EXISTENCE AND UNIQUENESS FOR ANISOTROPIC AND CRYSTALLINE MEAN CURVATURE FLOWS

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Abstract. An existence and uniqueness result, up to fattening, for crystalline mean curvature flows with forcing and arbitrary (convex) mobilities, is proven. This is achieved by introducing a new notion of solution to the corresponding level set formulation. Such solutions satisfy a comparison principle and stability properties with respect to the approximation by suitably regularized problems. The results are valid in any dimension and for arbitrary, possibly unbounded, initial closed sets. The approach accounts for the possible presence of a time-dependent bounded forcing term, with spatial Lipschitz continuity. As a result of our analysis, we deduce the convergence of a minimizing movement scheme proposed by Almgren, Taylor and Wang (1993), to a unique (up to fattening) "flat flow" in the case of general, including crystalline anisotropies, solving a long-standing open question.

Keywords: Geometric evolution equations, Minimizing movements, Crystalline mean curvature motion, level set formulation.

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Contents

1. Introduction

In this paper we deal with anisotropic and crystalline mean curvature flows; that is, flows of sets $t \mapsto E(t)$ (formally) governed by the law

$$V(x,t) = -\psi(\nu^{E(t)})(\kappa_\phi^{E(t)}(x) + g(x,t)), \tag{1.1}$$

where $V(x,t)$ stands for the (outer) normal velocity of the boundary $\partial E(t)$ at x, ϕ is a given norm on $\mathbb{R}^N$ representing the *surface tension*, $\kappa_\phi^{E(t)}$ is the *anisotropic mean curvature* of $\partial E(t)$ associated with the anisotropy ϕ, ψ is a norm evaluated at the outer unit normal $\nu^{E(t)}$ to $\partial E(t)$, and g is a bounded spatially Lipschitz continuous forcing term. The factor ψ plays the role of a *mobility*[1]. We recall that when ϕ is differentiable in $\mathbb{R}^N \setminus \{0\}$, then κ_ϕ^E is given by the surface divergence of a "Cahn-Hoffman" vector field [17, 48, 47]:

$$\kappa_\phi^E = \mathrm{div}_\tau \left(\nabla\phi(\nu^E)\right), \tag{1.2}$$

however in this work we will be interested mostly in the "crystalline case", which is whenever the level sets of ϕ are polytopes and (1.2) should be replaced with

$$\kappa_\phi^E \in \mathrm{div}_\tau \left(\partial\phi(\nu^E)\right), \tag{1.3}$$

and which we will describe later on.

Equation (1.1) is relevant in materials science and the study of crystal growth, see for instance [49, 48, 43] and the references therein. Its mathematical well-posedness is classical in the smooth setting, that is when ϕ, ψ, g and the initial set are sufficiently smooth (and ϕ satisfies suitable ellipticity conditions). However, it is also well-known that in dimensions $N \geq 3$ singularities may form in finite time even in the smooth case. When this occurs the strong formulation of (1.1) ceases to be applicable and one needs a weaker notion of solution leading to a (possibly unique) globally defined evolution.

Among the different approaches that have been proposed in the literature for the classical mean curvature flow (and for several other "regular" flows) in order to overcome this difficulty, we start by mentioning the so-called *level set approach* [45, 30, 31, 26, 37], which consists in embedding the initial set in the one-parameter family of sets given by the sublevels of some initial function u^0, and then in letting all these sets evolve according to the same geometric law. The evolving sets are themselves the sublevels of a time-dependent function $u(x,t)$, which turns out to solve a (degenerate) parabolic equation for u (with the prescribed initial datum u^0). The crucial point is that such a parabolic Cauchy problem is shown to admit a global-in-time unique viscosity solution for many relevant geometric motions: in fact, one only needs the continuity[2] of the Hamiltonian of the level-set equation which corresponds to (1.1) [26, 36]. When this happens, the evolution of the sublevels of u defines a *generalized motion* (with initial set given by the corresponding

[1]Strictly speaking, the mobility is $\psi(\nu^{E(t)})^{-1}$.

[2]Which is of course weaker than the requirements for the existence of strong solutions, which at least include ellipticity properties.

sublevels of u^0), which exists for all times and agrees with the classical one until the appearance of singularities (see [31]). Moreover, such a generalized motion satisfies the comparison principle and is unique whenever the level sets of u have an empty interior. Let us mention that the appearance of a nontrivial interior (the so called fattening phenomenon) may in fact occur even starting from a smooth set (see for instance [8]). On the other hand, such a phenomenon is rather rare: for instance, given any uniformly continuous initial function u^0, all its sublevels, with the exception of at most countably many, will not generate any fattening.

The second approach which is relevant for the present treatment is represented by the minimizing movements scheme devised by Almgren, Taylor and Wang [3] and, independently, by Luckhaus and Sturzenhecker [44]. It is variational in nature and hinges on the gradient flow structure of the geometric motion. More precisely, it consists in building a family of discrete-in-time evolutions by an iterative minimization procedure and in considering any limit of these evolutions (as the time step $\Delta t > 0$ vanishes) as an admissible global-in-time solution to the geometric motion, usually referred to as a *flat flow* (or ATW flat flow). The problem which is solved at each step has the form [3, §2.6]

$$\min_E P_\phi(E) + \frac{1}{\Delta t}\int_{E\triangle E^{n-1}} \operatorname{dist}(x, \partial E^{n-1})dx \tag{1.4}$$

where $E\triangle E^{n-1}$ denotes the symmetric difference of the two sets E and E^{n-1}, and the anisotropic perimeter $P_\phi(E) = \int_{\partial E}\phi(\nu)d\mathcal{H}^{N-1}$ is defined rigorously in (2.1) below. (We generalize slightly the scheme later on, in particular to deal with non-compact boundaries, of possibly infinite mass.) This scheme is studied in great details in [3] and many convergence properties are proven, including to the previously mentioned viscosity solutions, under some technical assumption. If ϕ (and the initial set) is smooth enough, also convergence to strong solutions are proven. However, except in dimension 2 [2], the convergence of this scheme in the crystalline case remained open.

In this paper, we show for the first time that, up to exceptional initial sets which might develop non-uniqueness, this discrete procedure converges to a unique motion in all cases, including crystalline. In practice, if we replace the distance in (1.4) with an anisotropic distance based on a norm "compatible" with ϕ ("ϕ-regular", Def. 4.1), it is relatively easy, though a bit technical, to extend our previous results in [22] and show convergence of the scheme. Our main result in this work is a stability result which shows additionally that even if this ϕ-regularity is lost (as is always the case when the distance in (1.4) is Euclidean and ϕ crystalline), the discrete-in-time flows remain close to ϕ-regular flows and their limit is still unique. While the stability for the limiting flow (only) is relatively simple [20], the stability at the discrete level, for $\Delta t > 0$, which only allows to derive the uniqueness of the flat flow, is quite technical and requires precise estimates on the minimizers of (1.4), established in Section 3.4. The remainder of this introduction describes more closely the technical content of this paper.

Practically, it is somewhat convenient to combine the variational approach with the level set point of view, by implementing the Almgren-Taylor-Wang scheme (ATW) for all the sublevels of the initial function u^0 (level set ATW). As already mentioned, it turns out that the two approaches produce in general the same solutions. A very simple proof of convergence of the level set ATW to the viscosity solution of the level set equation in the case of anisotropic mean curvature flows (with smooth anisotropy) is given in [23] (see also [3, 19]); such a result implies in turn the convergence of the ATW to the aforementioned generalized motion whenever fattening does not occur.

When the anisotropy is crystalline, all the results mentioned before for regular anisotropies become much more difficult, starting from the very definition of crystalline curvature which cannot be given by (1.2) anymore, but rather by (1.3): one has to consider a suitable selection z of the (multivalued) subdifferential map $x \mapsto \partial\phi(\nu^E(x))$ (of "Cahn-Hoffmann fields"), such that the tangential divergence $\mathrm{div}_\tau z$ has minimal L^2-norm among all possible selections. The crystalline curvature is then given by the tangential divergence $\mathrm{div}_\tau z_{\mathrm{opt}}$ of the optimal Cahn-Hoffman field (see [16, 35]) and thus, in particular, has a nonlocal character.

We briefly recall what is known about the mathematical well-posedness of (1.1) in the crystalline case. In two dimensions, the problem has been essentially settled in [34] (when g is constant) by developing a crystalline version of the viscosity approach for the level-set equation, see also [49, 48, 50, 2, 7, 33, 38] for important former work. The viscosity approach adopted in [34] applies in fact to more general equations of the form

$$V = f(\nu, -\kappa_\phi^E)\,, \tag{1.5}$$

with f continuous and non-decreasing with respect to the second variable, however without spatial dependence. Former studies were rather treating the problem as a system of coupled ODEs describing the relative motion of each facet of an initial crystal [49, 48, 2, 7]. We mention also the recent paper [25], where short time existence and uniqueness of strong solutions for initial "regular" sets (in a suitable sense) is shown.

In dimension $N \geq 3$ the situation was far less clear until very recently. Before commenting on the new developments, let us remark that before these, the only general available notion of global-in-time solution was that of a flat flow associated with the ATW scheme, defined as the limit of a converging subsequence of time discrete approximations. However, no general uniqueness and comparison results were available, except for special classes of initial data [18, 13, 39] or for very specific anisotropies [35]. As mentioned before, substantial progress in this direction has been made only very recently, in [41] and [22].

In [41], the authors succeed in extending the viscosity approach of [34] to $N = 3$. They are able to deal with very general equations of the general form (1.5) establishing existence and uniqueness for the corresponding level set formulations. In a preprint just appeared [42], they show how to extend their approach to any dimension, which is a major breakthrough (moreover the new proof is considerably simpler than before). It seems that their method, as far as we know, still requires a purely crystalline anisotropy ϕ (so mixed situations are not allowed), bounded initial sets, and the only possible forcing term is a constant.

In [22], the first global-in-time existence and uniqueness (up to fattening) result for the crystalline mean curvature flow valid in all dimensions, for arbitrary (possibly unbounded) initial sets, and for general (including crystalline) anisotropies ϕ was established, but under the particular choice $\psi = \phi$ (and $g = 0$) in (1.1). It is based on a new stronger distributional formulation of the problem in terms of distance functions, which is reminiscent of, but not quite the same as, the distance formulation proposed and studied in [46] (see also [28, 11, 6, 18, 4]). Such a formulation enables the use of parabolic PDE's arguments to prove comparison results, but of course makes it more difficult to prove existence. The latter is established by implementing the variant of the ATW scheme devised in [19, 18]. The methods of [22] yield, as a byproduct, the uniqueness, up to fattening, of the ATW flat flow for the equation (1.1) with $\psi = \phi$ and $g = 0$. But it leaves open the uniqueness issue for the general form of (1.1) and, in particular, for the constant mobility case

$$V = -\kappa_\phi^E\,, \tag{1.6}$$

originally appearing in [3], which is approximated by (1.4). The main reason is technical: the distributional formulation introduced in [22] becomes effective in yielding uniqueness results only if, roughly speaking, the level sets of the ψ°-distance function from any closed set (ψ° being the norm polar to the mobility ψ) have (locally) bounded crystalline curvatures. This is certainly the case when $\phi = \psi$ (and explains such a restriction in [22]).

As said, we remove in this paper the restriction $\phi = \psi$ and extend the existence and uniqueness results of [22] to the general equation (1.1). In order to deal with general mobilities, we cannot rely anymore on a distributional formulation in the spirit of [22], but instead we extend the notion of solution via an approximation procedure by suitable regularized versions of (1.1).

We now describe more in details the contributions and the methods of the paper. Before addressing the general mobilities, we consider the case where ψ may be different from ϕ but satisfies a suitable regularity assumption, namely we assume that the Wulff shape (2.3) associated with ψ (in short the ψ-Wulff shape) admits an inner tangent ϕ-Wulff shape at all points of its boundary. We call such mobilities *ϕ-regular* (see Definition 4.1). The ϕ-regularity assumption implies in turn that the level sets of the ψ°-distance function from any closed set have locally bounded crystalline curvatures and makes it possible to extend the distributional formulation (and the methods) of [22] to (1.1) (Definition 2.2), to show that such a notion of solution bears a comparison principle (Theorem 2.7) and that the ATW scheme converges to it (Theorem 4.3). As is classical, we then use these results to build a unique level set flow (and a corresponding generalized motion), which satisfies comparison and geometricity properties (Theorem 4.8).

