Linear vs. nonlinear selection for the propagation speed of the solutions of scalar reaction-diffusion equations invading an unstable equilibrium

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Abstract

We revisit the classical problem of speed selection for the propagation of disturbances in scalar reaction-diffusion equations with one linearly stable and one linearly unstable equilibrium. For a wide class of initial data this problem reduces to finding the minimal speed of the monotone traveling wave solutions connecting these two equilibria in one space dimension. We introduce a variational characterization of these traveling wave solutions and give a necessary and sufficient condition for linear vs. nonlinear selection mechanism. Easily verifiable sufficient conditions for the linear and nonlinear selection mechanisms are obtained. Our method also allows to obtain efficient lower and upper bounds for the propagation speed.

1 Introduction

In this paper, we revisit the problem of speed selection for the propagation of disturbances in scalar reaction-diffusion systems. To this end, we consider the following initial value problem:

$$u_t = u_{xx} + f(u), \qquad u(x,0) = u_0(x), \tag{1.1}$$

where $u = u(x,t)$ is a real-valued function of one spatial variable x and time t. For simplicity, we consider the problem on a real line, a straightforward generalization to the cylindrical geometry with Neumann boundary conditions or the entire $\mathbb{R}^n$ is possible. Furthermore, we will consider nonlinearities f possessing an unstable equilibrium (which without any loss of generality may be assumed to be zero) and a stable equilibrium of the space-independent dynamics governed by Eq. (1.1), with no other equilibria in between.

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This kind of equation is a prototypical model for a variety of applications in physics, chemistry, and biology (see, for example, [10,12,21,23]). One of the most notable examples is the Fisher equation with nonlinearity $f(u) = u(1-u)$ describing the spread of advantages genes in a population, whose studies go back to 1930's [14, 19, 23]. Equation (1.1) arises naturally in the context of autocatalytic reactions and combustion systems, for example, in the case of the Arrhenius nonlinearity one gets Eq. (1.1) with $f(u) = e^{-\frac{a}{u}}(1-u)$ with $a > 0$ [10, 12, 21]. Another important class of problems which leads to Eq. (1.1) with an unstable equilibrium arises from the analysis of amplitude equations describing the dynamics of the system near a bifurcation point [10]. As one such characteristic example, consider the subcritical quintic Ginzburg-Landau equation

$$u_t = u_{xx} + \mu u + u^3 - u^5, \tag{1.2}$$

which has an unstable equilibrium $u = 0$ for $\mu > 0$.

We are interested in the process of invasion of the unstable equilibrium by a stable one in reaction-diffusion systems described by Eq. (1.1). So, we will consider the solutions of Eq. (1.1) with the initial data decaying exponentially as $x \to +\infty$. These solutions are known to exhibit propagation with constant speed at long times. Back in the 30's, Kolmogorov, Petrovsky, and Piskunov showed that Eq. (1.1) admits a particular class of solutions in the form of *traveling waves*, $u(x,t) = \bar{u}(x-ct)$, moving with speed c, where the profile of the wave satisfies the ordinary differential equation

$$\bar{u}_{xx} + c\bar{u}_x + f(\bar{u}) = 0. \tag{1.3}$$

Under some extra assumptions on f, they were able to prove that there exists a continuous family of the traveling wave solutions with arbitrary speeds $c \geq c_0$, which decay exponentially with the rate depending on c [19]. This fact already indicates that the speed of propagation for the solutions of the considered initial value problem may depend on the way the initial data go to zero at plus infinity. However, as was shown already by Kolmogorov, Petrovsky, and Piskunov, for the initial data that decay sufficiently rapidly, in particular, if $u_0(x) \equiv 0$ for all $x > x_0$, for such nonlinearities the propagation speed turns out to be equal to c_0, which, furthermore, is easily calculated from the linearization of Eq. (1.1) around zero.

The problem of speed selection was discussed extensively in the physics literature (see, for example, [4,5,24,27–29]). It was observed that depending on the nonlinearity, the propagation speed c^* for sufficiently rapidly decaying initial conditions is either equal, or is greater than the speed c_0 obtained by Kolmogorov, Petrovsky, and Piskunov. Depending on whether the first or the second situation is realized, the selection mechanism is termed "linear" or "nonlinear" selection, respectively. Van Saarloos argued that whether the first or the second mechanism is realized depends on the existence of certain types of the traveling wave solutions [27–29]. Rigorous studies of this problem in the general context of Eqs. (1.1) were initiated by Aronson and Weinberger [1, 2]. They essentially resolved the problem of the speed selection with the help of the comparison arguments for a certain class of initial data. Central to their analysis is the construction of the monotone traveling wave solutions connecting the stable equilibrium at minus infinity with the unstable equilibrium at plus infinity. Aronson and Weinberger proved that under certain assumptions these solutions in fact exist for all $c \geq c^*$, where c^* is some constant which may be equal to or strictly greater than c_0. Furthermore, they proved that any nontrivial solution of the initial value problem in Eq. (1.1) with the initial data $u_0(x)$ between these equilibria and decaying exponentially, faster than the traveling wave solution with speed c^*, will propagate asymptotically with speed c^* in the sense that the position $R(t)$ of the *leading edge* of the solution of Eq. (1.1) behaves asymptotically as (for precise definitions and assumptions, see Section 2)

$$R(t) \sim c^* t, \qquad t \to +\infty. \tag{1.4}$$

Rothe, and more recently Roquejoffre, proved that when $c^* > c_0$ and under some mild assumptions any solution of the initial value problem in Eq. (1.1), with the initial data decaying sufficiently fast at plus infinity and bounded away from zero at minus infinity converges exponentially to the traveling wave solution with speed c^* [25, 26]. Thus, they proved not only that under these assumptions the position of the leading edge behaves asymptotically according to Eq. (1.4), but also that the wave profile approaches uniformly a traveling wave profile with speed c^* (this result remains valid in cylinders with Neumann boundary conditions, and even in the presence of advection terms [26]). Convergence results for this type of equations were also recently discussed from a variational perspective [22].

From the discussion above it is clear that under rather general assumptions on the initial data the problem of speed selection reduces to finding the minimal speed of the monotone traveling wave solutions connecting the two equilibria of f. While Aronson and Weinberger give a definitive answer about the existence of the propagation speed c^*, that is, the existence of the limit in Eq. (1.4), their techniques do not say when the value of $c^* = c_0$ and when $c^* > c_0$. In other words, they do not provide a way to establish whether the linear or nonlinear selection mechanism is realized. Note that this is also a necessary ingredient for applying the results of [25,26]. Therefore, the problem of characterizing the value of c^* is of fundamental importance for understanding the long-time behavior of the solutions of Eq. (1.1).

