Front propagation in infinite cylinders. II. The sharp reaction zone limit.

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Abstract

This paper applies the variational approach developed in part I of this work [22] to a singular limit of reaction-diffusion-advection equations which arise in combustion modeling. We first establish existence, uniqueness, monotonicity, asymptotic decay, and the associated free boundary problem for special traveling wave solutions which are minimizers of the considered variational problem in the singular limit. We then show that the speed of the minimizers of the approximating problems converges to the speed of the minimizer of the singular limit. Also, after an appropriate translation the minimizers of the approximating problems converge strongly on compacts to the minimizer of the singular limit. In addition, we obtain matching upper and lower bounds for the speed of the minimizers in the singular limit in terms of a certain area-type functional for small curvatures of the free boundary. The conclusions of the analysis are illustrated by a number of numerical examples.

1 Introduction

This paper continues the analysis of propagation phenomena for gradient reaction-diffusion-advection equations from a variational point of view [22]. Here we will treat an important special case arising in combustion modeling which leads to a singular limit and can be reformulated as a free boundary problem. We refer the reader to Section 2 for a review of the physical background of the problem.

For $0 < \varepsilon \ll 1$, consider the equation

$$\frac{\partial u_\varepsilon}{\partial t} + \mathbf{v} \cdot \nabla u_\varepsilon = \Delta u_\varepsilon + f_\varepsilon(u_\varepsilon), \qquad u_\varepsilon\big|_{\partial\Sigma_\pm} = 0, \qquad \nu \cdot \nabla u_\varepsilon\big|_{\partial\Sigma_0} = 0, \tag{1.1}$$

in $\Sigma = \Omega \times \mathbb{R} \subset \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^{n-1}$ is a bounded domain with boundary of class C^2 (not necessarily simply connected). As before [22], $u_\varepsilon = u_\varepsilon(x,t) \in \mathbb{R}$, and $x = (y,z) \in \Sigma$ denotes a point with coordinates $y \in \Omega$ on the cylinder cross-section and $z \in \mathbb{R}$ along the axis. Furthermore, $\partial\Sigma_{\pm,0} = \partial\Omega_{\pm,0} \times \mathbb{R}$, with

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$\partial\Omega$ being the union of disjoint portions $\partial\Omega_{\pm,0}$ corresponding to the inlets, the outlets, and the impermeable walls, respectively, and $\mathbf{v}$ is a transverse potential flow:

$$\mathbf{v} = (-\nabla_y \varphi, 0), \qquad \varphi \in C^{1,\gamma}(\overline{\Omega}) \tag{1.2}$$

for some $\gamma \in (0,1)$. For the nonlinearity f_ε, we assume:

(H4) The function $f_\varepsilon(u) = \varepsilon^{-1} g\left(\frac{1-u}{\varepsilon}\right)$, where $g \in C^0(\mathbb{R})$ has the following properties:

$$g \geq 0, \quad \mathrm{supp}(g) = [0,1], \quad g \in C^{1,\gamma}([0,1]), \quad \int_0^1 g(u)du = 1. \tag{1.3}$$

This assumption implies that $f_\varepsilon(u) \to \delta(u - 1^-)$ as $\varepsilon \to 0$ and leads to the *sharp reaction zone limit*, the main subject of this paper.

It is not difficult to see that under hypothesis (H4) the results of [22] apply to (1.1) for any $\varepsilon < 1$. In particular, under hypothesis (H3) of [22] one gets existence of the traveling wave solutions, i.e. solutions of (1.1) in the form $u_\varepsilon(x,t) = \bar{u}_\varepsilon(y, z - ct)$, where $\bar{u}_\varepsilon$ satisfies

$$\Delta \bar{u}_\varepsilon + c\frac{\partial \bar{u}_\varepsilon}{\partial z} + \nabla_y \varphi \cdot \nabla_y \bar{u}_\varepsilon + f_\varepsilon(\bar{u}_\varepsilon) = 0, \quad \bar{u}_\varepsilon\big|_{\partial\Sigma_\pm} = 0, \quad \nu \cdot \nabla \bar{u}_\varepsilon\big|_{\partial\Sigma_0} = 0 \tag{1.4}$$

with speed $c = c_\varepsilon^\dagger$, which are minimizers of the functional

$$\Phi_c^\varepsilon[u] = \int_\Sigma e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla u|^2 + V_\varepsilon(u)\right) dx, \qquad V_\varepsilon(u) = -\int_0^u f_\varepsilon(s)\, ds. \tag{1.5}$$

where $u \in H_c^1(\Sigma)$ (see [22, Definition 2.1]). Moreover, for $\varepsilon \in (0,1)$ we have $df_\varepsilon(0)/du \equiv 0$, hence the equilibrium $u = 0$ is a stable solution of (1.1) and $\nu_0 \geq 0$, where

$$\nu_0 = \min_{\substack{\psi \in H^1(\Omega) \\ \psi|_{\partial\Omega_\pm} = 0}} R(\psi), \qquad R(\psi) = \frac{\int_\Omega e^{\varphi(y)} |\nabla_y \psi|^2 dy}{\int_\Omega e^{\varphi(y)} \psi^2 dy}. \tag{1.6}$$

So, by [22, Theorem 3.9] existence of solutions of (1.4) is guaranteed if and only if

$$\inf_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega_\pm} = 0}} E_\varepsilon[v] < 0, \tag{1.7}$$

where

$$E_\varepsilon[v] = \int_\Omega e^{\varphi(y)} \left(\frac{1}{2}|\nabla_y v|^2 + V_\varepsilon(v)\right) dy. \tag{1.8}$$

Also, by [22, Theorem 5.9], the speed $c_\varepsilon^\dagger$ is the propagation speed of the solutions u_ε of the initial value problem for (1.1) for a broad class of front-like initial data.

Rigorous studies of this type of problems go back to the works of Berestycki and the co-workers [5–7]. In particular, the question of the limit behavior of the traveling wave solutions of (1.1) as $\varepsilon \to 0$ in cylinders was first analyzed by Berestycki, Caffarelli and Nirenberg in [4], who proved convergence of the solutions of equations of the type of (1.4) on a sequence $\varepsilon_n \to 0$ to a solution of the free boundary problem of Sec. 3. The studies of the singular limit itself go back to the works of Alt and Caffarelli [2], and more recently to the works by Weiss [27] and Caffarelli, Jerison and Kenig [11] (see also [10, 25]). The approximations in (H4) for the singular reaction term were also used to establish existence of solutions of the free boundary problem in cylinders with Neumann boundary conditions in the presence of a shear flow [4].

Here we revisit the problem of existence of traveling wave solutions and propagation phenomena for (1.1) from a variational point of view by studying the minimizers of the functional in (1.5). As we showed in part I of this work [22], these solutions are of special significance for the propagation phenomena governed by (1.1) with front-like initial data and, therefore, allow us to give a sharp characterization of the propagation velocity (in the sense of the average velocity of the leading edge [21, 22]) for a wide class of initial data. Our variational method becomes especially powerful in the sharp reaction zone limit of (1.4), since one can pass to the limit directly in (1.5) to obtain a free boundary problem that has been investigated earlier for this kind of problems [2, 4, 25].

Using our variational approach, we are able to obtain existence, uniqueness, monotonicity, and asymptotic decay ahead of the front for the minimizers of the limiting functional, see Theorem 3.1. The results of Sec. 3 generalize the work of Alt and Caffarelli [2] to the case of transverse potential flows and infinite cylinders. The latter aspect of our analysis is novel and is related to what was done in [19, 22] to overcome the lack of compactness associated with the translational symmetry in the considered problem. We also obtained a novel variational formulation, a kind of an area functional, which gives an upper bound for the propagation speed of the traveling waves, see Theorem 4.2. Importantly, together with a suitable choice of a trial function, this novel variational formulation also provides a matching lower bound for the propagation speed in the limit of vanishing front curvature, Theorem 4.3, thus giving a rigorous justification to the Markstein model of a flame front [20]. This formulation for the case of an attached flame is also related to the functional introduced by Joulin [17].

We also prove convergence of the minimizers of the approximating problems in (1.4) to the minimizers of the limit problem. Our convergence analysis of Theorems 5.1 and 5.3 is a counterpart of that of Berestycki, Caffarelli and Nirenberg [4]. Note that our analysis treats a more general class of boundary conditions and a transverse potential flow, instead of a shear axial flow. Moreover, our limit is a full limit as $\varepsilon \to 0$ and not a limit on a sequence $\varepsilon_n \to 0$, as in [4]. We also obtained explicit estimates on the propagation speed for the regularized problem in terms of that of the free boundary. Let us point out that our approach is substantially different from that of [4], in view of our a priori existence results for the free boundary problem. Instead of constructing solutions of the free boundary problem as a limit of solutions of the approximating problems, we first establish existence of solutions in the singular limit and then show that the solutions of the approximating problems are close to the limit solution.

We conclude by demonstrating that in practice our variational formulations can give very good numerical estimates for the propagation speed in combustion problems, see the numerical results and the estimates in Sec. 6.

This paper is organized as follows. In Sec. 2, we introduce a modeling setup which leads to the singular limit arising in combustion problems. In Sec. 3 we prove existence of traveling wave solutions for the free boundary problem arising in the sharp reaction zone limit which are minimizers of Φ_c^0. In Sec. 4 we introduce the area-type functional and establish a number of results about the upper and lower bounds for the propagation speed. Then, in Sec. 5 we prove convergence of the minimizers for the regularizing approximations to the minimizer of the free boundary problem in the sharp reaction zone limit. Finally, in Sec. 6, we illustrate our findings with a few numerical examples. For notation and various auxiliary results, see part I of this work [22].

2 Model

In this section we give the physical motivation for a particular modeling setup that leads to the problem analyzed in this paper. We note that, although the derivation is done in the context of a thermodiffusional model of combustion [9,14,28], similar modeling is applicable in a wider context, in particular, in the case of chemical reactions in gel reactors (see, e.g., [8]).

Let us recall briefly the thermodiffusional model of laminar flames. Let $n = n(x,t)$ be the fuel concentration and $T = T(x,t)$ the temperature of the gas mixture. The governing equations of this model take the following form:

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n = D\Delta n - \bar{\nu}_0 n e^{-E_a/T}, \tag{2.1}$$

$$c\rho\left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = \kappa\Delta T + \bar{\nu}_0 n E e^{-E_a/T}. \tag{2.2}$$

Here, $\mathbf{v}$ is the velocity field of the imposed advective flow; D is the fuel diffusion coefficient; c, ρ, κ are the specific heat, density, and heat conductance of the mixture, respectively; $\bar{\nu}_0$ is the frequency parameter, E is the reaction heat, and E_a is the activation energy (we use energy units to measure temperature).

The portion $\partial\Sigma_+$ corresponds to the fuel inlet ($\mathbf{v} \cdot \nu|_{\partial\Omega_+} < 0$), hence the fuel concentration there will be high and temperature low; the portion $\partial\Sigma_-$ corresponds to the products outlet ($\mathbf{v} \cdot \nu|_{\partial\Omega_-} > 0$), hence the fuel concentration there will be low and temperature high:

$$T(y,z,t)\big|_{\partial\Sigma_\pm} = T_\pm(y), \qquad n(y,z,t)\big|_{\partial\Sigma_\pm} = n_\pm(y). \tag{2.3}$$

On the other hand, the portion $\partial\Sigma_0$ is impermeable ($\nu \cdot \mathbf{v}|_{\partial\Sigma_0} = 0$):

$$\nu \cdot \nabla T(y,z,t)\big|_{\partial\Sigma_0} = 0, \qquad \nu \cdot \nabla n(y,z,t)\big|_{\partial\Sigma_0} = 0. \tag{2.4}$$

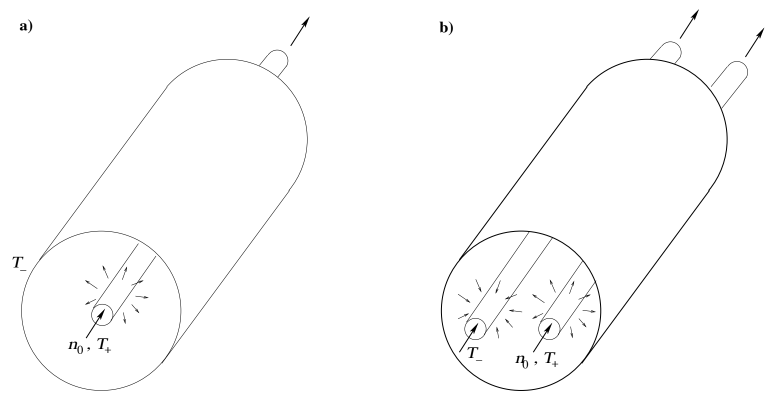

Figure 1: An illustration of a physical setup leading to (1.1).

