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A variational model of charged drops in dielectrically matched binary fluids: the effect of charge discreteness

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Abstract

This paper addresses the ill-posedness of the classical Rayleigh variational model of conducting charged liquid drops by incorporating the discreteness of the elementary charges. Introducing the model that describes two immiscible fluids with the same dielectric constant, with a drop of one fluid containing a fixed number of elementary charges together with their solvation spheres, we interpret the equilibrium shape of the drop as a global minimizer of the sum of its surface energy and the electrostatic repulsive energy between the charges under fixed drop volume. For all model parameters, we establish existence of generalized minimizers that consist of at most a finite number of components "at infinity". We also give several existence and non-existence results for classical minimizers consisting of only a single component. In particular, we identify an asymptotically sharp threshold for the number of charges to yield existence of minimizers in a regime corresponding to macroscopically large drops containing a large number of charges. The obtained non-trivial threshold is significantly below the corresponding threshold for the Rayleigh model, consistently with the ill-posedness of the latter and demonstrating a particular regularizing effect of the charge discreteness. However, when a minimizer does exist in this regime, it approaches a ball with the charge uniformly distributed on the surface as the number of charges goes to infinity, just as in the Rayleigh model. Finally, we provide an explicit solution for the problem with two charges and a macroscopically large drop.

Contents

1 Introduction

There has recently been a growing interest in geometric variational problems featuring a competition of attractive and repulsive interactions [6]. A prototypical model giving rise to the problems of this kind is the celebrated Gamow's liquid drop model of the atomic nucleus [15], in which a competition of the cohesive action of the surface tension with the Coulombic repulsion gives rise to delicate questions about the existence and the shape of minimizers, etc. There are now many studies of this model and its various generalizations and extensions that are too numerous to list here (for some recent works, see, e.g., [13,31,34] and references therein).

We focus on a closely related problem arising from the classical model introduced by Lord Rayleigh that describes the energetics of a perfectly conducting charged liquid drop [29] (for the technical details of the model, see section 2). In 1882, Rayleigh demonstrated that a spherical liquid drop becomes linearly unstable with respect to asymmetric distortions of its shape when the amount of charge on the droplet exceeds a critical value called the Rayleigh charge. Such an interfacial instability driven by the electric field was first observed experimentally by Zeleny [41, 42] and subsequently studied by great many authors (see, e.g., [1, 9, 10, 17, 21]), not least because of its important applications to analytical chemistry [16]. Surprisingly, however, the linear stability of the charged drop below the critical charge in the Rayleigh model was recently shown not to imply stability of a spherical drop with respect to arbitrarily small perturbations of its shape [33]. In fact, the

Rayleigh model leads to a problem that is variationally ill-posed [18, 20, 33]. Mathematically, this is because the regularizing action of the perimeter is not sufficient to control the electric charges at small scales [19]. Physically, it manifests itself in the formation of singularities in the form of Taylor cones and jets [12, 26, 37].

The variational ill-posedness of the above problem indicates that the Rayleigh model does not contain all the physics that is necessary to describe the equilibrium shapes of conducting charged drops. Several regularizing mechanisms have, therefore, been proposed, including thermal effects that restore existence of minimizers under certain conditions due to the spreading of the charges into a thin Debye layer beneath the droplet surface [7,32,33]. Nevertheless, in some situation such as cryogenic liquids or nanoscale droplets, in which the thermal motion of free charges is suppressed, another physical mechanisms may be necessary. One such mechanism relies on the fundamental discreteness of the electric charges [11, 25, 26, 28]. In this paper, we explore this possibility in the special case of dielectrically matched fluids, in which there is no dielectric contrast between the droplet and its surroundings (again, see section 2 for technical details).

For a model that keeps track of the positions of individual charges inside the droplet, we establish existence of generalized minimizers, a suitable notion of minimality for this kind of problems that accounts for a possibility of components that are infinitely far apart, first introduced in [27]. We also establish the regularity and connectedness of the components of the generalized minimizers. We then proceed to investigate under which conditions classical minimizers, consisting of only a single component, are possible in the physically important regime of sufficiently strong repulsion between the charges in comparison to the surface tension. Here we establish a sharp existence/non-existence criterion in the case of many charges, which yields a critical charge for existence that is significantly smaller than the Rayleigh charge. We also establish some structural information about the locations of the charges when the minimizers do exist and show that in a suitable continuum limit within the existence range the minimizer converges in an appropriate sense to a ball with the charges uniformly distributed on its surface. Lastly, we present an explicit solution of the variational problem in the case of only two point charges.

Our paper is organized as follows. In section 2, we introduce the model considered in this paper and discuss the relevant parameter ranges. In section 3, we state the main results of our paper. In section 4, we present the proof of Theorem 3.2 that gives existence of generalized minimizers. In section 5, we present the proofs of the existence result of Theorem 3.3, the non-existence result of Theorem 3.4, and the asymptotic characterization of minimizers with many charges in Theorem 3.5. Lastly, in section 6 we present the analysis of the two-charge problem that yields Theorem 3.6. This section also gives an explicit characterization of the energy minimizers.

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2 Model

We consider a system consisting of two immiscible fluids with matched dielectric constants, i.e., both fluids have the relative dielectric constant equal to ε_d. Because of this, we do not need to worry about the shape dependent dielectric polarization of the liquid drop in the presence of charges, which would otherwise considerably complicate the analysis [7]. In the following, we simply refer to the first fluid of finite volume surrounded by the second ambient fluid as the *liquid drop*. A notable example of such a fluid system is liquid helium in equilibrium with its vapor, which has been used to investigate the phenomenon of Wigner crystallization of charges at the liquid-vapor interface and is known to undergo charge-driven interfacial instabilities [2, 22, 23, 38]. More recently, charge-containing helium nanodroplets have been considered as a host medium for a variety of applications in molecular spectroscopy and quantum chemistry [5].

At the level of the continuum, the equilibrium shape of a charged, perfectly conducting liquid drop may be investigated with the help of a model that goes back over 140 years to Lord Rayleigh [29]. In this model, an equilibrium drop is viewed as a minimizer (at least local) of the energy

$$\mathcal{E}(\Omega) := \sigma P(\Omega) + \frac{Q^2}{2C(\Omega)}, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^3$ is the set occupied by the drop that carries the charge Q, with the volume of the drop $|\Omega| = m$. Here, σ is the surface tension of the liquid interface, $P(\Omega)$ is the perimeter of the set Ω defined by

$$P(\Omega) := \sup \left\{ \int_\Omega \nabla \cdot \phi(y)\, dy : \ \phi \in C_c^1(\mathbb{R}^3; \mathbb{R}^3), \ |\phi| \leq 1 \right\}, \tag{2.2}$$

which is a suitable measure-theoretic generalization of the surface measure for smooth sets, and C is the electrostatic capacity defined by

$$C^{-1}(\Omega) := \inf_{\mu(\Omega)=1} \int_\Omega \int_\Omega \frac{1}{4\pi\varepsilon_0\varepsilon_d|x-y|}\, d\mu(x)\, d\mu(y), \tag{2.3}$$

where ε_0 is the permeability of vacuum, and the minimization is carried out over probability measures μ supported on Ω. However, as was already mentioned, this model was recently shown to be variationally ill-posed [18, 33]. Thus, a regularization of the electrostatic problem is necessary to enable existence of even local energy minimizers in the natural classes of liquid configurations.

In this paper, we appeal to the discrete nature of electric charges as a possible physical regularizing mechanism [11, 25, 26, 28], while ignoring the entropic effects associated with thermal agitation of the charges (appropriate for nanoscale droplets or cryogenic fluids).

This amounts to restricting the measures appearing in (2.3) to those associated with N point charges:

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \tag{2.4}$$

where $x_i \in \mathbb{R}^3$ are the positions of the charges and δ_{x_i} are the Dirac delta-measures centered at x_i. Note that in doing so we must exclude the self-interaction of charges. Setting $x \neq y$ in the integral in (2.3) then yields the following expression for the energy:

$$\mathcal{E}_N(\Omega, X) := \sigma P(\Omega) + \frac{e^2}{8\pi\varepsilon_0\varepsilon_d} \sum_{i \neq j} \frac{1}{|x_i - x_j|}. \tag{2.5}$$

Here the set $\Omega \subset \mathbb{R}^3$ again denotes the domain occupied by the liquid drop, the discrete set $X = \cup_{i=1}^{N} \{x_i\} \subset \mathbb{R}^3$ specifies the positions of N point charges, and e is the elementary charge (positive), so that $|Q| = Ne$. For simplicity, we assume a single species of monovalent ions dissolved in the liquid drop, with the ambient fluid a perfect dielectric.

Notice that every charge in the liquid drop strongly attracts a cluster of liquid (solvent) molecules forming a *solvation shell* around the charge (ion). We model this effect by requiring that the liquid drop contains a ball of radius r_0, called the solvation radius, around each charge [25], i.e., we have $B_{r_0}(x_i) \subset \Omega$ for each $i = 1, \ldots, N$, with $B_{r_0}(x_i)$ mutually disjoint. The solvation radius of simple monoatomic ions in polar solvents like water usually measures to fractions of a nanometer.

To assess the relative strengths of the two terms in the energy and to carry out an appropriate non-dimensionalization, we introduce the molecular length scale

$$r_\sigma := \sqrt{\frac{k_B T}{\sigma}}, \tag{2.6}$$

where $k_B T$ is the temperature in the energy units, above which the interface may be considered as sharp and well defined in the presence of thermal noise. For low molecular weight liquids at room temperature, r_σ is on the order of a fraction of a nanometer. This scale may be compared with the Bjerrum length

$$r_B := \frac{e^2}{4\pi\varepsilon_0\varepsilon_d k_B T}, \tag{2.7}$$

which measures the scale at which the Coulombic energy of a pair of elementary charges in a dielectric liquid is comparable to the thermal energy. In polar solvents at room temperature, this length is on the order of a few nanometers. Rescaling lengths with r_σ and measuring the energy in the units of $k_B T$ then yields $\mathcal{E}_N(r_\sigma \Omega, r_\sigma X) = k_B T E_{\rho,\lambda,N}(\Omega, X)$, where

$$E_{\rho,\lambda,N}(\Omega, X) := P(\Omega) + \lambda \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{|x_i - x_j|}, \tag{2.8}$$

and we introduced the dimensionless parameters

$$\rho := \frac{r_0}{r_\sigma}, \qquad \lambda := \frac{r_B}{r_\sigma}. \tag{2.9}$$

From the basic physical considerations already mentioned, for typical liquids at room temperature both ρ and λ are expected to be of order one [25]. For example, for small monovalent ions in ethanol (a common solvent for electrospray) we have $\rho \approx 1$ and $\lambda \approx 5$. In contrast, for liquid helium at $T = 2$ K, for which $r_\sigma \approx 0.3$ nm and $r_B \approx 8\,\mu$m we get $\rho \sim 1$ and $\lambda \sim 10^6 \gg 1$. As a point of reference, let us note that for the parameters of liquid helium above our Theorem 3.6 yields existence of an equilibrium configuration only for droplets whose volume corresponds to a ball of radius greater than $\sim 10\,\mu$m even with just two point charges.

The case of the main physical interest corresponds to that of the volume of the charged drop becoming macroscopically large ($m \to \infty$), while the number of charges N simultaneously tends to infinity with a suitable rate. To study this regime, we can carry out another rescaling in which the volume is instead normalized to a constant while the radius of the solvation sphere vanishes. Introducing the parameter $\varepsilon > 0$ that will eventually be sent to zero, we have $E_{\rho,\lambda,N}(\varepsilon^{-1}\rho\Omega, \varepsilon^{-1}\rho X) = \varepsilon^{-2}\rho^2 E_\varepsilon(\Omega, X)$, where

$$E_\varepsilon(\Omega, X) := P(\Omega) + \gamma\varepsilon^3 \sum_{i=1}^{N_\varepsilon - 1} \sum_{j=i+1}^{N_\varepsilon} \frac{1}{|x_i - x_j|}, \tag{2.10}$$

$B_\varepsilon(x_i) \subset \Omega$ are disjoint for all $1 \le i \le N_\varepsilon$, and $\gamma := \lambda/\rho^3$ is a single dimensionless parameter that characterizes the physical properties of the liquid and is kept fixed throughout the analysis. The considerations following (2.9) motivate us to focus on the physically most relevant regime of $\gamma \gtrsim 1$. The assumptions on the dependence of $N_\varepsilon \to \infty$ on $\varepsilon \to 0$ that yield information about the equilibrium shape of the charged drops turn out to be non-trivial and will be specified in the following sections.

3 Main results

We now state the main results of our paper concerning the minimizers of the energy $E_{\rho,\lambda,N}$ and its rescaled version E_ε. We begin by defining the admissible class $\mathcal{A}_{m,N,\rho}$ of configurations consisting of a set of finite perimeter $\Omega \subset \mathbb{R}^3$ of volume $m > 0$ and $N \in \mathbb{N}$ non-overlapping charges of radius $\rho > 0$ contained in Ω, whose centers are collected into a discrete set $X \subset \mathbb{R}^3$:

$$\begin{aligned} \mathcal{A}_{m,N,\rho} := \{(\Omega, X) \ :\ & \\ & \Omega \subset \mathbb{R}^3 \text{ measurable}, \ |\Omega| = m, \ P(\Omega) < \infty, \\ & X = \cup_{i=1}^N \{x_i\}, \ (x_i)_{i=1}^N \in \mathbb{R}^3, \\ & |\Omega \cap B_\rho(x_i)| = |B_\rho(0)| \text{ for all } 1 \le i \le N, \\ & B_\rho(x_i) \cap B_\rho(x_j) = \varnothing \text{ for all } 1 \le i < j \le N\}. \end{aligned} \tag{3.1}$$

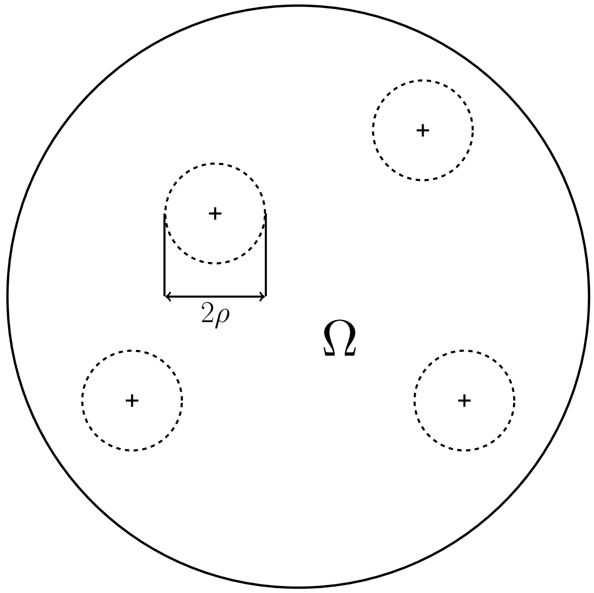

Figure 1: Schematics of an admissible configuration consisting of $N = 4$ charges indicated with "+" when Ω is a ball.

An example of an admissible configuration is shown in Fig. 1. Notice that the set $\cup_{i=1}^{N} B_\rho(x_i)$ representing the charges is assumed to be contained inside the set Ω in the measure theoretic sense.

We would like to investigate under which conditions the energy $E_{\rho,\lambda,N}$ admits a minimizer in the class $\mathcal{A}_{m,N,\rho}$. Notice that the question of existence of such minimizers is far from obvious because of the possibility of splitting of the set Ω into disjoint pieces that carry the charges apart to lower the Coulombic energy at the expense of increasing the interfacial energy. This issue is well known in the studies of geometric variational problems with competing interactions [6]. In the context of Gamow's liquid drop model, it was shown that an appropriate extension of the notion of minimizers for this kind of problems is given by *generalized minimizers* [27]. In our problem, these are defined as follows.

Definition 3.1. *Let ρ, $\lambda > 0$, $N \in \mathbb{N}$ and $m \geq \frac{4\pi}{3} N \rho^3$. Suppose there exists $K \in \mathbb{N}$, $m_k > 0$ and $N_k \in \mathbb{N} \cup \{0\}$ with $m = \sum_{k=1}^{K} m_k$, $N = \sum_{k=1}^{K} N_k$, and a family of minimizers $(\Omega_k, X_k) \in \mathcal{A}_{m_k,N_k,\rho}$ of E_{ρ,λ,N_k} which satisfies*

$$\sum_{k=1}^{K} E_{\rho,\lambda,N_k}(\Omega_k, X_k) = \inf_{(\Omega,X)\in\mathcal{A}_{m,N,\rho}} E_{\rho,\lambda,N}(\Omega, X). \tag{3.2}$$

Then the family of (Ω_k, X_k) is called a generalized minimizer of $E_{\rho,\lambda,N}$ over $\mathcal{A}_{m,N,\rho}$.

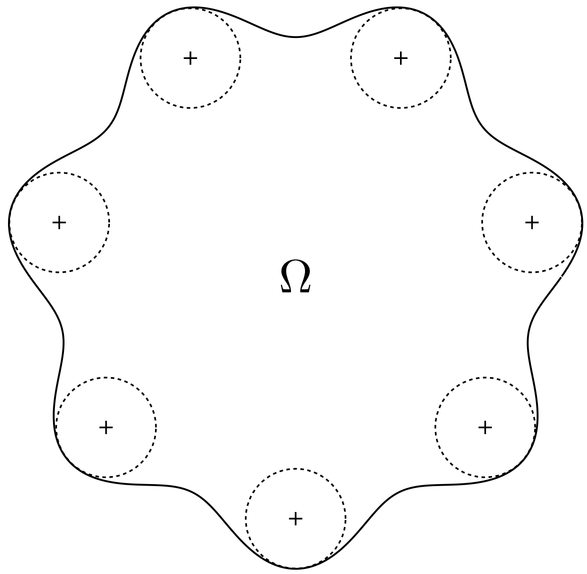

Figure 2: A schematic of a classical minimizer for $N = 7$.

Intuitively, a generalized minimizer can be thought of as a *finite* collection of droplets containing all the charges, with each droplet being a minimizer for the charge it contains and different droplets being "infinitely far apart" and thus not interacting. Each set Ω_k in a generalized minimizer is referred to as a *component*. Notice that a generalized minimizer is simply a minimizer if and only if it has only one component. An illustration of a classical minimizer of $E_{\rho,\lambda,N}$ with $N = 7$ is presented in Fig. 2, while a possible generalized minimizer is shown in Fig. 3. Our first result establishes existence of generalized minimizers for all nontrivial values of the parameters.

Theorem 3.2. *Let ρ, $\lambda > 0$, $N \in \mathbb{N}$ and $m \geq \frac{4\pi}{3} N\rho^3$. Then there exists a generalized minimizer of $E_{\rho,\lambda,N}$ over $\mathcal{A}_{m,N,\rho}$. Moreover, each component of the generalized minimizer has boundary of class $C^{1,1}$ and is connected.*

In view of the regularity of the components of generalized minimizers, in the following we always refer to the regular representatives when talking about the energy minimizing sets. In particular, we can choose these sets to be open.

We next establish a parameter regime in which the generalized minimizers are also classical, i.e., when there is a minimizer of $E_{\rho,\lambda,N}$ over $\mathcal{A}_{m,N,\rho}$. Naturally, as the most interesting case to consider is that of many charges, we will instead work with the energy E_ε defined in (2.10) and minimize it over the class $\mathcal{A}_\varepsilon$ obtained as a suitable modification of the definition in (3.1) corresponding to sets of volume $m = \frac{4\pi}{3}$ of a unit ball (without

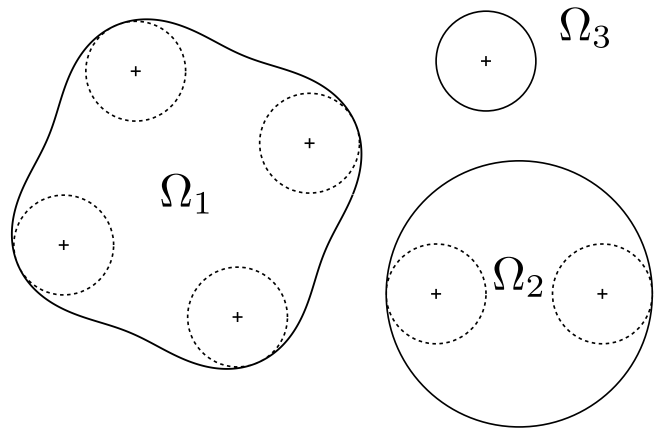

Figure 3: A schematic of a generalized minimizer with $K = 3$ for $N = 7$.

loss of generality) containing $N_\varepsilon \gg 1$ charges of radius $\varepsilon \ll 1$:

$$\mathcal{A}_\varepsilon := \mathcal{A}_{\frac{4\pi}{3}, N_\varepsilon, \varepsilon}. \tag{3.3}$$

Notice that existence vs. non-existence of classical minimizers in the class $\mathcal{A}_\varepsilon$ for $\varepsilon \ll 1$ must clearly depend on the rate of $N_\varepsilon \to \infty$ as $\varepsilon \to 0$. To begin with, due to the constraint $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \varnothing$ for $i \neq j$ we must have $N_\varepsilon \lesssim \varepsilon^{-3}$ in order for the admissible class $\mathcal{A}_\varepsilon$ to be non-empty, limiting the possible growth rate of N_ε. On the other hand, for Ω fixed and ε sufficiently small depending on $N_\varepsilon \gg 1$, one would be able to approximate $\inf_{X \subset \Omega} E_\varepsilon(\Omega, X)$ by

$$E_0(\Omega) := P(\Omega) + \frac{q^2}{2} \inf_{\mu(\Omega)=1} \int_\Omega \int_\Omega \frac{d\mu(x)\, d\mu(y)}{|x-y|}, \tag{3.4}$$

with $q = \gamma^{\frac{1}{2}} \varepsilon^{\frac{3}{2}} N_\varepsilon$, which is nothing but the dimensionless continuum energy in (2.1). Nevertheless, this energy is known to give $\inf_{|\Omega| = \frac{4\pi}{3}} E_0(\Omega) = 4\pi$ for all $q \geq 0$, thus failing to yield a minimizer for any $q > 0$ [18]. Therefore, it would be natural to expect existence of minimizers of E_ε over $\mathcal{A}_\varepsilon$ only for $N_\varepsilon \ll \gamma^{-\frac{1}{2}} \varepsilon^{-\frac{3}{2}}$. Still, the threshold N_ε for existence of minimizers in this regime is far from obvious.