Having accomplished this, we deal with the general case of ψ being any norm. As mentioned before, the idea here is to build a level set flow by means of approximation, after the easy observation that for any norm ψ there exists a sequence $\{\psi_n\}$ of ϕ-regular mobilities such that $\psi_n \to \psi$. More precisely, we say that u is a solution to the level set flow associated with (1.1) if there exists an approximating sequence $\{\psi_n\}$ of ϕ-regular mobilities such that the corresponding level set flows u_n constructed in Section 4 locally uniformly converge to u (Definition 5.6).

In Theorems 5.7 and 5.9 we establish the main results of the paper: we show that for any norm ψ a solution-via-approximation u always exists; moreover u satisfies the following properties:

(i) (Uniqueness and stability): The solution-via-approximation u is unique in that it is independent of the choice of the approximating sequence of ϕ-regular mobilities $\{\psi_n\}$. In fact, it is stable with respect to the convergence of any sequence of mobilities and anisotropies.
(ii) (Comparison): if $u^0 \le v^0$, then the corresponding level set solutions u and v satisfy $u \le v$.
(iii) (Convergence of the level set ATW): u is the unique limit of the level set ATW.
(iv) (Generic non-fattening): As in the classical case, for any given uniformly continuous initial datum u^0 all but countably many sublevels do not produce any fattening.
(v) (Comparison with other notions of solutions): Our solution-via-approximation u coincides with the classical viscosity solution in the smooth case and with the Giga-Požár viscosity solution [41, 42] whenever such a solution is well-defined, that is, when g is constant, ϕ is purely crystalline and the initial set is bounded.

(vi) (Phase-field approximation): When g is constant, a phase-field Allen-Cahn type approximation of u holds.

We finally mention that property (iii) implies the convergence of the ATW scheme, whenever no fattening occurs and thus settles the long-standing problem of the uniqueness (up to fattening) of the flat flow corresponding to (1.1) (and in particular for (1.6)) when the anisotropy is crystalline. In our later paper [20] we show that it is also possible to build crystalline flows by approximating the anisotropies with smooth ones and relying for existence on the standard viscosity theory of generalized solutions. However this variant, even if slightly simpler, does not show that flat flows are unique.

The plan of the paper is the following. In Section 2 we extend the distributional formulation of [22] to our setting and we study the main properties of the corresponding notions of sub and supersolutions. The main result of the section is the comparison principle established in Section 2.3.

In Section 3 we set up the minimizing movements algorithm and we start paving the way for the main results of the paper by establishing some preliminary results. In particular, the density estimates and the barrier argument of Section 3.4, which do not require any regularity assumption on the mobility ψ, will be crucial for the stability analysis of the ATW scheme needed to deal with the general mobility case and developed in Section 5.1.

In Section 4 we develop the existence and uniqueness theory under the assumption of ϕ-regularity for the mobility ψ. More precisely, we establish the convergence of the ATW scheme to a distributional solution of the flow, whenever fattening does not occur. Uniqueness then follows from the results of Section 2.

Finally, in Section 5 we establish the main results of the paper, namely the existence and uniqueness of a solution via approximation by ϕ-regular mobilities. As already mentioned, the approximation procedure requires a delicate stability analysis of the ATW scheme with respect to varying mobilities. Such estimates are established in Section 5.1 and represent the main technical achievement of Section 5.

2. A distributional formulation of curvature flows

In this section we generalize the approach introduced in [22] by introducing a suitable distributional formulation of (1.1) and we show that such a formulation yields a comparison principle and is equivalent to the standard viscosity formulation when the anisotropy ϕ and the mobility ψ are sufficiently regular.

The existence of the distributional solution defined in this section will be established in Section 4 under the additional assumption that the mobility ψ satisfies a suitable regularity assumption (see Definition 4.1 below).

2.1. **Preliminaries.** We introduce the main objects and notation used throughout the paper.

Given a norm η on $\mathbb{R}^N$ (a convex, even[3], one-homogeneous real-valued function with $\eta(\nu) > 0$ if $\nu \neq 0$), we define a polar norm η° by $\eta^\circ(\xi) := \sup_{\eta(\nu)\leq 1} \nu \cdot \xi$ and an associated anisotropic perimeter P_η as

$$P_\eta(E) := \sup\left\{ \int_E \mathrm{div}\zeta \, dx : \zeta \in C^1_c(\mathbb{R}^N;\mathbb{R}^N),\ \eta^\circ(\zeta) \leq 1 \right\}. \tag{2.1}$$

[3]For simplicity we develop the theory in the symmetric case; See Remark 6.3.

As is well known, $(\eta^\circ)^\circ = \eta$ so that when the set E is smooth enough one has

$$P_\eta(E) = \int_{\partial E} \eta(\nu^E) d\mathcal{H}^{N-1}\,,$$

which is the perimeter of E weighted by the surface tension $\eta(\nu^E)$. The notation $\mathcal{H}^s$, $s > 0$, stands for the s-dimensional Hausdorff measure. It is also useful to recall the notion of relative perimeter: given an open set $\Omega \subset \mathbb{R}^N$ we will denote by $P_\eta(E;\Omega)$ the η-perimeter of E relative to Ω; i.e.,

$$P_\eta(E;\Omega) := \sup\left\{ \int_E \mathrm{div}\zeta \, dx : \zeta \in C^1_c(\Omega;\mathbb{R}^N),\ \eta^\circ(\zeta) \le 1 \right\}.$$

As before, note that if E is sufficiently regular, then

$$P_\eta(E;\Omega) = \int_{\partial E\cap\Omega} \eta(\nu^E) d\mathcal{H}^{N-1}\,.$$

We will often use the following characterization:

$$\partial\eta(\nu) = \{\xi : \eta^\circ(\xi) \le 1 \text{ and } \xi\cdot\nu \ge \eta(\nu)\} \tag{2.2}$$

(and the symmetric statement for η°). In particular, if $\nu \neq 0$ and $\xi \in \partial\eta(\nu)$, then $\eta^\circ(\xi) = 1$, and $\partial\eta(0) = \{\xi : \eta^\circ(\xi) \le 1\}$. For $R > 0$ we denote

$$W^\eta(x,R) := \{y : \eta^\circ(y-x) \le R\}\,. \tag{2.3}$$

Such a set is called the *Wulff shape* (of radius R and center x) associated with the norm η and represents the unique (up to translations) solution of the anisotropic isoperimetric problem

$$\min\{P_\eta(E) : |E| = |W^\eta(0,R)|\}\,,$$

see for instance [32]. We let $W^\eta := W^\eta(0,1)$, $B_1 = W^{|\cdot|}$ the unit ball and more generally, for $r > 0$, $B_r = \{x : |x| \le r\}$.

We denote by $\mathrm{dist}^\eta(\cdot,E)$ the distance from E induced by the norm η, that is, for any $x \in \mathbb{R}^N$

$$\mathrm{dist}^\eta(x,E) := \inf_{y\in E} \eta(x-y)$$

if $E \neq \emptyset$, and $\mathrm{dist}^\eta(x,\emptyset) := +\infty$. Moreover, we denote by d^η_E the signed distance from E induced by η, i.e.,

$$d^\eta_E(x) := \mathrm{dist}^\eta(x,E) - \mathrm{dist}^\eta(x,E^c)\,. \tag{2.4}$$

so that $\mathrm{dist}^\eta(x,E) = d^\eta_E(x)^+$ and $\mathrm{dist}^\eta(x,E^c) = d^\eta_E(x)^-$, where we adopted the standard notation $t^+ := t \vee 0$ and $t^- := (-t)^+$. Note that by (2.2) we have $\eta(\nabla d^{\eta^\circ}_E) = \eta^\circ(\nabla d^\eta_E) = 1$ a.e. in $\mathbb{R}^N \setminus \partial E$. We will write $\mathrm{dist}(\cdot,E)$ and d_E without any superscript to denote the Euclidean distance and signed distance from E, respectively.

Finally we recall that a sequence of closed sets $(E_n)_{n\ge1}$ in $\mathbb{R}^m$ converges to a closed set E in the *Kuratowki sense* if the following conditions are satisfied:

(i) if $x_n \in E_n$ for each n, any limit point of $(x_n)_{n\ge1}$ belongs to E;
(ii) any $x \in E$ is the limit of a sequence $(x_n)_{n\ge1}$, with $x_n \in E_n$ for each n.

We write in this case:

$$E_n \xrightarrow{\mathcal{K}} E\,.$$

It is easily checked that $E_n \xrightarrow{\mathcal{K}} E$ if and only if (for any norm η) $\mathrm{dist}^\eta(\cdot, E_n) \to \mathrm{dist}^\eta(\cdot, E)$ locally uniformly in $\mathbb{R}^m$. In particular, Ascoli-Arzelà Theorem shows that any sequence of closed sets admits a converging subsequence in the Kuratowski sense.

2.2. **The distributional formulation.** In this subsection we give the precise formulation of the crystalline mean curvature flows we will deal with. Throughout the paper the norms ϕ and ψ will stand for the *anisotropy* and the *mobility*, respectively, appearing in (1.1). Note that we do not assume any regularity on ϕ (nor on ψ) and in fact we are mainly interested in the case when ϕ is crystalline, that is, when the associated unit ball is a polytope.

Moreover, we will assume throughout the paper that the forcing term $g : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ satisfies the following two hypotheses:

H1) $g \in L^\infty(\mathbb{R}^N \times (0, \infty))$;

H2) there exists $L > 0$ such that $g(\cdot, t)$ is L-Lipschitz continuous (with respect to the metric ψ°) for a.e. $t > 0$.

Remark 2.1. Assumption H1) can be weakened and replaced by

H1)' for every $T > 0$, $g \in L^\infty(\mathbb{R}^N \times (0, T))$.

Indeed under the weaker assumption H1)', all the arguments and the estimates presented throughout the paper continue to work in any time interval $(0, T)$, with some of the constants involved possibly depending on T. In the same way, if one restricts our study to the evolution of sets with compact boundary, then one could assume that g is only locally bounded in space. We assume H1) instead of H1)' to simplify the presentation.

In all that follows by the expression "admissible forcing term" we will mean a forcing term g satisfying H1) and H2) above.

We are now ready to provide a suitable distributional formulation of the curvature flow (1.1).

Definition 2.2. *Let $E^0 \subset \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$ and for each $t \geq 0$ denote $E(t) := \{x \in \mathbb{R}^N : (x, t) \in E\}$. We say that E is a* superflow *of* (1.1) *with initial datum E^0 if*

(a) Initial Condition: *$E(0) \subseteq E^0$;*

(b) Left Continuity: *$E(s) \xrightarrow{\mathcal{K}} E(t)$ as $s \nearrow t$ for all $t > 0$;*

(c) *If $E(t) = \emptyset$ for some $t \geq 0$, then $E(s) = \emptyset$ for all $s > t$.*

(d) Differential Inequality: *Set $T^* := \inf\{t > 0 : E(s) = \emptyset$ for $s \geq t\}$, and*

$$d(x, t) := \mathrm{dist}^{\psi^\circ}(x, E(t)) \qquad \textit{for all } (x, t) \in \mathbb{R}^N \times (0, T^*) \setminus E.$$

Then there exists $M > 0$ such that the inequality

$$\partial_t d \geq \mathrm{div} z + g - Md \tag{2.5}$$

holds in the distributional sense in $\mathbb{R}^N \times (0, T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0, T^*))$ such that $z \in \partial\phi(\nabla d)$ a.e., $\mathrm{div} z$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$, and $(\mathrm{div} z)^+ \in L^\infty(\{(x, t) \in \mathbb{R}^N \times (0, T^*) : d(x, t) \geq \delta\})$ for every $\delta \in (0, 1)$.

We say that A, open set in $\mathbb{R}^N \times [0,+\infty)$, is a subflow of (1.1) *with initial datum E^0 if A^c is a superflow of* (1.1) *with g replaced by $-g$ and with initial datum $(\mathring{E}^0)^c$.*

Finally, we say that E, closed set in $\mathbb{R}^N \times [0,+\infty)$, is a solution of (1.1) *with initial datum E^0 if it is a superflow and if $\mathring{E}$ is a subflow, both with initial datum E^0, assuming in addition that both E^0 and E coincide with the closure of their interior.*

In Subsection 4.2 we will prove the existence of solutions satisfying (2.5) with $M = L$. In our definition, "super"-flow refers to the fact that the distance function may grow *faster* than the distance to a solution of the mean curvature flow, which corresponds to a set shrinking also faster than expected with (1.1).