A simple illustrative example of the system's geometry would be a pair of coaxial perforated pipes, then the reactor Σ is the space between the pipes, see Fig. 1(a). The interior pipe carries cold fuel, with (constant) temperature $T = T_+$ and fuel concentration $n = n_0$, which then enters the reactor, the products (or unburned fuel) escape through the wall of the outer pipe which is in contact with inert gas mixture on the outside, with $T = T_-$ and $n = 0$. Or consider an isolated pipe with two smaller pipes inside, one supplying the fuel and the second used as the exhaust, in this case there are no-flux boundary conditions on the outermost pipe, see Fig. 1(b). It is also natural to assume in such a setup that the advective flow is incompressible, in this case φ is harmonic in Ω. Let us note that the problems of ignition in a slit burner [24] and propagation of an edge flame [12,18,26] fall naturally within our framework.

After an appropriate rescaling (2.1) and (2.2) can be written in the following dimensionless form

$$n_t + \mathbf{v} \cdot \nabla n = \mathrm{Le}^{-1} \Delta n - n e^{-a/T}, \tag{2.5}$$

$$T_t + \mathbf{v} \cdot \nabla T = \Delta T + n e^{-a/T}, \tag{2.6}$$

where we introduced dimensionless parameters

$$\mathrm{Le} = \frac{\kappa}{c\rho D}, \qquad a = \frac{c\rho E_a}{E n_0}, \tag{2.7}$$

where n_0 is the characteristic fuel density. As usual, when $\mathrm{Le} = 1$, we can add up these equation to eliminate one of the variables. Denoting $w = T + n$ and assuming a steady state for w, we find that $w = w(y)$ and satisfies

$$\mathbf{v} \cdot \nabla w = \Delta w, \qquad w\big|_{\partial\Sigma_\pm} = T_\pm + n_\pm, \quad \nu \cdot \nabla w\big|_{\partial\Sigma_0} = 0. \tag{2.8}$$

Substituting the solution of this equation back into (2.5) and introducing u defined as

$$u(y, z, t) = T(y, z, t) - T_0(y), \tag{2.9}$$

where $T_0 = T_0(y)$ is a solution of

$$\Delta T_0 - \mathbf{v} \cdot \nabla T_0 + (w - T_0)e^{-a/T_0} = 0, \ T_0\big|_{\partial\Sigma_\pm} = T_\pm, \ \nu \cdot \nabla T_0\big|_{\partial\Sigma_0} = 0, \quad (2.10)$$

we obtain (1.1), where f is the (generally, y-dependent) combustion-type nonlinearity

$$f(u, y) = (w(y) - T_0(y) - u)e^{-a/(T_0(y)+u)} - (w(y) - T_0(y))e^{-a/T_0(y)}. \quad (2.11)$$

The sharp reaction zone limit of $\varepsilon \to 0$ in (1.1) arises due to the specific Arrhenius nature of the nonlinearity in combustion at large activation energy, in which the dimensionless parameter $a \gg 1$ [9, 14, 28]. To see this, assume first that the quantities in (2.8) are $T_+ = T_- = 0$ (for simplicity, we do not consider the effect of finite temperature of the cold gas) and $n_+ = n_- = 1$, so both the inlet and the exhaust are maintained at the same fuel concentrations. These boundary conditions immediately imply that $w = 1$ and $T_0 = 0$ in Ω (we define the Arrhenius factor to be zero at $u = 0$). So, $f(u) = (1 - u)e^{-a/u}$ and is independent of y.

To proceed, let us set $\varepsilon = a^{-1}$, so that $\varepsilon \ll 1$, and introduce

$$f_\varepsilon(u) = C\varepsilon^{-2}(1 - u)e^{-\varepsilon^{-1}(1-u)(1+(1-u)/u)}, \quad (2.12)$$

with

$$C = \frac{2\varepsilon^4}{\varepsilon(\varepsilon + 1) - (2\varepsilon + 1)e^{\frac{1}{\varepsilon}}\Gamma\left(0, \frac{1}{\varepsilon}\right)}, \qquad \lim_{\varepsilon\to 0} C(\varepsilon) = 1, \quad (2.13)$$

where $\Gamma(a, z)$ is the incomplete Gamma-function. With these definitions we have

$$\int_0^1 f_\varepsilon(u)du = 1. \quad (2.14)$$

We also note that $f_\varepsilon(u)$ is an extremely rapidly decaying function of u. Therefore, unless u is very close to 1, for realistic values of $\varepsilon \ll 1$ the value of $f_\varepsilon(u)$ may be so small that the physical assumptions used to derive (1.1) are no longer valid. This motivates a common physical approximation used in the combustion models to truncate the function f_ε at some $u = u^* < 1$, called the "ignition temperature", and set $f_\varepsilon(u) \equiv 0$ for all $u \in [0, u^*]$. To be consistent with (2.12) and (2.14), one chooses $1 - u^* = O(\varepsilon)$. We note that the latter assumption turns out to be rather crucial for the analysis of regularity of the solutions in the limit $\varepsilon \to 0$ [4]. Finally, let us do a rescaling

$$x \to a\sqrt{C}e^{a/2}\ x, \qquad t \to Ca^2e^a\ t, \quad (2.15)$$

which leads to (1.1) together with hypothesis (H4).

Naturally, the dimension of physical interest is $n = 3$. Also, a problem on a two-dimensional strip is of special physical importance (our numerical examples will be from this category). And, of course, the case of a purely reaction-diffusion equation ($\varphi = 0$) with either Dirichlet or Neumann boundary conditions is included in our formulation.

3 Singular limit

In this section, we consider the singular limit of (1.4) as $\varepsilon \to 0$, when f_ε approaches a delta-function concentrated at $u = 1$. We note that in general it is difficult to assign a meaning to (1.1) or (1.4) with such a singular nonlinearity. Nevertheless, due to the localized character of the reaction it is possible to give a satisfactory interpretation for these equations in terms of a free boundary problem in which the reaction zone is described as a surface (in the physical case of $n = 3$) separating the so-called "preheat zone" from the "products" in combustion terminology [2, 4, 25].

On the other hand, it is possible to pass to the limit of $f_\varepsilon(u) \to \delta(u - 1^-)$ directly in (1.5). Introducing

$$V_0(u) = \begin{cases} 0 & u < 1, \\ -1 & u \geq 1, \end{cases} \tag{3.1}$$

we see that $V_\varepsilon \to V_0$ pointwise as $\varepsilon \to 0$. We note that V_0 defined in this way is lower semicontinuous, making further variational analysis of the problem possible.

Replacing V_ε with V_0 in (1.5), we introduce the functional

$$\Phi_c^0[u] = \int_\Sigma e^{cz+\varphi(y)} \left(\frac{1}{2}|\nabla u|^2 + V_0(u) \right) dx. \tag{3.2}$$

The results of [22] motivate us to analyze the minimizers of (3.2). Our main result here is a characterization of the uniformly translating solutions of the free boundary problem associated with (1.1) with the minimizers of (3.2). As in [22, Theorem 3.9], existence of the minimizers can be established in terms of the auxiliary functional $E_0[v]$, defined for all $v \in H^1(\Omega)$ that vanish on $\partial\Omega_\pm$, which for $V = V_0$ can be written simply as

$$E_0[v] = \frac{1}{2} \int_{\{v<1\}} e^{\varphi(y)} |\nabla v|^2 \, dy - \int_{\{v \geq 1\}} e^{\varphi(y)} \, dy. \tag{3.3}$$

We point out that a functional of this kind has been considered in [2]. Later, in Sec. 5, we prove that these minimizers are in fact limits of the corresponding approximation problems in (1.4).

Here is our main result concerning the minimizers of Φ_c^0.

Theorem 3.1. *Let $n \leq 3$, and assume that there exists $c > 0$ and $u \in H_c^1(\Sigma)$, $u \not\equiv 0$, such that $\Phi_c^0[u] \leq 0$. Then:*

(i) There exists a unique constant $c^\dagger \geq c$ and $\bar{u} \in H^1_{c^\dagger}(\Sigma) \cap W^{1,\infty}(\overline{\Sigma})$, $\bar{u} \not\equiv 0$, such that $\bar{u}$ is a minimizer of $\Phi^0_{c^\dagger}$. Moreover, $0 < \bar{u} \leq 1$ in Σ, and if

$$\omega_- = \{x \in \Sigma : \bar{u}(x) < 1\}, \qquad \omega_+ = \{x \in \Sigma : \bar{u}(x) = 1\}, \tag{3.4}$$

then the set ω_+ is non-empty, and $\bar{u}$ is a classical solution of

$$\Delta \bar{u} + c^\dagger \bar{u}_z + \nabla_y \varphi \cdot \nabla \bar{u} = 0, \qquad \bar{u}\big|_{\partial\Sigma_\pm} = 0, \quad \nu \cdot \nabla \bar{u}\big|_{\partial\Sigma_0} = 0, \tag{3.5}$$

in ω_-.

(ii) *The function* $\bar{u}(y,z)$ *is unique (up to translations), is strictly decreasing in* z *in* ω_-*, and* $\lim_{z\to+\infty}\bar{u}(\cdot,z)=0$ *in* $C^1(\overline{\Omega})$*.*

(iii) $\bar{u}(y,z) = a_0\psi_0(y)e^{-\lambda_+(c^\dagger,\nu_0)z} + O(e^{-\lambda z})$*, with some* $a_0 > 0$ *and* $\lambda > \lambda_+(c^\dagger,\nu_0)$*, uniformly in* $C^1(\overline{\Omega}\times[R,+\infty))$ *as* $R\to+\infty$*, where* $\psi_0>0$ *is a minimizer of* R *in (1.6) and*

$$\lambda_+(c,\nu) = \frac{c+\sqrt{c^2+4\nu}}{2}. \tag{3.6}$$

(iv) *The free boundary* $\partial\omega_\pm = \partial\omega_-\cap\partial\omega_+$ *is bounded from the right and has regularity* $C^{1,\alpha}$*, for some* $\alpha>0$*. Moreover* $\bar{u}\in C^{1,\alpha}(\overline{\omega}_-)$*, and the following boundary condition holds:*

$$\bar{u}\,|_{\partial\omega_\pm} = 1, \qquad \nu\cdot\nabla\bar{u}\,|_{\partial\omega_\pm} = -\sqrt{2}, \tag{3.7}$$

where ν *is the normal to* $\partial\omega_\pm$ *pointing into* ω_-*.*

(v) $\lim_{z\to-\infty}\bar{u}(\cdot,z)=v$ *uniformly in* Ω*, where* $v>0$ *is a critical point of* E_0 *such that* $E_0[v]<0$*. In particular, letting* $\omega_R=\omega_+\cap\{z=R\}$ *and*

$$\omega_0 = \bigcup_{R\in\mathbb{R}} \omega_R \subseteq \Omega, \tag{3.8}$$

we have $\partial\omega_R\to\partial\omega_0$ *in the Hausdorff sense as* $R\to-\infty$*,* $\partial\omega_0$ *is of class* $C^{1,\alpha}$*, for some* $\alpha>0$*, and* $\partial\omega_\pm$ *is a graph of a function* $h\in BV_{\mathrm{loc}}(\omega_0)$*.*

Proof of Part (i)

Existence of a minimizer, uniqueness of $c^\dagger$, and the fact that $\bar{u}(x)\in[0,1]$ for all $x\in\Sigma$ follow from the same arguments as in [22, Theorem 3.3], Part (i). The only difference is in the proof of the inclusion $\bar{u}\in W^{1,\infty}(\overline{\Sigma})$. The latter follows from the fact that $\bar{u}$ is a minimizer of $\Phi^0_{c^\dagger}$, reasoning as in [2, Corollary 3.3] (see also [4, Theorem 3.1]).