We begin with the following existence result, which shows that for $\gamma \gtrsim 1$ classical minimizers exist as soon as $N_\varepsilon \lesssim \gamma^{-1} \varepsilon^{-1}$ and all $\varepsilon > 0$ sufficiently small universal.

Theorem 3.3. *There exist universal constants $\varepsilon_0 > 0$, $\gamma_0 > 0$ and $C > 0$ such that for all $\gamma > \gamma_0$ and $1 < N_\varepsilon < \frac{C}{\varepsilon\gamma}$ there exists a minimizer of E_ε over $\mathcal{A}_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$. Furthermore, if $(\Omega, X) \in \mathcal{A}_\varepsilon$ is a minimizer of E_ε then $\operatorname{dist}(x_i, \partial\Omega) = \varepsilon$ and $\operatorname{dist}(x_i, X \backslash x_i) \geq c\gamma\varepsilon$ for all $x_i \in X$ with $1 \leq i \leq N_\varepsilon$ and $c > 0$ universal.*

We note that one of the conclusions of the above theorem is that all the balls $B_\varepsilon(x_i)$ containing the charges $x_i \in X$ in a minimizer touch the drop boundary $\partial\Omega$. This is consistent with the expectation at the level of the continuum that the measure μ minimizing the Coulombic energy in (3.4) is supported on $\partial\Omega$. Furthermore, we find that in this regime the charges are uniformly separated from one another at scale $O(\gamma\varepsilon)$ which exceeds that imposed by the constraint $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \varnothing$ for $i \neq j$.

Surprisingly, the existence threshold in Theorem 3.3 is considerably lower than $N_\varepsilon \sim \gamma^{-\frac{1}{2}}\varepsilon^{-\frac{3}{2}}$ for which the Coulombic energy matches the perimeter in the continuum as $\varepsilon \to 0$, see (3.4). Nevertheless, this is not simply a limitation of our analysis, as we demonstrate with our next non-existence result. To give some heuristics for the threshold appearing in Theorem 3.3, consider the basic mechanism in which a drop may lose its energy minimizing property by *evaporating* a single charge [11, 25, 26, 28]. If (Ω, X) is a minimizer of E_ε, then (Ω', X') with $\Omega' = (\Omega \backslash B_\varepsilon(x_i)) \cup B_\varepsilon(Re_1)$ and $X' = (X \backslash \{x_i\}) \cup \{Re_1\}$, obtained by cutting a single ball $B_\varepsilon(x_i)$ with a charge in its center and sending it far off, is an admissible configuration. Here e_1 is the unit vector along the first coordinate direction, $x_i \in X$ with $1 \leq i \leq N_\varepsilon$ arbitrary, and $R > 0$ is sufficiently large. Letting $R \to \infty$, we then conclude that

$$E_\varepsilon(\Omega, X) \leq E_\varepsilon(\Omega', X') \leq E_\varepsilon(\Omega, X) + 8\pi\varepsilon^2 - \gamma\varepsilon^3 \sum_{j \neq i} \frac{1}{|x_i - x_j|}, \tag{3.5}$$

which implies that

$$\operatorname{diam}(\Omega) \geq C\gamma\varepsilon N_\varepsilon, \tag{3.6}$$

for some $C > 0$ universal and all $N_\varepsilon > 1$.

We would expect that at least whenever the perimeter is not overwhelmed by the Coulombic energy the diameter of a minimizer of E_ε, if it exists, should not greatly exceed that of a unit ball corresponding to the mass constraint. From this and (3.6), we immediately get a contradiction if $N_\varepsilon \gg \gamma^{-1}\varepsilon^{-1}$, suggesting that in this regime the existence should fail, provided that the perimeter term indeed dominates the Coulombic energy. For the latter, we can consider a competitor of the form (Ω, X), where $\Omega = B_r(0) \cup_{i=1}^{N_\varepsilon} B_\varepsilon(iRe_1)$ and $X = \cup_{i=1}^{N_\varepsilon}\{iRe_1\}$, for $r^3 + \varepsilon^3 N_\varepsilon = 1$ with $\varepsilon \ll 1$ and $R \gg 1$, corresponding to *all* charges evaporated from the drop. This yields $\inf_{\mathcal{A}_\varepsilon} E_\varepsilon \leq 4\pi(r^2 + \varepsilon^2 N_\varepsilon)$ by sending $R \to \infty$. Thus, we have $\inf_{\mathcal{A}_\varepsilon} E_\varepsilon \lesssim 1$ whenever $N_\varepsilon \lesssim \varepsilon^{-2}$ and $\varepsilon \ll 1$ independently of γ, and the isoperimetric deficit becomes small when $N_\varepsilon \ll \varepsilon^{-2}$.

Under the condition of smallness of $\varepsilon^2 N_\varepsilon$, we now get our non-existence result that yields a sharp scaling for the threshold value of N_ε with $\gamma \gtrsim 1$ for $\varepsilon \ll 1$.

Theorem 3.4. *Let $\gamma > \gamma_0$, where γ_0 is as in Theorem 3.3. Then there exists a universal constant $C > 0$ and constants $\varepsilon_0, \delta_0 > 0$ depending only on γ such that if $\varepsilon \in (0, \varepsilon_0)$ and $\frac{C}{\gamma\varepsilon} < N_\varepsilon < \frac{\delta_0}{\varepsilon^2}$ then E_ε does not attain its infimum in $\mathcal{A}_\varepsilon$.*

We note that by (3.6) the minimizer is expected to be highly elongated for $N_\varepsilon \gtrsim \varepsilon^{-2}$, if it exists. Thus, although we do not believe minimizers could exist in this Coulombic dominated regime far beyond Rayleigh instability, i.e., for all $N_\varepsilon \gg \gamma^{-\frac{1}{2}}\varepsilon^{-\frac{3}{2}}$, a different approach would be needed to rule out existence of minimizers in this regime.

We now turn to the asymptotic behavior of the minimizers in the range of existence given by Theorem 3.3. In the next theorem, we show that when the minimizers of E_ε exist for $\varepsilon \ll 1$, they are always nearly spherical with a uniformly distributed charge over the boundary, as one would have expected on physical grounds. Notice that as was already mentioned, in the regime of Theorem 3.3 the isoperimetric deficit for minimizers vanishes as $\varepsilon \to 0$, which by the quantitative isoperimetric inequality implies that the minimizers converge to balls in the L^1 topology after suitable translations [14]. Nevertheless, a stronger control on the deviation of the minimizer Ω from a ball is necessary to establish convergence of the Coulombic energy and, as a result, of the charge density, which is given by the following theorem.

Theorem 3.5. *Let $\varepsilon_n > 0$ and $N_n \in \mathbb{N}$ be such that $\varepsilon_n \to 0$ and $N_n \to \infty$ as $n \to \infty$, and $N_n < \frac{C}{\gamma \varepsilon_n}$ for $\gamma > \gamma_0$, where C and γ_0 are as in Theorem 3.3. Then if $(\Omega_n, X_n) \in \mathcal{A}_{\varepsilon_n}$ are minimizers of E_{ε_n} and $X_n = \cup_{i=1}^{N_n}\{x_{i,n}\}$, we have, up to translations, $\Omega_n \subset B_{1+\delta}(0)$ for all $\delta > 0$ and all $n \in \mathbb{N}$ large enough, and*

$$\frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{x_{i,n}} \rightharpoonup \frac{1}{4\pi}\mathcal{H}^2(\partial B_1(0)), \tag{3.7}$$

in the sense of measures, as $n \to \infty$.

Lastly, we present an asymptotically sharp existence result for the minimization problem in the special case of $N_\varepsilon = 2$ charges and $\varepsilon \ll 1$. Actually, in this case the minimization problem admits and explicit solution in terms of the unduloid surfaces that span the space between the two charges. We present the rather technical details of these solutions in Sec. 6. Here instead we summarize our existence results for minimizers of E_ε with $N_\varepsilon = 2$ for $\varepsilon \ll 1$.

Theorem 3.6. *Let $N_\varepsilon = 2$ and $c > 0$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have:*

(i) if $c < 8\pi$ and $\gamma < \frac{c}{\varepsilon}$ then there exists a unique, up to translations and rotations, minimizer of E_ε in $\mathcal{A}_\varepsilon$.

(ii) if $c > 8\pi$ and $\gamma > \frac{c}{\varepsilon}$ then there is no minimizer of E_ε in $\mathcal{A}_\varepsilon$.

Note that the threshold for existence in the above theorem is consistent with the one found in Theorems 3.3 and 3.4, but without an a priori assumption on γ. A further quantitative characterization of these minimizers is presented in Theorem 6.15, with all the necessary notations defined in Sec. 6. The proof of the latter is rather technical and involves a careful asymptotic analysis of the exact global minimizers constructed in that case. Finally, we note that in the case $N_\varepsilon = 1$ the minimizers are trivially balls, so in the following we can always assume $N_\varepsilon \geq 2$ without loss of generality.

4 Existence of generalized minimizers

Lemma 4.1. *For every $P > 0$ there exists $\gamma > 0$ such that if $\Omega \subset \mathring{B}^c_R(0)$ for some $R > 0$ and*

$$|\Omega| \geq 1, \qquad P(\Omega) \leq P, \tag{4.1}$$

then there exists a vector field $\eta \in C^1_c(\mathring{B}^c_R(0))$ with $\|\eta\|_{C^1(\mathring{B}^c_R(0))} \leq 1$ such that

$$\int_\Omega \operatorname{div} \eta \, dx \geq \gamma. \tag{4.2}$$

Proof. We reason as in [7, Lemma 3.5] and assume by contradiction that there exist a sequence of radii $R_k > 0$ and a sequence of sets Ω_k satisfying (4.1) such that

$$\lim_{k\to\infty} \sup_{\eta \in \mathcal{A}_k} \int_{\Omega_k} \operatorname{div} \eta \, dx = 0, \tag{4.3}$$

where $\mathcal{A}_k := \{\eta \in C^1_c(\mathring{B}^c_{R_k}(0))$ such that $\|\eta\|_{C^1(\mathring{B}^c_{R_k}(0))} \leq 1\}$. By [30, Remark 29.11] for all $k \in \mathbb{N}$ there exists $x_k \in \mathbb{R}^3$ such that

$$|\Omega_k \cap B_1(x_k)| \geq \bar{\delta}, \tag{4.4}$$

with $\bar{\delta} = \bar{\delta}(P) > 0$. Letting $F_k = \Omega_k - x_k$, up to a subsequence we have that $R_k \to R \in [0, +\infty]$, $F_k \to F \subset \mathbb{R}^3$ in $L^1_{loc}(\mathbb{R}^3)$, with $P(F) \leq P$. We only deal with the case $R < +\infty$ and $x_k \to x \in \mathbb{R}^3$, since the other cases can be treated analogously and are easier.

Passing to the limit in (4.4), we get that $|F \cap B_1(0)| \geq \bar{\delta}$. In particular, by Almgren lemma [7, Lemma 3.4] (see also [3,30]) there exists $\eta_F \in C^1_c(\mathring{B}^c_R(-x))$ with $\|\eta_F\|_{C^1(\mathring{B}^c_R(-x))} \leq 1$ such that

$$\int_F \operatorname{div} \eta_F \, dx \geq \gamma_F, \tag{4.5}$$

for some $\gamma_F > 0$. Letting now $\eta_k := \eta_F(\cdot + x_k)$, which belongs to $C^1_c(\mathring{B}^c_{R_k}(0))$ for k large enough, we have that

$$\lim_{k\to\infty} \int_{\Omega_k} \operatorname{div} \eta_k \, dx \geq \gamma_F > 0, \tag{4.6}$$

thus contradicting (4.3). □

Remark 4.2. *It is not difficult to see that the conclusion of Lemma 4.1 in fact holds for general sets $\Omega \subset \mathbb{R}^n$ of finite perimeter and supported on a complement of a bounded open set $U \subset \mathbb{R}^n$, with the constant γ depending only on the perimeter of Ω and n.*

Following [7], from Lemma 4.1 we derive the following uniform density estimate.

Lemma 4.3. *For every $\eta, \delta > 0$ there exist a $c_0 > 0$ depending only on η such that if $\Omega \subset \mathbb{R}^3$ is a minimizer of*

$$\min \left\{ P(E, B^c_R(0)) \,:\, E \cap B_R(0) = \Omega \cap B_R(0),\, |E| = |\Omega| \right\}, \tag{4.7}$$

with $P(\Omega \cap B^c_R(0)) < \eta |\Omega \cap B^c_R(0)|^{2/3}$ and $|\Omega \cap B^c_R(0)| > \delta$, then

$$|\Omega \cap B_r(x)| \geq c r^3, \tag{4.8}$$

for all $x \in \overline{\Omega} \setminus B_R(0)$ and $r \in (0, c_0 \delta^{\frac{1}{3}})$ such that $B_r(x) \subset B^c_R(0)$, where the constant $c > 0$ is universal and $\overline{\Omega}$ is understood in the measure theoretic sense.

Proof. Up to a rescaling, we can assume that $\delta = 1$. Notice that by a projection argument we have

$$P(\Omega, B^c_R(0)) + \mathcal{H}^2(\Omega \cap \partial B_R(0)) = P(\Omega \cap B^c_R(0)) \geq 2\mathcal{H}^2(\Omega \cap \partial B_R(0)). \tag{4.9}$$

Then reasoning as in [7, Proposition 4.4] and applying Lemma 4.1 to the set $\Omega \cap B^c_R(0)$, we get that Ω is a (Λ, r_0)-minimizer of the perimeter in $B^c_R(0)$, where Λ, r_0 are positive constants depending only on P. The result then follows by [30, Theorem 21.11]. □

Proof of Theorem 3.2. Let (Ω_n, X_n) be a minimizing sequence and let $X_n = \cup_{i=1}^N \{x_{i,n}\}$. As the total number of charges is fixed, up to extraction of a subsequence (not relabeled) the charges segregate into $1 \leq K \leq N$ clusters moving apart as $n \to \infty$. More precisely, for each $k \in \{1, 2, \ldots, K\}$ there exist $N_k \in \mathbb{N}$ and an index set $I_k = \{i^k_1, i^k_2, \ldots, i^k_{N_k}\}$ such that $\cup_{k=1}^K I_k$ forms a disjoint partition of $\{1, \ldots, N\}$ for each $n \in \mathbb{N}$ and

$$\limsup_{n \to \infty} |x_{i,n} - x_{j,n}| < \infty \qquad \forall i \in I_k \text{ and } \forall j \in I_k, \tag{4.10}$$

$$\liminf_{n \to \infty} |x_{i,n} - x_{j,n}| = \infty \qquad \forall i \in I_k \text{ and } \forall j \notin I_k. \tag{4.11}$$

Consider now $\Omega^k_n := \Omega_n - x_{i^k_1,n}$ and $X^k_n := \cup_{i \in I_k} \{x_{i,n} - x_{i^k_1,n}\}$. By (4.10) and (4.11), there exists $R_0 \geq 1$ such that $B_\rho(x_{i,n}) \subset B_{R_0}(x_{i^k_1,n})$ for all $i \in I_k$ and all $1 \leq k \leq K$, and for every $\widetilde{R} > 0$ we have $B_\rho(x_{i,n}) \subset B^c_{\widetilde{R}}(x_{i^k_1,n})$ for all $i \notin I^k_n$ and all n large enough. Then, for $R_0 < R < \widetilde{R}$ and $L > 0$ we define a competitor set

$$\widetilde{\Omega}^{R,L}_n := \left(\bigcup_{k=1}^K (\Omega^k_n \cap B_R(0)) + e_1 k L \right) \bigcup \Omega^0_n, \tag{4.12}$$

where for $r := \left(\frac{3}{4\pi}|\Omega_n \setminus (\cup_{k=1}^K B_R(x_{i_1^k,n}))|\right)^{1/3}$ we have $\Omega_n^0 := \varnothing$ if $r = 0$ or $\Omega_n^0 := B_r(0)$ if $r > 0$, together with

$$\widetilde{X}_n^{R,L} = \bigcup_{k=1}^K (X_n^k + e_k kL). \tag{4.13}$$

By construction, $(\widetilde{\Omega}_n^{R,L}, \widetilde{X}_n^{R,L}) \in \mathcal{A}_{m,N,\rho}$ for all n and L large enough independent of R. Notice that

$$P(\widetilde{\Omega}_n^{R,L}) = \sum_{k=1}^K P(\Omega_n^k, B_R(0)) + \sum_{k=1}^K \mathcal{H}^2(\Omega_n^k \cap \partial B_R(0)) + 4\pi r^2, \tag{4.14}$$

for almost all $R_0 < R < \widetilde{R}$, and

$$\sum_{i\neq j} \frac{1}{|x_{i,n} - x_{j,n}|} \geq \sum_{k=1}^K \sum_{\substack{i,j\in I_k \\ i\neq j}} \frac{1}{|\tilde{x}_{i,n} - \tilde{x}_{j,n}|}, \tag{4.15}$$

where $\widetilde{X}_n^{R,L} = \cup_{i=1}^N \{\tilde{x}_{i,n}^{R,L}\}$. Thus, by the isoperimetric inequality we have

$$E_{\rho,\lambda,N}(\widetilde{\Omega}_n^{R,L}, \widetilde{X}_n^{R,L}) \leq E_{\rho,\lambda,N}(\Omega_n, X_n) + 2\sum_{k=1}^K \mathcal{H}^2(\Omega_n^k \cap \partial B_R(0)) + \frac{C}{L}, \tag{4.16}$$

for some $C > 0$ independent of n, L and R. Furthermore, since

$$\sum_{k=1}^K |\Omega_n^k \cap (B_{\widetilde{R}}(0) \backslash B_{R_0}(0))| = \sum_{k=1}^K \int_{R_0}^{\widetilde{R}} \mathcal{H}^2(\Omega_n^k \cap \partial B_R(0))\, dR \leq m, \tag{4.17}$$

for every $\widetilde{R} \geq 2R_0$ it is possible to choose $R \in (\widetilde{R}/2, \widetilde{R}) \subset (R_0, \widetilde{R})$ such that

$$\sum_{k=1}^K \mathcal{H}^2(\Omega_n^k \cap \partial B_R(0)) \leq \frac{2m}{\widetilde{R}}. \tag{4.18}$$

Therefore, up to a subsequence (again, not relabeled) we can choose $\widetilde{R} = \widetilde{R}_n \to \infty$ and $R = R_n \to \infty$ such that (4.18) holds, as well as $L = L_n \to \infty$ sufficiently fast, so that by (4.16) we have that $(\widetilde{\Omega}_n, \widetilde{X}_n) := (\widetilde{\Omega}_n^{R_n,L_n}, \widetilde{X}_n^{R_n,L_n})$ is also a minimizing sequence.