In this paper, we develop a variational characterization of the traveling wave solutions with speed $c^* > c_0$. We give a necessary and sufficient condition for existence of this solution and thus give a verifiable answer to the question of linear vs. nonlinear selection. Our method also allows to obtain efficient upper and lower bounds for c^* which can be easily implemented in practice.

Our paper is organized as follows. In section 2, we introduce some notation and summarize known results about propagation for solutions of Eq. (1.1). We also generalize the main results of Aronson and Weinberger for the limiting behavior of the propagation speed to initial data with exponential decay and state our main result. In section 3, we introduce our variational problem and demonstrate its relationship to the existence of some special traveling wave solutions which determine the limiting propagation speed. In section 4 we prove our main Theorem, in section 5 we consider a few applications of our results, and in section 6 we make concluding remarks.

2 Preliminaries and the main result

Here we give a few basic definitions and state our main result. Let us start by giving some known results that will be used throughout the paper. We look for the solutions of the initial value problem in Eq. (1.1)

$$u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, \quad \text{with} \quad u_0(x) \in C(\mathbb{R}). \tag{2.1}$$

We assume that the nonlinearity $f \in C^1([0,1])$ and satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0 > f'(1), \quad f(u) > 0 \text{ for } 0 < u < 1. \tag{2.2}$$

Thus, $u = 0$ is the unstable, and $u = 1$ is the stable equilibrium. We will further consider the initial data bounded to the strip

$$0 \leq u_0(x) \leq 1. \tag{2.3}$$

By maximum principle, the solutions of the initial value problem in Eq. (1.1) with these initial conditions will remain bounded between zero and one and therefore will exist for all $t > 0$ (see, for example, [13]).

Turn now to the traveling wave solutions. First, any bounded solution of Eq. (1.3) must connect the equilibria at infinity. In particular, with our assumptions on f we have

PROPOSITION 2.1 (Volpert *et al.* [30]) *Let $\bar{u}$ be a solution of Eq. (1.3) with $c > 0$ and $\bar{u}(x) \in [0,1]$ for all $x \in \mathbb{R}$. Then*

$$\lim_{x \to +\infty} \bar{u}(x) = 0, \tag{2.4}$$

$$\lim_{x \to -\infty} \bar{u}(x) = 1, \tag{2.5}$$

and, furthermore, $\bar{u}(x)$ is monotonically decreasing.

Naturally, because of reflection symmetry for any solution of Eq. (1.3) there exists a mirror-symmetric solution of this equation with speed $-c$ (hence, with the reversed order of the limits in Eqs. (2.4) and (2.5)). Therefore, in the following we will only consider the right-moving waves and propagation toward $x = +\infty$.

Let us define a positive constant

$$c_0 = 2\sqrt{f'(0)}. \tag{2.6}$$

It is easy to see that this constant plays a role of the minimal possible propagation speed for monotone traveling wave solutions. Indeed, the behavior ahead of the wave is determined by the linearization of Eq. (1.3) around zero. Assuming that $\bar{u}(x) \sim e^{-\lambda x}$, we obtain an equation

$$\lambda^2 - c\lambda + f'(0) = 0, \tag{2.7}$$

relating c and λ, whose solutions are $\lambda = \lambda_{\pm}(c)$, where

$$\lambda_{\pm}(c) = \frac{1}{2}\left(c \pm \sqrt{c^2 - 4f'(0)}\right). \tag{2.8}$$

From this equation follows that the decay of $\bar{u}(x)$ for $x \to +\infty$ is non-oscillatory only if $c \geq c_0$.

The existence of the traveling wave solutions that satisfy the conditions in Eqs. (2.4) and (2.5) for the nonlinearities satisfying Eq. (2.2) were proved by Aronson and Weinberger:

PROPOSITION 2.2 (Aronson and Weinberger [1, 2]) *Under the assumptions of Eq. (2.2), for each $c \geq c^*$, with some $c^* \geq c_0$, there exists a unique (up* *to translations) solution $\bar{u}(x)$ of Eq. (1.3) satisfying Eqs. (2.4) and (2.5). Furthermore, for $c > c^*$ we have*

$$\ln \bar{u}(x) \sim -\lambda_-(c)x, \qquad x \to +\infty, \tag{2.9}$$

whereas for $c = c^*$ we have

$$\ln \bar{u}(x) \sim -\lambda_+(c^*)x, \qquad x \to +\infty. \tag{2.10}$$

When $c < c^*$, there are no traveling wave solutions satisfying the assumptions of Proposition 2.1.

Let us now define the position of the leading edge $R(t)$ of the solution of the initial value problem in Eq. (1.1). Let $\alpha \in (0,1)$ be some fixed constant which is so small that $\sup_{x\in\mathbb{R}} u(x,t) \geq \alpha$ for $t \geq 0$. Clearly, such a constant necessarily exists at least for large time intervals, if the initial data are not zero identically. Denote

$$R(t) = \sup\{x \in \mathbb{R} : u(x,t) \geq \alpha\}. \tag{2.11}$$

In terms of $R(t)$, we have the following Theorem, which is a generalization of the classical results of Aronson and Weinberger.

THEOREM 2.3 *Let $u(x,t)$ be a solution of Eq. (1.1) with the initial condition not equal to zero identically, satisfying Eq. (2.3) and*

$$\limsup_{x\to+\infty} u_0(x)e^{\lambda x} < +\infty, \tag{2.12}$$

where $\lambda = \lambda_-(c^*)$. Then there exists a constant α such that Eq. (1.4) holds, with c^* the same as in Proposition 2.2.

PROOF: We first note that $R(t)$ is well-defined for all $t \geq 0$, if α is small enough. Indeed, by the results of Aronson and Weinberger, for any non-zero initial condition we have $\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} u(x,t) = 1$ (see [1,2]), so it is always possible to choose α such that $\sup_{x\in\mathbb{R}} u(x,t) \geq \alpha$ for all $t \geq 0$. Of course, $\sup_{x\in\mathbb{R}} u(x,t) > 0$ for all $t \geq 0$, otherwise by uniqueness of solutions of Eq. (1.1) we would have $u(x,t) \equiv 0$.

Let us first show that $R(t) \geq ct$ for any $c < c^*$ and large enough t. This follows immediately from the fact that by the results of Aronson and Weinberger $u \to 1$ pointwise in any reference frame moving with speed $0 < c < c^*$ [1,2].