Remark 2.3. Notice that the closedness of E yields that d is lower semicontinuous. Indeed, if $(x_k, t_k) \to (x, t)$, with $t_k, t \le T^*$, we can choose $y_k \in E(t_k)$ with $\psi^\circ(x_k - y_k) = d(x_k, t_k)$, then, since any limit point of (y_k, t_k) is in E, one deduces $d(x,t) \le \liminf_k d(x_k, t_k)$. On the other hand, condition (b) implies that $d(\cdot, t)$ is left-continuous. Moreover, by condition (d) of Definition 2.2, the distributional derivative $\partial_t d$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$, so that d is locally a function with bounded variation; using the fact that the distance functions are uniformly Lipschitz, we can deduce that for any $t \in [0, T^*)$, $d(\cdot, s)$ converges locally uniformly in $\{x : d(x,t) > 0\}$ as $s \searrow t$ to some function d^r with $d^r \ge d(\cdot, t)$ in $\{x : d(x,t) > 0\}$, while $d(\cdot, s)$ converges locally uniformly to $d(\cdot, t)$ as $s \nearrow t$ (*cf* [22, Lemma 2.4]).

Remark 2.4. Notice that the initial condition for subflows may be rewritten as $\mathring{E}^0 \subseteq A(0)$. In particular, if E is a solution according to the previous definition, then $E(0) = E^0$.

We now introduce the corresponding notion of sub- and supersolution to the level set flow associated with (1.1).

Definition 2.5 (Level set subsolutions and supersolutions)**.** *Let u^0 be a uniformly continuous function on $\mathbb{R}^N$. We will say that a lower semicontinuous function $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is a* supersolution to the level set flow *corresponding to* (1.1) *(*level set supersolution *for short), with initial datum u^0, if $u(\cdot, 0) \ge u^0$ and if for a.e. $\lambda \in \mathbb{R}$ the closed sublevel set $\{(x,t) : u(x,t) \le \lambda\}$ is a superflow of* (1.1) *in the sense of Definition 2.2, with initial datum $\{u_0 \le \lambda\}$.*

We will say that an upper-semicontinuous function $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is a subsolution to the level set flow *corresponding to* (1.1) *(*level set subsolution *for short), with initial datum u^0, if $-u$ is a level set supersolution in the previous sense, with initial datum $-u_0$ and with g replaced by $-g$.*

Finally, we will say that a continuous function $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is a solution *to the level set flow corresponding to* (1.1) *if it is both a level set subsolution and supersolution.*

2.3. **The comparison principle.** In this subsection we establish a comparison principle between sub- and superflows as defined in the previous subsection. A first technical result is a (uniform) left-continuity estimate for the distance function to a superflow.

Lemma 2.6. *Let E be a superflow in the sense of Definition 2.2, and $d(x,t) = \mathrm{dist}^{\psi^\circ}(x, E(t))$ the associated distance function. Then, there exist τ_0, χ depending on $N, \|g\|_\infty, M$ such that for any $x, t \ge 0$ and any $s \in [0, \tau_0]$,*

$$d(x, t+s) \ge d(x,t) e^{-5Ms} - \chi\sqrt{s}, \tag{2.6}$$

and (*for any* $s \in [0, \tau_0]$ *with* $s \le t$)

$$d(x, t - s) \le d(x, t)e^{5Ms} + \chi\sqrt{s}. \tag{2.7}$$

Proof. The proof follows the lines of the proof of [22, Lemma 3.2] up to minor changes that we will briefly describe in the following. By definition of a superflow we have

$$\partial_t d \ge \mathrm{div} z - Md - \|g\|_\infty,$$

wherever $d > 0$. Consider $(\bar{x}, \bar{t})$ with $d(\bar{x}, \bar{t}) = R > 0$. For $s > 0$, let $\tau(s) := \log(1 + Ms)/M$ and define

$$\delta(x, s) = d(x, \bar{t} + \tau(s))(1 + Ms) + \|g\|_\infty s \ge 0.$$

We have that $\delta(x, 0) = d(x, \bar{t})$, while, in $\{d > 0\}$, $\delta(x, \cdot)$ is BV in time, the singular part $\partial_s^s \delta$ is nonnegative (as the singular part $\partial_t^s d$ is nonnegative thanks to (2.5) and the assumption on $\mathrm{div} z$) and the absolutely continuous part satisfies

$$\partial_s^a \delta(x, s) = (\partial_t^a d(x, \bar{t}+\tau(s))\tau'(s)(1 + Ms) + Md(x, \bar{t}+\tau(s)) + \|g\|_\infty \ge \mathrm{div} z(x, \bar{t}+\tau(s)).$$

As $z(x, \tau(s)) \in \partial\phi(\nabla\delta(x, s))$, we obtain that δ is a supersolution of the ϕ-total variation flow starting from $d(\cdot, \bar{t})$, and we can reproduce the proof of [22, Lemma 3.2]: we find that there exists a constant χ_N such that $\delta(\bar{x}, s) \ge R - \chi_N\sqrt{s}$ for $s \ge 0$ as long as this bound ensures that $d(\bar{x}, \bar{t} + \tau(s)) > 3R/4$, which is as long as

$$\frac{R}{4} - \chi_N\sqrt{s} - \left(\|g\|_\infty + \frac{3MR}{4}\right) s > 0. \tag{2.8}$$

Now we prove that for any $s \ge 0$

$$d(\bar{x}, \bar{t} + \tau(s))(1 + Ms) \ge R - 4\chi_N\sqrt{s} - (4\|g\|_\infty + 3MR)s. \tag{2.9}$$

Indeed, as long as (2.8) holds true we have

$$\begin{aligned} R - 4\chi_N\sqrt{s} - (4\|g\|_\infty + 3MR)s &\le R - \chi_N\sqrt{s} - \|g\|_\infty s \\ &\le \delta(\bar{x}, s) - \|g\|_\infty s = d(\bar{x}, \bar{t} + \tau(s))(1 + Ms). \end{aligned}$$

On the other hand, for later times the left-hand side of (2.9) is (always) nonnegative and the right-hand side becomes nonpositive. Notice that (2.9) can be rewritten as

$$d(\bar{x}, \bar{t} + \tau(s))(1 + Ms) \ge d(\bar{x}, \bar{t})(1 - 3Ms) - 4\chi_N\sqrt{s} - 4\|g\|_\infty s,$$

and since this holds for any $s \ge 0$ and does not depend on the particular value of R, it holds in fact for any $\bar{x}, \bar{t}$ and we denote this point simply by x, t in the sequel.

Since $s = (e^{M\tau(s)} - 1)/M$, we deduce that for any $x, t \ge 0, \tau \ge 0$,

$$d(x, t + \tau) \ge d(x, t)(4e^{-M\tau} - 3) - 4\|g\|_\infty \frac{1 - e^{-M\tau}}{M} - 4\chi_N\sqrt{\frac{e^{M\tau} - 1}{M}}e^{-M\tau}.$$

Inequality (2.6) follows by Taylor expansion, while (2.7) is in fact equivalent to (2.6), up to a change of constants. □

We can now show the following important comparison result.

Theorem 2.7. *Let E be a superflow with initial datum E^0 and F be a subflow with initial datum F^0 in the sense of Definition 2.2. Assume that* $\mathrm{dist}^{\psi^\circ}(E^0, F^{0^c}) =: \Delta > 0$*. Then,*

$$\mathrm{dist}^{\psi^\circ}(E(t), F^c(t)) \geq \Delta e^{-Mt} \qquad \textit{for all } t \geq 0,$$

where $M > 0$ is as in (2.5) *for both E and F.*

Proof. Let T_E^* and T_F^* be the maximal existence time for E and F. For all $t > \min\{T_E^*, T_F^*\} =: T^*$ we have that either E or F^c is empty. For all such t's the conclusion clearly holds true.

Thus, we may assume without loss of generality that $T_E^*, T_F^* > 0$ and we consider the case $t \leq T^*$. By iteration (thanks to the left-continuity of d) it is clearly enough to show the conclusion of the theorem for a time interval $(0, t^*)$ for some $0 < t^* \leq T^*$.

Let us fix $0 < \eta_1 < \eta_2 < \eta_3 < \Delta/2$. We denote by z_E and z_{F^c} the fields appearing in the definition of superflow (see Definition 2.2), corresponding to E and F^c, respectively. Consider the set

$$S := \{x \in \mathbb{R}^N : d_E^{\psi^\circ}(x, 0) > \eta_1\} \cap \{x \in \mathbb{R}^N : d_{F^c}^{\psi^\circ}(x, 0) > \eta_1\}\,.$$

We now set

$$\begin{aligned} \widetilde{d}_E &:= d_E^{\psi^\circ} \vee (\eta_2 + Ct)\,, \\ \widetilde{d}_{F^c} &:= d_{F^c}^{\psi^\circ} \vee (\eta_2 + Ct)\,, \end{aligned}$$

with $C > 0$ to be chosen later. By our assumptions $(\widetilde{d}_E + \widetilde{d}_{F^c})(\cdot, 0) \geq \Delta$. Moreover, since by construction

$$\widetilde{d}_E + \widetilde{d}_{F^c} \geq \Delta + (\eta_2 - \eta_1) \text{ on } \partial S \times \{0\}\,,$$

it follows from Lemma 2.6 that there exists $t^* \in (0, 1 \wedge T^*)$ such that

$$\widetilde{d}_E + \widetilde{d}_{F^c} \geq \Delta \text{ on } \partial S \times (0, t^*)\,. \tag{2.10}$$

Relying again on Lemma 2.6 and arguing similarly we also have (for a possibly smaller t^*)

$$S \subset \left\{x \in \mathbb{R}^N : d_E^{\psi^\circ}(x, t) > \frac{\eta_1}{2}\right\} \cap \left\{x \in \mathbb{R}^N : d_{F^c}^{\psi^\circ}(x, t) > \frac{\eta_1}{2}\right\} \quad \text{for all } t \in (0, t^*)\,, \tag{2.11}$$

$$E(t) \subset\subset F(t) \qquad \text{for } t \in (0, t^*)$$

and:

$$\widetilde{d}_E = d_E^{\psi^\circ} \quad \text{and} \quad \widetilde{d}_{F^c} = d_{F^c}^{\psi^\circ} \qquad \text{in } S'' \times (0, t^*)\,, \tag{2.12}$$

where

$$S'' := \{x \in \mathbb{R}^N : d_E^{\psi^\circ}(x, 0) > \eta_3\} \cap \{x \in \mathbb{R}^N : d_{F^c}^{\psi^\circ}(x, 0) > \eta_3\}\,.$$

Since $d_E^{\psi^\circ}$ is Lipschitz continuous in space and $\partial_t d_E^{\psi^\circ}$ is a measure wherever $d_E^{\psi^\circ}$ is positive, it follows that $d_E^{\psi^\circ}$ (and in turn $\widetilde{d}_E$) is a function in $BV_{loc}(S \times (0, t^*))$ and its distributional time derivative has the form[4]

$$\partial_t d_E^{\psi^\circ} = \sum_{t \in J} \left(d_E^{\psi^\circ}(\cdot, t+0) - d_E^{\psi^\circ}(\cdot, t-0)\right) dx + \partial_t^d d_E^{\psi^\circ}$$

where J is the (countable) set of times where $d_E^{\psi^\circ}$ jumps and $\partial_t^d d_E^{\psi^\circ}$ is the diffuse part of the derivative. It turns out that (see Remark 2.3) $d_E^{\psi^\circ}(\cdot, t+0) - d_E^{\psi^\circ}(\cdot, t-0) \geq 0$ for each $t \in J$.

[4]with a slight abuse of notation, in the jump part at $t \in J$ we denote dx what should be the Hausdorff measure $\mathcal{H}^N$ on the hyperplane $\mathbb{R}^N \times \{t\}$.