We now modify the sets $\widetilde{\Omega}_n$ as follows to further reduce the energy: For each $1 \leq k \leq K$, we replace the set $(\widetilde{\Omega}_n - \tilde{x}_{i_1^k,n}) \cap B_{R_n}(0)$, where $\widetilde{X}_n = \cup_{i=1}^N \{\tilde{x}_{i,n}\}$, with the minimizer $\widetilde{\widetilde{\Omega}}_n^k$ of the perimeter among all sets supported in $B_{R_n}(0)$, containing $\cup_{\tilde{x}\in\widetilde{X}_n^k} B_\rho(\tilde{x})$, and satisfying

$|\widetilde{\Omega}_n^k| = |\Omega_n^k \cap B_{R_n}(0)|$. Existence of such a minimizer follows from the direct method of calculus of variations (see, e.g., [30, Section 12.5]). We may also assume that each set $\widetilde{\Omega}_n^k \cup B_{R_0}(0)$ is connected, since otherwise all the mass of the disconnected pieces of $\widetilde{\Omega}_n^k \backslash B_{R_0}(0)$ may be absorbed into the ball Ω_n^0 at the origin, producing a new set $\widetilde{\Omega}_n^0$ without increasing the perimeter while conserving the total mass. We denote by $\overline{\Omega}_n$ the set obtained by replacing Ω_n^k with $\widetilde{\Omega}_n^k$ in the definition of $\widetilde{\Omega}_n$. By construction we have $(\overline{\Omega}_n, \widetilde{X}_n) \in \mathcal{A}_{m,N,\rho}$ and $E_{\rho,\lambda,N}(\overline{\Omega}_n, \widetilde{X}_n) \leq E_{\rho,\lambda,N}(\widetilde{\Omega}_n, \widetilde{X}_n)$, so $(\overline{\Omega}_n, \widetilde{X}_n)$ is again a minimizing sequence.

Notice that if we compare the perimeter of $\widetilde{\Omega}_n^k$ with the one of $(\widetilde{\Omega}_n^k \cap B_r(0)) \cup B$, where $B \subset B_{R_n}^c(0)$ is a ball of volume $v(r) := |\widetilde{\Omega}_n^k \setminus B_r(0)|$, after some simple calculations we get that

$$P(\widetilde{\Omega}_n^k \setminus B_r(0)) \leq \bar{c} v^{\frac{2}{3}}(r) - 2\frac{dv(r)}{dr} \tag{4.19}$$

for a.e. $r \in (R_0, R_n)$, where $\bar{c} = (36\pi)^{\frac{1}{3}}$. It follows that if $v(R_0+1) > 0$, then for all large enough n we have

$$\int_{R_0}^{R_0+1} \frac{P(\widetilde{\Omega}_n^k \setminus B_r(0))}{v^{\frac{2}{3}}(r)}\, dr \leq \bar{c} + 6v^{\frac{1}{3}}(R_0) \leq \bar{c} + 6m^{\frac{1}{3}}. \tag{4.20}$$

In particular, there exists $R_0' \in (R_0, R_0+1)$, depending on n and k, such that

$$\frac{P(\widetilde{\Omega}_n^k \setminus B_{R_0'}(0))}{|\widetilde{\Omega}_n^k \setminus B_{R_0'}(0)|^{\frac{2}{3}}} \leq \bar{c} + 6m^{\frac{1}{3}}. \tag{4.21}$$

Then, by Lemma 4.3 applied with $\Omega = \widetilde{\Omega}_n^k$ and $R = R_0'$, the minimizer $\widetilde{\Omega}_n^k$ satisfies a uniform density estimate of the form

$$|\widetilde{\Omega}_n^k \cap B_r(x)| \geq cr^3, \tag{4.22}$$

for some universal $c > 0$ and for all $x \in \widetilde{\Omega}_n^k \backslash \overline{B}_{R_0'}(0)$ and $0 < r \leq r_0 := c(m)|\widetilde{\Omega}_n^k \backslash B_{R_0'}(0)|^{1/3}$. Moreover, we claim that $\widetilde{\Omega}_n^k \subset B_{R_\infty}(0)$ for some $R_\infty > 0$ independent of n. Indeed, if $|\widetilde{\Omega}_n^k \backslash B_{R_0+1}(0)| = 0$, there is nothing to prove. At the same time, in view of the connectedness of $\widetilde{\Omega}_n^k \cup B_{R_0}(0)$, the claim follows easily by applying the density estimate in (4.22) with $r = r_0$ to a sequence of $x = x_l \in \widetilde{\Omega}_n^k \cap \big(\partial B_{R_0+1+(3l-1)r_0}(0) \backslash B_{R_0+1+(3l-2)r_0}(0)\big)$, for $l \in \mathbb{N}$, and the fact that $|\widetilde{\Omega}_n^k \backslash B_{R_0}(0)|$ is bounded by m.

We now send $n \to \infty$. By compactness in $BV(B_{R_\infty}(0))$, upon extraction of a subsequence we have $\widetilde{\Omega}_n^k \to \Omega_\infty^k$ in the L^1-topology for all $1 \leq k \leq K$. Also, since by construction $\widetilde{\Omega}_n^0$ are balls containing the excess mass or are empty, we likewise have $\widetilde{\Omega}_n^0 \to \Omega_\infty^0$ in $L^1(\mathbb{R}^3)$ and $P(\widetilde{\Omega}_n^0) \to P(\Omega_\infty^0)$. Then, by the lower-semicontinuity of the perimeter we have

$\liminf_{n\to\infty} P(\widetilde{\Omega}_n^k) \geq P(\Omega_\infty^k)$ for all $1 \leq k \leq K$. Upon a further extraction of a subsequence we may also assume that $x_{i,n} - x_{i_1^k,n} \to x_{i,\infty}^k$ for all $i \in I_k$, and by continuity of the Coulombic energy we have

$$\lim_{n\to\infty} \sum_{\substack{i,j\in I_k \\ i\neq j}} \frac{1}{|x_{i,n} - x_{j,n}|} = \sum_{\substack{i,j\in I_k \\ i\neq j}} \frac{1}{|\tilde{x}_{i,\infty}^k - \tilde{x}_{j,\infty}^k|}. \tag{4.23}$$

Thus, letting $X_\infty^k := \cup_{i\in I_k}\{x_i^k\}$ we have

$$\begin{aligned}\inf_{(\Omega,X)\in\mathcal{A}_{m,N,\rho}} E_{\rho,\lambda,N} &= \liminf_{n\to\infty} E_{\rho,\lambda,N}(\Omega_n, X_n) \geq \liminf_{n\to\infty} E_{\rho,\lambda,N}(\overline{\Omega}_n, \widetilde{X}_n) \\ &\geq P(\Omega_\infty^0) + \sum_{k=1}^K \left(P(\Omega_\infty^k) + \frac{\lambda}{2} \sum_{\substack{i,j\in I_k \\ i\neq j}} \frac{1}{|\tilde{x}_{i,\infty}^k - \tilde{x}_{j,\infty}^k|} \right) \\ &= P(\Omega_\infty^0) + \sum_{k=1}^K E_{\rho,\lambda,N_k}(\Omega_\infty^k, X_\infty^k).\end{aligned} \tag{4.24}$$

Moreover, $(\Omega_\infty^k, X_\infty^k)$ minimize E_{ρ,λ,N_k} over $\mathcal{A}_{m_k,N_k,\rho}$, where $m_k := |\Omega_\infty^k|$, and by construction Ω_∞^0 minimizes the perimeter among all sets with mass $m_0 = m - \sum_{k=1}^K m_k$. Indeed, otherwise it would be possible to construct a test configuration of the form of (4.12) from those in $\mathcal{A}_{m_k,N_k,\rho}$ such that (4.24) is violated. Finally, using $(\Omega_\infty^k, X_\infty^k)$ to form a test function of the form of (4.12) and sending $L \to \infty$ yields equality in (4.24).

Finally, the regularity of $\partial\Omega_j$ follows by standard regularity theory for minimal surfaces with smooth obstacles (see for instance [30, Theorem 21.8]). □

5 Case of many charges

5.1 Preliminaries

From here on we are concerned with minimizing the energy given in (2.10) among $(\Omega, X) \in \mathcal{A}_\varepsilon$. Note that since by Theorem 3.2 generalized minimizers always exist whenever $\mathcal{A}_\varepsilon$ is non-empty, it is convenient to formulate our energy estimates in terms of the energy of such minimizers. Furthermore, as competitors we may consider finite collections of pairs (Ω_i, X_i), where $\Omega_i \subset \mathbb{R}^3$ are open sets with sufficiently smooth boundaries and $X_i \subset \mathbb{R}^3$ are finite discrete sets satisfying

$$\sum_i |\Omega_i| = \frac{4\pi}{3}, \qquad \sum_i |X_i| = N_\varepsilon. \tag{5.1}$$

With some abuse of notation, we will denote k copies of the component (Ω_i, X_i) of a competitor as $(\Omega_i, X_i)^k$, with the obvious convention that $(\Omega_i, X_i)^0 = (\varnothing, \varnothing)$. We also define the Coulombic interaction energy $V_\varepsilon(X)$ as

$$V_\varepsilon(X) := \gamma\varepsilon^3 \sum_{i=1}^{N_\varepsilon-1} \sum_{j=i+1}^{N_\varepsilon} \frac{1}{|x_i - x_j|}. \tag{5.2}$$

Lastly, we note that in the statements and proofs that follow we sometimes utilize explicit constants in the estimates, which, however, are not intended to be optimal.

As a starting point, we have the following basic upper bound on the minimal energy, which is obtained by considering a non-interacting configuration of one large ball and $N_\varepsilon - 1$ individual discrete charges. In particular, it gives a universal upper bound on the minimal energy for $N_\varepsilon \leq \frac{1}{\varepsilon^2}$.

Lemma 5.1. *If* $\{(\Omega_1, X_1), \ldots, (\Omega_k, X_k)\}$ *is a generalized minimizer then*

$$\sum_{i=1}^{k} E_\varepsilon(\Omega_i, X_i) < 4\pi(1 + \varepsilon^2 N_\varepsilon). \tag{5.3}$$

Proof. Testing the energy with the configuration of one charge in the center of a large ball and $N_\varepsilon - 1$ single charges in balls of radius ε, namely, taking as a candidate $\{(B_{r_1}(0), \{0\}), (B_\varepsilon(0), \{0\})^{N_\varepsilon - 1}\}$, where $r_1 = \sqrt[3]{1 - (N_\varepsilon - 1)\varepsilon^3} \leq 1$, we have

$$\sum_{i=1}^{k} E_\varepsilon(\Omega_i, X_i) \leq 4\pi r_1^2 + 4\pi\varepsilon^2(N_\varepsilon - 1), \tag{5.4}$$

which yields the desired inequality. □

Note that as a convention from here on we order the elements of a generalized minimizer $\{(\Omega_1, X_1), (\Omega_2, X_2), \ldots, (\Omega_k, X_k)\}$ in terms of the decreasing magnitude of $|\Omega_i|$.

Lemma 5.2. *There exists a universal constant* $C > 0$ *such that if* $\{(\Omega_1, X_1), \ldots, (\Omega_k, X_k)\}$ *is a generalized minimizer then* $|\Omega_1| \geq \frac{4\pi}{3} - C\varepsilon^3 N_\varepsilon^{\frac{3}{2}}$.

Proof. Without loss of generality let $k > 1$. Let $r_i := \left(\frac{3}{4\pi}|\Omega_i|\right)^{\frac{1}{3}}$, then by the isoperimetric inequality and positivity of V_ε we have

$$\sum_{i=1}^{k} E_\varepsilon(\Omega_i) \geq \sum_{i=1}^{k} 4\pi r_i^2. \tag{5.5}$$

Eliminating r_1 via the volume constraint $\sum_{i=1}^k r_i^3 = 1$ and using the fact that $t^{\frac{2}{3}} > t$ for $t \in (0,1)$, we obtain

$$\sum_{i=1}^k E_\varepsilon(\Omega_i) \geq \sum_{i=2}^k 4\pi r_i^2 + 4\pi \left(1 - \sum_{i=2}^k r_i^3\right)^{\frac{2}{3}} \geq 4\pi + \sum_{i=2}^k 4\pi r_i^2(1-r_i). \tag{5.6}$$

On the other hand, note that since $r_2 \leq \frac{1}{\sqrt[3]{2}}$, for all $1 < i \leq k$ we have

$$r_i^2(1-r_i) \geq \left(1 - \frac{1}{\sqrt[3]{2}}\right) r_i^2. \tag{5.7}$$

Therefore, by Lemma 5.1 we obtain

$$4\pi(1 + N_\varepsilon \varepsilon^2) > \sum_{i=1}^k E_\varepsilon(\Omega_i) \geq 4\pi + \sum_{i=2}^k 4\pi r_i^2(1-r_i) \geq 4\pi + C\sum_{i=2}^k r_i^2, \tag{5.8}$$

for some universal $C > 0$. Finally, by monotonicity of the l^p-norm in p, this implies

$$\sqrt[3]{\sum_{i=2}^k r_i^3} \leq \sqrt{\sum_{i=2}^k r_i^2} \leq C'\varepsilon\sqrt{N_\varepsilon}, \tag{5.9}$$

for some $C' > 0$ universal, yielding the claim. □

Our next lemma provides further information about the volume of the small components of generalized minimizers. Notice that the conditions on N_ε throughout the rest of this section tacitly imply that ε is small.

Lemma 5.3. *There exist universal constants $C, \delta > 0$ such that for $1 < N_\varepsilon < \frac{\delta}{\varepsilon^2}$, if $\{(\Omega_1, X_1), (\Omega_2, X_2), \ldots, (\Omega_k, X_k)\}$ is a generalized minimizer then $|\Omega_i| \leq C|X_i|^{\frac{3}{2}}\varepsilon^3$ for all $i > 1$.*

Proof. For $i > 1$ create a minimizing candidate $\{(c\Omega_1, cX_1), \ldots, (c\Omega_{i-1}, cX_{i-1}), (c\Omega_{i+1}, cX_{i+1}), \ldots, (c\Omega_k, cX_k), (B_\varepsilon(0), \{0\})^{|X_i|}\}$, which is obtained by deleting the i-th component, transferring its charges into $|X_i|$ non-interacting balls of radius ε and rescaling the remaining components to adjust for the volume change. Here

$$c = \sqrt[3]{\frac{\frac{4\pi}{3} - \frac{4\pi}{3}|X_i|\varepsilon^3}{\frac{4\pi}{3} - |\Omega_i|}} = \sqrt[3]{1 + \frac{|\Omega_i| - \frac{4\pi}{3}|X_i|\varepsilon^3}{\frac{4\pi}{3} - |\Omega_i|}} \leq \sqrt[3]{1 + \frac{3}{2\pi}\left(|\Omega_i| - \frac{4\pi}{3}|X_i|\varepsilon^3\right)} \tag{5.10}$$

$$\leq 1 + \frac{1}{2\pi}\left(|\Omega_i| - \frac{4\pi}{3}|X_i|\varepsilon^3\right), \tag{5.11}$$

where we used that $|\Omega_i| \leq \frac{2\pi}{3}$ for all $i > 1$. Furthermore, from Lemma 5.2, $|\Omega_i| \leq C\varepsilon^3 N_\varepsilon^{\frac{3}{2}} \leq C\delta^{\frac{3}{2}}$ for some universal constant $C > 0$. Thus, we can pick $\delta > 0$ so that $|\Omega_i| \leq 1$, which gives us

$$\sum_{j=1}^{k} E_\varepsilon(\Omega_j, X_j) \leq 4\pi|X_i|\varepsilon^2 + \sum_{j\neq i} \left(c^2 P(\Omega_j) + V_\varepsilon(X_j)\right)$$
$$\leq 4\pi|X_i|\varepsilon^2 + \left(1 + 2\left(|\Omega_i| - \frac{4\pi}{3}|X_i|\varepsilon^3\right)\right)\sum_{j\neq i} P(\Omega_j) + \sum_{j\neq i} V_\varepsilon(X_j). \tag{5.12}$$

From Lemma 5.1, we can pick $C' > 0$ so that $\sum_{j=1}^{k} P(\Omega_j) \leq 4\pi + 4\pi\varepsilon^2 N_\varepsilon \leq 4\pi(1+\delta) \leq C'$. This gives us that

$$\sum_{j=1}^{k} E_\varepsilon(\Omega_j, X_j) \leq 4\pi|X_i|\varepsilon^2 - P(\Omega_i) + 2C'\left(|\Omega_i| - \frac{4\pi}{3}|X_i|\varepsilon^3\right) + \sum_{j=1}^{k} E_\varepsilon(\Omega_j, X_j). \tag{5.13}$$

Thus, with the help of the isoperimetric inequality for Ω_i we have

$$\sqrt[3]{36\pi}|\Omega_i|^{\frac{2}{3}} - 2C'|\Omega_i| \leq \sqrt[3]{36\pi}|\Omega_i|^{\frac{2}{3}} - 2C'\left(|\Omega_i| - \frac{4\pi}{3}|X_i|\varepsilon^3\right) \leq 4\pi|X_i|\varepsilon^2. \tag{5.14}$$

Finally, since $|\Omega_i| \leq C\delta^{\frac{3}{2}}$, possibly decreasing δ we can ensure that $|\Omega_i|^{\frac{1}{3}} < \frac{1}{C'}$, yielding the desired inequality. □

Next we rule out the case where our generalized minimizer $\{(\Omega_1, X_1), \ldots, (\Omega_k, X_k)\}$ contains X_i's that are null, which means that each component of the generalized minimizer has to contain at least one charge, provided that N_ε is not too large and ε is sufficiently small. In this case, if a small component contains only one charge then it is a ball of radius ε.

Lemma 5.4. *There exist a universal constant $\delta > 0$ such that for $1 < N_\varepsilon < \frac{\delta}{\varepsilon^2}$, if $\{(\Omega_1, X_1), \ldots, (\Omega_k, X_k)\}$ is a generalized minimizer then $k \leq N_\varepsilon$, and each X_i for $1 \leq i \leq k$ is non-empty. Furthermore, if $|X_i| = 1$ for some $1 < i \leq k$, then $|\Omega_i| = \frac{4\pi}{3}\varepsilon^3$.*

Proof. First note that if X_i is empty, then Lemma 5.3 implies that $|\Omega_i| = 0$, a contradiction. Thus, all that remains, is to show that if $|X_i| = 1$ for some $1 < i \leq k$, then $|\Omega_i| = \frac{4\pi}{3}\varepsilon^3$. To do this, note that when $|X_i| = 1$, rearranging (5.14) provides

$$\sqrt[3]{36\pi}|\Omega_i|^{\frac{2}{3}} - 4\pi\varepsilon^2 \leq 2C'\left(|\Omega_i| - \frac{4\pi}{3}\varepsilon^3\right), \tag{5.15}$$

which implies that $|\Omega_i| = \frac{4\pi}{3}\varepsilon^3$ whenever δ is chosen to ensure that $|\Omega_i|^{\frac{1}{3}} < \frac{1}{C'}$. To see this, note that if $\frac{4\pi}{3}\varepsilon^3 < |\Omega_i| < \frac{1}{(C')^3}$, then

$$\int_{\frac{4\pi}{3}\varepsilon^3}^{|\Omega_i|} C' dt < \int_{\frac{4\pi}{3}\varepsilon^3}^{|\Omega_i|} t^{-\frac{1}{3}} dt < \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \int_{\frac{4\pi}{3}\varepsilon^3}^{|\Omega_i|} t^{-\frac{1}{3}} dt, \tag{5.16}$$

which implies

$$C'\left(|\Omega_i| - \frac{4\pi}{3}\varepsilon^3\right) < \frac{1}{2}\left(\sqrt[3]{36\pi}|\Omega_i|^{\frac{2}{3}} - 4\pi\varepsilon^2\right). \tag{5.17}$$

Thus, (5.17) contradicts (5.15), and we conclude that $|\Omega_i| = \frac{4\pi}{3}\varepsilon^3$. □

Lastly, we state a lower density estimate for generalized minimizers which will be useful for both the $N_\varepsilon = 2$ and the $N_\varepsilon \gg 1$ cases.