Let us rewrite the functional Φ^0_c as

$$\Phi^0_c[\bar{u}] = \frac{1}{2}\int_{\omega_-} e^{cz+\varphi(y)}|\nabla\bar{u}|^2dx - \int_{\omega_+} e^{cz+\varphi(y)}dx. \tag{3.9}$$

In view of [22, Lemma 2.2], if $\omega_+=\varnothing$, then $\Phi^0_{c^\dagger}[\bar{u}]\geq 0$ and $\Phi^0_{c^\dagger}[\bar{u}]=0$ if and only if $\bar{u}=0$, contradicting the existence of a nontrivial minimizer. Hence $\omega_+\neq\varnothing$.

Since $V_0=0$ in ω_-, the Gateaux derivative $D_\phi\Phi^0_{c^\dagger}[\bar{u}]$ exists and is equal to zero for all test functions ϕ vanishing on ω_+. So $\bar{u}$ solves the Euler-Lagrange equation for $\Phi^0_{c^\dagger}$ with $V_0=0$ in ω_-, which is precisely equation (3.5). Then, by the Strong Maximum Principle, we have $\bar{u}>0$ in ω_- and, therefore, in the whole of Σ.

Proof of Part (ii)

Uniqueness, monotonicity and uniform decay of $\bar{u}$ follow reasoning as in [22, Theorem 3.3], Parts (ii) and (v). We only point out a few modifications of the arguments above. Letting $\bar{u}_1, \bar{u}_2$ be as in Part (ii) of [22, Theorem 3.3], for a given translation $a > 0$, we have $\Phi_{c^\dagger}[\bar{u}_1] = \Phi_{c^\dagger}[\bar{u}_2] = 0$, hence $\bar{u}_1, \bar{u}_2$ are non-trivial minimizers of $\Phi^0_{c^\dagger}$. It follows that the difference $w = \bar{u}_2 - \bar{u}_1 \geq 0$ solves the equation

$$\Delta w + c^\dagger w_z + \nabla_y \varphi \cdot \nabla_y w = 0 \qquad \text{in the set} \qquad \{\bar{u}_2 < 1\}. \tag{3.10}$$

It follows that either $w = 0$ or $w > 0$ in $\{\bar{u}_2 < 1\}$. The first possibility would imply that $\bar{u}$ is independent of z and, hence, is zero, which is impossible. So, $w > 0$, implying that $\bar{u}(y, z - a) > \bar{u}(y, z)$ for all (y, z) such that $\bar{u}_2(y, z) < 1$. In view of the arbitrariness of $a > 0$, this implies that $\bar{u}$ is strictly monotone decreasing in ω_-.

Similarly, let $\bar{u}_3$ and $\bar{u}_4$ be as in Part (v) of [22, Theorem 3.3], with $\bar{u}_3(x^*) = \bar{u}_4(x^*) < 1$. Then, since $w = \bar{u}_4 - \bar{u}_3$ satisfies (3.10) in the set $\{\bar{u}_4 < 1\}$, with $w \geq 0$ on the boundary of $\{\bar{u}_4 = 1\}$, from the Strong Maximum Principle we conclude that $w \equiv 0$ (hence also on the boundary). Then, by monotonicity of the minimizers, $\bar{u}_3$ and $\bar{u}_4$ coincide in all of Σ.

Proof of Part (iii)

This is just a particular case of [22, Theorem 3.3], Part (iii), with $f \equiv 0$ for z large enough. Notice that in this case $\lambda_+(c^\dagger, \nu_0) \geq c^\dagger$.

Proof of Part (iv)

Notice first that $\partial\omega_\pm$ is bounded from the right since $\bar{u}(\cdot, z) \to 0$ uniformly, by Part (ii). The fact that $\bar{u} \in C^{1,\alpha}(\overline{\omega}_-)$ and $\partial\omega_\pm$ is of class $C^{1,\alpha}$, for some $\alpha > 0$, follows from [2, 11]. Here we use that $n \leq 3$, since otherwise the boundary set $\partial\omega_\pm$ could contain singular points [13]. Indeed, reasoning as in the proof of [23, Theorem 1.2], we have that $\partial\omega_\pm$ is *uniformly* of class $C^{1,\alpha}$, in the sense that there exists $\rho, C > 0$ such that $\partial\omega_\pm \cap B_\rho(x)$ is contained in the graph (along some direction) of a function with $C^{1,\alpha}$-norm bounded by C, for all $x \in \partial\omega_\pm$.

The free boundary condition in (3.7) is also obtained in [2]. For reader's convenience we present the argument here. The first condition follows from the definition of $\partial\omega_\pm$ and the continuity of $\bar{u}$ established in Part (i). Let us prove the second condition in (3.7). Fix $\varepsilon > 0$ and a function $\rho \in C^{1,\alpha}(\partial\omega_\pm)$. We perturb $\partial\omega_\pm$ by displacing each point of $\partial\omega_\pm$ by $\varepsilon\rho \leq 0$ along ν, where ν is the normal to $\partial\omega_\pm$ pointing into ω_-. In order to preserve the first boundary condition in (3.7), we also perturb the function $\bar{u}$ by adding to it $\varepsilon\phi$, where the function ϕ satisfies the same boundary conditions as $\bar{u}$ on $\partial\Sigma$ and solves in ω_- the following boundary value problem:

$$\Delta\phi + c\phi_z + \nabla_y \varphi \cdot \nabla_y \phi = 0, \qquad \phi\Big|_{\partial\omega_\pm} = -(\nu \cdot \nabla\bar{u})\,\rho + o(\varepsilon). \tag{3.11}$$

The derivative of Φ_c^0 with respect to ε becomes

$$0 = \frac{d\Phi_c^0[\bar{u}]}{d\varepsilon} = -\frac{1}{2}\int_{\partial\omega_\pm} e^{cz+\varphi(y)}|\nabla\bar{u}|^2\rho\, d\mathcal{H}^{n-1}$$
$$- \int_{\partial\omega_\pm} e^{cz+\varphi(y)}\rho\, d\mathcal{H}^{n-1} + \int_{\omega_-} e^{cz+\varphi(y)}\nabla\bar{u}\cdot\nabla\phi\, dx. \tag{3.12}$$

Integrating by parts and noting that on $\partial\omega_\pm$ we have $\nu\cdot\nabla\bar{u} = -|\nabla\bar{u}|$, after some algebra we obtain

$$0 = \frac{d\Phi_c^0[\bar{u}]}{d\varepsilon} = \frac{1}{2}\int_{\partial\omega_\pm} e^{cz+\varphi(y)}\left(|\nabla\bar{u}|^2 - 2\right)\rho\, d\mathcal{H}^{n-1}. \tag{3.13}$$

Therefore, the following condition defines the location of the free boundary $\partial\omega_\pm$:

$$|\nabla\bar{u}| = \sqrt{2} \quad \text{on} \quad \partial\omega_\pm. \tag{3.14}$$

In view of the fact that $\bar{u}$ decreases along ν, we obtain the statement.

Proof of Part (v)

The existence of a function $v\in W^{1,\infty}(\Omega)$, such that $\lim_{z\to-\infty}\bar{u}(\cdot,z) = v > 0$ uniformly in Ω, follows from the Lipschitz continuity and monotonicity of $\bar{u}$, proved in Parts (i) and (ii) respectively. Notice also that $v\equiv 1$ in $\overline{\omega}_0$.

Since $\omega_R\subseteq\omega_0$ for all $R\in\mathbb{R}$ and $|\omega_0\setminus\omega_R|\to 0$, as $R\to-\infty$, the Hausdorff convergence of $\partial\omega_R$ to $\partial\omega_0$ follows from the fact that $\partial\omega_R$ are uniformly of class $C^{1,\alpha}$, independently of R, as stated in the proof of Part (iv). It then follows that $\partial\omega_0$ is also of class $C^{1,\alpha}$.

We now show that the function $\bar{v}(y,z) = v(y)$ is a minimizer for Φ_c^0 on Σ, with respect to perturbations with *bounded* support. Indeed, letting $a,b,R\in\mathbb{R}$, with $a<b$, the function $\bar{u}_R(y,z) = \bar{u}(y,z-R)$ is a minimizer for Φ_c^0 restricted to $\Sigma_{a,b} = \Omega\times(a,b)$, with respect to perturbations vanishing on $\partial\Sigma_{a,b}\setminus\big(\partial\Omega_0\times(a,b)\big)$. Since $\|\bar{u}_R - \bar{v}\|_{L^\infty(\Sigma_{a,b})}\to 0$ as $R\to-\infty$, it follows that $\bar{v}$ is also a minimizer for Φ_c^0 restricted to $\Sigma_{a,b}$, with respect to such perturbations. In particular, since $\bar{u}_R$ satisfies equation (3.5) in $\big(\Omega\setminus\overline{\omega}_0\big)\times\mathbb{R}$, we obtain that v solves the linear equation (3.15) in $\{v<1\}$. In particular $v\in(0,1)$ in $\Omega\setminus\overline{\omega}_0$, by Strong Maximum Principle. Moreover, arguing as in Part (iv), we get that v satisfies the boundary condition (3.16). Equations (3.15) and (3.16) imply that v is a critical point of E_0. The inequality $E_0[v]<0$ follows as in [22, Theorem 3.3], Part (ii).

Finally, since $\partial\omega_\pm$ has locally finite perimeter in Σ (being of class $C^{1,\alpha}$) and $\bar{u}$ is monotone in the z-direction, we have that $\partial\omega_\pm$ is a graph of a function $h\in BV_{\mathrm{loc}}(\omega_0)$. □

Notice that, while Theorem 3.1 covers the physically relevant case $n\le 3$, most of its statements can be extended to arbitrary dimensions. The only difficulty in $n\ge 4$ is the lack of complete regularity theory for the free boundary $\partial\omega_\pm$ [2, 13]. It is currently known that the free boundary is regular, out of possibly a closed singular set $S_\pm \subset \partial\omega_\pm$ of Hausdorff dimension at most $n-4$ [27]. We note that, since in our case, the free boundary is a graph in the z-direction, we expect that the singular set be always empty, independently of the dimension [11, 27].

Remark 3.2. *The set $\overline{\omega}_0$ in Part (v) of Theorem 3.1 is the set on which $v = 1$, and*

$$\Delta_y v + \nabla_y \varphi \cdot \nabla_y v = 0, \qquad v|_{\partial\Omega_\pm} = 0, \quad \nu \cdot \nabla_y v|_{\partial\Omega_0} = 0 \tag{3.15}$$

in $\Omega \backslash \overline{\omega}_0$, and the free boundary conditions

$$v|_{\partial\omega_0 \backslash \partial\Omega} = 1, \qquad \nu \cdot \nabla_y v|_{\partial\omega_0 \backslash \partial\Omega} = -\sqrt{2}, \tag{3.16}$$

where ν is the normal to $\partial\omega_0$ pointing outside ω_0.

Arguing as in [22, Theorem 3.9], we obtain the following necessary and sufficient condition for the considered problem to have minimizers:

Corollary 3.3. *Minimizers of Φ_c^0 exist if and only if*

$$\inf_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega_\pm} = 0}} E_0[v] < 0. \tag{3.17}$$

We note that, conversely, existence of a solution of the free boundary problem in Theorem 3.1 implies existence of minimizers of Φ_c^0. Indeed, if u_c is a solution of the free boundary problem, it satisfies (1.4) with f_ε set to zero in ω_-, and by the arguments at the beginning of Sec. 3 of [22], we have $u_c \in H_c^1(\Sigma)$. So, repeating the arguments in the proof of Theorem 3.1, we conclude that u_c is a critical point of Φ_c^0. This, in turn, implies that $\Phi_c^0[u_c] = 0$, and so u_c can be used as a trial function in the assumptions of Theorem 3.1. Thus, non-existence of minimizers in Corollary 3.3 implies non-existence of solutions of the free boundary problem as well.

Let us also point out that the statement in part (v) of Theorem 3.1 includes the possibility that the free boundary has "vertical" portions (i.e. those with $\nu \cdot \hat{z} = 0$). However, one would expect that generally $\partial\omega_\pm$ does not have any such portions and thus is a graph of a $C^{1,\alpha}_{\mathrm{loc}}(\omega_0)$ function. In fact, when $n = 2$, it is easy to see that the possibility of vertical portions in the form of intervals is excluded, since otherwise $\bar{u}$ becomes independent of z in ω_- over such portions, contradicting strict monotonicity of $\bar{u}$ there.