Moreover, since the positive part of $\mathrm{div} z_E$ is absolutely continuous with respect to the Lebesque measure (*cf* Def. 2.2, (d)), (2.5) entails

$$\partial_t^d d_E^{\psi^\circ} \geq \mathrm{div} z_E + g - M d_E^{\psi^\circ}$$

in $S \times (0, t^*)$. Using the chain rule (see for instance [5]), in $S \times (0, t^*)$ we have

$$\partial_t^d \widetilde{d}_E = \begin{cases} C & \text{a.e. in } \{(x,t) : \eta_2 + Ct > d_E^{\psi^\circ}(x,t)\}\,, \\ \partial_t^d d_E^{\psi^\circ} & |\partial_t^d d_E^{\psi^\circ}|\text{-a.e. in } \{(x,t) : \eta_2 + Ct \leq d_E^{\psi^\circ}(x,t)\}\,. \end{cases}$$

An analogous formula holds for $\partial_t^d \widetilde{d}_{F^c}$. Recalling that $(\mathrm{div} z_E)^+$ and $(\mathrm{div} z_{F^c})^+$ belong to $L^\infty(S \times (0, t^*))$ it follows that

$$\partial_t^d \widetilde{d}_E \geq \mathrm{div} z_E + g - M \widetilde{d}_E \quad \text{and} \quad \partial_t^d \widetilde{d}_{F^c} \geq \mathrm{div} z_{F^c} - g - M \widetilde{d}_{F^c} \tag{2.13}$$

in the sense of measures in $S \times (0, t^*)$ provided that we have chosen (*cf* (2.11) and the conditions in Def. 2.2, (d))

$$C \geq \|(\mathrm{div} z_E)^+\|_{L^\infty(S\times(0,t^*))} + \|(\mathrm{div} z_{F^c})^+\|_{L^\infty(S\times(0,t^*))} + \|g\|_{L^\infty(S\times(0,t^*))}\,.$$

Note also that a.e. in $S \times (0, t^*)$

$$z_E \in \partial\phi(\nabla \widetilde{d}_E) \quad \text{and} \quad z_{F^c} \in \partial\phi(\nabla \widetilde{d}_{F^c})\,. \tag{2.14}$$

Fix $p > N$ and set $\Psi(s) := (s^+)^p$ and $w := \Psi(\Delta - e^{Mt}(\widetilde{d}_E + \widetilde{d}_{F^c}))$. By (2.10) we have

$$w = 0 \qquad \text{on } \partial S \times (0, t^*)\,. \tag{2.15}$$

Using as before the chain rule for BV functions, recalling (2.13) and the fact that the jump parts of $\partial_t \widetilde{d}_E$ and $\partial_t \widetilde{d}_{F^c}$ are nonnegative, in $S \times (0, t^*)$ we have

$$\begin{aligned} \partial_t w &\leq -\Psi'\big(\Delta - e^{Mt}(\widetilde{d}_E + \widetilde{d}_{F^c})\big) e^{Mt} \Big(M(\widetilde{d}_E + \widetilde{d}_{F^c}) + \partial_t^d(\widetilde{d}_E + \widetilde{d}_{F^c}) \Big) \\ &\leq -\Psi'\big(\Delta - e^{Mt}(\widetilde{d}_E + \widetilde{d}_{F^c})\big) e^{Mt} \mathrm{div}(z_E + z_{F^c})\,, \end{aligned} \tag{2.16}$$

where in the last inequality we have used (2.13). Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on B_1. For every $\varepsilon > 0$ we set $\eta_\varepsilon(x) := \eta(\varepsilon x)$. Using (2.15) and (2.16), we have

$$\begin{aligned} \partial_t \int_S w \eta_\varepsilon^p dx &\leq -e^{Mt} \int_S \eta_\varepsilon^p \Psi'\big(\Delta - e^{Mt}(\widetilde{d}_E + \widetilde{d}_{F^c})\big) \mathrm{div}(z_E + z_{F^c}) \\ &= -e^{Mt} \int_S \eta_\varepsilon^p \Psi''\big(\Delta - e^{Mt}(\widetilde{d}_E + \widetilde{d}_{F^c})\big)(\nabla \widetilde{d}_E + \nabla \widetilde{d}_{F^c}) \cdot (z_E + z_{F^c})\, dx \\ &\quad + p e^{Mt} \int_S \eta_\varepsilon^{p-1}\, \Psi'\big(\Delta - e^{Mt}(\widetilde{d}_E + \widetilde{d}_{F^c})\big) \nabla \eta_\varepsilon \cdot (z_E + z_{F^c})\, dx \\ &\leq p e^{Mt} \int_S \eta_\varepsilon^{p-1}\, \Psi'\big(\Delta - e^{Mt}(\widetilde{d}_E + \widetilde{d}_{F^c})\big) \nabla \eta_\varepsilon \cdot (z_E + z_{F^c})\, dx\,, \end{aligned}$$

where we have also used the inequality $(z_E + z_{F^c}) \cdot (\nabla \widetilde{d}_E + \nabla \widetilde{d}_{F^c}) \geq 0$, which follows from (2.14) and the convexity and symmetry of ϕ. By Hölder Inequality and using the explicit expression of Ψ and Ψ', we get

$$\partial_t \int_S w\, \eta_\varepsilon^p dx \leq C p^2 \|\nabla \eta_\varepsilon\|_{L^p(\mathbb{R}^N)} \left(\int_S w\, \eta_\varepsilon^p dx \right)^{1 - \frac{1}{p}}\,,$$

for some constant $C > 0$ depending only on the L^∞-norms of z_E and z_{F^c} and on t^*. Since $w = 0$ at $t = 0$, a simple ODE argument then yields

$$\int_S w\, \eta_\varepsilon^p dx \le \big(Cp\|\nabla\eta_\varepsilon\|_{L^p(\mathbb{R}^N)} t\big)^p$$

for all $t \in (0, t^*)$. Observing that $\|\nabla\eta_\varepsilon\|^p_{L^p(\mathbb{R}^N)} = \varepsilon^{p-N}\|\nabla\eta\|^p_{L^p(\mathbb{R}^N)} \to 0$ and $\eta_\varepsilon \nearrow 1$ as $\varepsilon \to 0^+$, we conclude that $w = 0$, and in turn $\widetilde{d}_E + \widetilde{d}_{F^c} \ge \Delta e^{-Mt}$ in $S \times (0, t^*)$. In particular, by (2.12), we have shown that $d_E^{\psi^\circ} + d_{F^c}^{\psi^\circ} \ge \Delta e^{-Mt}$ in $S'' \times (0, t^*)$. In turn, this easily implies that $\mathrm{dist}(E(t), F^c(t)) \ge \Delta e^{-Mt}$ for $t \in (0, t^*)$ (see the end of the proof of [22, Theorem 3.3]). This concludes the proof of the theorem. □

The previous theorem easily yields a comparison principle also between level set subsolutions and supersolutions.

Theorem 2.8. *Let u^0, v^0 be uniformly continuous functions on $\mathbb{R}^N$ and let u, v be respectively a level set subsolution with initial datum u^0 and a level set supersolution with initial datum v^0, in the sense of Definition 2.5. If $u^0 \le v^0$, then $u \le v$.*

Proof. Recall that by Definition 2.5 there exists a null set $N_0 \subset \mathbb{R}$ such that for all $\lambda \notin N_0$ the sets $\{(x,t) : u(x,t) < \lambda\}$ and $\{(x,t) : v(x,t) \le \lambda\}$ are respectively a subflow with initial datum $\{u^0 \le \lambda\}$ and a superflow with initial datum $\{v^0 \le \lambda\}$, in the sense of Definition 2.2. Fix now $\lambda \in \mathbb{R}$ and choose $\lambda < \lambda'' < \lambda'$, with λ', $\lambda'' \notin N_0$. Since $\{v^0 \le \lambda''\} \subset \{v^0 \le \lambda'\} \subset \{u^0 \le \lambda'\}$, we have

$$\mathrm{dist}^{\psi^\circ}(\{v^0 \le \lambda''\}, \{u^0 > \lambda'\}) \ge \mathrm{dist}^{\psi^\circ}(\{v^0 \le \lambda''\}, \{v^0 > \lambda'\}) := \Delta > 0\,,$$

where the last inequality follows from the uniform continuity of v^0. Thus, by Theorem 2.7, for all $t \ge 0$,

$$\{(x,t) : v(x,t) \le \lambda\} \subset \{(x,t) : v(x,t) \le \lambda''\} \subset \{(x,t) : u(x,t) < \lambda'\}\,.$$

Letting $\lambda' \searrow \lambda$, with $\lambda' \notin N_0$, we conclude that $\{(x,t) : v(x,t) \le \lambda\} \subseteq \{(x,t) : u(x,t) \le \lambda\}$ for all $\lambda \in \mathbb{R}$, which is clearly equivalent to $u \le v$. □

2.4. **Distributional versus viscosity solutions.** We show here that in the smooth cases, the notion of solution in Definition 2.2 coincides with the definition of standard viscosity solutions for geometric motions, as for instance proposed in [12].

Lemma 2.9. *Assume $\phi, \psi, \psi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, and assume that g is continuous also with respect to the time variable. Let E be a superflow in the sense of Definition 2.2. Then, $-\chi_E$ is a viscosity supersolution of*

$$u_t = \psi(\nabla u)\big(\mathrm{div}\nabla\phi(\nabla u) + g\big) \tag{2.17}$$

in $\mathbb{R}^N \times (0, T^*)$, and in fact in $\mathbb{R}^N \times (0, T^*]$ whenever $T^* < +\infty$, where T^* is the extinction time of E introduced in Definition 2.2.

A converse statement is also true, see [22, 20].

Proof. We follow the proof of a similar statement in [22, Appendix]. Let $\varphi(x,t)$ be a smooth test function and assume $-\chi_E - \varphi$ has a (strict) local minimum at $(\bar{x}, \bar{t})$, $0 < \bar{t} \le T^*$. In other words, we can assume that near $(\bar{x}, \bar{t})$, $-\chi_E(x,t) \ge \varphi(x,t)$, while $-\chi_E(\bar{x}, \bar{t}) = \varphi(\bar{x}, \bar{t})$. We can also assume that the latter quantity is -1 (i.e., $(\bar{x},\bar{t})\in E$), as if it is zero then we trivially deduce that $\nabla\varphi(\bar{x},\bar{t})=0$ while $\partial_t\varphi(\bar{x},\bar{t})\geq 0$.

If $\nabla\varphi(\bar{x},\bar{t})=0$, thanks to [10, Prop. 2.2] we can assume that also the spatial Hessian $D^2\varphi(\bar{x},\bar{t})=0$ (and then $D^3\varphi(\bar{x},\bar{t})=0$, $D^4\varphi(\bar{x},\bar{t})\leq 0$). As usual, if we assume that $-a=\partial_t\varphi(\bar{x},\bar{t})<0$ and choose $a'<a$, we observe that near $\bar{x}$, $\partial_t\varphi(x,\bar{t})<-a'$ and, for $t\leq\bar{t}$ close enough to $\bar{t}$ and x close enough to $\bar{x}$,

$$\varphi(x,t)=\varphi(x,\bar{t})+\partial_t\varphi(x,\bar{t})(t-\bar{t})+o(|t-\bar{t}|)\geq\varphi(x,\bar{t})+a'(\bar{t}-t).$$

Hence, one has that near $(\bar{x},\bar{t})$ (for $t\leq\bar{t}$), for some $\gamma>0$

$$\varphi(x,t)\geq -1+a'(\bar{t}-t)-\gamma|x-\bar{x}|^4.$$

It follows that for such t, $\mathcal{N}\cap\{x:\gamma|x-\bar{x}|^4<a'(\bar{t}-t)\}\cap E(t)=\emptyset$, where $\mathcal{N}$ is a neighborhood of $\bar{x}$. For $\bar{t}-t>0$ small enough we deduce that $B(\bar{x},(a'(\bar{t}-t)/\gamma)^{1/4})$ does not meet $E(t)$, in other words $d(\bar{x},t)\geq c((a'/\gamma)(\bar{t}-t))^{1/4}$ for constant c depending only on ψ. It then follows from Lemma 2.6, and more precisely from (2.6), that (provided $\bar{t}-t\leq\tau_0$ where τ_0 is as in Lemma 2.6)

$$d(\bar{x},\bar{t})\geq(\bar{t}-t)^{\frac14}\left(c\left(\frac{a'}{\gamma}\right)^{1/4}e^{-5M(\bar{t}-t)}-\chi(\bar{t}-t)^{\frac14}\right),$$

which is positive if t is close enough to $\bar{t}$, a contradiction. Hence $\partial_t\varphi(\bar{x},\bar{t})\geq 0$.

If, on the other hand, $\nabla\varphi(\bar{x},\bar{t})\neq 0$ then we can introduce the set $F=\{\varphi\leq -1\}$, and we have that $F(t)$ is a smooth set near $\bar{x}$, for $t\leq\bar{t}$ close to $\bar{t}$, which contains $E(t)$, with a contact at $(\bar{x},\bar{t})$. We then let $\delta(x,t)=\mathrm{dist}^{\psi^\circ}(x,F(t))$, which at least C^2 near $(\bar{x},\bar{t})$ (as ψ,ψ° are C^2) and is touching d from below at all the points $(\bar{x}+s\nabla\phi(\nu_{F(\bar{t})}),\bar{t})$ for $s>0$ small.