To prove that $R(t) \leq ct + R_0$ for all t, with arbitrary $c > c^*$ and some R_0, we construct an appropriate supersolution in the reference frame moving with speed c. To do that, let us note that for $c > c^*$ there is a unique trajectory starting at the origin and having a slope $-\lambda_+(c)$ in the phase plane [1, 2]. Since $c > c^*$, this trajectory intersects the line $u = 1$ at some $u_x = \nu_0 < 0$. By uniqueness of the above trajectory, any other phase plane trajectory going into the origin must have the slope $-\lambda_-(c)$. On the other hand, by Proposition 2.2, for $c > c^*$ there exists a traveling wave solution connecting $u = 0$ and $u = 1$, with slope $-\lambda_-(c)$. Therefore, any phase plane trajectory starting at $u = 1$ and $u_x = \nu$ with $\nu_0 < \nu < 0$ will terminate at the origin and will have the slope $-\lambda_-(c)$ as well. We now construct a supersolution $\bar{u}(x)$ in the reference frame moving with speed c by assuming that $\bar{u}(x)$ satisfies Eq. (1.3) for $x \geq 0$ with the initial conditions $\bar{u}(0) = 1$, $\bar{u}_x(0) = \nu$, and take $\bar{u} = 1$ for $x < 0$.

Now, observe that $\lambda_-(c) < \lambda_-(c^*)$ for $c > c^*$. Then, if Eq. (2.12) holds, we can always bound u_0 with a translate of $\bar{u}$ from above. Since also $\bar{u}$ is monotonically decreasing and $\bar{u}(x) \to 0$ as $x \to +\infty$, this implies that $R(t) \leq ct + R_0$ with some R_0. Finally, since c can be arbitrarily close to c^* in both the upper and lower bound for $R(t)$, we obtain Eq. (1.4). ∎

Thus, the asymptotic propagation speed for the solutions of Eq. (1.1) with the initial data satisfying the assumptions of Theorem 2.3 is c^* in the sense of the asymptotic average speed of the leading edge. Note that the speed c^* is also an asymptotic lower bound for the speed of the leading edge for *any* nontrivial initial condition (see the proof above). Furthermore, note that the assumption on the decay in Eq. (2.12) is almost necessary, since if $\lambda < \lambda_-(c^*)$, then solutions whose leading edge moves faster than c^* (like the traveling waves with speed $c > c^*$ in Proposition 2.2) are possible.

REMARK 2.4 *We point out that in view of [1] the function $R(t)$ is defined for any $\alpha \in (0, 1)$ when t is large enough (see also the proof above), so the statement of Theorem 2.3 remains valid for all such α. Also note that Eq. (1.4) holds independently of α.*

We have two possibilities which we need to discern, so, following van Saarloos [27–29], we introduce the following

DEFINITION 2.5 *Under the assumptions of Theorem 2.3, we call the case $c^* = c_0$ the* linear selection, *while the case $c^* > c_0$ the* nonlinear selection *mechanism.*

The purpose of what follows is to characterize linear vs. nonlinear selection for a given nonlinearity $f(u)$ obeying Eq. (2.2) within the context of Theorem 2.3. We start by defining the exponentially weighted Sobolev spaces we will be working in.

DEFINITION 2.6 *For $c > 0$, denote by $H^1_c(\mathbb{R})$ the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm*

$$||u||_{1,c} = ||u||_c + ||u_x||_c, \qquad ||u||_c^2 = \int_{\mathbb{R}} e^{cx} u^2 dx.$$

These are in fact natural spaces for working with the solutions of Eq. (1.3), since they provide control of the exponential decay of the solution at plus infinity for different choices of the constant c (see below, and also [22]). Also, a simple observation about the decay of the traveling wave solutions in Proposition 2.2, together with the fact that $|\bar{u}_x| \leq C\bar{u}$ with some constant C (see [1, 2]) gives the following

COROLLARY 2.7 *Let $c > c_0$, and let $\bar{u}$ be the traveling wave solution from Proposition 2.2. Then $\bar{u} \in H^1_c(\mathbb{R})$ if and only if $c = c^*$.*

Therefore, the question of linear vs. nonlinear selection is determined by whether or not there exists a traveling wave solution with speed $c > c_0$ and satisfying the assumptions of Proposition 2.1, which lies in $H^1_c(\mathbb{R})$. It is precisely the existence of this solution that we are going to characterize.

For $u \in H^1_c(\mathbb{R})$, define the functional

$$\Phi_c[u] = \int_{\mathbb{R}} e^{cx} \left(\frac{1}{2} u_x^2 + V(u) \right) dx, \tag{2.13}$$

where the function $V(u)$ is given by

$$V(u) = \begin{cases} \frac{1}{2}|f'(0)|u^2, & u < 0, \\ -\int_0^u f(s)ds, & 0 \leq u \leq 1, \\ -\int_0^1 f(s)ds + \frac{1}{2}|f'(1)|(u-1)^2, & u > 1. \end{cases} \tag{2.14}$$

From the assumptions on $f(u)$ in Eq. (2.2), it immediately follows that $V(u) \in C^1(\mathbb{R})$ and that $|V(u)| \leq Cu^2$ for some C, so $\Phi_c[u]$ is well-defined for all u in the considered class.

We now state our main result.

THEOREM 2.8 *Under the assumptions of Theorem 2.3, the nonlinear selection mechanism is realized if and only if there exists $u \in H^1_c(\mathbb{R})$, $u \not\equiv 0$, such that*

$$\Phi_c[u] \leq 0 \tag{2.15}$$

for some $c > c_0$.

Thus, the functional Φ_c provides a complete characterization of the speed selection mechanism within the framework of Aronson and Weinberger.

3 Constrained variational problem

We now formulate the problem of existence of the traveling wave solutions with speed c^* as a constrained variational problem. We point out that our method provides a very general way of constructing the traveling wave solutions and is not limited to the case of the nonlinearities specified in Eq. (2.2), or one-dimensional scalar reaction diffusion equations. The general treatment of this problem from the variational perspective will be presented elsewhere [18].

For $u \in H_c^1(\mathbb{R})$, introduce an auxiliary functional

$$\Gamma_c[u] = \frac{1}{2} \int_{\mathbb{R}} e^{cx} u_x^2 \, dx. \tag{3.1}$$

Note that both Φ_c and Γ_c transform similarly under translations.

LEMMA 3.1 *Let $u \in H_c^1(\mathbb{R})$ and $u_a(x) = u(x-a)$. Then,*

$$\Phi_c[u_a] = e^{ca} \Phi_c[u] \quad \text{and} \quad \Gamma_c[u_a] = e^{ca} \Gamma_c[u]. \tag{3.2}$$

Now, by setting

$$\mathcal{B}_c = \{u \in H_c^1(\mathbb{R}) : \Gamma_c[u] = 1\}, \tag{3.3}$$

we obtain the following constrained minimization problem:

(P) *Find $u_c \in \mathcal{B}_c$ satisfying: $\Phi_c[u_c] = \inf_{\mathcal{B}_c} \Phi_c[u]$.*

The connection between the solutions of problem (P) and the solutions of Eq. (1.3) is established by the following

PROPOSITION 3.2 *Let $u_c(x)$ be a solution of problem (P) with $\Phi_c[u_c] \leq 0$. Then*

$$\bar{u}(x) = u_c\left(x\sqrt{1 - \Phi_c[u_c]}\right) \tag{3.4}$$

is the traveling wave solution with speed $c^\dagger = c\sqrt{1 - \Phi_c[u_c]} \geq c$, which satisfies the assumptions of Proposition 2.1. Furthermore, $\bar{u} \in H_{c^\dagger}^1(\mathbb{R})$.