Note that in the case $\Sigma = \mathbb{R}$ we recover the classical result of combustion theory [9, 28]

$$\bar{u}(z) = \begin{cases} e^{-\sqrt{2}z}, & z > 0, \\ 1, & z \leq 0. \end{cases} \tag{3.18}$$

which is the minimizer with speed $c^\dagger = \sqrt{2}$. We note that by the same arguments as in [22, Proposition 3.4], this is also the minimizer in the case of purely Neumann boundary conditions (i.e. $\partial\Sigma_\pm = \varnothing$) and is the fastest variational traveling wave irrespectively of the choices of φ, Ω, and the boundary conditions.

Remark 3.4. *Using simple test functions, one can show that condition in (3.17) of Corollary 3.3 is satisfies whenever Ω contains a ball of radius R big enough.*

4 Area-type functional

Throughout this section we always assume that $n \geq 2$. We will now derive an area-type functional which can be used to obtain suitable bounds for the propagation speed of the minimizer. Integrating the first term in (3.9) by parts and the second term of (3.9) in z, and using (3.5) and (3.14), we obtain

$$\Phi^0_{c^\dagger}[\bar{u}] = \int_{\partial\omega_\pm} e^{c^\dagger z + \varphi(y)} \left(\frac{|\nabla \bar{u}|}{2} - \frac{\nu \cdot \hat{z}}{c^\dagger} \right) d\mathcal{H}^{n-1} = 0, \tag{4.1}$$

where the gradient is evaluated on the ω_- side of $\partial\omega_\pm$. Then, making use of the free boundary conditions (3.14), we find

$$\Pi_{c^\dagger}(\partial\omega_\pm) = \Phi^0_{c^\dagger}[\bar{u}] = 0, \tag{4.2}$$

where we introduced an area-type functional

$$\Pi_c(\partial\omega_\pm) = \int_{\partial\omega_\pm} e^{cz+\varphi(y)} \left(\frac{1}{\sqrt{2}} - \frac{\nu \cdot \hat{z}}{c} \right) d\mathcal{H}^{n-1}, \tag{4.3}$$

where $\hat{z}$ is the unit vector along the z-axis pointing to the right. It follows that, if the functional Φ^0_c has a minimizer, then $\inf \Pi_c \leq 0$ for all $0 < c \leq c^\dagger$. Therefore, if one can show that for some c we have $\Pi_c > 0$ for every surface contained in $\omega_0 \times \mathbb{R}$, then this automatically implies that $c^\dagger < c$.

Notice that the first term in (4.3) is an area term, whereas the second is a volume term, which is of lower order with respect to the first one. In particular, from the regularity theory for minimal surfaces (see [1, 16]), it follows that any minimizer of Π_c is smooth out of a closed singular set of zero $\mathcal{H}^{n-1}$-measure.

Before undertaking a more detailed analysis, let us make several general remarks regarding the functional Π_c. First, it is clear from (4.3) that $c^\dagger \leq \sqrt{2}$ independently of φ. Indeed, in (4.3) $\nu \cdot \hat{z} \leq 1$ so the integrand is strictly positive for all $c > \sqrt{2}$. On the other hand, the upper bound $c = \sqrt{2}$ is achieved only if the front is planar, hence, only when $\partial\Sigma_\pm = \varnothing$. In this case $\bar{u}$ depends only on the z-variable and is given explicitly by (3.18).

We now proceed with the analysis of (4.3). For $\zeta \in BV(\omega_0)$, we define

$$\Xi_c[\zeta] = \int_{\omega_0} e^{\varphi(y)} \left(\sqrt{\frac{c^2\zeta^2 + |\nabla_y \zeta|^2}{2}} - \zeta \right) dy. \tag{4.4}$$

Notice that there is a standard way to define the functional (4.4) on the whole of $BV(\omega_0)$ (see [3, Section 5.5]), as the lower semicontinuous relaxation of the same functional restricted to $H^1(\Omega)$, with Dirichlet boundary conditions on $\partial\omega_0\setminus\partial\Omega_0$. In particular, the functional Ξ_c takes into account possible jumps of ζ inside ω_0 and on $\partial\omega_0 \setminus \partial\Omega_0$.

A simple calculation shows that, if $\zeta > 0$, we have

$$\Pi_c(\Gamma_{\frac{1}{c}\log\zeta}) = \frac{1}{c}\Xi_c[\zeta], \tag{4.5}$$

where Γ_h denotes the graph $z = h(y)$ for any $h \in BV(\omega_0)$. In fact, there is a one-to-one correspondence between the functions on which Ξ_c is defined and the hypersurfaces in the domain of definition of Π_c. Therefore, in the following we will be using these two area-type functionals interchangeably.

Notice that Ξ_c is a one-homogeneous, convex, lower semicontinuous functional on $BV(\omega_0)$. Moreover, its gradient term corresponds to an anisotropic perimeter of the subgraph of ζ. Reasoning as in [16], it is possible to prove that $\bar\zeta$ is (locally) of class $C^{2,\alpha}$ in the open set where $\bar\zeta > 0$. We observe that any minimizer $\bar\zeta$ may be discontinuous (and jump to zero) on the boundary of such set.

The above arguments apply when the minimizer $\bar\zeta$ exists, this, of course, may not happen for all $c > 0$. In fact, the following statement holds.

Proposition 4.1. *Assume that* (3.17) *holds. Then, there exists a unique value of $c = c^\sharp$, for which Ξ_c admits a non-trivial minimizer $\bar\zeta \in BV(\omega_0)$, with $\bar\zeta \geq 0$ in ω_0. Furthermore, $\Xi_{c^\sharp}[\bar\zeta] = 0$.*

Proof. To construct a minimizer of Ξ_c, we consider the following constrained variational problem

$$\text{minimize } \Xi_c[\zeta] \qquad \text{subject to} \qquad \zeta \geq 0\,, \quad \int_{\omega_0} e^{\varphi(y)}\zeta\, dy = 1. \tag{4.6}$$

Indeed, letting ζ_n be a minimizing sequence, we have $\|\zeta_n\|_{BV(\omega_0)} \leq C$ for some $C > 0$, hence there exists a function ζ_c such that, up to a subsequence, $\zeta_n \rightharpoonup \zeta_c$ weakly in $BV(\omega_0)$. In particular $\zeta_n \to \zeta_c$ strongly in $L^1(\omega_0)$, and so ζ_c satisfies the constraints. Since Ξ_c is a lower-semicontinuous functional on $BV(\omega_0)$ [3], we get that ζ_c is a minimizer for the problem.

For shorthand we set $\mu_c = \Xi_c[\zeta_c]$. Theorem 3.1, (4.2) and (4.5) imply that

$$\inf\left\{\Xi_{c^\dagger}[\zeta] :\ \zeta \in BV(\omega_0),\ \zeta \geq 0\right\} \leq 0, \tag{4.7}$$

hence $\mu_{c^\dagger} \leq 0$. Moreover, from the discussion preceding (4.5), we have $\mu_c > 0$ for all $c > \sqrt{2}$. Furthermore, μ_c is an increasing function of c, hence $\mu_c < 0$, for all $c \in [0, c^\dagger)$. Indeed, $\Xi_{c'}[\zeta_c] < \Xi_c[\zeta_c]$ for any $0 \leq c' < c$, due to the monotonicity of the integrand in (4.4). Also, since $\zeta_{c'}$ is a minimizer of $\Xi_{c'}$, we have

$$\mu_{c'} = \Xi_{c'}[\zeta_{c'}] \leq \Xi_{c'}[\zeta_c] < \Xi_c[\zeta_c] = \mu_c. \tag{4.8}$$

Furthermore, by Mean Value Theorem applied pointwise to the integrand, with some $\tilde{c}(y) \in (c', c)$, we obtain

$$\mu_{c'} - \mu_c \geq -\frac{c - c'}{\sqrt{2}} \int_{\omega_0} e^{\varphi(y)} \frac{\tilde{c}\zeta_{c'}^2}{\sqrt{\tilde{c}^2\zeta_{c'}^2 + |\nabla_y \zeta_{c'}|^2}} dy \geq -\frac{c - c'}{\sqrt{2}}. \tag{4.9}$$

So, $c \mapsto \mu_c$ is continuous, and hence there exists a unique value of $c = c^\sharp$ such that $\mu_{c^\sharp} = 0$.

We now claim that $\bar{\zeta} = \zeta_{c^\sharp}$ is a minimizer of $\Xi_{c^\sharp}$ in the absence of the constraint. This follows from the fact that, for all $\zeta \in BV(\omega_0), \zeta \geq 0, \zeta \not\equiv 0$, we have

$$\Xi_c[\zeta] = a\Xi_c[\zeta/a], \qquad a = \int_{\omega_0} e^{\varphi(y)} \zeta \, dy > 0. \tag{4.10}$$

Hence, $\Xi_{c^\sharp} \geq 0$, and $\bar{\zeta}$ is a global minimizer of $\Xi_{c^\sharp}$. Moreover, if $c < c^\sharp$, then by (4.10) $\inf \Xi_c \leq a\mu_c \to -\infty$ as $a \to \infty$, and so the minimizer of Ξ_c does not exist. If, on the other hand, $c > c^\sharp$, then $\Xi_c[\zeta] = a\Xi_c[\zeta/a] \geq a\mu_c$, so that the only minimizer is the trivial one. □

We note that in general the support of $\bar{\zeta}$ (or even ω_0) does not have to be connected. However, on all connected portions of $\mathrm{supp}(\bar{\zeta})$ the functional $\Xi_{c^\sharp}$ must evaluate to zero, since otherwise it can be decreased by setting $\bar{\zeta}$ to zero in the one where it is positive. But this means that one can always choose a minimizer $\bar{\zeta}$ whose support is connected.

Let us now summarize the arguments leading from (4.3) and (4.5) to Proposition 4.1 in the following result:

Proposition 4.2. *Under the assumptions of Theorem 3.1, we have*

$$c^\dagger \leq c^\sharp, \tag{4.11}$$

where $c^\sharp$ is defined in Proposition 4.1.

Note that in the absence of information about the minimizers of E_0 it is still possible to use the functional Ξ_c to obtain a sufficient condition for non-existence of minimizers for Φ_c^0. Allowing the domain of the functions ζ to be the whole of Ω, we obtain that the condition

$$\inf_{\substack{\zeta \in BV(\Omega) \\ \zeta \geq 0}} \Xi_0[\zeta] = 0, \tag{4.12}$$

guarantees non-existence of minimizers for Φ_c^0 with any $c > 0$ in view of the monotonicity of Ξ_c with respect to Ω.

Let us point out that the minimizers of Ξ_c or, equivalently, of Π_c satisfy the Euler-Lagrange equation which reduces to the classical Markstein model of the dynamics of flame fronts [20]. This fact, for a thin flame in a potential flow was first noticed by Joulin [17], who introduced a functional which is essentially equivalent to Π_c. To see this, let us compute the first variation of $\Pi_c(\Gamma)$ with respect to infinitesimal displacements $\delta\rho$ of Γ along the unit normal vector ν pointing to the right. After simple manipulations, we arrive at

$$\delta\Pi_c(\Gamma) = \frac{1}{\sqrt{2}} \int_\Gamma e^{cz+\varphi(y)} \left(c\nu \cdot \hat{z} + \nu \cdot \nabla_y \varphi + (n-1)H - \sqrt{2} \right) \delta\rho \, d\mathcal{H}^{n-1}, \quad (4.13)$$

where H is the mean curvature of Γ, positive if Γ is convex towards $z = -\infty$. Therefore, if Γ is a minimizer of $\Pi_{c^\sharp}$, it satisfies the Euler-Lagrange equation

$$\nu \cdot (c^\sharp \hat{z} + \nabla_y \varphi) = \sqrt{2} - (n-1)H. \quad (4.14)$$

This is precisely the steady version of the Markstein equation, once it is realized that the term on the left is the normal velocity of the front relative to the flow. So, what we proved in Proposition 4.2 gives a rigorous justification for the Markstein model as giving a strict upper bound for the propagation speed of the flame front in the considered setup. On the other hand, the physical assumptions behind the Markstein model rely on the smallness of the front curvature and the flow variation compared to the width of the preheat zone [20]. Under this assumption, we can show that the minimizers of Π_c or, equivalently, of Ξ_c also give a matching *lower* bound for $c^\dagger$ which coincides with $c^\sharp$ in the limit of vanishing front curvature and advection velocity gradient. For clarity, we demonstrate this point under a simplifying assumption on the geometry of Ω.