Lemma 5.5. *There exists a universal constant $C > 0$ such that for any $M > 0$, if $N_\varepsilon < \frac{M}{\varepsilon^2}$, $\{(\Omega_1, X_1), \ldots, (\Omega_k, X_k)\}$ is a generalized minimizer, $x_0 \in \overline{\Omega_i \setminus \bigcup_{x_j \in X_i} B_\varepsilon(x_j)}$ for some $1 \leq i \leq k$, and $r < \min\left(R, \min_{x_j \in X_i} |x_0 - x_j| - \varepsilon\right)$, where $R > 0$ depends only on M, then*

$$|\Omega_i \cap B_r(x_0)| > Cr^3. \tag{5.18}$$

Proof. We may assume for convenience that $R < \frac{1}{2}$ and that $B_{R+\varepsilon}(x_0) \cap X_i = \varnothing$. Consider $\{(c\Omega_1, cX_1), \ldots, (c(\Omega_i \setminus B_r(x_0)), cX_i), \ldots, (c\Omega_k, cX_k)\}$, where $c > 1$ is defined as

$$c := \sqrt[3]{\frac{\frac{4\pi}{3}}{\frac{4\pi}{3} - |\Omega_i \cap B_r(x_0)|}} \leq 1 + |\Omega_i \cap B_r(x_0)|, \tag{5.19}$$

as a possible minimizing candidate. Then

$$\begin{aligned} &P(c(\Omega_i \setminus B_r(x_0))) + \sum_{j \neq i} P(c\Omega_j) \\ &\qquad \leq (1 + |\Omega_i \cap B_r(x_0)|)^2 \left(P(\Omega_i \setminus B_r(x_0)) + \sum_{j \neq i} P(\Omega_j)\right) \\ &\qquad \leq (1 + 3|\Omega_i \cap B_r(x_0)|) \left(P(\Omega_i \setminus B_r(x_0)) + \sum_{j \neq i} P(\Omega_j)\right). \end{aligned} \tag{5.20}$$

Furthermore, applying the isoperimetric inequality to the set $\Omega_i \cap B_r(x_0)$ we have that

$$P(\Omega_i \setminus B_r(x_0)) \leq P(\Omega_i) + 2\mathcal{H}^2(\Omega_i \cap \partial B_r(x_0)) - \sqrt[3]{36\pi}|\Omega_i \cap B_r(x_0)|^{\frac{2}{3}}. \tag{5.21}$$

Thus combining (5.20) and (5.21) and using Lemma 5.1, we get

$$
\begin{aligned}
&P(c(\Omega_i \setminus B_r(x_0))) + \sum_{j\neq i} P(c\Omega_j) \\
&\quad \leq (1+3|\Omega_i \cap B_r(x_0)|)\left(2\mathcal{H}^2(\Omega_i \cap \partial B_r(x_0)) - \sqrt[3]{36\pi}|\Omega_i \cap B_r(x_0)|^{\frac{2}{3}} + \sum_{j=1}^k P(\Omega_j)\right) \\
&\leq \left(-\sqrt[3]{36\pi} + 3|\Omega_i \cap B_r(x_0)|^{\frac{1}{3}} \sum_{j=1}^k P(\Omega_j)\right)|\Omega_i \cap B_r(x_0)|^{\frac{2}{3}} + C_1\mathcal{H}^2(\Omega_i \cap \partial B_r(x_0)) + \sum_{j=1}^k P(\Omega_j) \\
&\qquad \leq C_1\mathcal{H}^2(\Omega_i \cap \partial B_r(x_0)) - C_2|\Omega_i \cap B_r(x_0)|^{\frac{2}{3}} + \sum_{j=1}^k P(\Omega_j), \qquad (5.22)
\end{aligned}
$$

for some universal constants C_1, $C_2 > 0$ whenever $r < R$ is small enough depending only on M.

By (5.22) we have

$$
\begin{aligned}
\sum_{j=1}^k E_\varepsilon(\Omega_j, X_j) &\leq E_\varepsilon\left(c(\Omega_i \setminus B_r(x_0)), cX_i\right) + \sum_{j\neq i} E_\varepsilon(c\Omega_j, cX_j) \\
&\leq C_1\mathcal{H}^2(\Omega_i \cap \partial B_r(x_0)) - C_2|\Omega_i \cap B_r(x_0)|^{\frac{2}{3}} + \sum_{j=1}^k E_\varepsilon(\Omega_j, X_j), \qquad (5.23)
\end{aligned}
$$

which implies

$$
C_1\mathcal{H}^2(\Omega_i \cap \partial B_r(x_0)) \geq C_2|\Omega_i \cap B_r(x_0)|^{\frac{2}{3}}. \qquad (5.24)
$$

Finally, by letting $U(r) := |\Omega_i \cap B_r(x_0)| > 0$ and applying Fubini's theorem with the co-area formula to obtain $\frac{dU(r)}{dr} = \mathcal{H}^2(\Omega_i \cap \partial B_r(x_0))$ for a.e. $r \in (0, R)$, we arrive at

$$
\frac{dU(r)}{dr} \geq CU^{\frac{2}{3}}, \qquad (5.25)
$$

for some universal constant $C > 0$. Integrating this inequality yields the claim. □

5.2 Localizing the minimizers

In this subsection we perform a suitable localization of minimizers, which leads to outer convergence of minimizers to a unit ball as $\varepsilon \to 0$. For $x_0 \in \mathbb{R}^3$ and $r > 0$, we define the spherical cut of a set-charge pair $(\Omega, X) \in \mathcal{A}_{m,N,\varepsilon}$ by the ball $B_r(x_0)$ to be the two set-charge pairs $(\Omega^\pm_{x_0,r}, X^\pm_{x_0,r})$ defined as follows: If $\mathcal{H}^2\big(\partial B_r(x_0) \cap \big(\cup_{x\in X} B_\varepsilon(x)\big)\big) = 0$, then

$$
\Omega^+_{x_0,r} = \Omega \cap B_r(x_0), \qquad (5.26)
$$

$$X^+_{x_0,r} = \{x \in X \ : \ B_\varepsilon(x) \subset \Omega^+_{x_0,r}\}, \tag{5.27}$$

$$\Omega^-_{x_0,r} = \Omega \setminus \Omega^+_{x_0,r}, \tag{5.28}$$

and

$$X^-_{x_0,r} = X \setminus X^+_{x_0,r}. \tag{5.29}$$

If, on the contrary, $\mathcal{H}^2\big(\partial B_r(x_0) \cap \big(\cup_{x\in X} B_\varepsilon(x)\big)\big) > 0$, then we set

$$X^+_{x_0,r} = \left\{x \in X \ : \mathcal{H}^2(\partial B_\varepsilon(x) \cap B_r(x_0)) > \mathcal{H}^2(\partial B_\varepsilon(x) \cap B^c_r(x_0))\right\}, \tag{5.30}$$

$$X^-_{x_0,r} = X \setminus X^+_{x_0,r}, \tag{5.31}$$

$$\Omega^+_{x_0,r} = \Big((B_r(x_0) \cap \Omega) \cup \Big(\cup_{x\in X^+_{x_0,r}} B_\varepsilon(x)\Big)\Big) \setminus \Big(\cup_{x\in X^-_{x_0,r}} B_\varepsilon(x)\Big) \tag{5.32}$$

$$\Omega^-_{x_0,r} = \Omega \setminus \Omega^+_{x_0,r}. \tag{5.33}$$

For these spherical cuts we have the following result.

Lemma 5.6. *Let $\varepsilon, m > 0$, $N \in \mathbb{N}$, and let (Ω, X) be a classical minimizer of E_ε over $\mathcal{A}_{m,N,\varepsilon}$. Then if $x_0 \in \mathbb{R}^3$ and $r > 0$, we have*

$$P(\Omega^+_{x_0,r}) + P(\Omega^-_{x_0,r}) \le P(\Omega) + 4\mathcal{H}^2(\Omega \cap \partial B_r(x_0)). \tag{5.34}$$

Proof. By construction, we have

$$\begin{aligned} P(\Omega^+_{x_0,r}) = \mathcal{H}^2(\partial\Omega \cap B_r(x_0)) + \mathcal{H}^2(\partial B_r(x_0) \cap \Omega) - \sum_{x\in X} \mathcal{H}^2(\partial B_r(x_0) \cap B_\varepsilon(x)) \\ + \sum_{x\in X^+_{x_0,r}} \mathcal{H}^2(\partial B_\varepsilon(x) \cap B^c_r(x_0)) + \sum_{x\in X^-_{x_0,r}} \mathcal{H}^2(\partial B_\varepsilon(x) \cap B_r(x_0)), \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} P(\Omega^-_{x_0,r}) = \mathcal{H}^2(\partial\Omega \cap B^c_r(x_0)) + \mathcal{H}^2(\partial B_r(x_0) \cap \Omega) - \sum_{x\in X} \mathcal{H}^2(\partial B_r(x_0) \cap B_\varepsilon(x)) \\ + \sum_{x\in X^+_{x_0,r}} \mathcal{H}^2(\partial B_\varepsilon(x) \cap B^c_r(x_0)) + \sum_{x\in X^-_{x_0,r}} \mathcal{H}^2(\partial B_\varepsilon(x) \cap B_r(x_0)). \end{aligned} \tag{5.36}$$

Observe that for each $x \in X^+_{x_0,r}$ the set $\partial B_\varepsilon(x) \cap B^c_r(x_0)$ is either empty or a spherical cap with the base radius $a_x \in (0, \varepsilon]$ and height $h_x \in (0, \varepsilon]$. Therefore, we have $\mathcal{H}^2(\partial B_\varepsilon(x) \cap B^c_r(x_0)) = 2\pi\varepsilon h_x$ and $\mathcal{H}^2(\partial B_r(x_0) \cap B_\varepsilon(x)) \ge \pi a_x^2$. Noting that $(\varepsilon - h_x)^2 + a_x^2 = \varepsilon^2$, we then conclude that

$$\frac{\mathcal{H}^2(\partial B_\varepsilon(x) \cap B^c_r(x_0))}{\mathcal{H}^2(\partial B_r(x_0) \cap B_\varepsilon(x))} \le \frac{2\varepsilon}{2\varepsilon - h_x} \le 2. \tag{5.37}$$

By a similar argument, for every $x \in X^-_{x_0,r}$ we also have

$$\frac{\mathcal{H}^2(\partial B_\varepsilon(x) \cap B_r(x_0))}{\mathcal{H}^2(\partial B_r(x_0) \cap B_\varepsilon(x))} \leq 2. \tag{5.38}$$

Thus, we obtain

$$\begin{aligned} P(\Omega^+_{x_0,r}) + P(\Omega^-_{x_0,r}) &\leq P(\Omega) + 2\mathcal{H}^2(\partial B_r(x_0) \cap \Omega) + 2\sum_{x\in X} \mathcal{H}^2(\partial B_r(x_0) \cap B_\varepsilon(x)) \\ &\leq P(\Omega) + 4\mathcal{H}^2(\partial B_r(x_0) \cap \Omega), \end{aligned} \tag{5.39}$$

which is the desired inequality. □

Next we obtain an estimate for the L^1 convergence of classical minimizers to a ball as $\varepsilon \to 0$.

Lemma 5.7. *There exist universal constants C, C', δ_0 such that if $\delta < \delta_0$, $1 < N_\varepsilon < \frac{\delta}{\varepsilon^2}$, and (Ω, X) is a classical minimizer, then*

$$|\Omega \cap B^c_{r^*}(x_0)| < C N_\varepsilon^{\frac{3}{2}} \varepsilon^3, \tag{5.40}$$

for some $1 \leq r^* \leq 1 + C'\delta^{\frac{1}{6}}$ and some $x_0 \in \mathbb{R}^3$.

Proof. First note that Lemma 5.1 provides an upper bound on the isoperimetric deficit of Ω, which is given by

$$\frac{P(\Omega) - 4\pi}{4\pi} \leq N_\varepsilon \varepsilon^2. \tag{5.41}$$

In turn, by the quantitative isoperimetric inequality [14] this gives us an upper bound on the Fraenkel asymmetry of Ω, which tells us that there exists $x_0 \in \mathbb{R}^3$ and a universal constant $C_0 > 0$ such that

$$|\Omega \Delta B_1(x_0)| \leq C_0 \sqrt{N_\varepsilon}\varepsilon < C_0 \sqrt{\delta}. \tag{5.42}$$

Arguing by contradiction, assume that

$$|\Omega \cap B^c_r(x_0)| \geq C N_\varepsilon^{\frac{3}{2}} \varepsilon^3 \tag{5.43}$$

for all $1 \leq r \leq 1 + C'\delta^{\frac{1}{6}}$ and $C, C', \delta > 0$ arbitrary, provided that $1 < N_\varepsilon < \frac{\delta}{\varepsilon^2}$. Picking $C' = 15C^{\frac{1}{3}}$, for $1 \leq r \leq 1 + 15(C\sqrt{\delta})^{\frac{1}{3}}$ we then have that

$$|\Omega^-_{x_0,r}| \geq C N_\varepsilon^{\frac{3}{2}} \varepsilon^3 - \frac{4\pi}{3} N_\varepsilon \varepsilon^3 > \frac{C}{2} N_\varepsilon^{\frac{3}{2}} \varepsilon^3, \tag{5.44}$$

provided that C is sufficiently large universal. To construct a minimizing candidate we first cut (Ω, X) into $(\Omega^+_{x_0,r}, X^+_{x_0,r})$ and $(\Omega^-_{x_0,r}, X^-_{x_0,r})$ and then split off the individual charges from $(\Omega^-_{x_0,r}, X^-_{x_0,r})$ and move any remaining mass into $\Omega^+_{x_0,r}$. More precisely, this minimizing candidate is $\left\{(c\Omega^+_{x_0,r}, cX^+_{x_0,r}), (B_\varepsilon(0),\{0\})^k\right\}$, where

$$c = \sqrt[3]{\frac{|\Omega^+_{x_0,r}| + |\Omega^-_{x_0,r}| - \frac{4\pi}{3}k\varepsilon^3}{|\Omega^+_{x_0,r}|}}, \tag{5.45}$$

and $k = |X^-_{x_0,r}|$. Letting $\delta < \delta_0 < \frac{1}{C_0^2}$, from (5.42) we get that $|\Omega^+_{x_0,r}| \geq \frac{4\pi}{3} - 1 - \frac{4\pi}{3}N_\varepsilon \varepsilon^3 > 2$ for all ε sufficiently small universal, which gives

$$c \leq \sqrt[3]{1 + \frac{|\Omega^-_{x_0,r}|}{2}} \leq 1 + \frac{1}{6}|\Omega^-_{x_0,r}|. \tag{5.46}$$

Now, by cutting and re-scaling in this way we have that

$$\begin{aligned} E_\varepsilon(c(\Omega^+_{x_0,r}, X^+_{x_0,r})) + 4\pi k\varepsilon^2 &\leq V_\varepsilon(X^+_{x_0,r}) + c^2 P(\Omega^+_{x_0,r}) + 4\pi k \varepsilon^2 \\ &\leq V_\varepsilon(X) + P(\Omega^+_{x_0,r}) + |\Omega^-_{x_0,r}| P(\Omega^+_{x_0,r}) + 4\pi k\varepsilon^2. \end{aligned} \tag{5.47}$$

Furthermore, from Lemma 5.6 we have that

$$P(\Omega^+_{x_0,r}) \leq P(\Omega) + 4\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) - P(\Omega^-_{x_0,r}), \tag{5.48}$$

which together with the isoperimetric inequality applied to $\Omega^-_{x_0,r}$ gives

$$P(\Omega^+_{x_0,r}) \leq P(\Omega) + 4\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) - \sqrt[3]{36\pi}|\Omega^-_{x_0,r}|^{\frac{2}{3}}. \tag{5.49}$$

Thus, combining (5.47) and (5.49) and using Lemma 5.1 we get that

$$\begin{aligned} &E_\varepsilon\left(c(\Omega^+_{x_0,r}, X^+_{x_0,r})\right) + 4\pi k\varepsilon^2 \leq \\ &E_\varepsilon(\Omega, X) + 4\mathcal{H}^2(\Omega\cap\partial B_r(x_0)) - \sqrt[3]{36\pi}|\Omega^-_{x_0,r}|^{\frac{2}{3}} + 4\pi N_\varepsilon \varepsilon^2 + |\Omega^-_{x_0,r}|\left(P(\Omega) + 4\mathcal{H}^2(\Omega\cap\partial B_r(x_0))\right) \\ &\qquad \leq E_\varepsilon(\Omega, X) + 5\mathcal{H}^2(\Omega\cap\partial B_r(x_0)) - 3|\Omega^-_{x_0,r}|^{\frac{2}{3}} + 4\pi N_\varepsilon\varepsilon^2, \end{aligned} \tag{5.50}$$

provided that δ_0 and, hence, $|\Omega^-_{x_0,r}|$ is sufficiently small universal (see (5.42)).

Finally, by (5.44) we can pick C large enough so that $|\Omega^-_{x_0,r}|^{\frac{2}{3}} \geq 4\pi N_\varepsilon \varepsilon^2$. Hence from (5.50) and the minimality of (Ω, X) we obtain

$$E_\varepsilon(\Omega, X) \leq E_\varepsilon(c(\Omega^+_{x_0,r}, X^+_{x_0,r})) + 4\pi k\varepsilon^2 \leq E_\varepsilon(\Omega, X) + 5\mathcal{H}^2(\Omega\cap\partial B_r(x_0)) - 2|\Omega^-_{x_0,r}|^{\frac{2}{3}}, \tag{5.51}$$

which with the help of (5.43) gives us

$$\mathcal{H}^2(\Omega\cap\partial B_r(x_0)) \geq \frac{2}{5}|\Omega^-_{x_0,r}|^{\frac{2}{3}} \geq \frac{2}{5}|\Omega \cap B^c_r(x_0)|^{\frac{2}{3}} - \frac{8\pi}{15}N_\varepsilon\varepsilon^3 \geq \frac{1}{5}|\Omega\cap B^c_r(x_0)|^{\frac{2}{3}}, \tag{5.52}$$

whenever $1 \le r \le 1 + 15(C\sqrt{\delta})^{\frac{1}{3}}$ and C large enough. Now, letting $U(r) := |\Omega \cap B_r^c(x_0)|$ and applying Fubini's theorem and the co-area formula, from (5.52) we get that

$$U^{\frac{2}{3}}(r) \le -5\frac{dU(r)}{dr} \tag{5.53}$$

for a.e. $1 \le r \le 1 + 15(C\sqrt{\delta})^{\frac{1}{3}}$. Integrating over this interval gives us

$$0 \le U^{\frac{1}{3}}\left(1 + 15(C\sqrt{\delta})^{\frac{1}{3}}\right) \le U^{\frac{1}{3}}(1) - \left(C\sqrt{\delta}\right)^{\frac{1}{3}}. \tag{5.54}$$

But this is a contradiction for C large enough, since from (5.42) we know that $U(1) < C_0\sqrt{\delta}$. □

Using the L^1 convergence from Lemma 5.7 and the density estimate from Lemma 5.5, we have that classical minimizers converge to a ball from the outside.

Lemma 5.8. *For $\delta > 0$ there exists δ_0 depending only on δ and γ such that if $1 < N_\varepsilon < \frac{\delta_0}{\varepsilon^2}$, and (Ω, X) is a classical minimizer then $\Omega \subset B_{1+\delta}(\hat{x})$ for some $\hat{x} \in \mathbb{R}^3$.*

Proof. Without loss of generality, we may assume that δ is sufficiently small universal. From Lemma 5.7, the constant $\delta_0 > 0$ can be picked so that $|\Omega \cap B^c_{1+\frac{\delta}{2}}(\hat{x})| < CN_\varepsilon^{\frac{3}{2}}\varepsilon^3 < C\delta_0^{\frac{3}{2}}$, where $C > 0$ is a universal constant and $\hat{x} \in \mathbb{R}^3$. Now let $L = \frac{1}{6}\left(\sup_{x\in\Omega}|x - \hat{x}| - 1 - \frac{\delta}{2}\right)$, and, arguing by contradiction, assume that $L > \frac{\delta}{12}$.

For $r > 0$ and $y \in \mathbb{R}^3$, define $k_y(r)$ to be the number of charges inside of $\Omega^+_{y,r}$. Then there exists $x_0 \in \partial\Omega$ such that $B_L(x_0) \cap B_{1+\frac{\delta}{2}}(\hat{x}) = \varnothing$ and $k_{x_0}(L) \le \frac{N_\varepsilon}{3}$. Also, define $U(r) := |\Omega \cap B_r(x_0)|$, then since $U(r)$ is a continuous monotone increasing function and $k_{x_0}(r)$ is a lower semi-continuous piecewise-constant function with a finite number of jumps, we have that $S := \left\{r \in [0, L] : U(r) \ge (4\pi k_{x_0}(r)\varepsilon^2)^{\frac{3}{2}}\right\} = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_q, b_q]$.

