_d is water's dielectric constant, ϵ_0 is the vacuum permittivity, and χ_Ω is the characteristic function of Ω.

For a given bounded set Ω, this elliptic problem has a unique solution, which can be found by means of the Fourier transform with respect to the in-plane variables. Denoting

$$\widehat{U}_{\mathbf{k}}(z) := \int_{\mathbb{R}^2} e^{i\mathbf{k}\cdot\mathbf{r}} U(\mathbf{r}, z)\, d^2r \qquad \mathbf{k} \in \mathbb{R}^2, \tag{A.8}$$

and passing to the Fourier space in (A.4)–(A.7), after some simple algebra we obtain [3]

$$\widehat{U}_{\mathbf{k}}(z) = \frac{q\rho e^{z\sqrt{\kappa^2+|\mathbf{k}|^2}}\widehat{\chi}_\Omega(\mathbf{k})}{\epsilon_0(\epsilon_d\sqrt{\kappa^2+|\mathbf{k}|^2}+|\mathbf{k}|)}, \qquad z < 0, \tag{A.9}$$

$$\widehat{U}_{\mathbf{k}}(z) = \frac{q\rho e^{-z|\mathbf{k}|}\widehat{\chi}_\Omega(\mathbf{k})}{\epsilon_0(\epsilon_d\sqrt{\kappa^2+|\mathbf{k}|^2}+|\mathbf{k}|)}, \qquad z > 0, \tag{A.10}$$

where $\widehat{\chi}_\Omega(\mathbf{k})$ is the Fourier transform of χ_Ω. Notice that since $\epsilon_d \simeq 80$ is very large for water, with a very good accuracy one could neglect the $|\mathbf{k}|$ term compared to $\epsilon_d\sqrt{\kappa^2+|\mathbf{k}|^2}$ in the expression for $\widehat{U}_{\mathbf{k}}(0)$. Thus, we have

$$\widehat{U}_{\mathbf{k}}(0) \simeq \frac{q\rho\widehat{\chi}_\Omega(\mathbf{k})}{\epsilon_0\epsilon_d\sqrt{\kappa^2+|\mathbf{k}|^2}}, \tag{A.11}$$

and returning to the real space, we get

$$\bar{U}(\mathbf{r}) \simeq \frac{q\rho}{2\pi\epsilon_0\epsilon_d} \int_\Omega \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d^2r' \tag{A.12}$$

at the air-water interface. Thus, the non-local part of the energy is, to the leading order in $\epsilon_d \gg 1$:

$$\mathcal{E}_{\text{long-range}}(\Omega) = \frac{q^2\rho^2}{4\pi\epsilon_0\epsilon_d} \int_\Omega \int_\Omega \frac{e^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d^2r\, d^2r'. \tag{A.13}$$

We now carry out a non-dimensionalization, introducing

$$E(\Omega) := P(\Omega) + \frac{1}{4\pi} \int_\Omega \int_\Omega \frac{e^{-\alpha|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, d^2r\, d^2r', \tag{A.14}$$

and noting that $\mathcal{E}(\ell\Omega) = \gamma\ell E(\Omega)$ with the choices of the scale and the dimensionless screening parameter, respectively:

$$\ell = \frac{\sqrt{\epsilon_0\epsilon_d\gamma}}{q\rho}, \qquad \alpha = \frac{\kappa\sqrt{\epsilon_0\epsilon_d\gamma}}{q\rho}. \tag{A.15}$$

Taking into account that $\int_{\mathbb{R}^2} r^{-1}e^{-\alpha r} d^2r = 2\pi/\alpha$, we can then rewrite the energy $E(\Omega)$ as

$$E(\Omega) = E_\alpha(\Omega) + \frac{|\Omega|}{2\alpha}, \tag{A.16}$$

and so, up to an additive constant the energy $E(\Omega)$ coincides with that in (2.1).

We note that the kernel appearing in (A.12) exhibits exponential decay due to the fact that we neglected the $|\mathbf{k}|$ term in the Fourier transform of $\bar{U}$ for large ϵ_d. This, however, becomes invalid for arbitrarily large separations, for which the kernel can be shown to exhibit an algebraic decay of the form $q^2\rho^2/(2\pi\epsilon_0\epsilon_d^2\kappa^2|\mathbf{r}-\mathbf{r}'|^3)$, up to an additive constant. Therefore, in agreement with the conventional wisdom the limit of large droplets should be described by the model in which the long-range part of the energy is of dipolar type [5, 33]. This model corresponds to the case of strong ionic solutions and was first studied rigorously in Ref. [40]. Nevertheless, for $\epsilon_d \gg 1$ the model in (A.14) is appropriate in a certain range of droplet sizes, which corresponds to the case of weak ionic solutions [3].

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