Assume that

$$\partial_t\delta<\psi(\nabla\delta)(\mathrm{div}\nabla\phi(\nabla\delta)+g)=D^2\phi(\nabla\delta):D^2\delta+g \tag{2.18}$$

at $(\bar{x},\bar{t})$. Then, by continuity, we can find $\bar{s}>0$ small and a neighborhood $B=\{|x-\bar{x}|<\rho,\bar{t}-\rho<t\leq\bar{t}\}$ of $(\bar{x},\bar{t})$ in $\mathbb{R}^N\times(0,\bar{t}]$ where

$$\partial_t\delta<D^2\phi(\nabla\delta):D^2\delta+g-M\bar{s}.$$

Possibly reducing ρ and using (*cf* Rem. 2.3) the left-continuity of d, since $d(\bar{x},\bar{t})=0$, we can also assume that $d\leq\bar{s}$ in B.

We choose then $s<\bar{s}$ small enough so that $\bar{x}^s=\bar{x}+s\nabla\phi(\nu_{F(\bar{t})})$ is such that $|\bar{x}^s-\bar{x}|<\rho$, and for $\eta>0$ small we define $\delta^\eta(x,t)=\delta(x,t)-\eta(|x-\bar{x}^s|^2+|t-\bar{t}|^2)/2$. Then $d-\delta^\eta$ has a unique strict minimum point at $(\bar{x}^s,\bar{t})$ in B. Moreover if η is small enough, by continuity, we still have that

$$\partial_t\delta^\eta<D^2\phi(\nabla\delta^\eta):D^2\delta^\eta+g-M\bar{s}.$$

in B.

Then we continue as in [22, Appendix]: given $\Psi\in C^\infty(\mathbb{R})$ nonincreasing, convex, vanishing on $\mathbb{R}_+$ and positive on $(-\infty,0)$, we introduce $w=\Psi(d-\delta^\eta-\varepsilon)\chi_B$ for $\varepsilon<d(\bar{x}^s,\bar{t}-\rho)-\delta^\eta(\bar{x}^s,\bar{t}-\rho)$

small enough. We then show that, thanks to (2.5), for $\bar t-\rho<t<\bar t$,

$$\begin{aligned}\partial_t\int w dx &\le \int_B \Psi'(d-\delta^\eta-\varepsilon)(\mathrm{div} z+g-Md-\mathrm{div}\nabla\phi(\nabla\delta^\eta)-g+M\bar s)dx\\ &\le -\int_B \Psi''(d-\delta^\eta-\varepsilon)(\nabla d-\nabla\delta^\eta)\cdot(z-\nabla\phi(\nabla\delta^\eta))dx+M\int_B \Psi'(d-\delta^\eta-\varepsilon)(\bar s-d)dx\le 0\end{aligned}$$

as we have assumed $d\le\bar s$ in B. This is in contradiction with $\Psi(d(\bar x^s,\bar t)-\delta^\eta(\bar x^s,\bar t)-\varepsilon)=\Psi(-\varepsilon)>0$, and it follows that (2.18) cannot hold: one must have

$$\partial_t\delta\ge\psi(\nabla\delta)(\mathrm{div}\nabla\phi(\nabla\delta)+g)$$

at $(\bar x,\bar t)$. Since this equation is geometric and the level set $\{\delta\le 0\}$ is F, which is the level -1 of φ (near $(\bar x,\bar t)$), we also deduce that at the same point,

$$\partial_t\varphi\ge\psi(\nabla\varphi)(\mathrm{div}\nabla\phi(\nabla\varphi)+g)$$

so that $-\chi_E$ is a supersolution of (2.17). □

3. Minimizing movements

As in [22], in order to build solutions to our geometric evolution problem, we implement a variant of the Almgren-Taylor-Wang [3] minimizing movements scheme (1.4) (in short the ATW scheme) introduced in [19, 18]. In Section 3.2 we adapt this construction to take into account the forcing term, as in [24]. We start by presenting some preliminary properties of the incremental problem.

3.1. **The incremental problem.** Given $z\in L^\infty(\mathbb{R}^N;\mathbb{R}^N)$ with $\mathrm{div} z\in L^2_{loc}(\mathbb{R}^N)$ and $w\in BV_{loc}(\mathbb{R}^N)\cap L^2_{loc}(\mathbb{R}^N)$, let $z\cdot Dw$ denotes the Radon measure associated with the linear functional

$$L\varphi := -\int_{\mathbb{R}^N} w\,\varphi\mathrm{div} z\,dx-\int_{\mathbb{R}^N} w\,z\cdot\nabla\varphi\,dx \qquad \text{for all } \varphi\in C^\infty_c(\mathbb{R}^N),$$

see [9]. We recall the following result:

Proposition 3.1. *Let $p>\max\{N,2\}$, $f\in L^p_{loc}(\mathbb{R}^N)$ and $h>0$. There exist a field $z\in L^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and a unique function $u\in BV_{loc}(\mathbb{R}^N)\cap L^p_{loc}(\mathbb{R}^N)$ such that the pair (u,z) satisfies*

$$\begin{cases} -h\,\mathrm{div} z+u=f & \text{in } \mathcal{D}'(\mathbb{R}^N),\\ \phi^\circ(z)\le 1 & \text{a.e. in } \mathbb{R}^N,\\ z\cdot Du=\phi(Du) & \text{in the sense of measures.}\end{cases} \tag{3.1}$$

Moreover, for any $R>0$ and $v\in BV(B_R)$ with $\mathrm{Supp}\,(u-v)\Subset B_R$,

$$\phi(Du)(B_R)+\frac{1}{2h}\int_{B_R}(u-f)^2\,dx\le\phi(Dv)(B_R)+\frac{1}{2h}\int_{B_R}(v-f)^2\,dx,$$

and for every $s\in\mathbb{R}$ the set $E_s:=\{x\in\mathbb{R}^N: u(x)\le s\}$ solves the minimization problem

$$\min_{F\Delta E_s\Subset B_R} P_\phi(F;B_R)+\frac{1}{h}\int_{F\cap B_R}(f(x)-s)\,dx.$$

If $f_1\le f_2$ and if u_1, u_2 are the corresponding solutions to (3.1) *(with f replaced by f_1 and f_2, respectively), then $u_1\le u_2$.*

Finally if in addition f is Lipschitz with $\psi(\nabla f) \leq 1$ for some norm ψ, then the unique solution u of (3.1) *is also Lipschitz and satisfies $\psi(\nabla u) \leq 1$ a.e. in $\mathbb{R}^N$. As a consequence,* (3.1) *is equivalent to*

$$\begin{cases} -h\,\mathrm{div} z + u = f & \text{in } \mathcal{D}'(\mathbb{R}^N), \\ z \in \partial\phi(\nabla u) & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.2}$$

Proof. See [18, Theorem 2], [1, Theorem 3.3]. □

The comparison property in the previous proposition has a "local" version, which results from the geometric character of (3.1):

Lemma 3.2. *Let f_1, f_2 be Lipschitz functions and let (u_i, z_i), $i = 1, 2$, be solutions to* (3.1) *with f replaced by f_i. Assume also that for some $\lambda \in \mathbb{R}$,*

$$|(\{u_1 \leq \lambda\} \cup \{u_2 \leq \lambda\}) \setminus \{f_1 \leq f_2\}| = 0 \qquad \text{for } i = 1, 2\,. \tag{3.3}$$

Then, $\min\{u_1, \lambda\} \leq \min\{u_2, \lambda\}$ *a.e.*

Proof. Let us set $v_i := \min\{u_i, \lambda\}$ for $i = 1, 2$ and observe that

$$z_i \in \partial\phi(\nabla v_i) \quad \text{a.e.}. \tag{3.4}$$

Writing (3.1) for u_i and subtracting the equations we get

$$-h\mathrm{div}(z_1 - z_2) + (u_1 - u_2) = f_1 - f_2\,. \tag{3.5}$$

Let ψ be a smooth, increasing, nonnegative function with support in $(0, +\infty)$ and $\eta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}_+)$, and let $p > N$. First notice that

$$\int_{\mathbb{R}^N} (v_1 - v_2)\psi(v_1 - v_2)\eta^p\,dx = \int_{\{v_1 > v_2\}} (v_1 - v_2)\psi(v_1 - v_2)\eta^p\,dx \leq \int_{\mathbb{R}^N} (u_1 - u_2)\psi(v_1 - v_2)\eta^p\,dx$$

since it can be easily checked that $v_1 - v_2 \leq u_1 - u_2$ in $\{v_1 > v_2\}$. Thus, from (3.5) we deduce that

$$\begin{aligned} ph \int_{\mathbb{R}^N} (z_1 - z_2) \cdot \nabla\eta\, \psi(v_1 - v_2)\eta^{p-1} dx + h \int_{\mathbb{R}^N} (z_1 - z_2) \cdot (\nabla v_1 - \nabla v_2)\psi'(v_1 - v_2)\eta^p\,dx \\ + \int_{\mathbb{R}^N} (v_1 - v_2)\psi(v_1 - v_2)\eta^p\,dx \leq \int_{\mathbb{R}^N} (f_1 - f_2)\psi(v_1 - v_2)\eta^p\,dx. \end{aligned}$$

Notice that the last integral in this equation is nonpositive, since by (3.3), the set $\{f_1 > f_2\}$ is contained (up to a negligible set) in $\{u_1 > \lambda\} \cap \{u_2 > \lambda\} \subseteq \{v_1 = v_2\}$. Hence, using also that $(z_2 - z_1) \cdot (\nabla v_2 - \nabla v_1) \geq 0$ thanks to (3.4), we deduce

$$ph \int_{\mathbb{R}^N} (z_1 - z_2) \cdot \nabla\eta\, \psi(v_1 - v_2)\eta^{p-1} dx + \int_{\mathbb{R}^N} (v_1 - v_2)\psi(v_1 - v_2)\eta^p\,dx \leq 0.$$

Letting $\psi(s) \to (s^+)^{p-1}$ we obtain

$$\begin{aligned} \|(v_1 - v_2)^+\eta\|^p_{L^p(\mathbb{R}^N)} &\leq -ph \int_{\mathbb{R}^N} (z_1 - z_2) \cdot \nabla\eta\, \left((v_1 - v_2)^+\eta\right)^{p-1} dx \\ &\leq ph\|(z_1 - z_2)\nabla\eta\|_{L^p(\mathbb{R}^N)}\|(v_1 - v_2)^+\eta\|^{p-1}_{L^p(\mathbb{R}^N)} \end{aligned}$$

so that

$$\|(v_1 - v_2)^+\eta\|_{L^p(\mathbb{R}^N)} \leq 2phC\|\nabla\eta\|_{L^p(\mathbb{R}^N)}$$

where $C = \max_{\partial\phi(0)} |z|$. Replacing now $\eta(\cdot)$ with $\eta(\cdot/R)$, $R > 0$, assuming $\eta(0) = 1$, we obtain

$$\|(v_1 - v_2)^+\eta(\cdot/R)\|_{L^p(\mathbb{R}^N)} \le 2phCR^{d/p-1}\|\nabla\eta\|_{L^p(\mathbb{R}^N)} \xrightarrow{R\to\infty} 0$$

as we have assumed $p > N$. It follows that $(v_1 - v_2)^+ = 0$ a.e., which is the thesis of the Lemma. □

Remark 3.3. The result remains true if $f_i \in L^p_{loc}(\mathbb{R}^N)$, $p > N$, are not assumed to be Lipschitz. This is because again, in this case, the pairing $(z_1 - z_2) \cdot D(\psi(v_1 - v_2))$ is non-negative, which may be shown for instance by first approximating the functions f_i with Lipschitz functions.

We conclude this subsection with the following useful computation, proved in [18, page 1576].

Lemma 3.4. *Let $R > 0$ and u the solution to* (3.2)*, with $f := c_1(\phi^\circ - R) \vee c_2(\phi^\circ - R)$, where $0 < c_1 \le c_2$. Then u is given by*

$$u(x) = \begin{cases} \sqrt{c_1 h}\frac{2N}{\sqrt{N+1}} - c_1 R & \text{if } \phi^\circ(x) \le \sqrt{\frac{h}{c_1}(N+1)}, \\ f(x) + h\frac{N-1}{\phi^\circ(x)} & \text{otherwise,} \end{cases}$$

as long as $h/c_1 \le R^2/(N+1)$.