We prove this proposition via a sequence of lemmas.

LEMMA 3.3 *Let $u_c(x)$ be a solution of problem (P) with $\Phi_c[u_c] \leq 0$. Then $0 \leq u_c(x) \leq 1$.*

PROOF: We argue by contradiction. First note that $u \in H^1_c(\mathbb{R})$ implies $u \in C(\mathbb{R})$. Observe that for any $u \in H^1_c(\mathbb{R})$ we can define

$$\tilde{u}(x) = \begin{cases} 0, & u(x) < 0, \\ u(x), & 0 \leq u(x) \leq 1, \\ 1, & u(x) > 1. \end{cases} \tag{3.5}$$

Since $V(u)$ is strictly increasing outside the interval $u \in [0,1]$, we have

$$\Phi_c[\tilde{u}] < \Phi_c[u] \leq 0, \tag{3.6}$$

unless $u \in [0,1]$. We also have $\Gamma_c[\tilde{u}] \leq 1$. Furthermore, $\Gamma_c[\tilde{u}] > 0$, since otherwise $\tilde{u} = 0$ and hence $\Phi_c[\tilde{u}] = 0$, contradicting Eq. (3.6). Therefore, by Lemma 3.1 it is always possible to find a value of $a \geq 0$ that $\tilde{u}_a(x) = \tilde{u}(x-a) \in \mathcal{B}_c$, too. So, if u is the solution of problem (P), then so is $\tilde{u}_a$, which by Lemma 3.1 and Eq. (3.6) gives lower value of Φ_c, leading to contradiction. ∎

LEMMA 3.4 *Let $u = u_c(x)$ be a solution of problem (P). Then $u \in C^2(\mathbb{R})$ and satisfies*

$$(1-\mu)(u_{xx} + cu_x) + f(u) = 0. \tag{3.7}$$

Moreover,

$$\mu = \Phi_c[u_c]. \tag{3.8}$$

PROOF: We have Φ_c and Γ_c of class C^1. Let $D\Gamma_c[u]v$ be the Frechet derivative of Γ_c at u acting on v. Since

$$D\Gamma_c[u]u = \int_{\mathbb{R}} e^{cx} u_x^2 dx = 2 \quad \forall u \in \mathcal{B}_c,$$

we get $D\Gamma_c[u] \not\equiv 0$ on $\mathcal{B}_c$. Thus, applying the Lagrange Multiplier Theorem (see, for example, [9, Section 3.5]), we obtain

$$\int_{\mathbb{R}} e^{cx} \{(1-\mu) u_x \varphi_x + V'(u)\varphi\} dx = 0 \qquad \forall \varphi \in H^1_c(\mathbb{R}), \tag{3.9}$$

where μ is the Lagrange multiplier.

To proceed, we first show that $\mu \neq 1$. Indeed, if $\mu = 1$, from Eq. (3.9) we get $V'(u) \equiv 0$. Since also u is continuous, according to Eq. (2.2) this means that $u \equiv 0 \notin \mathcal{B}_c$, leading to contradiction.

So, $\mu \neq 1$, and from elliptic regularity theory (see, for example, [15]) and Lemma 3.3 we deduce that $u \in C^2(\mathbb{R})$ and satisfies Eq. (3.7). From this equation and the fact that $|f(u)| \leq C|u|$ then follows that $u_x \in H^1_c(\mathbb{R})$, and, therefore, we can use $\varphi = u_x$ in Eq. (3.9). Integrating by parts, we obtain

$$0 = \int_{\mathbb{R}} e^{cx}\{(1-\mu)u_x u_{xx} + V'(u)u_x\}dx$$
$$= -c\int_{\mathbb{R}} e^{cx}\left(\frac{1}{2}(1-\mu)u_x^2 + V(u)\right) = -c(\Phi_c[u] - \mu\Gamma_c[u]).$$

So, taking into account that $u \in \mathcal{B}_c$, we obtain Eq. (3.8). ∎

Proof of Proposition 3.2: We are now ready to prove Proposition 3.2. By Lemma 3.4, the solution $u = u_c$ of problem (P) satisfies Eq. (3.7) with $\mu \leq 0$ given by Eq. (3.8). Introducing $\bar{u}(x) = u_c(x\sqrt{1-\mu})$ and using Eq. (3.7), we obtain that $\bar{u}$ is a traveling wave with speed $c^\dagger = c\sqrt{1-\mu} \geq c$. Furthermore, by Lemma 3.3 we have $\bar{u} \in [0,1]$ and $\bar{u} \in H^1_{c^\dagger}(\mathbb{R})$. Indeed, with $u = u_c$ we have

$$\int_{\mathbb{R}} e^{c^\dagger x}\bar{u}^2 dx = \int_{\mathbb{R}} e^{cx\sqrt{1-\mu}}u^2(x\sqrt{1-\mu})dx = \frac{1}{\sqrt{1-\mu}}\int_{\mathbb{R}} e^{cx}u^2 dx < \infty,$$
$$\int_{\mathbb{R}} e^{c^\dagger x}\bar{u}_x^2 dx = \int_{\mathbb{R}} e^{cx\sqrt{1-\mu}}u_x^2(x\sqrt{1-\mu})dx = \sqrt{1-\mu}\int_{\mathbb{R}} e^{cx}u_x^2 dx < \infty.$$

∎

Remark 3.5 *We point out that the speed $c^\dagger$ of the obtained traveling wave is independent of the value of c in Proposition 3.2. This is in fact a general property of the considered variational problem (for more details, see [18, 22]).*

To show that the assumption of Theorem 2.8 is necessary, we note simply that if $c^* > c_0$ in Proposition 2.2, then by Corollary 2.7 we have $\bar{u} \in H^1_{c^*}(\mathbb{R})$, where $\bar{u}$ is a traveling wave solution with speed c^*. Furthermore, $\bar{u}$ satisfies Eq. (3.7) with $c = c^*$ and $\mu = 0$. This implies that $\Phi_{c^*}[\bar{u}] = 0$, which gives the desired result.

4 Existence of minimizers

We now show that under the assumption of Theorem 2.8 problem (P) always has a solution. To prove that, we need to establish a few auxiliary results regarding the properties of the functional Φ_c. First, the following Lemma is of fundamental importance to the study of problem (P) and gives the analog of the Poincaré inequality for spaces $H^1_c(\mathbb{R})$.