Proposition 4.3. *Assume Theorem 3.1 holds, and, in addition, that* $\bar{\Gamma} = \Gamma_{\frac{1}{c}\log\bar{\zeta}}$*, where* $\bar{\zeta}$ *is a minimizer of* Ξ_c *obtained in Proposition 4.1, has all principal curvatures bounded by* $\varepsilon > 0$*, that* $\operatorname{dist}(\omega_0, \partial\Omega) = O(\varepsilon^{-1})$*, and that* $|(\nabla_y \otimes \nabla_y)\varphi| \leq M\varepsilon$ *for some* $M > 0$*. Then*

$$c^\dagger \geq c^\sharp - C_1\varepsilon^2 - C_2 M\varepsilon, \quad (4.15)$$

for some $C_{1,2} > 0$ *independent of* ε*, when* ε *is small enough.*

Proof. We prove this statement by constructing an appropriate trial function for Φ_c^0 from $\bar{\Gamma}$, based on the one-dimensional minimizer, see (3.18). Introduce the signed distance function

$$d(x) = \pm\operatorname{dist}(x, \bar{\Gamma}), \quad (4.16)$$

which is positive if x is to the right of $\bar{\Gamma}$ and negative otherwise. We can then define a trial function

$$u(x) = \begin{cases} 1, & d(x) \leq 0, \\ e^{-\sqrt{2}d(x)}, & 0 < d(x) < d_0 - 1, \\ e^{-\sqrt{2}(d_0-1)}(d_0 - d(x)), & d_0 - 1 \leq d(x) < d_0, \\ 0, & d(x) \geq d_0, \end{cases} \quad (4.17)$$

where we introduced a constant d_0 such that $1 < d_0 < \mathrm{dist}(\omega_0, \partial\Omega)$. Clearly, u lies in $H^1_c(\Sigma)$ and satisfies the boundary conditions on $\partial\Sigma$.

Let us now evaluate $\Phi^0_c[u]$ for some $c < c^\sharp$. To proceed, observe that the second term in (3.2) coincides with the second term in (4.3):

$$\int_\Sigma e^{cz+\varphi(y)} V_0(u)dx = -\frac{1}{c}\int_{\bar\Gamma} e^{cz+\varphi(y)} \nu \cdot \hat z \, d\mathcal{H}^{n-1}, \tag{4.18}$$

where, as before, ν is the unit normal vector to $\bar\Gamma$ pointing to the right. So, it remains to evaluate the first integral in (3.2). Let us write this integral in curvilinear coordinates associated with $\bar\Gamma$, which is justified when $d_0 \le \varepsilon^{-1}$:

$$\int_\Sigma e^{cz+\varphi(y)} |\nabla u|^2 dx = \int_{\bar\Gamma}\int_0^{d_0} e^{cz+\varphi(y)} |\nabla u(x)|^2 \left(\prod_{i=1}^{n-1} (1 + k_i(x')\rho) \right) d\rho \, d\mathcal{H}^{n-1}(x'). \tag{4.19}$$

Here k_i are the principal curvatures on $\bar\Gamma$, assumed to be positive if the set enclosed by $\bar\Gamma$ (i.e. the set on the left of $\bar\Gamma$) is convex, and x' is the projection of x on $\bar\Gamma$, so that $x = x' + \rho\nu(x')$. Now we estimate

$$\begin{aligned}
&\int_\Sigma e^{cz+\varphi(y)} |\nabla u|^2 dx \\
&\quad \le \int_{\bar\Gamma}\int_0^{d_0} e^{cz+\varphi(y)} |\nabla u(x)|^2 (1 + (n-1)H(x')\rho + C\varepsilon^2\rho^2) \, d\rho \, d\mathcal{H}^{n-1}(x') \\
&\quad \le \int_{\bar\Gamma} e^{cz'+\varphi(y')} \int_0^{d_0} e^{c\rho\nu\cdot\hat z + \rho\nu\cdot\nabla\varphi(y') + \varepsilon C M \rho^2} \\
&\qquad\qquad \times |\nabla u(x)|^2 (1 + (n-1)H(x')\rho + C\varepsilon^2\rho^2) \, d\rho \, d\mathcal{H}^{n-1}(x') \\
&\quad \le \int_{\bar\Gamma} e^{cz'+\varphi(y')} \int_0^{d_0} e^{\sqrt{2}\rho + \varepsilon C M \rho^2} |\nabla u(x)|^2 (1 + C\varepsilon^2\rho^2) \, d\rho \, d\mathcal{H}^{n-1}(x'),
\end{aligned} \tag{4.20}$$

where $H(x')$ is the mean curvature of $\bar\Gamma$ at x', and C denotes a generic positive constant. In writing the last line in the estimate above we took into account the Euler-Lagrange equation (4.14) for $\bar\Gamma$ and the fact that $e^{(c-c^\sharp)\rho\nu\cdot\hat z} \le 1$. Substituting the expression for u from (4.17) and choosing $d_0 = K \log \varepsilon^{-1}$, with

$K > 0$ sufficiently large, we get

$$\begin{aligned}
&\int_\Sigma e^{cz+\varphi(y)}|\nabla u|^2 dx \\
&\quad \le 2\int_{\bar\Gamma} e^{cz'+\varphi(y')}\int_0^{d_0-1} e^{-\sqrt{2}\rho+\varepsilon CM\rho^2}(1+C\varepsilon^2\rho^2)\,d\rho\,d\mathcal{H}^{n-1}(x') \\
&\quad + \int_{\bar\Gamma} e^{cz'+\varphi(y')}\int_{d_0-1}^{d_0} e^{\sqrt{2}(2-d_0)+\varepsilon CMd_0^2}(1+C\varepsilon^2 d_0^2)\,d\rho\,d\mathcal{H}^{n-1}(x') \\
&\quad \le C\varepsilon^2\int_{\bar\Gamma} e^{cz'+\varphi(y')}d\mathcal{H}^{n-1}(x') \qquad\qquad (4.21)\\
&\quad +2\int_{\bar\Gamma} e^{cz'+\varphi(y')}\int_0^{d_0-1} e^{-\sqrt{2}\rho+\varepsilon CM\rho^2}\,d\rho\,d\mathcal{H}^{n-1}(x') \\
&\quad \le (C_1\varepsilon^2+C_2M\varepsilon)\int_{\bar\Gamma} e^{cz'+\varphi(y')}d\mathcal{H}^{n-1}(x') \\
&\quad +2\int_{\bar\Gamma} e^{cz'+\varphi(y')}\int_0^{\infty} e^{-\sqrt{2}\rho}\,d\rho\,d\mathcal{H}^{n-1}(x').
\end{aligned}$$

Integrating the last term with respect to ρ, we finally obtain

$$\int_\Sigma e^{cz+\varphi(y)}|\nabla u|^2 dx \le \sqrt{2}(1+C_1\varepsilon^2+C_2M\varepsilon)\int_{\bar\Gamma} e^{cz+\varphi(y)}d\mathcal{H}^{n-1}. \qquad (4.22)$$

Now, observe that from (4.5), Proposition 4.1, and the monotonicity of Ξ_c with respect to c it follows that

$$\frac{1}{\sqrt{2}}\int_{\bar\Gamma} e^{cz+\varphi(y)}d\mathcal{H}^{n-1} \le \frac{1}{c^\sharp}\int_{\bar\Gamma} e^{cz+\varphi(y)}\nu\cdot\hat z\,d\mathcal{H}^{n-1}, \qquad (4.23)$$

for all $0 < c \le c^\sharp$. Combining this with the estimate in (4.22), we have

$$\begin{aligned}
\Phi_c^0[u] &= \frac{1}{2}\int_\Sigma e^{cz+\varphi(y)}|\nabla u|^2 dx - \frac{1}{c}\int_{\bar\Gamma} e^{cz+\varphi(y)}\nu\cdot\hat z\,d\mathcal{H}^{n-1} \\
&\le \frac{c-c^\sharp+C_3\varepsilon^2+C_4M\varepsilon}{c\sqrt{2}}\int_{\bar\Gamma} e^{cz+\varphi(y)}d\mathcal{H}^{n-1} = 0, \qquad (4.24)
\end{aligned}$$

if $c = c^\sharp - C_3\varepsilon^2 - C_4M\varepsilon$. Therefore, u satisfies the assumptions of Theorem 3.1, and so there exists a minimizer with speed $c^\dagger \ge c$. □

Let us make a few remarks about the result in Proposition 4.3 before concluding this section. The main assumption of Proposition 4.3 was that of uniform smallness of the curvature of $\bar\Gamma$, which is at the heart of the idea of approximating the free boundary $\partial\omega_\pm$ of the minimizer of Φ_c^0 with that of Π_c and is, therefore, essential here. We note that the assumption $\mathrm{dist}(\omega_0, \partial\Omega) = O(\varepsilon^{-1})$ does not contradict the assumption on the curvature, and may even be replaced by the weaker assumption $\mathrm{dist}(\omega_0, \partial\Omega) \gg \log\varepsilon^{-1}$ (see also Sec. 6). On the other hand, as follows from evaluating (4.14) at a point where $\nu = \hat z$, if the curvature of $\bar{\Gamma}$ is uniformly $O(\varepsilon)$, then the speed $c^\sharp$ has an estimate $c^\sharp = \sqrt{2}+O(\varepsilon)$ as well. But since $c^\sharp - c^\dagger = O(\varepsilon^2)+O(M\varepsilon)$, the speed $c^\sharp$ of the minimizer of Π_c captures, as it should, the leading order curvature corrections to $c^\dagger$ and so should give a good approximation for the propagation velocity $c^\dagger$ in practice. On the other hand, if $\bar{\Gamma}$ is allowed to approach $\partial\Sigma_\pm$ (the cold boundaries), then the curvature will not be uniformly bounded near the boundary, and the propagation speed can have an $O(1)$ difference from $c = \sqrt{2}$, the speed of the planar front, or propagation failure may occur altogether.

We also note, that if $|\nabla_y \varphi| \ll 1$, we get into the situation of a weakly perturbed planar front. Assuming that $|\nabla_y \zeta| \ll \zeta$ and $\sqrt{2} - c \ll 1$, Taylor-expanding (4.4) in $|\nabla_y \zeta|$, and introducing $\psi = \sqrt{\zeta}$, we obtain (up to an additive constant)

$$\Xi_c[\psi^2] = \int_{\omega_0} e^{\varphi(y)} \left(|\nabla_y \psi|^2 - \frac{\sqrt{2}-c}{\sqrt{2}} \psi^2 \right) dy + \text{h.o.t.} \tag{4.25}$$

Thus, in this situation finding $c^\sharp$ amounts to computing the smallest eigenvalue of the Schrödinger-type operator generated by (4.25), which is easier to study than the minimizers of (4.4).

5 Approximating problems

Now we study the question of how well the minimizers of (3.2) approximate the minimizers of (1.5) in the limit $\varepsilon \to 0$. Our main results in this section are the estimates for the wave velocity $c^\dagger_\varepsilon$ of the approximating problem in terms of the speed $c^\dagger_0$ of the limit free boundary problem and strong convergence of the (appropriately translated) minimizers of the approximating problem to the minimizer of the limit problem.

Observe that by definition

$$V_\varepsilon(u) \le V_0(u), \qquad \lim_{\varepsilon\to 0} V_\varepsilon(u) = V_0(u), \qquad \forall u \in \mathbb{R}. \tag{5.1}$$

Therefore, under the assumption of existence of minimizers for the limit functional Φ^0_c in (3.2) existence of minimizers for Φ^ε_c is guaranteed for all $\varepsilon < 1$. Indeed, by Corollary 3.3, we have $\inf E_0[v] < 0$, and, from the first inequality in (5.1), that $\inf E_\varepsilon[v] < 0$ as well. So, by [22, Theorem 3.9] the minimizer $\bar{u}_\varepsilon$ of Φ^ε_c exists and has all the properties guaranteed by [22, Theorem 3.3].