3.2. **The ATW scheme.** Let ϕ, ψ and g satisfy all the assumptions stated in Subsection 2.2. Set

$$G(\cdot, t) := \int_0^t g(\cdot, s)\, ds\,.$$

Let $E^0 \subset \mathbb{R}^N$ be closed. Fix a time-step $h > 0$ and set $E^0_h := E^0$. We then inductively define E^{k+1}_h (for all $k \in \mathbb{N}$) according to the following procedure: If $E^k_h \notin \{\emptyset, \mathbb{R}^N\}$, then let $(u^{k+1}_h, z^{k+1}_h) : \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N$ satisfy

$$\left\{ \begin{array}{ll} -h\,\mathrm{div} z^{k+1}_h + u^{k+1}_h = d^{\psi^\circ}_{E^k_h} + G(\cdot, (k+1)h) - G(\cdot, kh) & \text{in } \mathcal{D}'(\mathbb{R}^N), \\ z^{k+1}_h \in \partial\phi(\nabla u^{k+1}_h) & \text{a.e. in } \mathbb{R}^N, \end{array} \right. \tag{3.6}$$

and set $E^{k+1}_h := \{x : u^{k+1}_h \le 0\}$.[5] If either $E^k_h = \emptyset$ or $E^k_h = \mathbb{R}^N$, then set $E^{k+1}_h := E^k_h$. We denote by T^*_h the first discrete time hk such that $E^k_h = \emptyset$, if such a time exists; otherwise we set $T^*_h = +\infty$. Analogously, we denote by ${T'}^*_h$ the first discrete time hk such that $E^k_h = \mathbb{R}^N$, if such a time exists; otherwise we set ${T'}^*_h = +\infty$.

Remark 3.5. In the following, when changing mobilities, forcing terms, and initial data, we will sometimes write $(E^0)^{\psi,k}_{g,h}$ in place of E^k_h in order to highlight the dependence of the scheme on ψ, g, and E^0. More generally, given any closed set H, $H^{\psi,k}_{g,h}$ will denote *the k-th minimizing movements starting from H with mobility ψ, forcing term g and time-step h*, as described by the algorithm above.

Remark 3.6 (Monotonicity of the scheme). From the comparison property stated in Proposition 3.1 it easily follows that if $E^0 \subseteq F^0$ are closed sets, then (with the notation introduced in the previous remark) $(E^0)^{\psi,k}_{g,h} \subseteq (F^0)^{\psi,k}_{g,h}$ for all $k \in \mathbb{N}$. In addition, note that $\overline{\left((E^0)^{\psi,k}_{g,h}\right)^c} = \left(\overline{(E^0)^c}\right)^{\psi,k}_{-g,h}$ for all k. Thus, if $\mathrm{dist}(E^0, F^0) > 0$, then we may apply Lemma 5.2 below with $g_1 = g_2 = g$, $c = 0$, and with η the Euclidean norm, to deduce that $\mathrm{dist}\left((E^0)^{\psi,k}_{g,h}, (F^0)^{\psi,k}_{-g,h}\right) > 0$ for all $k \in \mathbb{N}$.

[5]Choosing $E^{k+1}_h = \overline{\{u^{k+1}_h < 0\}}$ might provide a different (smaller) solution, which would enjoy exactly the same properties as the one we (arbitrarily) choose.

The assumption that the sets are at positive distance is necessary, otherwise one could only conclude, for instance, that the smallest solution of the ATW scheme from E^0 is in the complement of any solution from F^0, etc.

We now study the space regularity of the functions u_h^k constructed above. In the following computations, given any function $f : \mathbb{R}^N \to \mathbb{R}^m$ and $\tau \in \mathbb{R}^N$, we denote $f_\tau(\cdot) := f(\cdot + \tau)$. Then, the function $(u_h^{k+1})_\tau$ satisfies

$$
\begin{aligned}
(3.7) \qquad - h \operatorname{div}(z_h^{k+1})_\tau + (u_h^{k+1})_\tau = (d^{\psi^\circ}_{E_h^k})_\tau + (G(\cdot,(k+1)h))_\tau - (G(\cdot,kh))_\tau \leq \\
d^{\psi^\circ}_{E_h^k} + G(\cdot,(k+1)h) - G(\cdot,kh) + \psi^\circ(\tau)(1+Lh)\,,
\end{aligned}
$$

where in the last inequality we also used the Lipschitz-continuity of g. By the comparison property stated in Proposition 3.1 we deduce that $(u_h^{k+1})_\tau - (1+Lh)\psi^\circ(\tau) \leq u_h^{k+1}$. By the arbitrariness of $\tau \in \mathbb{R}^N$, we get $\psi(\nabla u_h^{k+1}) \leq 1 + Lh$, and in turn

$$
(3.8) \qquad
\begin{aligned}
u_h^{k+1} &\leq (1+Lh) d^{\psi^\circ}_{E_h^{k+1}} \quad \text{in } \left\{x : d^{\psi^\circ}_{E_h^{k+1}}(x) > 0\right\}, \\
u_h^{k+1} &\geq (1+Lh) d^{\psi^\circ}_{E_h^{k+1}} \quad \text{in } \left\{x : d^{\psi^\circ}_{E_h^{k+1}}(x) < 0\right\}.
\end{aligned}
$$

We are now in a position to define the discrete-in-time evolutions constructed via minimizing movements. Precisely, we set

$$
(3.9) \qquad
\begin{cases}
E_h(t) := E_h^{[t/h]}, \\
\mathcal{E}_h := \{(x,t) : x \in E_h(t)\}, \\
d_h(x,t) := d^{\psi^\circ}_{E_h(t)}(x), \\
u_h(x,t) := u_h^{[t/h]}(x), \\
z_h(x,t) := z_h^{[t/h]}(x),
\end{cases}
$$

where $[\cdot]$ stands for the integer part.

We conclude this subsection with the following remark.

Remark 3.7 (Discrete comparison principle)*.* Remark 3.6 now reads as follows: If $E^0 \subseteq F^0$ are closed sets and if we denote by E_h and F_h the discrete evolutions with initial datum E^0 and F^0, respectively, then $E_h(t) \subseteq F_h(t)$ for all $t \geq 0$. Analogously, if $\operatorname{dist}(E^0, F^0) > 0$, E_h is defined with a forcing g and F_h with the forcing $-g$, then $\operatorname{dist}(E_h(t), F_h(t)) > 0$ for all $t \geq 0$.

3.3. **Evolution of ϕ-Wulff shapes.** We start paving the way for the convergence analysis of the scheme, by deriving some estimates on the minimizing movements starting from a Wulff shape. We consider as initial set the ϕ-Wulff shape $W^\phi(0,R)$, for $R > 0$. First, thanks to Lemma 3.4 (with $c_1 = c_2 = 1$) (*cf* also [18, Appendix B, Eq. (39)]), the solution of (3.1) with $f = d^{\phi^\circ}_{W^\phi(0,R)} = \phi^\circ - R$ is given by $\phi_h^\circ - R$, where

$$
(3.10) \qquad \phi_h^\circ(x) :=
\begin{cases}
\sqrt{h}\frac{2N}{\sqrt{N+1}} & \text{if } \phi^\circ(x) \leq \sqrt{h(N+1)}, \\
\phi^\circ(x) + h\frac{N-1}{\phi^\circ(x)} & \text{else.}
\end{cases}
$$

Observe then that there exist two positive constants $c_1 \leq c_2$ such that

$$
(3.11) \qquad c_1 \phi^\circ \leq \psi^\circ \leq c_2 \phi^\circ\,,
$$

and in particular

$$d^{\psi^\circ}_{W^\phi(0,R)} \le c_1(\phi^\circ - R) \vee c_2(\phi^\circ - R). \tag{3.12}$$

Thus, for any $k \in \mathbb{N}$ we have

$$d^{\psi^\circ}_{W^\phi(0,R)} + G(\cdot,(k+1)h) - G(\cdot,kh) \le c_1(\phi^\circ - R) \vee c_2(\phi^\circ - R) + \|g\|_\infty h =: f\,. \tag{3.13}$$

Denoting by u the solution to (3.2), with f defined above, then Lemma 3.4 yields

$$u(x) = \|g\|_\infty h + \begin{cases} \sqrt{c_1 h}\frac{2N}{\sqrt{N+1}} - c_1 R & \text{if } \phi^\circ(x) \le \sqrt{\frac{h}{c_1}(N+1)}, \\ f(x) + h\frac{N-1}{\phi^\circ(x)} & \text{otherwise,} \end{cases}$$

provided that $h/c_1 \le C(N)R^2$ (here and in the following $C(N)$ denotes a positive constant that depends only on the dimension N and may change from line to line). Notice that $\{u \le 0\} = W^\phi(0,\bar r)$ for

$$\bar r := \frac{R - \frac{h}{c_1}\|g\|_\infty + \sqrt{\big(R - \frac{h}{c_1}\|g\|_\infty\big)^2 - 4\frac{h}{c_1}(N-1)}}{2}.$$

Taking into account (3.13), we may apply the comparison principle stated in Proposition 3.1 to infer that if $k = [t/h]$ and $E^k_h = E_h(t) = W^\phi(0,R)$, then

$$W^\phi(0,\bar r) = \{u \le 0\} \subseteq \{u^{k+1}_h \le 0\} = E_h(t+h)\,,$$

provided that $h/c_1 \le C(N)R^2$. Since

$$\bar r \ge \sqrt{\big(R - \tfrac{h}{c_1}\|g\|_\infty\big)^2 - 4\tfrac{h}{c_1}(N-1)} \overset{(R\le 1)}{\ge} \sqrt{R^2 - 2\tfrac{h}{c_1}\big(2(N-1) + \|g\|_\infty\big)}$$

and setting for $0 \le s - t \le \frac{c_1 R^2}{4\left(2(N-1)+\|g\|_\infty\right)}$

$$r^R(s) := \sqrt{R^2 - 2\frac{s-t}{c_1}\big(2(N-1) + \|g\|_\infty\big)} \ge \frac{R}{\sqrt 2}\,, \tag{3.14}$$

by iteration we deduce that

$$W^\phi(0, r^R(s)) \subseteq E_h(s) \tag{3.15}$$

for all $0 \le s - t \le \frac{c_1 R^2}{4\left(2(N-1)+\|g\|_\infty\right)}$ and $h \le c_1 C(N) R^2$.

In particular, there exist a constant C depending only on $\|g\|_\infty$, c_1 and the dimension N, and $h_0 > 0$, depending also on R, such that for any $y \in \mathbb{R}^N$ and for all $h \le h_0$

$$W^\phi(y, R - \tfrac{C}{R}h) \subseteq \big(W^\phi(y,R)\big)^{\psi,1}_{g,h} \subseteq W^\phi(y,R). \tag{3.16}$$

3.4. **Density estimates and barriers.** In this section we collect some preliminary estimates on the incremental problem that will be crucial for the stability properties established in Section 5.1. The following density lemma and the subsequent corollary show that the solution to the incremental problem starting from a closed set E cannot be too "thin" in $\mathbb{R}^N \setminus E$. The main point is that the estimate turns out to be independent of h and ψ, see also Lemma 1.3 and Remark 1.4 in [44]. We observe that there exist positive constants a_1, a_2 such that

$$a_1|\xi| \le \phi(\xi) \le a_2|\xi| \quad \text{for all } \xi \in \mathbb{R}^N. \tag{3.17}$$

Lemma 3.8. *Let $E \subset \mathbb{R}^N$ be a closed set, $h > 0$, and let $g_h \in L^\infty(\mathbb{R}^N)$ with $\|g_h\|_\infty \le Gh$ for some $G > 0$. Let E' be a solution to*

$$\min_{F \Delta E' \subset\subset B_R} P_\phi(F; B_R) + \frac{1}{h} \int_{F \cap B_R} (d_E^{\psi^\circ}(x) + g_h(x))\, dx \tag{3.18}$$

for all positive R. Then, there exist $\sigma > 0$, depending only on N, a_1, and $r_0 > 0$ depending on N, G, with the following property: if $\bar{x}$ is such that $|E' \cap W^\phi(\bar{x}, s)| > 0$ for all $s > 0$ and $W^\phi(\bar{x}, r) \cap E = \emptyset$ with $r \le r_0$, then

$$|E' \cap W^\phi(\bar{x}, r)| \ge \sigma r^N.$$

Proof. We adapt to our context a classical argument from the regularity theory of the (quasi) minimizers of the perimeter [44, Lem. 1.3, Rem. 1.4]. As mentioned before, the main point is to use the fact that the Wulff shapes $W^\phi(\bar{x}, r)$ lies outside E to deduce that the constants σ, r_0 are independent of h and ψ°. Let $\bar{x}$ and $W^\phi(\bar{x}, r)$ be as in the statement. Fix $R > 0$ such that $W^\phi(\bar{x}, r) \subset B_R$. For all $s \in (0, r)$, set $E'(s) := E' \setminus W^\phi(\bar{x}, s)$. For a.e. s, we have:[6]

$$P_\phi(E'(s); B_R) = P_\phi(E'; B_R) - P_\phi(E' \cap W^\phi(\bar{x}, s)) + 2 \int_{E' \cap \partial W^\phi(\bar{x}, s)} \phi(\nu) d\mathcal{H}^{N-1}$$

with ν the outer normal vector to $\partial W^\phi(\bar{x}, s)$. Using also the fact that

$$\int_{E'(s) \cap B_R} d_E^{\psi^\circ}\, dx \le \int_{E' \cap B_R} d_E^{\psi^\circ}\, dx$$