Now by cutting and rescaling we will show that the set S is small whenever δ is small. Let $r \in S$. Create a minimizing candidate $\left\{\left(c\Omega^-_{x_0,r}, cX^-_{x_0,r}\right), (B^1_\varepsilon(0), \{0\})^{k_{x_0}(r)}\right\}$, where

$$c = \sqrt[3]{\frac{\frac{4\pi}{3} - \frac{4\pi}{3}k_{x_0}(r)\varepsilon^3}{\frac{4\pi}{3} - |\Omega^+_{x_0,r}|}} \le \sqrt[3]{\frac{\frac{4\pi}{3}}{\frac{4\pi}{3} - |\Omega^+_{x_0,r}|}} \le 1 + \frac{3}{2\pi}|\Omega^+_{x_0,r}|, \tag{5.55}$$

where from Lemma 5.7, δ_0 is chosen so that $|\Omega^+_{x_0,r}| < C\delta_0^{\frac{3}{2}}$ is small universal. Then by the minimality of (Ω, X) we have

$$\begin{aligned} P(\Omega\cap B_r(x_0)) + P(\Omega\cap B_r^c(x_0)) - 2\mathcal{H}^2(\partial B_r(x_0)\cap\Omega) + V_\varepsilon(X) \le P(\Omega) + V_\varepsilon(X) = E_\varepsilon(\Omega, X) \\ \le c^2 P(\Omega^-_{x_0,r}) + 4\pi k_{x_0}(r)\varepsilon^2 + V_\varepsilon(X). \quad (5.56) \end{aligned}$$

Furthermore, from the argument in the proof of Lemma 5.6 we have that $P(\Omega^-_{x_0,r}) < P(\Omega \cap B_r^c(x_0)) + 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0))$, and from (5.56) and (5.55) we get that

$$\begin{aligned} &P(\Omega \cap B_r(x_0)) + P(\Omega \cap B_r^c(x_0)) - 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) \\ &\qquad \le \left(1 + \frac{3}{2\pi}|\Omega^+_{x_0,r}|\right)^2 \big(P(\Omega \cap B_r^c(x_0)) + 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0))\big) + 4\pi k_{x_0}(r)\varepsilon^2 \\ &\qquad \le \big(1 + 2|\Omega^+_{x_0,r}|\big) \big(P(\Omega \cap B_r^c(x_0)) + 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0))\big) + 4\pi k_{x_0}(r)\varepsilon^2. \end{aligned} \tag{5.57}$$

Since $|\Omega^+_{x_0,r}| \le U(r) + \frac{4\pi}{3}k_{x_0}(r)\varepsilon^3$, we get that

$$\begin{aligned} &P(\Omega \cap B_r(x_0)) + P(\Omega \cap B_r^c(x_0)) - 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) \\ &\quad \le \left(1 + 2U(r) + \frac{8\pi}{3}k_{x_0}(r)\varepsilon^3\right) \big(P(\Omega \cap B_r^c(x_0)) + 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0))\big) + 4\pi k_{x_0}(r)\varepsilon^2, \end{aligned} \tag{5.58}$$

and using the assumption that $r \in S$ gives

$$\begin{aligned} &P(\Omega \cap B_r(x_0)) + P(\Omega \cap B_r^c(x_0)) - 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) \\ &\qquad \le (1 + 3U(r)) \big(P(\Omega \cap B_r^c(x_0)) + 2\mathcal{H}^2(\Omega \cap \partial B_r(x_0))\big) + 4\pi k_{x_0}(r)\varepsilon^2 \\ &\le P(\Omega \cap B_r^c(x_0)) + 2\mathcal{H}^2(\partial B_r(x_0) \cap \Omega) + 3U(r)P(\Omega) + 9U(r)\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) + 4\pi k_{x_0}(r)\varepsilon^2 \\ &\qquad \le P(\Omega \cap B_r^c(x_0)) + 3\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) + 15\pi U(r) + 4\pi k_{x_0}(r)\varepsilon^2, \end{aligned} \tag{5.59}$$

after possibly decreasing the value of δ_0.

We now apply the isoperimetric inequality to $\Omega \cap B_r(x_0)$ in (5.59) to obtain

$$\sqrt[3]{36\pi}U^{\frac{2}{3}}(r) - 15\pi U(r) - 4\pi k_{x_0}(r)\varepsilon^2 \le 5\mathcal{H}^2(\Omega \cap \partial B_r(x_0)). \tag{5.60}$$

Since $U(r) < C\delta_0^{\frac{3}{2}}$, we can pick δ_0 to give

$$\frac{\sqrt[3]{36\pi}}{2}U^{\frac{2}{3}}(r) - 4\pi k_{x_0}(r)\varepsilon^2 \le 5\mathcal{H}^2(\Omega \cap \partial B_r(x_0)). \tag{5.61}$$

Finally, since $r \in S$ we have that $\frac{\sqrt[3]{36\pi}}{4}U^{\frac{2}{3}}(r) \ge U^{\frac{2}{3}}(r) \ge 4\pi k_{x_0}(r)\varepsilon^2$. Thus

$$U^{\frac{2}{3}}(r) \le 5\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) \qquad \forall r \in S. \tag{5.62}$$

Noting that $dU(r)/dr = \mathcal{H}^2(\Omega \cap \partial B_r(x_0))$ for a.e. r and integrating this expression for $r \in [a_i, b_i]$ then gives us that

$$U^{\frac{1}{3}}(b_i) - U^{\frac{1}{3}}(a_i) \ge \frac{1}{15}(b_i - a_i). \tag{5.63}$$

However, since by monotonicity of $U(r)$ we have $U(a_i) \geq U(b_{i-1})$, it holds that

$$U^{\frac{1}{3}}(b_q) \geq \frac{1}{15}\sum_{i=1}^{q}(b_i - a_i). \tag{5.64}$$

At the same time, since we also have $U^{\frac{1}{3}}(b_q) < C\sqrt{\delta_0}$ for some $C > 0$ universal, we obtain that

$$\sum_{i=1}^{q}(b_i - a_i) \leq 15C\sqrt{\delta_0} \leq \frac{\delta}{24} < \frac{L}{2}, \tag{5.65}$$

whenever $\delta_0 \leq \left(\frac{\delta}{360C}\right)^2$, where $C > 0$ is a universal constant. In particular, the set S has a small measure controlled by δ, as claimed.

Now let $r < L$ and $r \in S^c$. Then from Lemma 5.6 and a comparison of the energy of (Ω, X) with that of $\{(\Omega^+_{x_0,r}, X^+_{x_0,r}), (\Omega^-_{x_0,r}, X^-_{x_0,r})\}$ we get

$$\begin{aligned} P(\Omega^+_{x_0,r}) + P(\Omega^-_{x_0,r}) - 4\mathcal{H}^2(\Omega \cap \partial B_r(x_0)) + V_\varepsilon(X) &\leq P(\Omega) + V_\varepsilon(X) = E_\varepsilon(\Omega, X) \\ &\leq P(\Omega^+_{x_0,r}) + P(\Omega^-_{x_0,r}) + V_\varepsilon(X) - \frac{\gamma\varepsilon^3}{(3+12L)}k_{x_0}(r)(N_\varepsilon - k_{x_0}(r)), \end{aligned} \tag{5.66}$$

since from the definition of L the diameter of Ω is less or equal than $2 + \delta + 12L < 3 + 12L$. Thus, (5.66) implies that

$$\frac{\gamma\varepsilon^3}{(3+12L)}k_{x_0}(r)(N_\varepsilon - k_{x_0}(r)) \leq 4\mathcal{H}^2(\Omega \cap \partial B_r(x_0)). \tag{5.67}$$

However, we chose x_0 and L so that $k_{x_0}(L) \leq \frac{N_\varepsilon}{3}$, which gives us that

$$\frac{\gamma\varepsilon^3 N_\varepsilon}{1+L}k_{x_0}(r) \leq C'\mathcal{H}^2(\Omega \cap \partial B_r(x_0)). \tag{5.68}$$

for some new universal constant $C' > 0$ that will change from line to line in the remainder of the proof. Furthermore, since $r \in S^c$, $U(r) < \left(4\pi k_{x_0}(r)\varepsilon^2\right)^{\frac{3}{2}}$, which implies that $\frac{U^{\frac{2}{3}}(r)}{4\pi\varepsilon^2} \leq k_{x_0}(r)$. Thus

$$\frac{\gamma\varepsilon N_\varepsilon U^{\frac{2}{3}}(r)}{1+L} \leq C'\mathcal{H}^2(\Omega \cap \partial B_r(x_0)). \tag{5.69}$$

Integrating this expression from $[b_i, a_{i+1}]$, with $a_{q+1} := L$, gives us that

$$C'\left(U^{\frac{1}{3}}(a_{i+1}) - U^{\frac{1}{3}}(b_i)\right) \geq \frac{\gamma\varepsilon N_\varepsilon}{1+L}(a_{i+1} - b_i), \tag{5.70}$$

which again by monotonicity of $U(r)$ implies that

$$U^{\frac{1}{3}}(L) \geq \frac{\gamma\varepsilon N_\varepsilon}{C'(1+L)}\sum_{i=1}^{q}(a_{i+1} - b_i). \tag{5.71}$$

From (5.65), we have that $\sum_{i=1}^{q}(a_{i+1}-b_i) \geq \frac{L}{2}$, which gives that

$$U^{\frac{1}{3}}(L) \geq \frac{\gamma\varepsilon N_\varepsilon L}{C'(1+L)}. \tag{5.72}$$

However, from Lemma 5.7 we have that $U(L) < CN_\varepsilon^{\frac{3}{2}}\varepsilon^3$, which implies that

$$N_\varepsilon \leq C'\left(\frac{1+L}{\gamma L}\right)^2. \tag{5.73}$$

Since $L > \frac{\delta}{12}$, this gives that $N_\varepsilon \leq \frac{C'}{\delta^2\gamma^2}$. Then by Lemma 5.7 we have

$$|\Omega \cap B^c_{1+\frac{\delta}{2}}(\hat{x})| \leq \frac{C'\varepsilon^3}{\delta^3\gamma^3} \tag{5.74}$$

Finally, to arrive at a contradiction observe that the set $\Omega \cap B^c_r(\hat{x})$ must have a connected component whose diameter exceeds $\frac{1}{2}\delta$. A slicing argument at scale ε then gives

$$|\Omega \cap B^c_{1+\frac{\delta}{2}}(\hat{x})| \geq C\varepsilon^2\delta, \tag{5.75}$$

for some universal constant $C > 0$. Indeed, if a slice contains a charge then it trivially contains a volume of at least of order ε^3. If, however, the slice does not contain any charges, then it still contains at least that much volume by Lemma 5.5. Together, the above two inequalities give a contradiction for $\varepsilon < C\delta^4\gamma^3$, with $C > 0$ universal. □

5.3 Existence results

We now proceed to proving our existence and non-existence results for $\varepsilon \ll 1$. We begin by adapting the arguments in the proof of Lemma 5.8 to show that for not too large values of N_ε the diameter of all but the first component of a generalized minimizer must be small.

Lemma 5.9. *There exist universal constants $C, \delta > 0$ such that for $1 < N_\varepsilon < \frac{\delta}{\varepsilon^2}$, if $\{(\Omega_1, X_1), ...(\Omega_k, X_k)\}$ is a generalized minimizer and $i > 1$, then* $\operatorname{diam}(\Omega_i) \leq C\max\left(r_i, \frac{\varepsilon}{\gamma^3}\right)$, *where $r_i := \left(\frac{3}{4\pi}|\Omega_i|\right)^{\frac{1}{3}}$.*

Proof. Let $i > 1$. First note that Lemma 5.3 provides an upper bound on $|\Omega_i|$, which implies smallness of $|\Omega_i|$ for δ sufficiently small universal. If $L_i := \operatorname{diam}(\Omega_i)$, then arguing by contradiction we may assume that $L_i > C\max\left(r_i, \frac{\varepsilon}{\gamma^3}\right)$, where $C > 0$ is arbitrary. Arguing exactly as in the proof of Lemma 5.8 for the components Ω_i, we may then obtain the estimate

$$L_i < \frac{C'\varepsilon}{\gamma^3}, \tag{5.76}$$

for some $C' > 0$ universal: we can first find a point $x_0 \in \partial\Omega_i$ such that $(\Omega_i)^+_{x_0,r}$ contains only a universally small fraction of the total number of charges $|X_i|$, then by cutting and rescaling we get an analog of the estimate in (5.65) in which the right-hand side is instead bounded by a universal multiple of $|\Omega_i|^{\frac{1}{3}}$, and finally by cutting and separating the pieces we get an analog of the estimate in (5.71) with $C'\gamma\varepsilon|X_i|/L_i$ multiplying the sum in the right-hand side instead. This provides a contradiction by choosing $C > C'$. □

Our next lemma gives a precise characterization of the generalized minimizers when γ is sufficiently large universal.

Lemma 5.10. *There exist universal constants $\delta, \gamma_0 > 0$ such that for $1 < N_\varepsilon < \frac{\delta}{\varepsilon^2}$ and $\gamma > \gamma_0$, if $\{(\Omega_1, X_1), ...(\Omega_k, X_k)\}$ is a generalized minimizer then up to translations it has the form $\{(\Omega_1, X_1), (B_\varepsilon(0), \{0\})^{k-1}\}$.*

Proof. Without loss of generality, assume that $k > 1$ and consider (Ω_i, X_i) for $i > 1$. Arguing by contradiction, assume that $|X_i| > 1$ (the case of $|X_i| \leq 1$ is taken care by Lemma 5.4). First note that from the definition of generalized minimizers and Lemma 5.3 we have that $c\varepsilon^3 < |\Omega_i| < C|X_i|^{\frac{3}{2}}\varepsilon^3$ for some universal $c, C > 0$. Therefore, from Lemma 5.9 there exists $\gamma_0 > 0$ such that if $\gamma > \gamma_0$ then $\operatorname{diam}(\Omega_i) \leq C|\Omega_i|^{\frac{1}{3}}$ for some universal $C > 0$. Thus

$$\operatorname{diam}(\Omega_i) \leq C\varepsilon\sqrt{|X_i|}. \tag{5.77}$$

Now construct a minimizing candidate by cutting one charge at $x_0 \in X_i$ together with the ball $B_\varepsilon(x_0)$ from Ω_i and adding a new component $(B_\varepsilon(0), \{0\})$, i.e., consider a competitor $\{(\Omega_1, X_1), \ldots, (\Omega_i \setminus B_\varepsilon(x_0), X_i \setminus x_0), \ldots, (\Omega_k, X_k), (B_\varepsilon(0), \{0\})\}$. Comparing the energies then yields

$$\begin{aligned} 8\pi\varepsilon^2 - \frac{\varepsilon^2\gamma(|X_i| - 1)}{C\sqrt{|X_i|}} &\geq 8\pi\varepsilon^2 - \frac{\varepsilon^3\gamma(|X_i| - 1)}{\operatorname{diam}(\Omega_i)} \\ &\geq E_\varepsilon(\Omega_i \backslash B_\varepsilon(x_0), X_i \backslash x_0) + E_\varepsilon(B_\varepsilon(0), \{0\}) - E_\varepsilon(\Omega_i, X_i) \geq 0. \end{aligned} \tag{5.78}$$

Since $|X_i| \geq 2$, this gives $\gamma \leq 16\pi C$, contradicting the assumption on γ. Thus, for $i > 1$ we have that $|X_i| = 1$. □

Proof of Theorem 3.3. First note that from Lemma 5.10, without loss of generality we may assume that our generalized minimizer takes the form $\{(\Omega_1, X_1), (B_\varepsilon(0), \{0\})^{k-1}\}$. Arguing by contradiction, assume that $k > 1$. We will construct a competitor by bringing one of the isolated charges into (Ω_1, X_1) and reducing the total energy. Since (Ω_1, X_1) is a classical minimizer of E_ε in the admissible class $\mathcal{A}_{|\Omega_1|, |X_1|, \varepsilon}$, by considering a ball with $|X_1|$ approximately hexagonally packed charges, from [40, Theorem C] we obtain

$$E_\varepsilon(\Omega_1, X_1) \leq \sqrt[3]{36\pi}|\Omega_1|^{\frac{2}{3}} + \frac{\gamma\varepsilon^3}{2\left(\sqrt[3]{\frac{3}{4\pi}|\Omega_1|} - \varepsilon\right)}|X_1|^2. \tag{5.79}$$

Note that in view of Lemma 5.2 the distance between the charges in this construction is at least of order $\frac{1}{\sqrt{N_\varepsilon}} > \sqrt{\gamma\varepsilon} \gg \varepsilon$, for $N_\varepsilon < \frac{1}{\gamma\varepsilon}$, $\gamma > 1$ and ε small enough universal. Thus, letting $X_1 = \{x_1, x_2, \dots, x_{|X_1|}\}$, by isoperimetric inequality we get that

$$V_\varepsilon(X_1) = \frac{\gamma\varepsilon^3}{2} \sum_{i=1}^{|X_1|} \sum_{j \neq i} \frac{1}{|x_i - x_j|} \leq \frac{\gamma\varepsilon^3}{2\left(\sqrt[3]{\frac{3}{4\pi}|\Omega_1|} - \varepsilon\right)} |X_1|^2 \leq \gamma\varepsilon^3 |X_1|^2, \tag{5.80}$$

in view of

$$\sqrt[3]{\frac{3}{4\pi}|\Omega_1|} - \varepsilon \geq \frac{1}{2}, \tag{5.81}$$

for all ε sufficiently small universal. This implies that there exists $x_{i^*} \in X_1$ such that

$$\sum_{j \neq i^*} \frac{1}{|x_{i^*} - x_j|} \leq 2|X_1|. \tag{5.82}$$

Consider $d := \frac{1}{2} \min_{j \neq i^*} |x_{i^*} - x_j|$. First note that arguing as in (3.5) we have that

$$\frac{\gamma\varepsilon^3}{2d} \leq \gamma\varepsilon^3 \sum_{j \neq i^*} \frac{1}{|x_{i^*} - x_j|} \leq 8\pi\varepsilon^2, \tag{5.83}$$

which implies that

$$d \geq \frac{\gamma\varepsilon}{16\pi} \geq 4\varepsilon, \tag{5.84}$$

whenever $\gamma_0 \geq 64\pi$. In addition, (3.5) implies $|x_i - x_j| > c\gamma\varepsilon$ for $i \neq j$ and some universal $c > 0$, which proves the statement about charge separation.

Let $x_0 \in \Omega_1 \cap \partial B_d(x_{i^*})$, which exists since by Theorem 3.2 the set Ω_1 is connected. Now our hope is to place a charge inside of Ω_1 at x_0 and lower the energy. Using (5.84), we have that x_0 is sufficiently far away from all the charges:

$$|x_0 - x_j| \geq 4\varepsilon \qquad \forall x_j \in X_1. \tag{5.85}$$

Therefore, by Lemma 5.5 we have that

$$\frac{4\pi}{3}\varepsilon^3 \geq |\Omega_1 \cap B_\varepsilon(x_0)| > C\varepsilon^3, \tag{5.86}$$

for some universal constant $C > 0$. Thus, for ε small enough universal we can create a new minimizing candidate $\{c(\Omega_1 \cup B_\varepsilon(x_0), X_1 \cup \{x_0\}), (B_\varepsilon(0), \{0\})^{k-2}\}$, where

$$c = \sqrt[3]{\frac{|\Omega_1| + \frac{4\pi}{3}\varepsilon^3}{|\Omega_1| - |\Omega_1 \cap B_\varepsilon(x_0)| + \frac{4\pi}{3}\varepsilon^3}} \leq \sqrt[3]{\frac{1}{1 - 8\varepsilon^3}} < 1 + 3\varepsilon^3, \tag{5.87}$$

in which the first inequality is due to (5.81). Thus, (5.86) with the isoperimetric inequality and (5.87) gives us the following upper bound on the perimeter of $c(\Omega_1 \cup B_\varepsilon(x_0))$:

$$\begin{aligned}P(c(\Omega_1 \cup B_\varepsilon(x_0))) = c^2 P(\Omega_1 \cup B_\varepsilon(x_0)) &\le c^2\Big(P(\Omega_1) + 4\pi\varepsilon^2 - P(\Omega_1 \cap B_\varepsilon(x_0))\Big)\\ &\le (1+10\varepsilon^3)\left(P(\Omega_1) + 4\pi\varepsilon^2 - \sqrt[3]{36\pi C^2}\varepsilon^2\right)\\ &\le P(\Omega_1) + 4\pi\varepsilon^2 - \sqrt[3]{36\pi C^2}\varepsilon^2 + 40\pi\varepsilon^3(1+N_\varepsilon\varepsilon^2)\\ &\le P(\Omega_1) + 4\pi\varepsilon^2 - C'\varepsilon^2, \quad (5.88)\end{aligned}$$

for ε small enough universal, where $C' > 0$ is a universal constant and in the third inequality we used Lemma 5.1.

Lastly, we will obtain an upper bound on $V_\varepsilon(c(X_1 \cup \{x_0\})) \le V_\varepsilon(X_1 \cup \{x_0\})$. To do this, first note that from the definition of d we have

$$|x_0 - x_j| \ge \frac{1}{2}|x_{i^*} - x_j| \quad (5.89)$$

for all $j \ne i^*$, while

$$|x_0 - x_{i^*}| = \frac{1}{2}|x_{j^*} - x_{i^*}| \quad (5.90)$$

for some $j^* \ne i^*$. This gives us that

$$\begin{aligned}V_\varepsilon(X_1 \cup \{x_0\}) &= V_\varepsilon(X_1) + \gamma\varepsilon^3 \sum_{x_j \in X_1} \frac{1}{|x_0 - x_j|}\\ &\le V_\varepsilon(X_1) + \frac{2\gamma\varepsilon^3}{|x_{j^*} - x_{i^*}|} + 2\gamma\varepsilon^3 \sum_{j \ne i^*} \frac{1}{|x_{i^*} - x_j|}\\ &\le V_\varepsilon(X_1) + 4\gamma\varepsilon^3 \sum_{j \ne i^*} \frac{1}{|x_{i^*} - x_j|}. \quad (5.91)\end{aligned}$$

Now using (5.82), this gives that

$$V_\varepsilon(X_1 \cup \{x_0\}) \le V_\varepsilon(X_1) + 8\gamma\varepsilon^3|X_1|. \quad (5.92)$$

Finally, combining the bound on the perimeter of $c(\Omega_1 \cup B_\varepsilon(x_0))$ given in (5.88) with the bound on $V_\varepsilon(c(X_1 \cup \{x_0\})$ given in (5.92), we obtain

$$\begin{aligned}&E_\varepsilon\left(c(\Omega_1 \cup B_\varepsilon(x_0)), c(X_1 \cup \{x_0\})\right) + 4\pi(k-2)\varepsilon^2\\ &\le P(\Omega_1) + 4\pi(k-1)\varepsilon^2 - C'\varepsilon^2 + V_\varepsilon(X_1) + 8\gamma\varepsilon^3|X_1|\\ &= E_\varepsilon(\Omega_1, X_1) + 4\pi(k-1)\varepsilon^2 - C'\varepsilon^2 + 8\gamma\varepsilon^3|X_1| < \sum_{j=1}^{k} E_\varepsilon(\Omega_j, X_j) \quad (5.93)\end{aligned}$$

whenever $|X_1| < N_\varepsilon < \frac{C'}{8\gamma\varepsilon}$, a contradiction. Thus, generalized minimizers that are not classical minimizers cannot exist for these values of N_ε, with ε sufficiently small and γ sufficiently large universal, and the existence statement of the theorem holds.