We now show that the speed $c^\dagger_0$ is in fact the limiting speed of the minimizers $\bar{u}_\varepsilon$ for the approximating problems.

Theorem 5.1. *Under the assumption of Corollary 3.3, let $c^\dagger_0$ and $c^\dagger_\varepsilon$ be the speeds of the minimizers of Φ^0_c and Φ^ε_c, respectively. Then we have*

$$c^\dagger_0 \le c^\dagger_\varepsilon \le c^\dagger_0 + \frac{32\varepsilon}{c^\dagger_0}, \qquad 0 < \varepsilon < \frac{1}{2}. \tag{5.2}$$

Proof. Since $V_\varepsilon \leq V_0$, we immediately conclude the lower bound in (5.2). Let us now prove the upper bound. It is easy to see that by the assumptions on f_ε we have $V_\varepsilon(u) \geq V_0\left(\frac{u}{1-\varepsilon}\right)$. Let us introduce $\tilde{u}(y,z) = \frac{1}{1-\varepsilon} u\left(y, \frac{c_0^\dagger}{c} z\right)$. Then, clearly for any $u \in H^1_c(\Sigma)$ we have $\tilde{u} \in H^1_{c_0^\dagger}(\Sigma)$. So, evaluating Φ_c^ε on u, we get

$$\Phi_c^\varepsilon[u] \geq \int_\Sigma e^{cz+\varphi(y)} \left\{ \frac{u_z^2}{2} + \frac{|\nabla_y u|^2}{2} + V_0\left(\frac{u}{1-\varepsilon}\right) \right\} dx$$

$$\geq \left(\frac{c_0^\dagger}{c}\right) \int_\Sigma e^{c_0^\dagger z + \varphi(y)} \left\{ (1-\varepsilon)^2 \frac{|\nabla_y \tilde{u}|^2}{2} + (1-\varepsilon)^2 \left(\frac{c}{c_0^\dagger}\right)^2 \frac{u_z^2}{2} + V_0(\tilde{u}) \right\} dx$$

$$\geq (1-\varepsilon)^2 \left(\frac{c_0^\dagger}{c}\right) \left(\Phi^0_{c_0^\dagger}[\tilde{u}] \right.$$

$$\left. + \int_\Sigma e^{c_0^\dagger z+\varphi(y)} \left\{ \frac{c^2 - {c_0^\dagger}^2}{2{c_0^\dagger}^2} \tilde{u}_z^2 + \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2} V_0(\tilde{u}) \right\} dx \right). \quad (5.3)$$

Now, using the fact that $V_0(u) \geq -u^2$, the Poincaré inequality from [22, Lemma 2.2], and that by definition of $c_0^\dagger$ we have $\Phi^0_{c_0^\dagger}[\tilde{u}] \geq 0$ for all $\tilde{u} \in H^1_{c_0^\dagger}(\Sigma)$, we can proceed to estimate the last line in the inequality above as

$$\Phi_c^\varepsilon[u] \geq (1-\varepsilon)^2 \left(\frac{c_0^\dagger}{c}\right) \int_\Sigma e^{c_0^\dagger z+\varphi(y)} \left(\frac{c^2 - {c_0^\dagger}^2}{8} - \frac{2\varepsilon}{(1-\varepsilon)^2} \right) \tilde{u}^2 \, dx. \quad (5.4)$$

Then, from this inequality it follows that $\Phi_c^\varepsilon[u] \geq \delta \int_\Sigma e^{cz+\varphi(y)} u^2 \, dx$, with some $\delta > 0$, if $\varepsilon < \frac{1}{2}$ and $c > c_0^\dagger + \frac{32\varepsilon}{c_0^\dagger}$, so only trivial minimizers exist for these values of c. In view of this, we have the second inequality in (5.2).

□

Let us recall the following uniform gradient estimate for both the minimizers $\bar{u}_\varepsilon$ of Φ_c^ε and the minimizer $\bar{u}_0$ of the limit functional Φ_c^0, which were obtained in [4, Theorem 3.1] (see also [10, Chapter 1]).

Proposition 5.2. *There exists a constant $C > 0$ independent of ε such that*

$$\|\bar{u}_\varepsilon\|_{W^{1,\infty}(\Sigma)} \leq C, \qquad \|\bar{u}_0\|_{W^{1,\infty}(\Sigma)} \leq C. \quad (5.5)$$

With the help of these estimates, we are now ready to prove our convergence result for the sequence of minimizers Φ_c^ε of the approximating problem to a minimizer of Φ_c^0.

Theorem 5.3. *There exists a translation $a_\varepsilon \in \mathbb{R}$, such that if $u_\varepsilon(y,z) = \bar{u}_\varepsilon(y, z - a_\varepsilon)$, then we have*

$$u_\varepsilon \to u_0 \in C^{0,1}(\overline{\Sigma}) \quad \text{in} \quad H^1_{c_0^\dagger}(\Sigma), \quad (5.6)$$

as $\varepsilon \to 0$, and

$$\Phi^0_{c_0^\dagger}[u_0] = 0, \qquad u_0 \not\equiv 0. \tag{5.7}$$

The convergence is also uniform on compact subsets of $\overline{\Sigma}$.

Proof. Let $\varepsilon < \frac{1}{2}$, and observe that we have $\sup_{x\in\Sigma} \bar{u}_\varepsilon(x) > \frac{1}{2}$, since otherwise $V_\varepsilon(\bar{u}_\varepsilon) \equiv 0$ and so $\Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] > 0$, which contradicts the fact that u_ε is a minimizer of $\Phi^\varepsilon_{c_\varepsilon^\dagger}$. Recalling also that $\bar{u}_\varepsilon(\cdot, z) \to 0$ uniformly as $z \to +\infty$, we can choose $a_\varepsilon \in \mathbb{R}$ such that, by letting $u_\varepsilon(y,z) = \bar{u}_\varepsilon(y, z - a_\varepsilon)$, we have

$$\max_{y\in\overline{\Omega}} u_\varepsilon(y, 0) = \frac{1}{2} \qquad \text{and} \qquad u_\varepsilon(y,z) \le \frac{1}{2}, \quad \forall (y,z) \in \Omega \times [0, +\infty). \tag{5.8}$$

Now, notice that by Proposition 5.2 and the Arzelà-Ascoli Theorem the functions u_ε converge (on a sequence of $\varepsilon \to 0$) uniformly on compact subsets of $\overline{\Sigma}$ to a function $u_0 \in C^{0,1}(\overline{\Sigma})$, which satisfies (5.8) (hence, in particular, $u_0 \not\equiv 0$). Moreover, from [22, Lemma 2.3] and Proposition 5.1 we know that $c_\varepsilon^\dagger \ge c_0^\dagger$ and

$$u_\varepsilon \in H^1_{c_0^\dagger}(\Sigma), \qquad \forall \varepsilon > 0. \tag{5.9}$$

Let us show that u_ε are also equibounded in $H^1_{c_\varepsilon^\dagger}(\Sigma)$, and, as a consequence, in $H^1_{c_0^\dagger}(\Sigma)$ as well. Thanks to [22, Lemma 2.2], it is enough to prove that

$$\int_\Sigma e^{c_\varepsilon^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 \, dx \le C, \tag{5.10}$$

for some constant $C > 0$ independent of ε. Since $V_\varepsilon(u) \ge -1$ for all u and, by construction, $V_\varepsilon(u_\varepsilon(\cdot, z)) = 0$ for all $z > 0$, we have

$$\begin{aligned} 0 = \Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] &= \frac{1}{2}\int_\Sigma e^{c_\varepsilon^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 \, dx + \int_{-\infty}^0 \int_\Omega e^{c_\varepsilon^\dagger z + \varphi(y)} V_\varepsilon(u_\varepsilon) \, dy dz \\ &\ge \frac{1}{2}\int_\Sigma e^{c_\varepsilon^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 \, dx - \frac{M|\Omega|}{c_\varepsilon^\dagger}, \end{aligned} \tag{5.11}$$

for some $M > 0$, which proves the inequality in (5.10) with $C = 2M|\Omega|/c_0^\dagger$. Now, to pass to the norm in $H^1_{c_0^\dagger}(\Sigma)$, we observe that

$$\begin{aligned} \int_\Sigma e^{c_0^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 dx &\le \int_0^\infty \int_\Omega e^{c_\varepsilon^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 dy dz \\ &+ \int_{-\infty}^0 \int_\Omega e^{c_0^\dagger z + \varphi(y)} |\nabla u_\varepsilon|^2 dy dz \le C + \frac{M|\Omega|}{c_0^\dagger} ||\, |\nabla u_\varepsilon|^2 ||_{L^\infty(\Sigma)}. \end{aligned} \tag{5.12}$$

In view of the result in Proposition 5.2, we conclude that u_ε are equibounded in $H^1_{c_0^\dagger}(\Sigma)$ also. So, it follows that $u_\varepsilon \rightharpoonup u_0$ also weakly in $H^1_{c_0^\dagger}(\Sigma)$.

Let us now prove that

$$\Phi^0_{c_0^\dagger}[u_0] = 0. \tag{5.13}$$

Since we already know that $\Phi^0_{c_0^\dagger}[u] \geq 0$ for all $u \in H^1_{c_0^\dagger}(\Sigma)$, in order to obtain (5.13) it is enough to prove that

$$\Phi^0_{c_0^\dagger}[u_0] \leq \lim_{\varepsilon \to 0} \Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] = 0. \tag{5.14}$$

Recalling (5.8), for $\varepsilon < \frac{1}{2}$ we can write

$$\begin{aligned} 0 = \Phi^\varepsilon_{c_\varepsilon^\dagger}[u_\varepsilon] \quad \geq \quad & \int_{-\infty}^0 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{(c_\varepsilon^\dagger - c_0^\dagger)z} e^{c_0^\dagger z + \varphi(y)}\, dydz \\ & + \int_0^{+\infty} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{c_0^\dagger z + \varphi(y)}\, dydz \\ & + \int_{-\infty}^0 \int_\Omega V_\varepsilon(u_\varepsilon) e^{c_\varepsilon^\dagger z + \varphi(y)}\, dydz. \end{aligned} \tag{5.15}$$

Now, when $u_0 < 1$, by definition of V_ε we have $V_\varepsilon(u_\varepsilon) \to 0 = V_0(u_0)$. Since also $V_\varepsilon(u_\varepsilon) \geq -1 = V_0(u_0)$ whenever $u_0 = 1$, this implies that $V_0(u_0) \leq \liminf_{\varepsilon \to 0} V_\varepsilon(u_\varepsilon) \leq 0$ in Σ. Then, in view of $e^{c_\varepsilon^\dagger z} \to e^{c_0^\dagger z}$ by Proposition 5.1, it follows that

$$e^{c_0^\dagger z + \varphi(y)} V_0(u_0) \leq \liminf_{\varepsilon \to 0} e^{c_\varepsilon^\dagger z + \varphi(y)} V_\varepsilon(u_\varepsilon) \quad \text{ in } \Omega \times (-\infty, 0). \tag{5.16}$$

Notice also that $e^{c_\varepsilon^\dagger z + \varphi(y)} V_\varepsilon(u_\varepsilon) \geq -e^{c_0^\dagger z + \varphi(y)} \in L^1(\Omega \times (-\infty, 0))$. By monotonicity of u_0, we have $V_0(u_0) = V_\varepsilon(u_\varepsilon) = 0$ in $\Omega \times (0, +\infty)$. So, by Fatou's Lemma we finally obtain

$$\int_\Sigma V_0(u_0) e^{c_0^\dagger z + \varphi(y)}\, dx \leq \liminf_{\varepsilon \to 0} \int_\Sigma V_\varepsilon(u_\varepsilon) e^{c_\varepsilon^\dagger z + \varphi(y)}\, dx. \tag{5.17}$$

Similarly, since $e^{\frac{c_\varepsilon^\dagger - c_0^\dagger}{2} z} \to 1$ in $L^2_{c_0^\dagger}\big(\Omega \times (-\infty, 0)\big)$, and

$$\nabla u_\varepsilon \quad \rightharpoonup \quad \nabla u_0 \quad \text{weakly in } L^2_{c_0^\dagger}\big(\Omega \times (0, +\infty); \mathbb{R}^n\big), \tag{5.18}$$

we have

$$\nabla u_\varepsilon e^{\frac{c_\varepsilon^\dagger - c_0^\dagger}{2} z} \quad \rightharpoonup \quad \nabla u_0 \quad \text{weakly in } L^2_{c_0^\dagger}\big(\Omega \times (-\infty, 0); \mathbb{R}^n\big), \tag{5.19}$$

which implies

$$\begin{aligned} \int_\Sigma \frac{|\nabla u_0|^2}{2} e^{c_0^\dagger z + \varphi(y)}\, dx \leq \liminf_{\varepsilon \to 0} & \int_0^{+\infty} \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{c_0^\dagger z + \varphi(y)}\, dydz \\ + \liminf_{\varepsilon \to 0} & \int_{-\infty}^0 \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} e^{(c_\varepsilon^\dagger - c_0^\dagger)z} e^{c_0^\dagger z + \varphi(y)}\, dydz. \end{aligned} \tag{5.20}$$

Taking the lim inf in (5.15) and recalling (5.17) and (5.20), we get the equality (5.13). Finally, in view of (5.8), we then obtain that u_0 is a nontrivial minimizer of Φ^0_c.