LEMMA 4.1 *For all $u \in H^1_c(\mathbb{R})$, we have*

$$\frac{c^2}{4}\int_R^\infty e^{cx}u^2dx \le \int_R^\infty e^{cx}u_x^2dx \qquad \forall R \in \mathbb{R}, \tag{4.1}$$

$$u^2(R) \le \frac{e^{-cR}}{c}\int_R^\infty e^{cx}u_x^2dx \qquad \forall R \in \mathbb{R}. \tag{4.2}$$

PROOF: Let us first prove Eq. (4.1). We have

$$\begin{aligned}\frac{c}{2}\int_R^\infty e^{cx}u^2dx &= -\frac{1}{2}e^{cR}u^2(R) - \int_R^\infty e^{cx}uu_xdx \\ &\le \left(\int_R^\infty e^{cx}u^2dx\right)^{1/2}\left(\int_R^\infty e^{cx}u_x^2dx\right)^{1/2},\end{aligned}$$

which implies (4.1).

Let us now prove Eq. (4.2). Since, $\int_R^\infty e^{cx}\left(\sqrt{c}u + \frac{1}{\sqrt{c}}u_x\right)^2 dx \ge 0$, we get

$$\begin{aligned}\frac{1}{c}\int_R^\infty e^{cx}u_x^2dx + c\int_R^\infty e^{cx}u^2dx &\ge -2\int_R^\infty e^{cx}uu_xdx \\ &= u^2(R)e^{cR} + c\int_R^\infty e^{cx}u^2dx,\end{aligned}$$

which gives (4.2). ∎

An immediate consequence of Eq. (4.1) is

COROLLARY 4.2 *If $u \in H^1_c(\mathbb{R})$, then*

$$\frac{c^2}{4}\int_{\mathbb{R}} e^{cx}u^2dx \le \int_{\mathbb{R}} e^{cx}u_x^2dx. \tag{4.3}$$

Observe that since under our assumptions $V(u) \ge -Cu^2$ with some $C \ge 0$, the functional Φ_c will be positive for all non-zero $u \in H^1_c(\mathbb{R})$ for sufficiently large c. More precisely, we have (see also [22])

LEMMA 4.3 *Let $c_{\max}$ be defined as*

$$c_{\max} = \min\left\{c \geq 0 : \quad \frac{1}{8}c^2 s^2 + V(s) \geq 0, \quad \forall s \in \mathbb{R}\right\}. \tag{4.4}$$

Then, for any $c > c_{\max}$ we have $\Phi_c[u] > 0$ for all $u \in H^1_c(\mathbb{R})$, such that $u \not\equiv 0$.

Combined with Theorem 2.8, this result gives an upper bound for the value of c^*.

PROPOSITION 4.4 *The value of $c_{\max}$ in Eq. (4.4) gives an upper bound for the propagation speed c^* in Theorem 2.3.*

PROOF: First of all, we note that $c_{\max} < \infty$. Secondly, we must have $c_{\max} \geq c_0$. Indeed, we have

$$V(s) \sim -\frac{1}{2}f'(0)s^2, \qquad s \to 0^+,$$

with $f'(0) > 0$. Recalling the definition of c_0 in Eq. (2.6), we see that $c_{\max} \geq c_0$ in order for the inequality in Eq. (4.4) to hold for small s. Now, if $c^* > c_{\max}$, then $c^* > c_0$, so by the argument at the end of section 3 we have $\Phi_{c^*}[\bar{u}] = 0$, where $\bar{u}$ is the traveling wave solution with speed c^*, contradicting Lemma 4.3. ∎

REMARK 4.5 *The statement of Proposition 4.4 remains valid in a much more general context (see [22]).*

Before proceeding further to the proof of existence of solutions of problem (P), let us introduce the following notation. For given $-\infty \leq a < b \leq +\infty$, define

$$\Phi_c[u,(a,b)] = \int_a^b e^{cx}\left(\frac{1}{2}u_x^2 + V(u)\right)dx. \tag{4.5}$$

LEMMA 4.6 *Assume $c > c_0$, then there exists $R > 0$ such that*

1. *$\Phi_c[u,(R,+\infty)] \geq 0$ for all $u \in \mathcal{B}_c$;*
2. *letting $u_n \rightharpoonup u$ in $H^1_c(\mathbb{R})$, then $\liminf_{n\to\infty} \Phi_c[u_n,(-\infty,R)] \geq \Phi_c[u,(-\infty,R)]$.*

PROOF: 1) Since $c > c_0$, we can choose $0 < \epsilon < \frac{c^2}{4} - f'(0)$. By the definition of V, there exists some $s_0 > 0$ such that

$$V(s) \geq -\frac{1}{2}(f'(0)+\epsilon)s^2, \qquad |s| \leq s_0.$$

From Lemma 4.1, there exists R_0 such that

$$|u(x)| < s_0 \qquad \forall x > R_0, \quad \forall u \in \mathcal{B}_c.$$

Therefore, given $R > R_0$, we get

$$\int_R^\infty e^{cx} V(u) dx \geq -\frac{1}{2}(f'(0) + \epsilon) \int_R^\infty e^{cx} u^2 dx. \tag{4.6}$$

Thus, from Eqs. (4.1) and (4.6) and the choice of ϵ, we obtain

$$\Phi_c[u, (R, +\infty)] \geq \frac{1}{2}\left(\frac{c^2}{4} - f'(0) - \epsilon\right) \int_R^\infty e^{cx} u^2 dx \geq 0,$$

which concludes the proof.

2) Since $V(u)$ is bounded from below and $\int_{-\infty}^R e^{cx} dx < \infty$, this follows by standard semicontinuity results, see, for example, [11, Propositions 2.1, 2.2]. ∎

We are now ready to prove our existence result. We note that our method has a number of features in common with the technique developed by Berestycki and Lions for scalar field equations [6]. Similar techniques were also used by Heinze in [17].