(since $d_E^\psi > 0$ in E^c), by minimality of E' in (3.18) we find:

$$P_\phi(E' \cap W^\phi(\bar{x}, s)) + \frac{1}{h} \int_{E' \cap B_R} g_h(x) dx \le 2 \int_{E' \cap \partial W^\phi(\bar{x}, s)} \phi(\nu) d\mathcal{H}^{N-1} + \frac{1}{h} \int_{E'(s) \cap B_R} g_h(x) dx.$$

The above inequality and the (anisotropic, see for instance [32]) isoperimetric inequality yield:

$$\begin{aligned} 2 \int_{E' \cap \partial W^\phi(\bar{x}, s)} \phi(\nu) d\mathcal{H}^{N-1} &\ge N |W_\phi|^{\frac{1}{N}} |E' \cap W^\phi(\bar{x}, s)|^{\frac{N-1}{N}} - G |E' \cap W^\phi(\bar{x}, s)| \\ &\ge (N - Gs) |W_\phi|^{\frac{1}{N}} |E' \cap W^\phi(\bar{x}, s)|^{\frac{N-1}{N}}, \end{aligned} \tag{3.19}$$

where we have used that $|E' \cap W^\phi(\bar{x}, s)|^{\frac{1}{N}} \le |W_\phi|^{\frac{1}{N}} s$. We observe also that (using $\phi(\nabla \phi^\circ) = 1$ and the co-area formula):

$$|E' \cap W^\phi(\bar{x}, s)| = \int_{E' \cap \{\phi^\circ(\cdot - \bar{x}) \le s\}} \phi\left(\frac{\nabla \phi^\circ}{|\nabla \phi^\circ|}\right) |\nabla \phi^\circ| dx = \int_0^s \int_{E' \cap \partial W^\phi(\bar{x}, t)} \phi(\nu) d\mathcal{H}^{N-1} dt$$

so that for a.e. s, $\int_{E' \cap \partial W^\phi(\bar{x}, s)} \phi(\nu) d\mathcal{H}^{N-1} = \frac{d}{ds} |E' \cap W^\phi(\bar{x}, s)|$. Using that $|E' \cap W^\phi(\bar{x}, s)| > 0$ for all $s > 0$, the inequality (3.19) implies in turn that

$$\frac{d}{ds} |E' \cap W^\phi(\bar{x}, s)|^{\frac{1}{N}} \ge \frac{1}{2} |W_\phi|^{\frac{1}{N}} \left(1 - \tfrac{G}{N} s\right) \text{ for a.e. } s \ge 0.$$

If $r \le r_0 := N/G$ we obtain, integrating the above inequality on $(0, r)$,

$$|E' \cap W^\phi(\bar{x}, r)|^{\frac{1}{N}} \ge \frac{r}{4} |W_\phi|^{\frac{1}{N}} \ge \frac{a_1 r}{4} |B_1|^{\frac{1}{N}}$$

where we have used (3.17) for the last inequality. The thesis follows. □

[6] We use here that ϕ is even, otherwise the constant 2 in the formula should be replaced by $1 + c_\phi$ where c_ϕ is such that $\phi(\xi) \le c_\phi \phi(-\xi)$; see also Remark 6.3.

Remark 3.9. The same argument shows that a similar but h-dependent density estimate holds inside E.

We introduce the following notation: For any set $A \subset \mathbb{R}^N$, for any norm η, and for $\rho \in \mathbb{R}$ we denote

$$(A)^\eta_\rho := \{x \in \mathbb{R}^N : d^\eta_A(x) \le \rho\}\,, \tag{3.20}$$

and we will omit η in the notation if η is the Euclidean norm. We also recall the notation $E^{\psi,k}_{g,h}$ introduced in Remark 3.5 to denote the k-th minimizing movement starting from E, with mobility ψ, forcing term g and time-step h.

Corollary 3.10. *Let g and ψ be an admissible forcing term and a mobility, respectively, and let $h > 0$. Denote by $E^{\psi,1}_{g,h}$ the corresponding (single) minimizing movement starting from E (see Section 3.2 and Remark 3.5). Let σ and r_0 be the constants provided by Lemma 3.8 for $G := \|g\|_\infty + 1$. If $\bar{x} \in E^{\psi,1}_{g,h}$ and $W^\phi(\bar{x}, r) \cap E = \emptyset$ with $r \le r_0$, then*

$$|E^{\psi,1}_{g,h} \cap W^\phi(\bar{x}, r)| \ge \sigma r^N.$$

Proof. Recall that $E^{\psi,1}_{g,h} = \{u(\cdot) \le 0\}$, where u solves

$$\begin{cases} -h\,\mathrm{div} z + u = d^{\psi^\circ}_E + \int_0^h g(\cdot, s)\,ds & \text{in } \mathcal{D}'(\mathbb{R}^N), \\ \phi^\circ(z) \le 1 & \text{a.e. in } \mathbb{R}^N, \\ z \cdot Du = \phi(Du) & \text{in the sense of measures.} \end{cases}$$

Thus, by virtue of Proposition 3.1, setting $E'_\eta := \{u(\cdot) \le \eta\}$ for $\eta \in (0, h)$, we have that E'_η solves (3.18) with $g_h := \int_0^h g(\cdot, s) - \eta$. Since $\bar{x}$ belongs to the interior of E'_η and $\|g_h\|_\infty \le Gh$, from Lemma 3.8 we deduce that

$$|E'_\eta \cap W^\phi(\bar{x}, r)| \ge \sigma r^N. \tag{3.21}$$

The thesis follows by monotone convergence by letting $\eta \searrow 0^+$. □

Lemma 3.11. *Let $F \subset \mathbb{R}^N$ be a convex set and let $r > 0$. Then,*

$$|((F)_\varepsilon \setminus F) \cap B_r| \le C(N)\varepsilon r^{N-1} \qquad \text{for all } \varepsilon \ge 0,$$

where $C(N)$ depends only on the dimension N.

Proof. Notice that $(F)_s$ is convex for all positive ε, so that $(F)_s \cap B_r$ is a convex set contained in B_r, and

$$\mathcal{H}^{N-1}(\partial((F)_s \cap B_r)) \le \mathcal{H}^{N-1}(\partial B_r) = C(N)r^{N-1} \qquad \text{for all } s > 0.$$

Therefore, thanks to the coarea formula,

$$|((F)_\varepsilon \setminus F) \cap B_r(0)| = \int_0^\varepsilon \mathcal{H}^{N-1}(\partial(F)_s \cap B_r)\,ds \le \int_0^\varepsilon \mathcal{H}^{N-1}(\partial((F)_s \cap B_r))\,ds \le C(N)\varepsilon r^{N-1}.$$

□

The next lemma provides a crucial estimate on the "expansion" of any closed set E under a single minimizing movement, provided that E satisfies a uniform exterior Wulff shape condition. The result is achieved by combining a barrier argument with the the density estimate established in Corollary 3.10.

Lemma 3.12. *For any β, G, $\Delta > 0$, there exists $h_0 > 0$, depending on the previous constants, on the anisotropy ϕ and the dimension N, and there exists $M_0 > 0$ depending on the same quantities but Δ, with the following property: Let ψ be a mobility satisfying*

$$\psi \leq \beta\phi \tag{3.22}$$

and let g be an admissible forcing term with $\|g\|_\infty \leq G$. Then for any closed set $E \subseteq \mathbb{R}^N$ such that $\mathbb{R}^N \setminus E = \bigcup_{W\in\mathcal{G}} W$, where $\mathcal{G}$ is a family of (closed) ϕ-Wulff shapes of radius Δ, and for all $h \leq h_0$, we have $E^{\psi,1}_{g,h} \subset (E)^{\phi^\circ}_{\frac{M_0 h}{\Delta}}$.

Proof. First notice that (3.22) is equivalent to $\frac{1}{\beta}\phi^\circ \leq \psi^\circ$. Hence, recalling (3.16), there exists a constant C, depending only on $\|g\|_\infty$, β $(= 1/c_1$ in (3.16)), and N, such that

$$W^\phi(y, \Delta - \tfrac{C}{\Delta}h) \subseteq \big(W^\phi(y,\Delta)\big)^{\psi,1}_{-g,h} \subseteq W^\phi(y,\Delta)$$

for all $h \leq h_0$. By Lemma 3.11 it follows that for a possibly different constant C, depending only on $\|g\|_\infty$, β, and N, we have

$$\Big|\big(W^\phi(y,\Delta) \setminus \big(W^\phi(y,\Delta)\big)^{\psi,1}_{-g,h}\big) \cap B_r(x)\Big| \leq \frac{C}{\Delta} h r^{N-1} \quad \text{for all } x, y \in \mathbb{R}^N,\ r > 0,\ h \leq h_0\,. \tag{3.23}$$

Observe now that there exists $\theta > 0$, depending only on $W^\phi(0,1)$, such that

$$\frac{|W^\phi(y,R) \cap Q_r(x)|}{|Q_r(x)|} \geq \theta \text{ for all } y \in \mathbb{R}^N,\ x \in W^\phi(y,R),\ R \geq 1, \text{ and } r \in (0,1), \tag{3.24}$$

where $Q_r(x)$ stands for the cube of side r centered at x. Let $\sigma > 0$ be the constant provided by Corollary 3.10, and let $\mathcal{N}$ be the constant provided by the covering Lemma 3.13 below, corresponding to the constant θ in (3.24), $\delta = \sigma/4$ and $\mathcal{C} = W^\phi$. Set $M_0 := (2C\mathcal{N})/\sigma$, where C is the constant in (3.23) and note that for $r = \frac{M_0 h}{\Delta}$ we have

$$\mathcal{N}\frac{C}{\Delta} h r^{N-1} = \mathcal{N}\frac{C}{\Delta^N} h^N M_0^{N-1} = \frac{\sigma}{2}\left(\frac{M_0 h}{\Delta}\right)^N = \frac{\sigma}{2} r^N. \tag{3.25}$$

Let $x \in \mathbb{R}^N$ be such that $W^\phi\left(x, \frac{M_0 h}{\Delta}\right) \cap E = \emptyset$ for some $h \leq h_0$, and assume by contradiction that $x \in E^{\psi,1}_{g,h}$. Without loss of generality we may assume $x = 0$.