Notice that by (5.13) the inequalities in (5.17) and (5.20) are in fact equalities, therefore we also have $u_\varepsilon \to u_0$ strongly in $H^1_{c_0^\dagger}(\Sigma)$, as $\varepsilon \to 0$. Also, in view of the uniqueness of the minimizer of Φ_c^0 subject to (5.8) (recall that by Theorem 3.1 the minimizer u_0 is strictly decreasing whenever $u_0 < 1$), the limit is a full limit and is not restricted to a sequence of $\varepsilon \to 0$. □

6 Numerical examples

In this section, we illustrate the applicability of our theory with a few numerical examples. We will concentrate on the results of Sec. 3 as, on one hand, the sharp reaction zone limit is important for combustion applications, and, on the other, because in this case both upper and lower bounds for propagation speed of the minimizers are available, and so it is possible to check how well they fit the propagation speed both for the limit problem and its regularizing approximations.

For the sake of clarity, we will consider the simplest possible, yet non-trivial situation, namely that of front propagation along a two-dimensional strip: $\Sigma = (0, 2L) \times \mathbb{R}$, where $L > 0$, with Dirichlet boundary conditions. We note that in the case of a bistable nonlinearity and $\varphi = 0$ existence of traveling waves on a strip with Dirichlet boundary conditions was first proved by Gardner in [15].

Let us start by considering the minimizers in the sharp reaction zone limit in the absence of a flow, $\varphi = 0$. Here we only need to consider the problem on half of the domain: $(0, L) \times \mathbb{R}$, due to the obvious symmetry of the solution with respect to the transformation $y \to 2L - y$. According to Corollary 3.3, the minimizers of Φ_c^0 exist if and only if (3.17) holds. Here we have explicitly, according to Remark 3.2,

$$v = \begin{cases} \sqrt{2}\, y, & 0 \le y \le L_0, \\ 1, & L_0 \le y \le L, \end{cases} \tag{6.1}$$

where $L_0 = 1/\sqrt{2}$, and $E_0[v] < 0$ whenever

$$L > \sqrt{2}. \tag{6.2}$$

So, the minimizers of Φ_c^0 exist if and only if the value of L is greater than this critical value. Also note that for every L satisfying (6.2) the critical point of E_0 is unique and is given by (6.1). Therefore, a pair $(c^\dagger, \bar{u})$, where $\bar{u}$ is a minimizer of $\Phi_{c^\dagger}^0$ is, in fact, the only (up to translations) traveling wave solution in the sharp reaction zone limit. In particular, the speed of the wave is unique and is given by $c^\dagger > 0$.

To obtain a lower bound for the propagation speed $c^\dagger$, we introduce a trial function $u = u_{\lambda,\mu,l}$, where

$$u_{\lambda,\mu,l}(y,z) = \begin{cases} 1, & l \le y \le L,\ z \le \frac{\mu}{\lambda}(y-L), \\ e^{-\lambda z+\mu(y-L)}, & l \le y \le L,\ z \ge \frac{\mu}{\lambda}(y-L), \\ \frac{y}{l}, & 0 \le y \le l,\ z \le \frac{\mu}{\lambda}(l-L), \\ \frac{y}{l}e^{-\lambda z+\mu(l-L)}, & 0 \le y \le l,\ z \ge \frac{\mu}{\lambda}(l-L). \end{cases} \tag{6.3}$$

This function is characterized by 3 parameters, λ, μ, l. We must have $0 < l < L$, as well as $2\lambda > c$ in order for $u_{\lambda,\mu,l}$ to lie in $H^1_c((0,L)\times\mathbb{R})$. Substituting this u into Φ^0_c, after straightforward algebra we obtain

$$\begin{aligned} \Phi^0_c[u_{\lambda,\mu,l}] &= \frac{\left(-1+e^{\frac{c(l-L)\mu}{\lambda}}\right)\left(\lambda^2+\mu^2\right)\lambda}{2c(c-2\lambda)\mu} \\ &+ \frac{\left(-1+e^{\frac{c(l-L)\mu}{\lambda}}\right)\lambda}{c^2\mu} - \frac{e^{\frac{c(l-L)\mu}{\lambda}}\left(l^2\lambda^2+3\right)}{6l(c-2\lambda)} + \frac{e^{\frac{c(l-L)\mu}{\lambda}}}{2cl}. \end{aligned} \tag{6.4}$$

With $c > 0$ and $L > \sqrt{2}$, this expression can be minimized numerically, and the sign of the minimum be evaluated. Then, one can find the largest value of c for which this minimum still remains reliably negative (to numerical precision). For example, when $L = \frac{5}{2}$ and $c = 0.925$, we found that $\Phi^0_c[u_{\lambda,\mu,l}]$ is minimized with $\lambda \simeq 1.237, \mu \simeq 0.5150, l \simeq 0.8870$ and attains the value of $\simeq -1.15 \times 10^{-3} < 0$. The level curves of $u_{\lambda,\mu,l}$ corresponding to these values are shown in Fig. 2(a). So, from Theorem 3.1 we conclude that $c^\dagger \ge 0.925$. Of course, it is no trouble at all to make this estimate completely rigorous, if need be.

Now we compute the upper bound $c^\sharp$ for the minimizer above. For that, we need to find a non-trivial minimizer ζ for the functional Ξ_c in (4.4), with $\omega_0 = (L_0, L)$, $\zeta_y(L) = 0$, and $\zeta(L_0) = 0$. In the case of a general potential $\varphi(y)$, the Euler-Lagrange equation for Ξ_c is

$$\frac{d}{dy}\left(\frac{e^{\varphi(y)}\zeta_y}{\sqrt{c^2\zeta^2+\zeta_y^2}}\right) = e^{\varphi(y)}\left(\frac{c^2\zeta}{\sqrt{c^2\zeta^2+\zeta_y^2}} - \sqrt{2}\right). \tag{6.5}$$

Actually, this equation can be solved in closed form in the special case when φ is a linear function of y and, in particular, when $\varphi = 0$ (of course, this equation can also be straightforwardly integrated numerically for arbitrary φ to any desired accuracy). However, since the algebra becomes too messy in the case $\varphi = \alpha y$ with $\alpha \ne 0$, we will only analyze the case $\varphi = 0$ explicitly, and will instead use a numerical solution of (6.5) in other cases.

When $\varphi = 0$, the first integral of (6.5) is

$$H = \sqrt{2}\,\zeta - \frac{c^2\zeta^2}{\sqrt{c^2\zeta^2+\zeta_y^2}}. \tag{6.6}$$

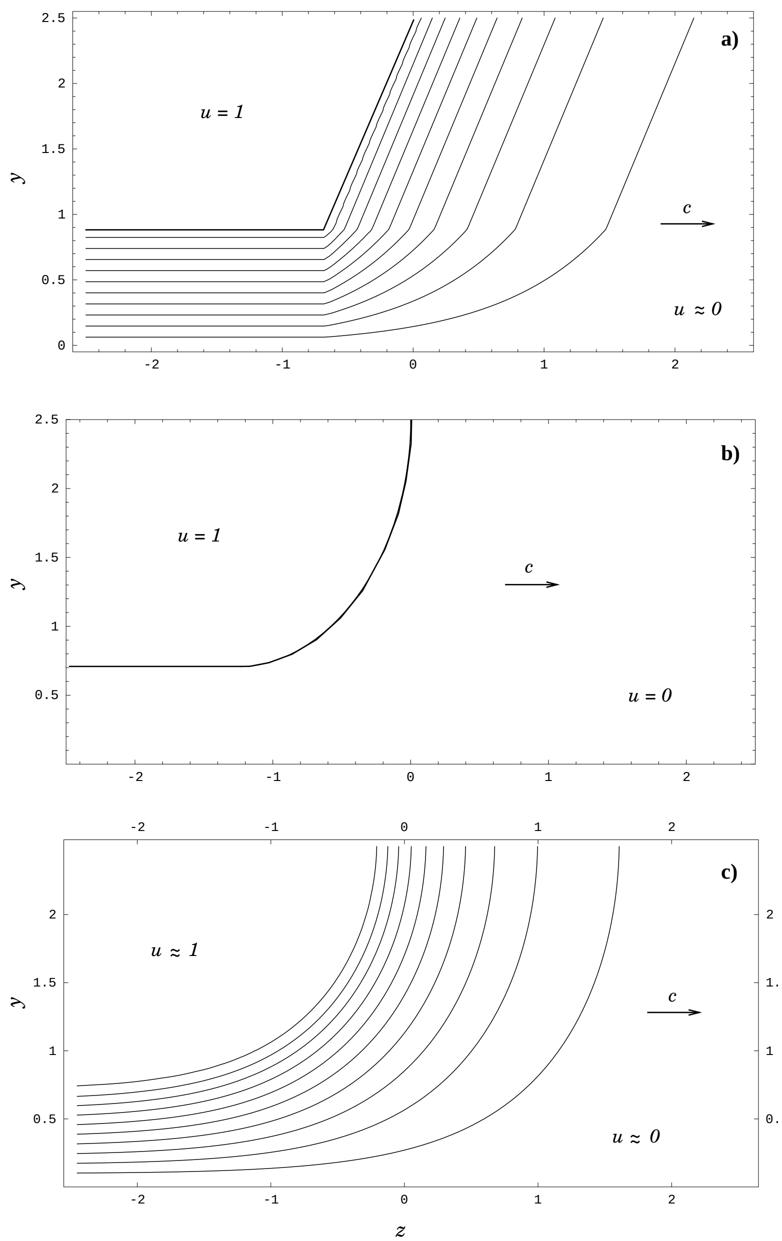

Figure 2: Comparison of the front profiles obtained from different approximations in the case $\varphi = 0$ and $L = \frac{5}{2}$. (a) The level curves of the trial function $u_{\lambda,\mu,l}$ for $c = 0.925$ and the parameters λ, μ, l obtained from minimizing $\Phi_c^0[u_{\lambda,\mu,l}]$. (b) The curve minimizing Π_c with $c = c^\sharp$. (c) The level curves of the numerical solution of (1.4) and (6.12) for $\varepsilon = 0.2$. Only the lower half of u is shown in all cases.

Also, given H, the value of the functional on the solution of (6.6) is

$$\Xi_c[\zeta] = \frac{1}{\sqrt{2}} \left(\zeta(L_0 + 0) + \int_{L_0}^{L} \frac{\zeta_y^2 \, dy}{\sqrt{c^2\zeta^2 + \zeta_y^2}} - H(L - L_0) \right), \qquad (6.7)$$

where we took into account a jump discontinuity in ζ at $y = L_0$. Now, note that in view of (6.7), we will get $\Xi_c[\zeta] > 0$ unless $\zeta = 0$ when $H \leq 0$. Therefore, we need to consider only the case $H > 0$. In fact, because the right-hand side of (6.6) is a one-homogeneous function of ζ and ζ_y, without the loss of generality we can set $H = 1$. Let us also recall that the non-trivial minimizers exist only for $c < \sqrt{2}$.