PROPOSITION 4.7 *Suppose there exists $u \in \mathcal{B}_c$ such that $\Phi_c[u] \leq 0$ for some $c > c_0$. Then problem (P) has a solution.*

PROOF: Let (u_n) be a minimizing sequence of Problem (P), i.e $u_n \in \mathcal{B}_c$ with $\Phi_c[u_n] \to \inf_{\mathcal{B}_c} \Phi_c$. By assumption, $\inf_{\mathcal{B}_c} \Phi_c \leq 0$. Since $\Gamma_c[u_n] = 1$, from inequality (4.3), we get that $\int_\mathbb{R} e^{cx} u_n^2 dx \leq \frac{8}{c^2}$. Thus, (u_n) is bounded in $H_c^1(\mathbb{R})$ and therefore it converges weakly to some $u \in H_c^1(\mathbb{R})$. Furthermore,

$$\begin{aligned}
\inf_{\mathcal{B}_c} \Phi_c &= \liminf_{n\to\infty} \Phi_c[u_n] \\
&\geq \liminf_{n\to\infty}\{\Phi_c[u_n, (-\infty, R)]\} + \liminf_{n\to\infty}\{\Phi_c[u_n, (R, +\infty)]\} \\
&\geq \Phi_c[u, (-\infty, R)] \\
&= \Phi_c[u] - \Phi_c[u, (R, +\infty)], \qquad (4.7)
\end{aligned}$$

for large enough R. Now, by letting $R \to +\infty$ and noting that $\lim_{R\to+\infty} \Phi_c[u, (R, +\infty)] = 0$, Eq. (4.7) leads to

$$0 \geq \inf_{\mathcal{B}_c} \Phi_c \geq \Phi_c[u].$$

If $\inf_{\mathcal{B}_c} \Phi_c = 0$, we deduce that u in the assumption of this proposition is a minimizer. Therefore, let us assume that $\inf_{\mathcal{B}_c} \Phi_c < 0$. Then $u \not\equiv 0$, and by standard semicontinuity results [11]

$$1 = \liminf_{n\to\infty} \Gamma_c[u_n] \geq \Gamma_c[u] > 0.$$

Then we can, by using Lemma 3.1, choose $a \geq 0$ such that

$$\Gamma_c[u_a] = 1 \quad \text{with} \quad u_a(x) = u(x-a).$$

Since $\inf_{\mathcal{B}_c} \Phi_c \leq 0$ and $a \geq 0$, we derive

$$\Phi_c[u_a] = e^{ca}\Phi_c[u] \leq \Phi_c[u] \leq \inf_{\mathcal{B}_c} \Phi_c.$$

Therefore, $\Phi_c[u_a] = \inf_{\mathcal{B}_c} \Phi_c$, and u_a solves Problem (P).

∎

Theorem 2.8 then follows by noting that if $\Phi_c[u] \leq 0$ for some $u \not\equiv 0$, then, according to Lemma 3.1, we can make $u \in \mathcal{B}_c$ by an appropriate shift. Therefore, by Propositions 3.2 and 4.7 there exists a traveling wave solution with speed $c^\dagger$ that lies in $H^1_{c^\dagger}(\mathbb{R})$ and satisfies the assumptions of Proposition 2.1, and hence is one of the solutions from Proposition 2.2. Then, by Corollary 2.7 we have $c^* = c^\dagger \geq c > c_0$, which completes the proof.

REMARK 4.8 *It is not difficult to see that Propositions 3.2 and 4.7 remain valid when $f(u)$ is not necessarily positive for all $u \in (0,1)$.*

Thus, our method also provides a general method of constructing the traveling wave solutions, as well as obtaining upper and lower bounds for their speed. Moreover, in the case of $f'(0) \leq 0$ there is no need for any assumptions on c (since $c^2 - 4f'(0) > 0$ in Lemma 4.6 for all $c > 0$), and the assumption of Proposition 4.7 is always satisfied as long as $V(u) < 0$ for some $0 < u \leq 1$ (see section 5). We also note that by the result of Rothe the existence of the traveling wave solution with $c^* > c_0$ established above implies that under essentially the same assumptions as those in Theorem 2.3 the solutions of the initial value problem for Eq. (1.1) will converge uniformly to a translate of this traveling wave as $t \to +\infty$ [25].

REMARK 4.9 *Observe that by Proposition 3.2 we have $c^* \geq c$, so the value of c in Theorem 2.8 also provides the lower bound for the propagation speed.*

5 Some applications

In this section, we first give two results concerning sufficient conditions for linear and nonlinear selection mechanisms, respectively. We then perform a variational study of a particular example, namely Eq. (1.2).

Let us first formulate the general sufficient condition for the linear selection mechanism.

THEOREM 5.1 *If for all* $0 < u \leq 1$

$$\frac{2}{u^2}\int_0^u f(s)ds \leq f'(0), \tag{5.1}$$

then the linear selection is realized.

PROOF: From Eq. (5.1) we obtain

$$V(u) \geq -\frac{1}{2}f'(0)u^2 = -\frac{1}{8}c_0^2u^2,$$

so $c_{\max} = c_0$ in Eq. (4.4). Therefore, by Lemma 4.3 we have $\Phi_c[u] > 0$ for all $u \not\equiv 0$ and $c > c_0$. The result then follows from the "only if" statement of Theorem 2.8. ∎

Naturally, this implies that $c^* = c_0$. Observe that this result is the generalization of that of Kolmogorov, Petrovsky, and Piskunov, who required that $f'(u) \leq f'(0)$ [19], as well as the result of Aronson and Weinberger, who obtained an upper bound for c^*, which is equal to c_0 if $f(u)/u \leq f'(0)$ [2].

We now give a sufficient criterion of the nonlinear selection mechanism. We note that better criteria can be obtained for a given nonlinearity by using suitable trial functions (see below). This criterion, nevertheless, gives a more precise meaning to the "ZFK" case considered in [26] and is relevant to combustion. We also point out that this criterion was obtained earlier by Berestycki and Nirenberg as a sufficient condition for $c^* > c_0$ in Proposition 2.2 [7].

THEOREM 5.2 *If*

$$f'(0) \leq \frac{1}{2}\int_0^1 f(u)du, \tag{5.2}$$

then the nonlinear selection mechanism is realized.

PROOF: Let us take

$$u_\lambda(x) = \begin{cases} 1, & x \le 0, \\ e^{-\lambda x}, & x > 0. \end{cases}$$

Then for $u = u_\lambda(x)$ we have

$$\Phi_c[u] < \frac{1}{2}\int_0^\infty e^{cx} u_x^2 dx + \int_{-\infty}^0 e^{cx} V(u) dx$$
$$= \frac{\lambda^2}{2(2\lambda - c)} + \frac{V(1)}{c},$$

as long as $\lambda > \frac{c}{2}$. Minimizing this expression with respect to λ, we obtain that the minimum is achieved at $\lambda = \lambda_{\min} = c$. This means that

$$\Phi_c[u_{\lambda_{\min}}] < \frac{c}{2} + \frac{V(1)}{c}.$$

Recalling that $V(1) < 0$, we see that the expression above is negative whenever

$$c = c_{\min} = \sqrt{-2V(1)} = \sqrt{2\int_0^1 f(u) du}. \tag{5.3}$$

By continuity, there exists $c > c_{\min}$ such that $\Phi_c[u_{\lambda_{\min}}] \le 0$, so $c > c_0$ (see Eq. (2.6)) and $u_{\lambda_{\min}}$ satisfies the assumption of Theorem 2.8. ∎

Note that the expression in the right-hand side of Eq. (5.3) tends to c^* in the limit of narrow reaction zone (that is, when $f(u)$ is concentrated around $u = 1$). Of course, it also provides a lower bound for c^*. Also note that for general $f(u)$ (not necessarily positive) this proof can be modified to obtain an analogous estimate, as long as $V(1) < 0$, which is a necessary condition for the existence of traveling waves with $c > 0$ (see, for example, [30]).