To conclude the proof of Theorem 3.3, let $X = \cup_{i=1}^{N_\varepsilon}\{x_i\}$ be the minimizing set of the positions of the charges. To prove that each $B_\varepsilon(x_i)$ touches $\partial\Omega$, suppose that, to the contrary, there exists $1 \le i^* \le N_\varepsilon$ such that $B_\varepsilon(x_{i^*}) \Subset \Omega$. Therefore, x_{i^*} is a local minimizer of $v_{i^*}(x) := \sum_{j\neq i^*} |x - x_j|^{-1}$, since otherwise it would be possible to lower the energy by slightly displacing the ball $B_\varepsilon(x_{i^*}) \subset \Omega$ without touching the other balls $B_\varepsilon(x_j)$ with $j \neq i^*$. However, v_{i^*} is a harmonic function in some neighborhood of x_{i^*} and must, therefore, be constant there, which is impossible as v_{i^*} is a real analytic function in $\mathbb{R}^3\backslash\{X\backslash x_{i^*}\}$ that goes to infinity at $X\backslash x_{i^*}$. □

Proof of Theorem 3.4. Assume that (Ω, X) is a classical minimizer. First note that for a fixed $\gamma > 0$, from Lemma 5.8 we have that $\varepsilon_0, \delta > 0$ can be chosen to make $\Omega \subset B_2(x)$ for some $x \in \mathbb{R}$. Thus, Lemma 5.1, and [39, Theorem 2] imply that there exists a universal constant $C_0 > 0$ such that

$$4\pi + 4\pi N_\varepsilon \varepsilon^2 \ge E_\varepsilon(\Omega, X) \ge 4\pi + \frac{\gamma\varepsilon^3}{2}\left(\frac{N_\varepsilon^2}{2} - C_0 N_\varepsilon^{\frac{3}{2}}\right) \ge 4\pi + \frac{1}{8}\gamma N_\varepsilon^2 \varepsilon^3, \tag{5.94}$$

whenever $N_\varepsilon > \frac{C}{\gamma\varepsilon}$ is large enough universal. Thus

$$N_\varepsilon \le \frac{32\pi}{\varepsilon\gamma}, \tag{5.95}$$

a contradiction.

□

5.4 Convergence

Proof of Theorem 3.5. The fact that $\Omega_n \subset B_{1+\delta}(0)$ for any $\delta > 0$ and all $n \in \mathbb{N}$ large enough, after suitable translations, follows from Lemma 5.8. Now, for N distinct points $x_i \in \mathbb{R}^3$ define

$$F_N(\mu) := \begin{cases} \frac{2}{N^2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{1}{|x_i - x_j|} & \text{if } \mu = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, \\ +\infty & \text{otherwise,} \end{cases} \tag{5.96}$$

Then by [36, Proposition 2.8] we have that $\Gamma - \lim_{N\to\infty} F_N = F_\infty$ with respect to the weak convergence of probability measures in $\mathbb{R}^3$, where

$$F_\infty(\mu) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{d\mu(x)\,d\mu(y)}{|x-y|}. \tag{5.97}$$

Therefore, for $\mu_n := \frac{1}{N_n}\sum_{x_i\in X_n}\delta_{x_i}$ we have that, upon suitable translations and extraction of subsequences, $\mu_n \to \mu_\infty$ in the sense of measures as $n\to\infty$, where μ_∞ is a probability measure supported on $\overline{B}_1(0)$, and $\liminf_{n\to\infty} F_{N_n}(\mu_n) \geq F(\mu_\infty)$. At the same time, testing the energy with $\Omega = B_1(0)$ and uniformly distributed N_n points supported on $\partial B_{1-\varepsilon_n}(0)$ (the existence of the latter follows from the construction of the recovery sequence for the above Γ-convergence, together with a scaling argument and the fact that $N_n \ll \varepsilon_n^{-2}$, or an explicit construction in [40]), with the help of the isoperimetric inequality we obtain

$$4\pi + \tfrac{1}{2}\gamma\varepsilon_n^3 N_n^2\left(F_\infty(\tfrac{1}{4\pi}\mathcal{H}^2\lfloor_{\partial\Omega}) + o_n(1)\right) \geq E_{\varepsilon_n}(\Omega_n, X_n) \geq 4\pi + \tfrac{1}{2}\gamma\varepsilon_n^3 N_n^2 F_\infty(\mu_\infty). \quad (5.98)$$

Thus $F_\infty(\frac{1}{4\pi}\mathcal{H}^2\lfloor_{\partial\Omega}) \geq F_\infty(\mu_\infty)$. However, $\mu = \frac{1}{4\pi}\mathcal{H}^2\lfloor_{\partial\Omega}$ is the unique minimizer of F_∞ among all probability measures supported on $\overline{B}_1(0)$. Hence $\mu_\infty = \frac{1}{4\pi}\mathcal{H}^2\lfloor_{\partial\Omega}$ and the limit is in fact a full limit. □

6 Case of two charges

In this section, we give an explicit characterization of the minimizers in the simplest non-trivial case of $N = 2$ point charges. Note that when $N = 1$, the minimizer of E_ε is always a unit ball with the charge located anywhere inside.

6.1 Existence results

For $N = 2$ and $X = \{x_1, x_2\}$, the energy in (2.10) becomes simply

$$E_\varepsilon(\Omega, X) = P(\Omega) + \frac{\gamma\varepsilon^3}{|x_2 - x_1|}. \quad (6.1)$$

In this case, the energy of a generalized minimizer that is not classical is known explicitly and satisfies the estimate below.

Lemma 6.1. *There exists a universal constant $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and a classical minimizer of the energy in* (6.1) *does not exist, then the generalized minimizer has the form $\{(\Omega_1, \{x_1\}), (\Omega_2, \{x_2\})\}$ with $x_1, x_2 \in \mathbb{R}^3$, and*

$$4\pi(1 + \varepsilon^2 - \varepsilon^3) < E_\varepsilon(\Omega_1, \{x_1\}) + E_\varepsilon(\Omega_2, \{x_2\}) < 4\pi(1 + \varepsilon^2). \quad (6.2)$$

Proof. From Lemma 5.4, we have that all the components of a generalized minimizer have at least one charge. Hence in the absence of classical minimizers the generalized minimizer consists of precisely two components, each of which has exactly one charge. Thus each component of the generalized minimizer is a ball, and again by Lemma 5.4 we have $|\Omega_2| = \frac{4\pi}{3}\varepsilon^3$. Then

$$E_\varepsilon(\Omega_1, \{x_1\}) + E_\varepsilon(\Omega_2, \{x_2\}) = 4\pi\left(\varepsilon^2 + (1 - \varepsilon^3)^{\frac{2}{3}}\right), \quad (6.3)$$

and the statement follows. □

Using the density estimates from Lemma 5.5, we have that if (Ω, X) is a minimizer to (6.1) then it is contained inside a ball of radius close to one.

Lemma 6.2. *There exist universal constants* $\varepsilon_0, C > 0$ *such that if* $\varepsilon < \varepsilon_0$ *and* (Ω, X) *is a minimizer to* (6.1) *then* $\Omega \subset B_{1+C\sqrt[3]{\varepsilon}}(x_0)$ *for some* $x_0 \in \mathbb{R}^3$.

Proof. Let (Ω, X) be a minimizer. Then from Lemma 6.1 we have that

$$P(\Omega) < E_\varepsilon(\Omega, X) < 4\pi(1+\varepsilon^2). \tag{6.4}$$

By quantitative isoperimetric inequality [14], this gives us a bound on the Fraenkel asymmetry of Ω, namely, that there exists $x_0 \in \mathbb{R}^3$ and a universal constant $C_0 > 0$ such that

$$|\Omega \Delta B_1(x_0)| < C_0 \varepsilon. \tag{6.5}$$

Now assume that the claim of the lemma is false, i.e., that Ω is not contained in $B_{1+C\sqrt[3]{\varepsilon}}(x_0)$ for arbitrary $C > 0$ and ε small enough. Then Lemma 5.5 tells us that there exist universal constants $C_1 > 0$ and $\varepsilon_0 > 0$ such that when $\varepsilon < \varepsilon_0$ we have

$$|\Omega \Delta B_1(x_0)| > C_1 C^3 \varepsilon, \tag{6.6}$$

which contradicts (6.5) for a suitable choice of C. □

From the above lemmas, we have the following existence/non-existence result for classical minimizers.

Lemma 6.3. *There exist universal constants* $\varepsilon_0, C > 0$ *such that if* $\varepsilon < \varepsilon_0$ *and* $\gamma < \frac{8\pi}{\varepsilon} - C$ *then there exists a minimizer* (Ω, X) *to* (6.1) *among all* (Ω, X) *admissible, and if* $\gamma > \frac{8\pi}{\varepsilon} + \frac{C}{\varepsilon^{\frac{2}{3}}}$ *then there is no minimizer.*

Proof. If $\gamma < \frac{8\pi}{\varepsilon} - C$ and $\varepsilon \ll 1$, then consider an admissible test configuration in the form of a ball with two charges at the opposite extremes, $(\Omega, X) = \big(B_1(0), \{-(1-\varepsilon)e_1, (1-\varepsilon)e_1\}\big)$, for which we have

$$E_\varepsilon(\Omega, X) = 4\pi + \frac{\gamma\varepsilon^3}{2-2\varepsilon} < 4\pi + \frac{8\pi\varepsilon^2 - C\varepsilon^3}{2-2\varepsilon}. \tag{6.7}$$

Picking $C > 16\pi$, from (6.7) we then get that

$$E_\varepsilon(\Omega, X) < 4\pi + 4\pi\varepsilon^2 - \frac{8\pi\varepsilon^3}{2-2\varepsilon} < 4\pi + 4\pi\varepsilon^2 - 4\pi\varepsilon^3. \tag{6.8}$$

Thus from Theorem 3.2 and Lemma 6.1, we have that the energy in (6.1) must have a classical minimizer.

To prove non-existence, note that when $\gamma > \frac{8\pi}{\varepsilon} + \frac{C}{\varepsilon^{\frac{2}{3}}}$, from Lemma 6.2 we have that for (Ω, X) admissible and $\varepsilon \ll 1$ there exists a universal constant $C_0 > 0$ such that

$$E_\varepsilon(\Omega, X) \geq 4\pi + \frac{\gamma\varepsilon^3}{2 + C_0\sqrt[3]{\varepsilon}} > 4\pi + \frac{8\pi\varepsilon^2 + C\varepsilon^{\frac{7}{3}}}{2 + C_0\sqrt[3]{\varepsilon}} > 4\pi + 4\pi\varepsilon^2, \tag{6.9}$$

whenever $C > 4\pi C_0$ and ε is sufficiently small. However, Lemma 6.1 implies that (Ω, X) cannot be a minimizer and the Lemma is proved. □

In the case $N = 2$, we can use rotational symmetry of the problem about the axis passing through the two charges to explicitly solve for global minimizers. Without loss of generality, let $X = \{x_1, x_2\}$, with x_1 and x_2 located on the x-axis. Furthermore, let $\mathcal{T}\Omega$ denote the Schwarz symmetrization of Ω with respect to the x-axis, i.e. let

$$\mathcal{T}\Omega = \{(x, y, z) \in \mathbb{R}^3 : (y, z) \in A_x^*\}, \tag{6.10}$$

where $A_x = \{(y, z) \in \mathbb{R}^2 : (x, y, z) \in \Omega\}$ and $A_x^* = B_r(0) \in \mathbb{R}^2$ such that $\mathcal{L}^2(A_x^*) = \mathcal{L}^2(A_x)$ denotes the two dimensional symmetric rearrangement of A_x.

Lemma 6.4. *Let $N = 2$, let $x_1, x_2 \in \mathbb{R} \times (0, 0)$ and let $(\Omega, \{x_1, x_2\}) \in \mathcal{A}_{m,N,\rho}$ be a minimizer of $E_{\rho,\lambda,N}$. Then $\Omega = \mathcal{T}\Omega$.*

Proof. Note that from Fubini's theorem we have that $|\mathcal{T}\Omega| = |\Omega| = m$. Furthermore, $|\mathcal{T}\Omega \cap B_\rho(x_1)| = |\mathcal{T}\Omega \cap B_\rho(x_2)| = |B_\rho(0)|$. To see this, note that $|\Omega \cap B_\rho(x_1)| = |B_\rho(x_1)|$ implies that $\mathcal{L}^2(A_x) \geq \mathcal{L}^2(\{(y, z) \in \mathbb{R}^2 : (x, y, z) \in B_\rho(x_1)\})$ for almost every $x \in \mathbb{R}$. Since $\{(y, z) \in \mathbb{R}^2 : (x, y, z) \in B_\rho(x_1)\}$ is also a ball in $\mathbb{R}^2$, we have that $\{(y, z) \in \mathbb{R}^2 : (x, y, z) \in B_\rho(x_1)\} \subset A_x^*$ for almost every $x \in \mathbb{R}$. Thus, by Fubini's theorem $|\mathcal{T}\Omega \cap B_\rho(x_1)| = |B_\rho(x_1)|$, which gives that $(\mathcal{T}\Omega, \{x_1, x_2\})$ is also admissible.

By [4, Theorem 1.1], we have that $P(\mathcal{T}\Omega) \leq P(\Omega)$, so $\mathcal{T}\Omega$ is also a minimizer, and by minimality of the energy this inequality is in fact an equality. Therefore, by [4, Theorem 1.2] the sets Ω and $\mathcal{T}\Omega$ are equal up to a translation in the yz-plane. The latter follows from the fact that as a minimizer the set $\mathcal{T}\Omega$ is open and connected, and away from $B_\rho(x_1)$ and $B_\rho(x_2)$ the set Ω is a local volume-constrained minimizer of the perimeter, implying that $\partial\Omega$ is analytic [30] and, hence, that the non-degeneracy assumptions of [4, Theorem 1.2] are satisfied.

Finally, assume by contradiction that $\Omega = \mathcal{T}\Omega + v$ for some vector $v \neq 0$ contained in the yz-plane. Since $\mathcal{T}\Omega$ contains the two balls $B_\rho(x_{1,2})$, it follows that Ω also contains the translated balls $B_\rho(x_{1,2}) + \lambda v$, for all $\lambda \in [-1, 1]$. Therefore, each $\partial B_\rho(x_{1,2})$ could touch $\partial(\mathcal{T}\Omega)$ only at a point lying on the x-axis. But that is also impossible, since in that case $\partial\Omega$ would be flat near those points, contradicting once again the analyticity of $\partial\Omega$. Thus, $B_\rho(x_{1,2})$ are both strictly contained in $\mathcal{T}\Omega$. However, the latter contradicts the minimizing property of $(\mathcal{T}\Omega, \{x_1, x_2\})$, since one could reduce the energy by moving $B_\rho(x_{1,2})$ slightly further apart while still keeping them in $\mathcal{T}\Omega$. □

Since, according to Lemma 6.4, every minimizer to (6.1) coincides with its Schwarz symmetrization around the axis connecting the two charges, it can be defined with the help of a profile function $\varphi : \mathbb{R} \to [0, \infty)$, defining Ω, up to a rotation, as

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : 0 < \sqrt{y^2 + z^2} < \varphi(x) \right\}. \tag{6.11}$$

By Theorem 3.2, the function φ is of class $C^{1,1}$ on the set $\{x \in \mathbb{R} : \varphi(x) > 0\}$. Furthermore, the support of φ is a single bounded interval.

Lemma 6.5. *There exists a universal constant* $\varepsilon_0 > 0$ *such that if* $\varepsilon < \varepsilon_0$, $N = 2$, $(\Omega, \{x_1, x_2\}) \in \mathcal{A}_{\frac{4\pi}{3}, N, \varepsilon}$, *and* $(\Omega, \{x_1, x_2\})$ *is a minimizer to* (6.1), *then its free surface* $\partial\Omega \setminus (\partial B_\varepsilon(x_1) \cup \partial B_\varepsilon(x_2))$ *is a single section of an unduloid.*

Proof. By Lemma 6.4, every minimizer $(\Omega, \{x_1, x_2\})$ to (6.1) is rotationally symmetric, and away from the obstacles $\partial B_\varepsilon(x_{1,2})$ the surface $\partial\Omega$ has constant mean curvature as a local minimizer of the perimeter [30]. Then the profile function φ defined in (6.11) must satisfy the Euler-Lagrange equation [8, 35, Section 3.6]

$$\varphi' = \pm\sqrt{\frac{\varphi^2}{(H\varphi^2 + C_0)^2} - 1} \tag{6.12}$$

away from the obstacles. Here H is the mean curvature (the average of the principal curvatures) and $C_0 \in \mathbb{R}$. Furthermore, when the free surface $\partial\Omega\setminus(\partial B_\varepsilon(x_1) \cup \partial B_\varepsilon(x_2))$ touches the obstacle, say, $\partial B_\varepsilon(x_1)$ at height h (distance from the x-axis), then from the $C^{1,1}$ regularity of the minimizers the tangency condition at the point of contact gives

$$C_0 = \frac{\pm 1 - H\varepsilon}{\varepsilon} h^2. \tag{6.13}$$

Rewriting this equation by solving for the positive height of contact h, gives that h is unique for a fixed C_0, H, and ε. This tells us that a segment of φ satisfying (6.12) must leave and connect to the obstacles at the same height.

If $C_0 < 0$, then (6.12) gives that the graph of φ is an arc of a nodary curve. However, this is impossible. To see why, without loss of generality let the nodary arc touch $B_\varepsilon(x_1)$ at a least one point, then the maximum height of the nodary arc is either h or a local maximum of a complete nodary curve. If the maximum is a local maximum, then the nodary arc must attain a local maximum, which contradicts the analyticity of $\partial\Omega$, since the height of contact given in (6.13) is below the point of infinite slope given in (6.12), which in turn is below the local maxima of a nodary curve. This implies that the nodary arc stays below $h \le \varepsilon$ and thus must contact both of the two charges, which either contradicts our volume constraint or Lemma 6.2 when ε is sufficiently small universal. Thus, $C_0 \ge 0$.

If $C_0 = 0$, we get that Ω is a ball of radius one, which is the limit case of $C_0 > 0$ and for which the charges would touch the boundary of the ball from inside at the diametrically opposite points. However, it is not difficult to see that this is impossible, using a first variation type argument by displacing the charges further away by distance $0 < \delta \ll 1$ and gaining $O(\delta)$ in the Coulombic energy, while losing only $O(\delta^2)$ in the perimeter. Hence $C_0 > 0$, and we get that the graph of φ is an arc of an elliptic catenary creating a corresponding unduloid section as its surface of revolution. Up to translations we will characterize an unduloid by its minimum height a and its maximum height c. Thus, Ω is the graph of φ that consists of arcs of elliptic catenary curves.

To show that φ contains only one section of an elliptic catenary arc, note that the maximum height c_1 of at least one elliptic catenary arc contained in φ satisfies

$$c_1 = 1 + o(1), \tag{6.14}$$

where $o(1)$ is with respect to $\varepsilon \ll 1$. This follows directly from our volume constraint and Lemma 6.2. Furthermore, let $a_1 \leq \varepsilon$ denote the minimum height of this elliptic catenary. Thus, the mean curvature of the unduloid formed by this elliptic catenary arc is given by

$$H_1 = \frac{1}{a_1 + c_1}. \tag{6.15}$$

Since the mean curvature of the free surface is constant, this implies that

$$H = 1 + o(1). \tag{6.16}$$

Assume that the graph of φ contains more than one elliptic catenary arc, then at least one arc must contact the same charge at two distinct points. Furthermore, (6.13) implies that both contact points happen at the same height $0 < h < \varepsilon$. Now let a_2 and c_2 denote the minimum and maximum of this elliptic catenary, then for ε sufficiently small (6.16) gives us that $c_2 = 1 + o(1)$. However, for sufficiently small ε this is impossible, since the elliptic catenary contacts the same charge at two distinct points. □

Here we state the following parametrization for an elliptic catenary, which is obtained from [24].