Solving (6.6) with $H = 1$, we obtain a first-order equation

$$\frac{d\zeta}{dy} = \frac{c\zeta\sqrt{(c^2-2)\zeta^2 + 2\sqrt{2}\,\zeta - 1}}{\sqrt{2}\,\zeta - 1}, \qquad L_0 < y < L, \qquad (6.8)$$

which can be solved implicitly for y. After some tedious algebra, we find (up to an additive constant)

$$y = \frac{1}{c} \left(\frac{\sin^{-1}\left(\frac{(2-c^2)\zeta - \sqrt{2}}{c}\right)}{\sqrt{1 - \frac{c^2}{2}}} - \sin^{-1}\left(\frac{\sqrt{2}\,\zeta - 1}{c\,\zeta}\right) \right), \qquad (6.9)$$

where

$$\frac{1}{\sqrt{2}} < \zeta < \frac{1}{\sqrt{2} - c}. \qquad (6.10)$$

These limits are chosen from the requirements that $dy/d\zeta = 0$ and $dy/d\zeta = \infty$ at the endpoints of the interval. Now, recalling that $\zeta = e^{cz}$, where $z = h(y)$ is the function whose graph is a minimizer of Π_c, see (4.5), we conclude that we have obtained a parametric representation of this minimizer, once the value of $c = c^\sharp$ is known.

Finally, to find the value of $c^\sharp$, we equate the total variation of y in (6.9) over the interval in (6.10) to $L - L_0$:

$$L - L_0 = \frac{\pi\left(\sqrt{2} - \sqrt{2 - c^2}\right) + 2\sqrt{2}\sin^{-1}\left(\frac{c}{\sqrt{2}}\right)}{2c\sqrt{2 - c^2}} \qquad (6.11)$$

The solution of this equation gives $c^\sharp$. We computed the value of $c^\sharp$ for $L = \frac{5}{2}$ numerically and found that $c^\sharp \simeq 1.010$. Therefore, we conclude that for this value of L we have $0.925 \leq c^\dagger \leq 1.011$. Thus, a variational characterization of the traveling wave solutions in the sharp reaction zone limit allowed us to bracket the value of the wave speed within a 5% accuracy, with a minimal computational effort. Also, the curve that minimizes Π_c in this case is shown in

Fig. 2(b). Observe the similarity of the main characteristics of the two profiles in Fig. 2, parts (a) and (b).

We now would like to compare these sharp reaction zone limit estimates with the numerical solution of the approximating problem in (1.4). For the purposes of the numerics, we chose the following form of $g(u)$:

$$g(u) = 12u(1-u)^2. \tag{6.12}$$

Fixing $\varepsilon \in (0,1)$, we obtain numerical approximations to the traveling wave solutions on the strip $(0,2L)\times\mathbb{R}$ with Dirichlet boundary conditions by solving the corresponding parabolic PDE on a sufficiently large rectangle with a localized initial condition (using simple explicit in time, centered in space, finite difference scheme) and waiting sufficient time for an (approximate) traveling wave to form. We find, for example, that when $\varepsilon = 0.2$, the traveling wave has a speed $c^\dagger_\varepsilon \simeq 1.095$. The profile of the wave front for this value of ε is also presented in Fig. 2(c). Note, once again, the similarity between all three profiles in Fig. 2. We also performed a series of simulation in the range $0.1 \leq \varepsilon \leq 0.5$ and extrapolated the value of $c^\dagger_\varepsilon$ to $\varepsilon = 0$, finding $c^\dagger_0 \simeq 0.987$, see Fig. 3, in agreement with the estimates obtained earlier for the sharp reaction zone limit. We note that for $\varepsilon < 0.2$ all three estimates obtained by us are within $\sim$10% of each other. In particular, the value of $c^\sharp$, corresponding to the Markstein model of flame propagation [20], gives a very good approximation for the propagation speed even in the presence of "heat loss" through the walls and curvature comparable to the "flame" size.

Note that from the phase plane analysis it follows that the positive equilibrium of (1.4) is unique for each $\varepsilon < 1$, thus, there exists a unique traveling wave solution for each $\varepsilon \in (0,1)$, which is the minimizer we found. Similarly, the results of [22] apply for each $\varepsilon \in (0,1)$, and so propagation with speed $c^\dagger_\varepsilon$ is guaranteed for the initial data that approach v_ε as $t \to \infty$ on compacts as $t \to \infty$. In particular, the propagation speed for the parabolic problem will tend to $c^\dagger$ estimated in the first part of this section in the limit $\varepsilon \to 0$.

We conclude this section by presenting a few results for the case when $\varphi \neq 0$. In particular, for $n = 2$ an important special case is that of a linear function $\varphi = \alpha y$, corresponding to a divergence-free flow across the strip. We solved (1.4) numerically with $\alpha = 1$, $L = \frac{5}{2}$, and $\varepsilon = 0.2$, to find a propagation speed $c^\dagger_\varepsilon = 0.698$. The profile of the front in this case is also shown in Fig. 4(a). The value of $c^\dagger_\varepsilon$ is compared with the numerical solution of (6.5) on the domain $\omega_0 = (\log(1+\frac{1}{\sqrt{2}}), 2L+\log(1-\frac{1}{\sqrt{2}}))$. We obtained $c^\sharp \simeq 0.5776$, which, once again, is close to the value of $c^\dagger_\varepsilon$ obtained earlier. Also, the profile of the corresponding minimizer of Π_c is shown in Fig. 4(b). Again, extrapolating the values of $c^\dagger_\varepsilon$ obtained in the interval $0.1 \leq \varepsilon \leq 0.5$ to $\varepsilon = 0$ as before, we obtained $c^\dagger \simeq 0.554$ for the sharp reaction zone limit, in agreement with the above upper estimate. To summarize, the value of $c^\sharp$ approximates the value of $c^\dagger$ within 5%, despite the fact that the domain size is comparable with the minimal size in (6.2) for which propagation is possible, and for which the curvature of the front is not small.

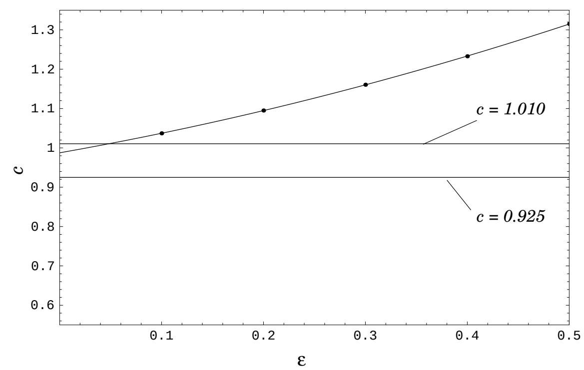

Figure 3: Dependence of $c^\dagger_\varepsilon$ obtained from the numerical solution of (1.4) and (6.12) with $\varphi = 0$. The dots are the results of the simulations, the curve is a fit using a quadratic polynomial $c^\dagger_\varepsilon \simeq 0.987 + 0.458\varepsilon + 0.393\varepsilon^2$. Numerical solutions of (1.4) and the propagation speeds are obtained by solving the associated parabolic problem in (1.1) on a rectangle $(0, 5) \times (0, 20)$ with the initial data $u(y, z) = \cosh^{-2}(\frac{1}{2}\sqrt{(y - \frac{5}{2})^2 + z^2})$ discretized on the 100×400 grid (except for $\varepsilon = 0.1$ when a 200×800 grid was used), with Dirichlet boundary conditions everywhere except at $z = 0$, where Neumann boundary conditions are used.

Finally, let us illustrate the assumptions of Proposition 4.3 with a numerical example with $L = 10$ and $\varphi_y = -2\cos\left(\frac{\pi y}{2L}\right)$. Since this expression is greater than $\sqrt{2}$ in absolute value outside of the interval $5 \leq y \leq 15$, the minimizer of Π_c cannot come closer than distance $L/2 \gg 1$ to the boundary, as required by the assumptions of Proposition 4.3. Similarly, since φ_y varies on the length scale of L, the minimizer of Π_c has curvature of order L^{-1}. For this choice of φ, this minimizer is shown in Fig. 5(a). For comparison, Fig. 5(b) shows the numerical solution of (1.4) with $\varepsilon = 0.2$. The value $c^\dagger_\varepsilon \simeq 1.13$ found here is, again, in good agreement with $c^\sharp \simeq 1.016$ found from solving (6.5) numerically. The corresponding extrapolated value $c^\dagger \simeq 0.99$ for $\varepsilon = 0$ limit is, once again, very close to the upper bound. We note that the solution just analyzed is also related to the front solutions found in the edge flame problem (see e.g. [26]), these will be studied in more detail elsewhere.

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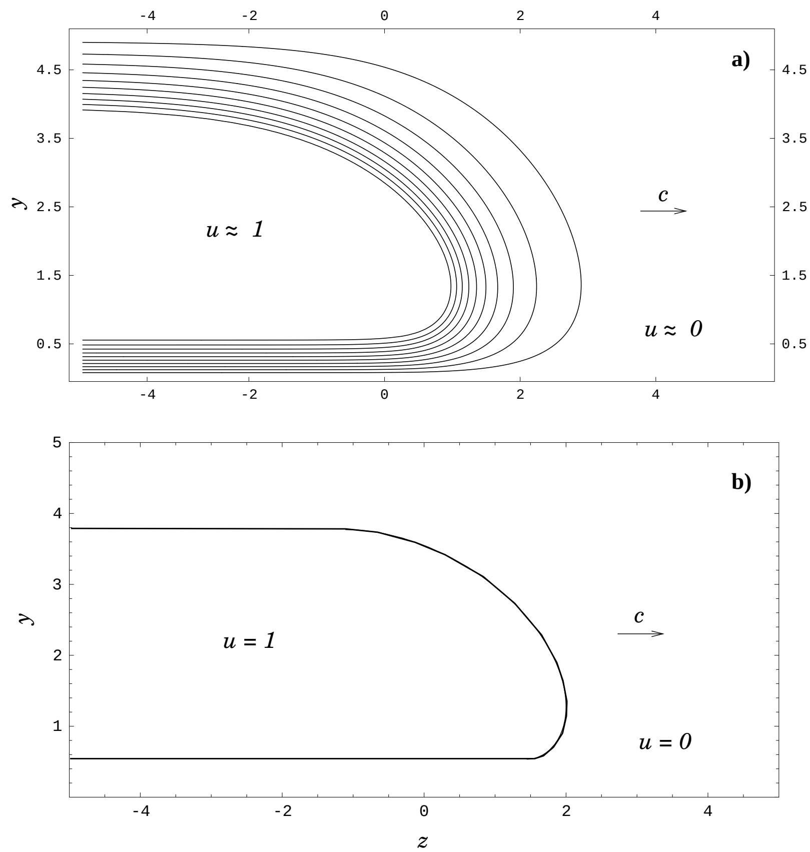

Figure 4: Comparison of the front profiles for $\varphi = y$ and $L = \frac{5}{2}$. (a) The level curves of the numerical solution of (1.4) and (6.12) with $\varepsilon = 0.2$. (b) The curve minimizing Π_c with $c = c^\sharp$.

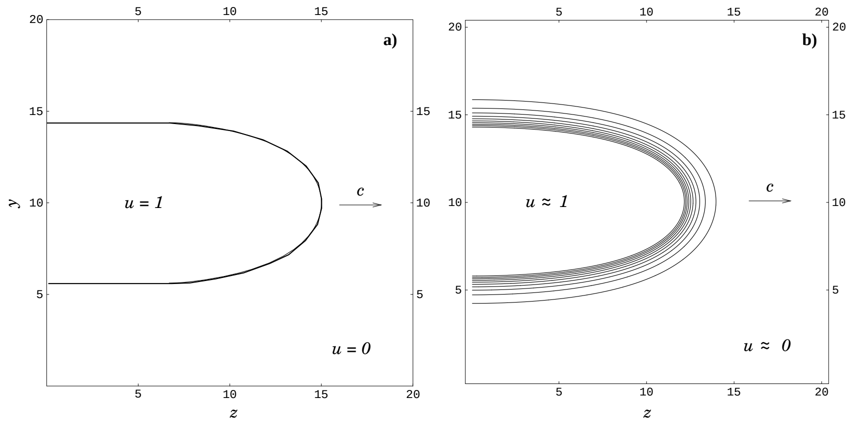

Figure 5: Comparison of the front profiles for $\varphi_y = -2\cos\left(\frac{\pi y}{2L}\right)$ and $L = 10$. (a) The minimizer of Π_c for $c = c^\sharp$. (b) The level curves of the numerical solution of (1.4) and (6.12) with $\varepsilon = 0.2$.

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