Let us now perform a variational study of Eq. (1.2). This equation with $\mu > 0$ satisfies the assumptions in Eq. (2.2) after a trivial rescaling of u. The stable positive equilibrium $u = u_{\max}$ is given by

$$u_{\max} = \frac{\sqrt{1 + \sqrt{1 + 4\mu}}}{\sqrt{2}}. \tag{5.4}$$

The nice thing about Eq. (1.2) is that it admits a traveling wave solution with speed $c > 0$ for $\mu > -\frac{3}{16}$ which can be found exactly [24, 28].

Furthermore, for $0 < \mu < \frac{3}{4}$ this is precisely the solution with speed c^*, which is explicitly given by

$$c^* = \frac{2\sqrt{1+4\mu} - 1}{\sqrt{3}}. \tag{5.5}$$

Numerical evidence also suggests that for $\mu \geq \frac{3}{4}$ we have $c^* = c_0 = 2\sqrt{\mu}$. Note that the expression in Eq. (5.5) for the speed of the wave also remains valid for $\mu < 0$, it corresponds to the unique (up to translations) traveling wave solution in this case [1]. Also, when $\mu = 0$, it corresponds to the unique traveling wave solution which decays exponentially at plus infinity (for an existence proof in this degenerate case, see [8,20]). Since our proof of existence of the traveling wave solution in fact extends to the case of $f'(0) \leq 0$, we will treat Eq. (1.2) for all $\mu > -\frac{3}{16}$. In short, the availability of an analytical expression for the speed c^* makes it possible to assess the effectiveness of our variational method, as well as study the parametric dependence of c^*.

We start by looking for the value of $c_{\max}$ as a function of μ. Observe that for $0 \leq u \leq 1$ we have

$$\frac{1}{8}c^2u^2 + V(u) = \frac{1}{2}\left(\frac{c^2}{4} - \mu - \frac{1}{2}u^2 + \frac{1}{3}u^4\right)u^2.$$

So, minimizing the expression in the brackets and demanding that it remains positive for all u, we obtain

$$c_{\max} = \frac{1}{2}\sqrt{3+16\mu}. \tag{5.6}$$

Note that the value of $c_{\max}$ in Eq. (5.6) provides an upper bound for c^* and is typically within $\sim 20\%$ of the exact value in Eq. (5.5). This equation also shows that $c_{\max} > 0$ only for $\mu > -\frac{3}{16}$, as expected from the exact solution.

We are now going to analyze the existence of the traveling wave solutions with speed $c^* > c_0$ and obtain the lower bounds for the speed, using our variational approach. To proceed, let us choose a very simple trial function that looks like a front:

$$u_\lambda(x) = \begin{cases} \frac{1}{2}u_{\max}e^{-\lambda x}, & x \geq 0, \\ \frac{1}{2}u_{\max}\left(2 - e^{\lambda x}\right), & x < 0, \end{cases} \tag{5.7}$$

where $u_{\max}$ is given by Eq. (5.4). So, u_λ is a C^1 function connecting the unstable equilibrium $u = 0$ at plus infinity with the stable equilibrium

$u = u_{\max}$ at minus infinity, characterized by just a single parameter λ that gives the rate of exponential decay at plus infinity. What we are going to show below is that this choice of the trial function already allows to determine the value of c^* with an accuracy of just a few per cent in the entire range of μ.

Let us substitute u_λ into the functional. It is straightforward, although tedious, to perform the integration, the resulting expression is a rational function of $\lambda > \frac{c}{2}$ (the algebraic calculations were performed using MATHEMATICA software):

$$
\begin{aligned}
&\Phi_c[u_\lambda] = \\
&-(\lambda^2(16c^7\lambda(1+\sqrt{1+4\mu}) - 92160\lambda^6(1+\sqrt{1+4\mu}+\mu(6+4\sqrt{1+4\mu})) \\
&-8c^4\lambda^2(9(1+\sqrt{1+4\mu}) + 832\lambda^2(1+\sqrt{1+4\mu}) - 3\mu(13+19\sqrt{1+4\mu})) \\
&+192c\lambda^5(53(1+\sqrt{1+4\mu}) + 720\lambda^2(1+\sqrt{1+4\mu}) - 2\mu(222+275\sqrt{1+4\mu})) \\
&+48c^2\lambda^4(295(1+\sqrt{1+4\mu}) + 1536\lambda^2(1+\sqrt{1+4\mu}) + \mu(973+383\sqrt{1+4\mu})) \\
&-12c^3\lambda^3(-187(1+\sqrt{1+4\mu}) + 272\lambda^2(1+\sqrt{1+4\mu}) - \mu(1153+779\sqrt{1+4\mu})) \\
&-c^5\lambda(592\lambda^2(1+\sqrt{1+4\mu}) + 9(5+27\mu+5\sqrt{1+4\mu}+17\mu\sqrt{1+4\mu})) \\
&-c^6(-128\lambda^2(1+\sqrt{1+4\mu}) + 3(1+\sqrt{1+4\mu}+\mu(7+5\sqrt{1+4\mu})))))/ \\
&(64c(c-6\lambda)(c-4\lambda)(c-2\lambda)(c+2\lambda)(c+3\lambda)(c+4\lambda)(c+5\lambda)(c+6\lambda)).
\end{aligned}
$$

The situation is complicated somewhat by the μ-dependence, so before we apply our techniques we need to make sure that $\Phi_c[u_\lambda]$ can in fact be negative in a certain range of μ for $c > c_0 = 2\sqrt{\mu}$. To do this, we will use the following trick. Take a sufficiently small $\epsilon > 0$ and consider $\lambda_\epsilon = \sqrt{\mu} + \epsilon$ and $c = 2\sqrt{\mu} + \epsilon^2$. Then clearly $u_{\lambda_\epsilon} \in H^1_{c_\epsilon}(\mathbb{R})$. A direct calculation then shows that

$$
(5.8) \quad \lim_{\epsilon \to 0^+} \Phi_{c_\epsilon}[u_{\lambda_\epsilon}] = \frac{\mu\,(331 + 5785\sqrt{1+4\mu}) - 2727\,(1+\sqrt{1+4\mu})}{215040\sqrt{\mu}}.
$$