Lemma 6.6. *Up to translations, one period of an elliptic catenary with minimum height a and maximum height c has the following parametrization:*

$$x(t) = aF\left(\frac{t}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right) + cE\left(\frac{t}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right), \tag{6.17}$$

and

$$z(t) = \sqrt{\frac{c^2 - a^2}{2}\sin(t) + \frac{c^2 + a^2}{2}}, \tag{6.18}$$

where $-\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$, $F(u,k)$ is the elliptic integral of the first kind, and $E(u,k)$ is the elliptic integral of the second kind which are defined as,

$$F(u,k) := \int_0^u \frac{1}{\sqrt{1 - k\sin^2(\theta)}}d\theta, \tag{6.19}$$

and

$$E(u,k) := \int_0^u \sqrt{1-k\sin^2(\theta)}d\theta. \tag{6.20}$$

From the parametrization given in Lemma 6.6 and Lemmas 6.5 and 6.2 we have the following result.

Lemma 6.7. *There exists a universal constant $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and if $(\Omega, \{x_1, x_2\})$ is a minimizer to* (6.1)*, then the profile function of the free surface $\partial\Omega\backslash(\partial B_\varepsilon(x_1) \cup \partial B_\varepsilon(x_2))$ is a graph of a single arc of an elliptic catenary that has exactly one maximum.*

Proof. Let φ be the profile function defined in (6.11). Then by Lemma 6.5 the graph of φ must consist of a single arc of an elliptic catenary. Thus, we must show that φ has exactly one maximum. To do this, define M to be the number of maximum points of φ. First, note that M is nonzero, since if $M = 0$ then Lemma 6.2 implies that we cannot satisfy our volume constraint.

Now assume that $M > 1$, and let h be the height of contact between φ and the charges. Using the parametrization given in Lemma 6.6, we can find a lower bound on the distance between the two contact points, which gives us that

$$\begin{aligned}\operatorname{diam}(\Omega) \geq 2&\left(aF\left(\frac{t_0}{2}-\frac{\pi}{4},\frac{c^2-a^2}{c^2}\right)+cE\left(\frac{t_0}{2}-\frac{\pi}{4},\frac{c^2-a^2}{c^2}\right)\right)\\ &+2(M-1)\left(aF\left(\frac{\pi}{2},\frac{c^2-a^2}{c^2}\right)+cE\left(\frac{\pi}{2},\frac{c^2-a^2}{c^2}\right)\right),\end{aligned} \tag{6.21}$$

where

$$t_0 = \pi - \arcsin\left(\frac{2h^2-(c^2+a^2)}{c^2-a^2}\right). \tag{6.22}$$

However, as $\varepsilon \to 0$ we have that $a \to 0$ and $c \to 1$. Thus, (6.21) implies that

$$\operatorname{diam}(\Omega) \geq 2M - o(1), \tag{6.23}$$

and we have that ε_0 can be chosen small enough universal to provide a contradiction to Lemma 6.2. □

Lastly, we state some expansions for elliptic integrals.

Lemma 6.8. *Let $F(u,k)$ and $E(u,k)$ be the incomplete elliptic integrals of the first and second kind as defined in* (6.19) *and* (6.20)*, then*

$$F\left(\frac{\pi}{2},z\right) = -\frac{1}{2}\log(1-z) + \mathrm{O}(1), \tag{6.24}$$

$$E\left(\frac{\pi}{2},z\right) = 1 + \frac{z-1}{4}\log(1-z) + \mathrm{O}((z-1)), \tag{6.25}$$

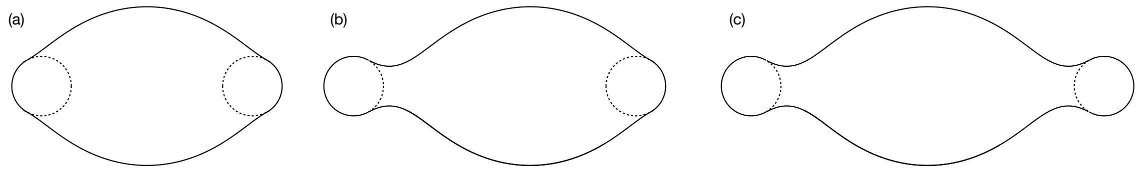

Figure 4: Three possibilities of the energy minimizing candidates for $N = 2$: (a) case 1, (b) case 2, (c) case 3.

$$F(\arcsin(u), z) = \operatorname{arctanh}(u) - \mathrm{O}\left(\frac{z-1}{u-1}\right), \tag{6.26}$$

$$E(\arcsin(u), z) = u - \frac{z-1}{2}\operatorname{arctanh}(u) + \mathrm{O}\left(z-1, \frac{(z-1)^2}{u-1}\right), \tag{6.27}$$

$$F(\arcsin(u), z) = -\frac{1}{2}\log(1-z) + \mathrm{O}\left(1, \frac{\sqrt{1-u}}{\sqrt{1-z}}\right), \tag{6.28}$$

and

$$E(\arcsin(u), z) = 1 + \frac{z-1}{4}\log(1-z) + \mathrm{O}\left((z-1), \sqrt{(1-u)(1-zu^2)}\right). \tag{6.29}$$

Here and below we are using the notation $\mathrm{O}(a, b)$, etc., to denote the quantities that are bounded by universal multiples of $\max(|a|, |b|)$ for $|a|, |b| \ll 1$.

6.2 Classification of cases

Let $(\Omega^*, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$ be a minimizer to (6.1), then from Lemma 6.7, we have that Ω^* falls into one of three cases. In case 1, we have that the free surface of Ω^* has the profile function whose graph is an arc of an elliptic catenary that does not attain its minimum value a. In case 2, we have that this elliptic catenary arc attains its minimum value a at exactly one point. Lastly, in case 3, this elliptic catenary arc obtains its minimum value a at exactly two points. These three alternatives are illustrated in Figure 4. Thus, from here on we only need to consider test configurations $(\Omega, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$, which fall into one of the three cases defined above. Furthermore, without loss of generality we define the maximum of the profile function to be located at $(0, c)$, which is consistent with the parametrization given in Lemma 6.6 and fixes translations along the x-axis.

In case 1, the unduloid arc joining the two charges does not attain its minimum and the minimizer is symmetric about the z-axis. In this case, we have that our minimizer is of the form $\left(\Omega^*, \{(-\frac{L}{2}, 0, 0), (\frac{L}{2}, 0, 0)\}\right)$, where L is the distance between the charges. For a case 1 test configuration we have the following lemma, which implicitly expresses the energy as a function of the contact height h (see Figure 5 for a schematic of a case 1 test configuration).

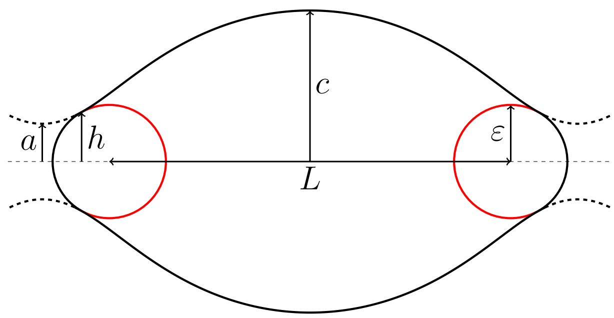

Figure 5: The schematics of the cross-section of the case 1 candidate for a minimizer.

Lemma 6.9. *Let* $(\Omega, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$ *be a case 1 test configuration with contact height* h. *Then*

$$E_\varepsilon(\Omega, X) = \hat{E}_\varepsilon(h) = 4\pi\left((a+c)cE\left(\frac{t_0}{2}-\frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)+\varepsilon\left(\varepsilon-\sqrt{\varepsilon^2-h^2}\right)\right)$$
$$+\frac{\gamma\varepsilon^3}{2\left(aF\left(\frac{t_0}{2}-\frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)+cE\left(\frac{t_0}{2}-\frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)-\sqrt{\varepsilon^2-h^2}\right)}, \tag{6.30}$$

where

$$a = \frac{ch^2-\varepsilon h^2}{c\varepsilon - h^2}, \tag{6.31}$$

$$t_0 = \pi - \arcsin\left(\frac{2h^2-(c^2+a^2)}{c^2-a^2}\right), \tag{6.32}$$

and c *is given implicitly by*

$$2 = 2\varepsilon^3 - \sqrt{\varepsilon^2-h^2}(2\varepsilon^2+h^2) + \frac{h(c^2-a^2)}{2}\sqrt{1-\left(\frac{2h^2-c^2-a^2}{c^2-a^2}\right)^2} -$$
$$a^2cF\left(\frac{t_0}{2}-\frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)+(2c(a^2+c^2)+3ac^2)E\left(\frac{t_0}{2}-\frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right). \tag{6.33}$$

Proof. Let the unduloid section that joins the two charges have minimum height a and maximum height c. Since the charges contact the unduloid section at height h, we can use the parameterization given in (6.18) to find that the contact between the unduloid and the right charge happens when $t = t_0$, with t_0 defined in (6.32). Then from (6.17) we obtain that

$$L = 2\left(aF\left(\frac{t_0}{2}-\frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)+cE\left(\frac{t_0}{2}-\frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)-\sqrt{\varepsilon^2-h^2}\right), \tag{6.34}$$

where L is the distance between the charges. Now, since our unduloid has mean curvature $H = \frac{1}{a+c}$, from (6.12) we have that each monotone arc of our elliptic catenary is given by the equation,

$$\varphi' = \pm\sqrt{\frac{(a+c)^2\varphi^2}{(\varphi^2+ac)^2} - 1}. \tag{6.35}$$

Thus, our tangency condition between the charges and the elliptic catenary implies (6.31). In addition, calculating the volume of the unduloid section, which is given in [24], and accounting for the volume of the excess charges gives (6.33). Finally, (6.34) explains the interaction energy given in (6.30), and the perimeter term is derived directly from accounting for the surface area of the unduloid section (given in [24]) and the surface area over the charges. □

Lemma 6.10. *Let* $(\Omega, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$ *be a case 2 test configuration with contact height* h. *Then*

$$E_\varepsilon(\Omega, X) = \hat{E}_\varepsilon(h) = 4\pi c(a+c)E\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) + 4\pi\varepsilon^2$$
$$+\lambda\varepsilon^3\left(2aF\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) + 2cE\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right)\right)^{-1}, \tag{6.36}$$

where a *is given by* (6.31) *and* c *is given implicitly by*

$$2 = (2(a^2+c^2)c + 3ac^2)E\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) - a^2cF\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) + 2\varepsilon^3. \tag{6.37}$$

Proof. Note that the equation for a comes from the tangency condition, as in Lemma 6.9. Furthermore, (6.36) and (6.37) follow directly from [24]. □

Lemma 6.11. *Let* $(\Omega, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$ *be a case 3 test configuration with contact height* h. *Then*

$$E_\varepsilon(\Omega, X) = \hat{E}_\varepsilon(h) = 8\pi c(a+c)E\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right)$$
$$-4\pi c(a+c)E\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right) + 4\pi\varepsilon(\varepsilon + \sqrt{\varepsilon^2-h^2})$$
$$+\lambda\varepsilon^3\Bigg(2\Bigg(\sqrt{\varepsilon^2-h^2} + 2aF\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) + 2cE\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) - \tag{6.38}$$
$$aF\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right) - cE\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)\Bigg)\Bigg)^{-1},$$

where the equations for a and t_0 are given by (6.31) *and* (6.32)*, respectively, and c is given implicitly by*

$$\begin{aligned}2 = (2(a^2+c^2)c + 3ac^2)\left(2E\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) - E\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)\right)\\ -a^2c\left(2F\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) - F\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right)\right)\\ -\frac{(c^2-a^2)h}{2}\sqrt{1-\left(\frac{2h^2-c^2-a^2}{c^2-a^2}\right)^2} + 2\varepsilon^3 + \sqrt{\varepsilon^2-h^2}(2\varepsilon^2+h^2).\end{aligned} \tag{6.39}$$

Proof. Note that for case 3 candidates our tangency condition between the charges and the elliptic catenary remains unchanged. Furthermore, the distance between charges and the volume and surface area of the unduloid section follow directly from [24]. □

Proposition 6.12. *There exists a universal $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $(\Omega, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$ is a case 1 test configuration with contact height $h \le \varepsilon$ then*

$$E_\varepsilon(\Omega, X) > 4\pi - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{\varepsilon}\right) + \mathrm{O}\left(\varepsilon^2\right), \tag{6.40}$$

whenever $h > \frac{\varepsilon}{2}$. Furthermore, we have

$$E_\varepsilon(\Omega, X) = 4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) - \left(\frac{2\pi h^4}{\varepsilon^2} - \frac{\gamma h^2\varepsilon^2}{2}\right)\log(h) + \mathrm{O}\left(\gamma\varepsilon^6, (\gamma+1)h^2\varepsilon^2, \frac{h^4}{\varepsilon^2}\right), \tag{6.41}$$

whenever $h \le \frac{\varepsilon}{2}$.

Proof. Let (Ω, X) be a case 1 test configuration with contact height h. Now we will expand the expressions for the energy of (Ω, X) and volume of Ω given in Lemma 6.9 in terms of ε and the contact height $0 < h \le \varepsilon$. To do this, first we will obtain a lower bound on the energy in the regime where $h > \frac{\varepsilon}{2}$.

Let $h > \frac{\varepsilon}{2}$. Using the notation from Lemma 6.9, we have that the minimum a of the extended unduloid section of Ω is given by,

$$a = \frac{ch^2 - \varepsilon h^2}{c\varepsilon - h^2} = \frac{h^2}{\varepsilon} - \frac{h^2}{c} + \frac{h^4}{c\varepsilon^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right). \tag{6.42}$$

This gives us that

$$\begin{aligned}\frac{t_0}{2} - \frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{2}\arcsin\left(\frac{2h^2-(c^2+a^2)}{c^2-a^2}\right) = \arcsin\left(\sqrt{\frac{c^2-h^2}{c^2-a^2}}\right)\\ = \arcsin\left(1 - \frac{h^2}{2c^2} + \frac{h^4}{2c^2\varepsilon^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}, \frac{h^6}{\varepsilon^3}\right)\right).\end{aligned} \tag{6.43}$$

Since $h > \frac{\varepsilon}{2}$, from (6.42), (6.43), and (6.28) we get that

$$F\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right) = F\left(\frac{t_0}{2} - \frac{\pi}{4}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) = -\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}\left(1\right), \quad (6.44)$$

and from (6.42), (6.43), and (6.29) we get

$$\begin{aligned} E\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right) &= E\left(\frac{t_0}{2} - \frac{\pi}{4}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) \\ &= 1 - \frac{h^4}{2c^2\varepsilon^2}\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}\left(h^2\right). \quad (6.45) \end{aligned}$$

Expanding our volume constraint given in (6.33) we obtain

$$2 = 2c^3 + \frac{3h^2c^2}{\varepsilon} + \mathrm{O}\left(\varepsilon^2\right). \quad (6.46)$$

Thus, (6.46) implies

$$c > 1 - \frac{h^2}{2\varepsilon} + \mathrm{O}\left(\varepsilon^2\right). \quad (6.47)$$

Lastly, using (6.44), (6.45), and (6.47) to expand the contribution of the perimeter to our energy given in (6.30), we obtain

$$\begin{aligned} E_\varepsilon\left(\Omega, X\right) &> 4\pi c^2 + 4\pi\frac{h^2 c}{\varepsilon} - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{c\varepsilon}\right) + \mathrm{O}\left(\varepsilon^2\right) \\ &> 4\pi - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{c\varepsilon}\right) + \mathrm{O}\left(\varepsilon^2\right), \quad (6.48) \end{aligned}$$

for $\varepsilon < \varepsilon_0$ with ε_0 small enough. This proves (6.40).

Now we move on to the case where

$$h \leq \frac{\varepsilon}{2}. \quad (6.49)$$

First note that from (6.26), (6.42), (6.43), and (6.49) we get that

$$\begin{aligned} F\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right) &= F\left(\frac{t_0}{2} - \frac{\pi}{4}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) \\ &= \operatorname{arctanh}\left(1 + \frac{h^2}{2c^2}\left(-1 + \frac{h^2}{\varepsilon^2}\right)\right) + \mathrm{O}\left(1\right), \quad (6.50) \end{aligned}$$

and from (6.27), (6.42), (6.43), and (6.49) we also have that

$$\begin{aligned} &E\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right) = E\left(\frac{t_0}{2} - \frac{\pi}{4}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) = \\ &1 + \frac{h^2}{2c^2}\left(-1 + \frac{h^2}{\varepsilon^2}\left(1 + \operatorname{arctanh}\left(1 + \frac{h^2}{2c^2}\left(-1 + \frac{h^2}{\varepsilon^2}\right)\right)\right)\right) + \mathrm{O}\left(\frac{h^4}{\varepsilon^2}\right). \quad (6.51) \end{aligned}$$

Thus, from (6.50), (6.51), and (6.30) we get that

$$E_\varepsilon(\Omega, X) = 4\pi c^2 + 4\pi\frac{h^2 c}{\varepsilon} - 4\pi h^2 + \frac{2\pi h^4}{\varepsilon^2}\operatorname{arctanh}\left(1 - \frac{h^2}{2c^2}\left(1 - \frac{h^2}{\varepsilon^2}\right)\right) + \mathrm{O}\left(\frac{h^4}{\varepsilon^2}\right)$$
$$+\gamma\varepsilon^3\left(\frac{1}{2c} - \frac{h^2}{2c^2\varepsilon}\operatorname{arctanh}\left(1 - \frac{h^2}{2c^2}\left(1 - \frac{h^2}{\varepsilon^2}\right)\right) + \frac{\sqrt{\varepsilon^2 - h^2}}{2c^2} + \frac{\varepsilon^2}{2c^3} + \mathrm{O}\left(\varepsilon^3, \frac{h^2}{\varepsilon}\right)\right). \tag{6.52}$$

Furthermore, from expanding our volume constraint given in (6.33) we get that

$$2 = 2c^3 + \frac{3h^2c^2}{\varepsilon} - 3h^2 c + \mathrm{O}\left(\frac{h^4}{\varepsilon^2}\right). \tag{6.53}$$

Expanding (6.53) we get that

$$h = \sqrt{\frac{2\varepsilon}{3}\left(\frac{1}{c^2} - c\right)\left(1 + \frac{\varepsilon}{c} + \frac{\varepsilon^2}{c^2} + \mathrm{O}\left(\frac{h^2}{\varepsilon}, \varepsilon^3\right)\right)}. \tag{6.54}$$

Now we introduce the new variable α, which is defined via

$$c = 1 - \alpha\varepsilon. \tag{6.55}$$

Thus, (6.54) allows us to express h in terms of α,

$$h = \sqrt{2\alpha\varepsilon^2 + 2\alpha\varepsilon^3 + 2\alpha\varepsilon^4 + \mathrm{O}\left(\alpha\varepsilon^5, \alpha^2\varepsilon^3, \alpha\varepsilon h^2\right)} \tag{6.56}$$

Finally, plugging (6.56) into (6.52) we obtain the following leading order equation for the energy:

$$E_\varepsilon(\Omega, X) = 4\pi + 8\pi\alpha^2\varepsilon^2\operatorname{arctanh}\left(1 - \alpha\varepsilon^2\right) + \mathrm{O}\left(\alpha\varepsilon^4, \alpha^2\varepsilon^2, \alpha h^2\right)$$
$$+\gamma\varepsilon^3\left(\frac{1}{2} - \alpha\varepsilon\operatorname{arctanh}\left(1 - \alpha\varepsilon^2\right) + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} + \mathrm{O}\left(\varepsilon^3, \alpha\varepsilon\right)\right). \tag{6.57}$$

Simplifying further gives

$$E_\varepsilon(\Omega, X) = 4\pi + \frac{\gamma\varepsilon^3}{2}(1 + \varepsilon + \varepsilon^2) + (8\pi\alpha^2\varepsilon^2 - \gamma\alpha\varepsilon^4)\operatorname{arctanh}\left(1 - \alpha\varepsilon^2\right)$$
$$+\mathrm{O}\left(\gamma\varepsilon^6, (\gamma+1)\alpha\varepsilon^4, \alpha^2\varepsilon^2, \alpha h^2\right), \tag{6.58}$$

which gives us

$$E_\varepsilon(\Omega, X) = 4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) - (4\pi\alpha^2\varepsilon^2 - \frac{\gamma\alpha\varepsilon^4}{2})\log\left(\alpha\varepsilon^2\right) + \mathrm{O}\left(\gamma\varepsilon^6, (\gamma+1)\alpha\varepsilon^4, \alpha^2\varepsilon^2, \alpha h^2\right). \tag{6.59}$$

Furthermore, we use (6.56) to convert (6.59) into

$$E_\varepsilon(\Omega, X) = 4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) - \left(\frac{2\pi h^4}{\varepsilon^2} - \frac{\gamma h^2\varepsilon^2}{2}\right)\log(h) + \mathrm{O}\left(\gamma\varepsilon^6, (\gamma+1)h^2\varepsilon^2, \frac{h^4}{\varepsilon^2}\right), \tag{6.60}$$

which completes the proof. □

Proposition 6.13. *There exists a universal $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $(\Omega, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$ is a case 2 test configuration then (Ω, X) cannot be a minimizer to* (6.1).