Analyzing this expression as a function of μ, we see that it is negative as long as $0 < \mu < \mu_{\max}$, where

$$
(5.9) \qquad \mu_{\max} = \frac{23212224}{33466225} \simeq 0.6936.
$$

Now, since the limit in Eq. (5.8) exists, for all $0 < \mu < \mu_{\max}$ it is possible to choose ϵ such that $\Phi_{c_\epsilon}[u_{\lambda_\epsilon}] \leq 0$, so u_{λ_ϵ} satisfies the assumption of Theorem 2.8. Thus, we have just proved, using the trial function in Eq. (5.7), that the nonlinear selection mechanism is realized at least for

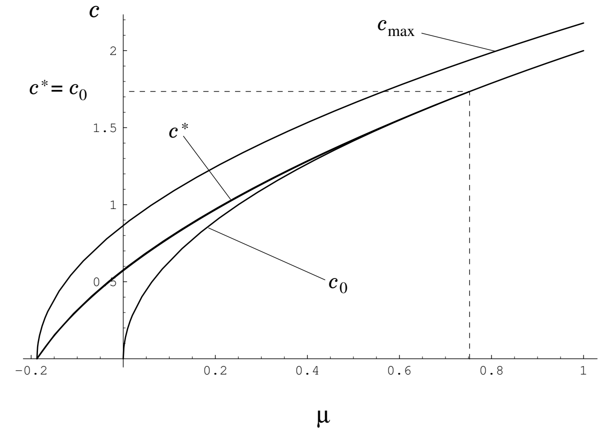

Figure 5.1. Speed of the traveling wave as a function of μ.

$0 < \mu < \mu_{\max}$, with $\mu_{\max}$ given explicitly by Eq. (5.9). Note that numerical analysis of the expression for $\Phi_{c_\epsilon}[u_{\lambda_\epsilon}]$ shows that the value of $\mu_{\max}$ obtained above is in fact optimal. Comparing this with the known exact result, we see that $\mu_{\max}$ slightly underestimates the maximal value of $\mu = \frac{3}{4}$ below which the nonlinear selection mechanism is realized [24, 28].

Let us now look at how closely our variational procedure allows to estimate the propagation speed c^* from below. First of all, note that $c_0 = 2\sqrt{\mu}$ is a natural lower bound for the speed c^* for $\mu > 0$. In fact, for $\mu \geq \mu_{\max}$ the difference between c^* and c_0 is less than 0.03%. Now, for $\mu < \mu_{\max}$ and fixed c, $\Phi_c[u_\lambda]$ as a function of λ has a minimum for $\lambda > \frac{c}{2}$, since $\Phi_c[u_\lambda] \to +\infty$ as $\lambda \to \frac{c}{2} + 0$ (since in this case the integral is dominated by the quadratic terms at plus infinity, which for $c > c_0$ are positive) or $\lambda \to +\infty$. Therefore, we look for the values of c and λ that solve

$$\Phi_c[u_\lambda] = 0, \qquad \frac{\partial \Phi_c[u_\lambda]}{\partial \lambda} = 0.$$

The solution is obtained numerically for all $-\frac{3}{16} < \mu < \mu_{\max}$. The results are shown in Fig. 5.1. This figure shows the speed c obtained above, together with the exact speed c^* given by Eq. (5.5), as well as $c_{\max}$ given by Eq. (5.6) and c_0, all as functions of μ. Note that the dependences $c(\mu)$ and $c^*(\mu)$ are virtually indistinguishable. To see the agreement between c and c^*, we plot the difference between the two as a function of μ in Fig. 5.2. For $\mu \geq \mu_{\max}$, we use $c = c_0$ as the lower bound for c^*. One can see that the error is within 1% of the true value for all $\mu > 0$ and rapidly

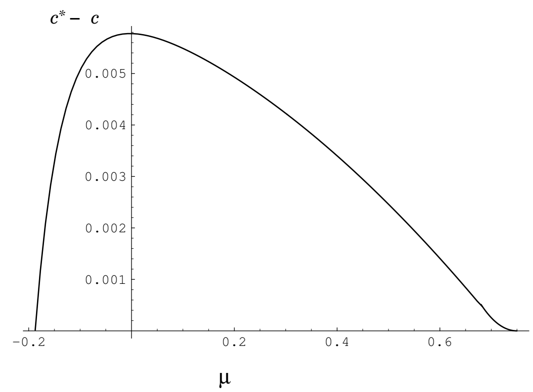

Figure 5.2. The absolute error $c^* - c$ as a function of μ.

decreases as μ increases. We also verified that the speed c remains within 2.5% of the exact value in the *entire* range of μ (including $-\frac{3}{16} < \mu \leq 0$). This accuracy is quite remarkable, especially considering the fact that it is achieved by tuning only one parameter, λ, in the trial function. Note that we experimented with a number of different choices of the trial functions, they all give comparable results. Thus, our method provides a very efficient way of estimating the propagation speed, as well as establishing the ranges of parameters for the nonlinear selection mechanism.

6 Concluding remarks

To conclude, we have obtained a variational characterization of certain traveling wave solutions for scalar reaction-diffusion equations. These traveling waves are special in the sense that they have a non-generic exponential decay at plus infinity (see Eq. (2.10)). In the context of the theory of Aronson and Weinberger these solutions determine the propagation speed for the sufficiently localized initial conditions, so our method gives an easily verifiable answer to the problem of linear vs. nonlinear selection mechanism. We showed that our method allows to obtain very accurate lower bounds for the propagation speed, as well as simple upper bounds.

We remark that various other variational characterizations of the speed c^* exist. One method is based on the minimax characterization of the traveling wave solutions (see, for example, [16, 30]). Another technique, introduced by Benguria and Depassier, uses integral variational principles in the phase plane [3,4]. We point out, that these methods are formulated within phase plane and may therefore be rather sensitive to the choices of the trial functions. We, on the other hand, demonstrated that in our variational approach there is very little sensitivity to the choice of the trial functions, what allows very accurate estimates of the propagation speed, at least for the example considered above. We also point out that our method does not just characterize the speed c^*, but also gives an alternative proof of existence of the traveling wave solutions of certain types, and is in fact much more general than the original setup (see also [18]).

Concerning the problem of propagation, the framework of Aronson and Weinberger allows to associate the speed c^* with the average asymptotic speed of the leading edge for the solutions of Eq. (1.1) with suitable initial conditions. An interesting question arising here is whether this remains true for more general classes of initial conditions, for example, in the case of Eq. (1.2), for initial conditions that are not necessarily positive (see, for example, [24]). Such problems have recently been addressed in [22], where the generalization of the functional from Eq. (2.13) was used to study questions of propagation and convergence to traveling wave solutions for reaction-diffusion systems of gradient type in cylinders in a very general setting. It is interesting to see whether the selection problem discussed in this paper is meaningful in this more general context and in fact reduces to the problem of existence of the special traveling waves whose existence can be captured by our variational procedure.

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