Proof. We will argue by contradiction. Thus, first assume that (Ω, X) is a case 2 minimizer with contact height h. Now we will expand the expressions for the energy of (Ω, X) and volume of Ω given in Lemma 6.10 in terms of ε and the contact height $0 < h \le \varepsilon$. To do this, first note that from (6.24) and (6.31) we have

$$F\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) = F\left(\frac{\pi}{2}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) = -\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}(1), \tag{6.61}$$

and from (6.25) and (6.31) we get

$$E\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) = E\left(\frac{\pi}{2}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) = 1 - \frac{h^4}{2\varepsilon^2 c^2}\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}\left(\frac{h^4}{\varepsilon^2}\right). \tag{6.62}$$

Expanding the the volume constraint given in (6.37) gives us

$$2 = 2c^3 + \frac{3c^2h^2}{\varepsilon} + \mathrm{O}\left(h^2, \varepsilon^3\right). \tag{6.63}$$

Using (6.61) and (6.62) we expand the energy given in (6.36), to obtain

$$E_\varepsilon(\Omega, X) > 4\pi c^2 + \frac{4\pi c h^2}{\varepsilon} - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{\varepsilon c}\right) + 4\pi\varepsilon^2 + \frac{\gamma\varepsilon^3}{2} + \mathrm{O}\left(h^2, \gamma\varepsilon^2 h^2 \log\left(\frac{h^2}{\varepsilon c}\right)\right). \tag{6.64}$$

Furthermore, (6.63) implies that

$$c > 1 - \frac{c^2h^2}{2\varepsilon} + \mathrm{O}\left(h^2, \varepsilon^3\right), \tag{6.65}$$

and from (6.64) and (6.65) we get

$$E_\varepsilon(\Omega, X) > 4\pi - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{\varepsilon c}\right) + 4\pi\varepsilon^2 + \frac{\gamma\varepsilon^3}{2} + \mathrm{O}\left(\varepsilon^3, h^2, \gamma\varepsilon^2 h^2\log\left(\frac{h^2}{\varepsilon c}\right)\right). \tag{6.66}$$

Lastly, since we assumed that (Ω, X) is a minimizer, from considering the test configuration given by $\big(B_1(0,0,0), \{(\varepsilon-1,0,0),(1-\varepsilon,0,0)\}\big)$ we obtain

$$4\pi + \frac{\gamma\varepsilon^3}{2(1-\varepsilon)} \geq E_\varepsilon(\Omega, X) = 4\pi - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{\varepsilon c}\right)$$
$$+4\pi\varepsilon^2 + \frac{\gamma\varepsilon^3}{2} + \mathrm{O}\left(\varepsilon^3, h^2, \gamma\varepsilon^2 h^2 \log\left(\frac{h^2}{\varepsilon c}\right)\right). \tag{6.67}$$

Thus, for $\varepsilon < \varepsilon_0$ small enough, (6.67) implies that

$$\gamma\varepsilon^4 \geq 2\pi\varepsilon^2 + \mathrm{O}\left(\gamma\varepsilon^2 h^2 \log\left(\frac{h^2}{\varepsilon c}\right)\right). \tag{6.68}$$

However, this implies that $\gamma > -\frac{c_0}{\varepsilon^2\log(\varepsilon)}$, where $c_0 > 0$ is a universal constant, which contradicts Lemma 6.3 for $\varepsilon < \varepsilon_0$ small enough. Thus, (Ω, X) cannot be a minimizer. □

Proposition 6.14. *There exists a universal $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $(\Omega, X) \in \mathcal{A}_{\frac{4\pi}{3},2,\varepsilon}$ is a case 3 test configuration then (Ω, X) cannot be a minimizer to (6.1).*

Proof. Assume that (Ω, X) is a case 3 minimizer with contact height h. Now we will expand the expressions for the energy of (Ω, X) and volume of Ω given in Lemma 6.11 in terms of ε and the contact height $0 < h \leq \varepsilon$.

However, first we will eliminate the regime where $h > \frac{\varepsilon}{2}$. To do this, assume that $h > \frac{\varepsilon}{2}$, then from (6.24) and (6.31) we get

$$F\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) = F\left(\frac{\pi}{2}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) = -\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}(1), \tag{6.69}$$

and from (6.25) and (6.31) we get

$$E\left(\frac{\pi}{2}, \frac{c^2-a^2}{c^2}\right) = E\left(\frac{\pi}{2}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right)$$
$$= 1 - \frac{h^4}{2\varepsilon^2 c^2}\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}\left(\frac{h^4}{\varepsilon^2}\right). \tag{6.70}$$

Since by assumption $h > \frac{\varepsilon}{2}$, from (6.28), (6.31) and (6.32) we get that

$$F\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right) = F\left(\frac{t_0}{2} - \frac{\pi}{4}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right) = -\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}(1), \tag{6.71}$$

and from (6.29), (6.31) and (6.32) we get

$$E\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2-a^2}{c^2}\right) = E\left(\frac{t_0}{2} - \frac{\pi}{4}, 1 - \frac{h^4}{\varepsilon^2 c^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon}\right)\right)$$
$$= 1 - \frac{h^4}{2c^2\varepsilon^2}\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}\left(h^2\right). \tag{6.72}$$

Thus, using (6.69), (6.70), (6.71), and (6.72) to expand the contribution of the perimeter to (6.38), we obtain

$$E_{\varepsilon}\left(\Omega, X\right) > 4\pi c^2 + \frac{4\pi c h^2}{\varepsilon} - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}\left(\varepsilon^2\right), \tag{6.73}$$

and from our volume constraint given in (6.39) we obtain

$$2 = 2c^3 + \frac{3h^2c^2}{\varepsilon} + \mathrm{O}\left(\varepsilon^2\right), \tag{6.74}$$

which implies that

$$c > 1 - \frac{c^2h^2}{2\varepsilon} + \mathrm{O}\left(\varepsilon^2\right). \tag{6.75}$$

Thus, from (6.73) and (6.75) we get that

$$E_{\varepsilon}\left(\Omega, X\right) > 4\pi - \frac{2\pi h^4}{\varepsilon^2}\log\left(\frac{h^2}{\varepsilon c}\right) + \mathrm{O}\left(\varepsilon^2\right), \tag{6.76}$$

which contradicts Lemma 6.1 whenever $\varepsilon < \varepsilon_0$ with ε_0 small, and we conclude that (6.49) holds.

Now from (6.26), (6.27), (6.31), (6.32) and (6.49) we obtain

$$F\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right) = -\frac{1}{2}\log\left(\frac{h^2}{2c^2}\left(1 - \frac{h^2}{\varepsilon^2}\right)\right) + \mathrm{O}\left(1\right), \tag{6.77}$$

and

$$E\left(\frac{t_0}{2} - \frac{\pi}{4}, \frac{c^2 - a^2}{c^2}\right) = 1 - \frac{h^2}{2c^2} - \frac{h^4}{4c^2\varepsilon^2}\log\left(\frac{h^2}{2c^2}\left(1 - \frac{h^2}{\varepsilon^2}\right)\right) + \mathrm{O}\left(\frac{h^4}{\varepsilon^2}\right). \tag{6.78}$$

Using (6.69), (6.70), (6.77), and (6.78) to expand the contribution of perimeter to (6.38) we obtain

$$\begin{aligned} E_{\varepsilon}\left(\Omega, X\right) > 4\pi c^2 + 4\pi c\frac{h^2}{\varepsilon} - 2\pi h^2 + \frac{\pi h^4}{\varepsilon^2}\log\left(\frac{\varepsilon^4 c^2}{2h^6}\left(1 - \frac{h^2}{\varepsilon^2}\right)\right) \\ +4\pi\varepsilon^2 + 4\pi\varepsilon\sqrt{\varepsilon^2 - h^2} + \mathrm{O}\left(\frac{h^4}{\varepsilon^2}, h^2\right). \end{aligned} \tag{6.79}$$

and from our volume constraint given in (6.39) we obtain

$$2 = 2c^3 + \frac{3h^2c^2}{\varepsilon} + \mathrm{O}\left(\varepsilon^3, h^2\right). \tag{6.80}$$

Thus, from (6.80) we have

$$c \geq 1 - \frac{h^2c^2}{2\varepsilon} + \mathrm{O}\left(\varepsilon^3, h^2\right). \tag{6.81}$$

Lastly, from (6.81), (6.49), and (6.79) we obtain

$$E_\varepsilon(\Omega, X) > 4\pi + \frac{\pi h^4}{\varepsilon^2}\log\left(\frac{\varepsilon^4 c^2}{2h^6}\left(1-\frac{h^2}{\varepsilon^2}\right)\right)$$
$$+4\pi\varepsilon^2 + 4\pi\varepsilon\sqrt{\varepsilon^2-h^2} + \mathrm{O}\left(\varepsilon^3, h^2, \frac{h^4}{\varepsilon^2}\right) \tag{6.82}$$
$$\geq 4\pi + 4\pi\varepsilon^2, \tag{6.83}$$

for $\varepsilon < \varepsilon_0$ with ε_0 small. Thus, from Lemma 6.1 we conclude that (Ω, X) cannot be a minimizer. □

Theorem 6.15. *There exist universal constants C, C_1, $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $\gamma < \frac{8\pi}{\varepsilon} - C$ then there exists a unique, up to translations and rotations, minimizer $(\Omega^*, \{(-\frac{L^*}{2}, 0, 0), (\frac{L^*}{2}, 0, 0)\})$ to (6.1). Furthermore, it is a case 1 configuration, and if $\frac{C_1}{\log\varepsilon^{-1}} < \gamma < \frac{8\pi}{\varepsilon} - C$, then*

$$E_\varepsilon\left(\Omega^*, \left\{\left(-\frac{L^*}{2}, 0, 0\right), \left(\frac{L^*}{2}, 0, 0\right)\right\}\right)$$
$$= 4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) + \frac{\gamma^2\varepsilon^6}{64\pi}\log(\gamma\varepsilon^4) + \mathrm{O}\left(\varepsilon^6\gamma^{\frac{7}{4}}(1+\gamma)^{\frac{1}{4}}(\log\varepsilon^{-1})^{\frac{3}{4}}\right), \tag{6.84}$$
$$L^* = 2 - 2\varepsilon - \frac{\gamma\varepsilon^3}{8\pi}\log(\gamma\varepsilon^4) + \mathrm{O}\left(\gamma^{\frac{3}{4}}(1+\gamma)^{\frac{1}{4}}\varepsilon^3(\log\varepsilon^{-1})^{\frac{3}{4}}\right), \tag{6.85}$$

and $(\Omega^*, \{(-\frac{L^*}{2}, 0, 0), (\frac{L^*}{2}, 0, 0)\})$ has contact height h^* satisfying

$$h^* = \sqrt{\frac{\gamma\varepsilon^4}{8\pi}} + \mathrm{O}\left(\frac{\varepsilon^2\gamma(\gamma+1)}{\log\varepsilon^{-1}}^{\frac{1}{4}}\right). \tag{6.86}$$

Proof. First note that the existence and rotational symmetry of a minimizer follows directly from Theorem 3.2, Lemma 6.3 and Lemma 6.4. Furthermore, Lemma 6.7 implies that this minimizer is either a case 1, case 2, or case 3 configuration. Thus, Propositions 6.13 and 6.14 rule out all but a case 1 minimizer $(\Omega^*, \{(-\frac{L^*}{2}, 0, 0), (\frac{L^*}{2}, 0, 0)\})$.

Let h^* be the contact height of the minimizer above. Then from (6.40) and Lemma 6.1 we conclude that

$$h^* \leq \frac{\varepsilon}{2}. \tag{6.87}$$

Thus, from (6.41) we conclude that

$$E_\varepsilon\left(\Omega^*,\left\{\left(-\frac{L^*}{2},0,0\right),\left(\frac{L^*}{2},0,0\right)\right\}\right)$$
$$= 4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) - \left(\frac{2\pi h^{*4}}{\varepsilon^2} - \frac{\gamma h^{*2}\varepsilon^2}{2}\right)\log(h^*)$$
$$+ \mathrm{O}\left(\gamma\varepsilon^6,(\gamma+1)h^{*2}\varepsilon^2,\frac{h^{*4}}{\varepsilon^2}\right). \quad (6.88)$$

Now let $\left(\Omega,\left\{\left(-\frac{L}{2},0,0\right),\left(\frac{L}{2},0,0\right)\right\}\right)$ be a case 1 candidate minimizer with a contact height h given by

$$h = \sqrt{\frac{\gamma\varepsilon^4}{8\pi}}. \quad (6.89)$$

From the minimality of $\left(\Omega^*,\left\{\left(-\frac{L^*}{2},0,0\right),\left(\frac{L^*}{2},0,0\right)\right\}\right)$ and from (6.41) and (6.88) we obtain

$$4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) - \left(\frac{2\pi h^{*4}}{\varepsilon^2} - \frac{\gamma h^{*2}\varepsilon^2}{2}\right)\log(h^*) + \mathrm{O}\left(\gamma\varepsilon^6,(\gamma+1)h^{*2}\varepsilon^2,\frac{h^{*4}}{\varepsilon^2}\right)$$
$$\leq 4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) + \frac{\gamma^2\varepsilon^6}{32\pi}\log\left(\sqrt{\frac{\gamma\varepsilon^4}{8\pi}}\right) + \mathrm{O}\left(\gamma(\gamma+1)\varepsilon^6\right). \quad (6.90)$$

Thus

$$-\left(\frac{2\pi h^{*4}}{\varepsilon^2} - \frac{\gamma h^{*2}\varepsilon^2}{2}\right)\log(h^*) + \mathrm{O}\left(\gamma(\gamma+1)\varepsilon^6,(\gamma+1)h^{*2}\varepsilon^2,\frac{h^{*4}}{\varepsilon^2}\right) \leq \frac{\gamma^2\varepsilon^6}{32\pi}\log\left(\sqrt{\frac{\gamma\varepsilon^4}{8\pi}}\right). \quad (6.91)$$

Now note that (6.91) implies that

$$h^* \geq \frac{\sqrt{\gamma\varepsilon^4}}{8}, \quad (6.92)$$

since otherwise (6.91) implies

$$\frac{\gamma^2\varepsilon^6}{128}\log\left(\frac{\sqrt{\gamma\epsilon^4}}{8}\right) + \mathrm{O}\left(\gamma(\gamma+1)\varepsilon^6\right) \leq \frac{\gamma^2\varepsilon^6}{32\pi}\log\left(\sqrt{\frac{\gamma\varepsilon^4}{8\pi}}\right). \quad (6.93)$$

However, this implies

$$\frac{\gamma^2\varepsilon^6}{256}\log\left(\gamma\varepsilon^4\right) + \mathrm{O}\left(\gamma(\gamma+1)\varepsilon^6\right) \leq \frac{\gamma^2\varepsilon^6}{64\pi}\log\left(\gamma\varepsilon^4\right). \quad (6.94)$$

Now pick $\varepsilon_0 > 0$ so that $|\log(\gamma\varepsilon^4)| > \mathrm{O}\left(\frac{\gamma+1}{\gamma}\right)$, then (6.94) provides a contradiction. Thus, (6.92) holds.

Finally, let

$$h^* = \sqrt{\frac{\gamma\varepsilon^4}{8\pi}} + h_e, \tag{6.95}$$

then from (6.91) we obtain

$$\begin{aligned} &-\left(\gamma\varepsilon^2 h_e^2 + \sqrt{8\pi\gamma} h_e^3 + \frac{2\pi}{\varepsilon^2} h_e^4\right) \log h^* \\ &\qquad \leq -\frac{\gamma^2\varepsilon^6}{32\pi} \log\left(1 + \sqrt{\frac{8\pi}{\gamma\varepsilon^4}} h_e\right) + \mathrm{O}\left(\gamma(\gamma+1)\varepsilon^6, (\gamma+1)h^{*2}\varepsilon^2, \frac{h^{*4}}{\varepsilon^2}\right) \end{aligned} \tag{6.96}$$

If $h_e > 0$, then (6.95) implies $h^* > \sqrt{\frac{\gamma\varepsilon^4}{8\pi}}$, and (6.96) gives

$$-\frac{2\pi}{\varepsilon^2} h_e^4 \log h^* \leq \mathrm{O}\left(\frac{h^{*4}(\gamma+1)}{\gamma\varepsilon^2}\right). \tag{6.97}$$

Thus,

$$h_e \leq \mathrm{O}\left(h^*\left(\frac{\gamma+1}{\gamma\log\left(h^{*-1}\right)}\right)^{\frac{1}{4}}\right). \tag{6.98}$$

Now (6.98) and (6.95) imply that

$$h^* \leq \sqrt{\frac{\gamma\varepsilon^4}{8\pi}}\left(1 + \mathrm{O}\left(\left(\frac{\gamma+1}{\gamma\log\left(h^{*-1}\right)}\right)^{\frac{1}{4}}\right)\right), \tag{6.99}$$

whenever $h^* \leq \varepsilon < \varepsilon_0$ is sufficiently small.

If $h_e < 0$, then from (6.95) we have $0 < h^* < \sqrt{\frac{\gamma\varepsilon^4}{8\pi}}$ and (6.96) implies

$$-\frac{2\pi}{\varepsilon^2} h_e^4 \log h^* \leq -\frac{\gamma^2\varepsilon^6}{32\pi} \log\left(1 + \sqrt{\frac{8\pi}{\gamma\varepsilon^4}} h_e\right) + \mathrm{O}\left(\gamma(\gamma+1)\varepsilon^6\right). \tag{6.100}$$

Furthermore, (6.92) implies that $h_e \geq \frac{\sqrt{\gamma\varepsilon^4}}{8} - \frac{\sqrt{\gamma\varepsilon^4}}{\sqrt{8\pi}}$. Thus, from (6.100) we obtain

$$h_e \geq \mathrm{O}\left(\varepsilon^2\left(\frac{\gamma(\gamma+1)}{\log\left(h^{*-1}\right)}\right)^{\frac{1}{4}}\right). \tag{6.101}$$

Hence, (6.99) and (6.101) imply that

$$h^* = \sqrt{\frac{\gamma\varepsilon^4}{8\pi}} + \mathrm{O}\left(\varepsilon^2\left(\frac{\gamma(\gamma+1)}{\log\varepsilon^{-1}}\right)^{\frac{1}{4}}\right). \tag{6.102}$$

Thus, from (6.88) and (6.102) we obtain

$$E_\varepsilon\left(\Omega^*, \left\{\left(-\frac{L^*}{2}, 0, 0\right), \left(\frac{L^*}{2}, 0, 0\right)\right\}\right)$$
$$= 4\pi + \frac{\gamma\varepsilon^3}{2}(1+\varepsilon+\varepsilon^2) + \frac{\gamma^2\varepsilon^6}{64\pi}\log(\gamma\varepsilon^4) + \mathrm{O}\left(\varepsilon^6\gamma^{\frac{7}{4}}(1+\gamma)^{\frac{1}{4}}(\log\varepsilon^{-1})^{\frac{3}{4}}\right). \quad (6.103)$$

Lastly, from (6.50), (6.51), and (6.30) we obtain

$$L^* = 2 - 2\varepsilon - \frac{\gamma\varepsilon^3}{8\pi}\log(\gamma\varepsilon^4) + \mathrm{O}\left(\gamma^{\frac{3}{4}}(1+\gamma)^{\frac{1}{4}}\varepsilon^3(\log\varepsilon^{-1})^{\frac{3}{4}}\right). \quad (6.104)$$

To conclude the proof, observe that $(\Omega^*, \{(-\frac{L^*}{2}, 0, 0), (\frac{L^*}{2}, 0, 0)\})$ is unique, since expanding the second derivative of (6.30) gives

$$\hat{E}''_\varepsilon(h) = \left(\frac{24\pi h^2}{\varepsilon^2} - \gamma\varepsilon^2\right)\log h^{-1} + \mathrm{O}\left(\frac{h^2}{\varepsilon^2}, \gamma\varepsilon^2\right), \quad (6.105)$$

which implies that $\hat{E}_\varepsilon(h)$ is strictly convex in the neighborhood of the minimum. □

Proof of Theorem 3.6. The proof is obtained by combining the statements of Propositions 6.13 and 6.14 with that of Theorem 6.15. □

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