By taking h_0 smaller if needed, we can also assume that $\frac{\Delta^2}{M_0 h} \geq 1$ and $\frac{M_0 h}{\Delta} \leq r_0$ for all $h \leq h_0$, where r_0 is the radius provided by Corollary 3.10. Thus, recalling also (3.24), for $h \leq h_0$, applying Lemma 3.13 below (with $\delta = \sigma/4$) to the family

$$\mathcal{F} = \frac{\Delta}{M_0 h}\mathcal{G} = \left\{\frac{\Delta}{M_0 h} W : W \in \mathcal{G}\right\}$$

which by assumption on $\mathcal{G}$ and E covers $\mathcal{C} = W^\phi(0,1)$, we find a finite subfamily

$$\mathcal{F}' = \left\{\frac{\Delta}{M_0 h} W^\phi(x_1,\Delta), \ldots, \frac{\Delta}{M_0 h} W^\phi(x_{\mathcal{N}},\Delta)\right\}$$

of $\mathcal{N}$ elements such that

$$\left|W^\phi(0,1) \setminus \bigcup_{i=1}^{\mathcal{N}} \frac{\Delta}{M_0 h} W^\phi(x_i,\Delta)\right| \leq \frac{\sigma}{4}\,.$$

By scaling back we obtain

$$\left|W^\phi\Big(0, \frac{M_0 h}{\Delta}\Big) \setminus \bigcup_{i=1}^{\mathcal{N}} W^\phi(x_i,\Delta)\right| \leq \frac{\sigma}{4}\left(\frac{M_0 h}{\Delta}\right)^N. \tag{3.26}$$

Note now that by the comparison principle (see Remark 3.6)

$$E^{\psi,1}_{g,h} \cap W^\phi\Big(0, \frac{M_0 h}{\Delta}\Big) \subset W^\phi\Big(0, \frac{M_0 h}{\Delta}\Big) \setminus \bigcup_{i=1}^{\mathcal{N}} \big(W^\phi(x_i,\Delta)\big)^{\psi,1}_{-g,h}.$$

Thus, using also (3.23), (3.25) and (3.26) we deduce that

$$\begin{aligned}
\Big|E^{\psi,1}_{g,h} \cap W^\phi\Big(0, \frac{M_0 h}{\Delta}\Big)\Big| \le& \Big|W^\phi\Big(0, \frac{M_0 h}{\Delta}\Big) \setminus \bigcup_{i=1}^{\mathcal{N}} \big(W^\phi(x_i,\Delta)\big)^{\psi,1}_{-g,h}\Big| \\
\le& \Big|W^\phi\Big(0, \frac{M_0 h}{\Delta}\Big) \setminus \bigcup_{i=1}^{\mathcal{N}} W^\phi(x_i,\Delta)\Big| + \sum_{i=1}^{\mathcal{N}} \Big|\big(W^\phi(x_i,\Delta) \setminus \big(W^\phi(x_i,\Delta)\big)^{\psi,1}_{-g,h}\big) \cap W^\phi\Big(0, \frac{M_0 h}{\Delta}\Big)\Big| \\
\le& \frac{\sigma}{4}\left(\frac{M_0 h}{\Delta}\right)^N + \mathcal{N}\frac{C}{\Delta} h \left(\frac{M_0 h}{\Delta}\right)^{N-1} = \frac{3}{4}\sigma\left(\frac{M_0 h}{\Delta}\right)^N,
\end{aligned}$$

which contradicts the density estimate provided by Corollary 3.10. □

We conclude with the following covering lemma, which we used in the previous proof:

Lemma 3.13. *Let $\mathcal{F}$ be a family of closed convex sets in $\mathbb{R}^N$ which covers a closed convex set $\mathcal{C}$. Assume that there exists $\theta > 0$ such that for all $W \in \mathcal{F}$, $x \in W$, and $r \le 1$,*

$$\frac{|W \cap Q_r(x)|}{|Q_r(x)|} \ge \theta, \tag{3.27}$$

where $Q_r(x) = x + [-r/2, r/2]^N$. Then for every $\delta > 0$ there exists $\mathcal{N} = \mathcal{N}(\delta, \theta, \mathcal{C}, N)$ and sets $W_i \in \mathcal{F}$, $i = 1, \dots, n \le \mathcal{N}$ such that

$$\Big|\mathcal{C} \setminus \bigcup_{i=1}^{n} W_i\Big| \le \delta. \tag{3.28}$$

Proof. Given $\varepsilon \in \{2^{-i} : i \in \mathbb{N}\}$, let $\mathcal{Q}_\varepsilon$ be the set of closed cubes of size ε, centered at $\varepsilon\mathbb{Z}^N$, which are included in $\mathcal{C}$. Consider ε_1 the largest dyadic value for which, letting $I_1 = \bigcup_{Q \in \mathcal{Q}_{\varepsilon_1}} Q$, one has $|\mathcal{C} \setminus I_1| \le 1/2$. We denote $\mathcal{N}_1$ the cardinality of $\mathcal{Q}_{\varepsilon_1}$.

For each $Q = Q_{\varepsilon_1}(x) \subset I_1$, we choose $W_Q \in \mathcal{F}$ with $x \in W_Q$ and let $F_1 = \bigcup_{Q \in \mathcal{Q}_{\varepsilon_1}} Q \cap W_Q \subseteq I_1$, $\widetilde{F}_1 = \bigcup_{Q \in \mathcal{Q}_{\varepsilon_1}} W_Q$. By construction and (3.27), $|F_1| \ge \theta |I_1|$. Observe moreover that the number $\mathcal{N}_1$ of sets $Q \in \mathcal{Q}_{\varepsilon_1}$ with $Q \subset I_1$ depends only on the initial convex set $\mathcal{C}$.

We now assume we have built sets $I_i, F_i, \widetilde{F}_i$, $i = 1, \dots, k-1$, such that

i) I_i is the union of $\mathcal{N}_i$ dyadic cubes $Q_{i,1}, \dots, Q_{i,\mathcal{N}_i} \in \mathcal{Q}_{\varepsilon_i}$ where $\varepsilon_i, \mathcal{N}_i$ depend only on $\theta, \mathcal{C}$ and the dimension N;
ii) $F_i = \bigcup_{l=1}^{\mathcal{N}_i} W_{Q_{i,l}} \cap Q_{i,l} \subseteq I_i$ and $\widetilde{F}_i = \bigcup_{l=1}^{\mathcal{N}_l} W_{Q_{i,l}}$ where $W_{Q_{i,l}} \in \mathcal{F}$ contains the center of the cube $Q_{i,l}$;
iii) For $j = 1, \dots, k-1$, $I_j \subset \mathcal{C} \setminus (\bigcup_{i=1}^{j-1} \mathring{F}_i)$ (in particular the sets F_i have disjoint interior) and $|\big(\mathcal{C} \setminus (\bigcup_{i=1}^{j-1} F_i)\big) \setminus I_j| \le 2^{-j}$.

We claim that we can build $I_k, F_k, \widetilde{F}_k$ which satisfy the same conditions, with I_k made of $\mathcal{N}_k$ cubes of size ε_k, the numbers $\mathcal{N}_k, \varepsilon_k$ depending only on $\theta, \mathcal{C}, N$.

In order to do this, we show that we can find $\varepsilon_k < \varepsilon_{k-1}$ depending only on $\theta, \mathcal{C}, N$ such that if I_k is the union of all the cubes in $\mathcal{Q}_{\varepsilon_k}$ not intersecting $\cup_{i=1}^{k-1} \mathring{F}_i$, then $|\big(\mathcal{C} \setminus (\cup_{i=1}^{k-1} F_i)\big) \setminus I_k| \le 2^{-k}$. The set $\big(\mathcal{C} \setminus (\cup_{i=1}^{k-1} \mathring{F}_i)\big) \setminus I_k$ is covered by the dyadic cubes of size ε_k centered at $\varepsilon_k \mathbb{Z}^N$ which either

- intersect $\partial\mathcal{C}$: the total measure of such cubes is bounded by $c\mathcal{H}^{N-1}(\partial\mathcal{C})\varepsilon_k$;

- intersect, for some $i = 1, \dots, k-1$, $\partial(W_{Q_{i,l}} \cap \mathring{Q}_{i,l})$ for some $l \in \{1, \dots, \mathcal{N}_i\}$, where $Q_{i,l}$ is a dyadic cube of size ε_i in I_i: the total measure of such small cubes for a given $Q_{i,l}$ is bounded, thanks to Lemma 3.11, by $c(\varepsilon_i)^{N-1}\varepsilon_k = c|Q_{i,l}|(\varepsilon_k/\varepsilon_i) \le c(\varepsilon_k/\varepsilon_i)|W_{Q_{i,l}} \cap Q_{i,l}|/\theta$, with c depending only on N, hence the total measure of this region is bounded by
$$\frac{c}{\theta}\sum_{i=1}^{k-1}\sum_{l=1}^{\mathcal{N}_k}\frac{\varepsilon_k}{\varepsilon_i}|W_{Q_{i,l}} \cap Q_{i,l}| = \frac{c}{\theta}\sum_{i=1}^{k-1}\frac{\varepsilon_k}{\varepsilon_i}|F_i| \le \frac{c|\mathcal{C}|}{\theta}\frac{\varepsilon_k}{\varepsilon_{k-1}}.$$

We see that volume of the union of all these "bad" cubes is less than
$$\frac{c|\mathcal{C}|}{\theta}\frac{\varepsilon_k}{\varepsilon_{k-1}} + c\mathcal{H}^{N-1}(\partial\mathcal{C})\varepsilon_k.$$
Again, one can find ε_k depending only on $\mathcal{C}, N, \theta$ such that this quantity is less than 2^{-k}. It follows that the total number of cubes in I_k, $\mathcal{N}_k$, depends also only on $\mathcal{C}, N, \theta$. We denote $Q_{k,l}$, $l = 1, \dots, \mathcal{N}_k$, the corresponding cubes.

As before we build then $F_k \subset I_k$ as the union of $Q_{k,l} \cap W_{Q_{k,l}}$ where $W_{Q_{k,l}} \in \mathcal{F}$ is a convex set containing the center of $Q_{k,l}$, and $\widetilde{F}_k = \bigcup_{l=1}^{\mathcal{N}_k} W_{Q_{k,l}}$. We have then proved that i), ii) and iii) hold true with $k-1$ replaced by k.

Notice that, thanks to (3.27), $|I_k| \le |F_k|/\theta$. Recall that the sets I_k and $\mathring{F}_1, \dots, \mathring{F}_{k-1}$ are disjoint. Hence $|\big(\mathcal{C} \setminus (\cup_{i=1}^{k-1}F_i)\big) \setminus I_k| = |\mathcal{C} \setminus (\cup_{i=1}^{k-1}F_i)| - |I_k|$ so that
$$|\mathcal{C} \setminus (\cup_{i=1}^{k-1}F_i)| \le \frac{|F_k|}{\theta} + 2^{-k}.$$
Let $K \ge 1$. As the sets $\mathcal{C} \setminus (\cup_{i=1}^{k-1}F_i)$ are decreasing (with k), one has for every $k \le K+1$ that $|\mathcal{C} \setminus (\cup_{i=1}^{k-1}F_i)| \ge |\mathcal{C} \setminus (\cup_{i=1}^{K}F_i)|$. Using also that the sets $\mathring{F}_k \subset \mathcal{C}$ are pairwise disjoint, it follows:
$$K|\mathcal{C} \setminus (\cup_{i=1}^{K}F_i)| \le \sum_{k=2}^{K+1}|\mathcal{C} \setminus (\cup_{i=1}^{k-1}F_i)| \le \frac{|\mathcal{C}|}{\theta} + 1$$
hence
$$|\mathcal{C} \setminus (\cup_{i=1}^{K}\widetilde{F}_i)| \le |\mathcal{C} \setminus (\cup_{i=1}^{K}F_i)| \le \frac{\frac{|\mathcal{C}|}{\theta} + 1}{K}.$$
If $K \ge (|\mathcal{C}|/\theta + 1)/\delta$, (3.28) holds true and the maximum number of sets $W \in \mathcal{F}$ used in the construction, which is bounded by $\sum_{k=1}^{K}\mathcal{N}_k$, depends only on $N, \delta, \theta, \mathcal{C}$. □

Remark 3.14. A careful study of the previous proof shows that $\mathcal{N}$ only depends on the convex set $\mathcal{C}$ through $|\mathcal{C}|$ and $\mathcal{H}^{N-1}(\partial\mathcal{C})$. Hence, as these quantities, for $\mathcal{C} = W^\phi(0,1)$, as well as θ in (3.24) depend continuously on ϕ, one can deduce that the constants M_0, h_0 in Lemma 3.12 can be chosen as depending on ϕ only through the ellipticity constants a_1, a_2 of (3.17).

4. The crystalline mean curvature flow with a ϕ-regular mobility

Throughout this section we assume the mobility ψ satisfies the following regularity assumption with respect to the metric induced by ϕ°:

Definition 4.1. *We will say that a norm ψ is ϕ-regular if the associated Wulff shape $W^\psi(0,1)$ satisfies a uniform interior ϕ-Wulff shape condition, that is, if there exists $\varepsilon_0 > 0$ with the following property: for every $x \in \partial W^\psi(0,1)$ there exists $y \in W^\psi(0,1)$ such that $W^\phi(y,\varepsilon_0) \subseteq W^\psi(0,1)$ and $x \in \partial W^\phi(y,\varepsilon_0)$.*

Notice that it is equivalent to saying that $W^\psi(0,1)$ is the sum of a convex set and $W^\phi(0,\varepsilon_0)$, or equivalently that $\psi(\nu)=\psi_0(\nu)+\varepsilon_0\phi(\nu)$ for some convex function ψ_0. We will show that, under the above additional regularity assumption, the ATW scheme converges to a (generically) unique solution of the flow in the sense of Definition 2.2, see Subsections 4.2 and 4.3 below. We start with some preliminary estimates.

4.1. **Evolution of ψ-Wulff shapes and preliminary estimates.** We now analyze the minimizing movement of a ψ-Wulff shape $W^\psi(0,R)$, with ψ ϕ-regular, that is, we assume $E_h(t)=W^\psi(0,R)$ for some time $t\ge 0$.

By the regularity assumption in Definition 4.1, setting $\overline{R}:=(\varepsilon_0 R)\wedge 1$, we have (with the notation introduced in (3.20)) that for every $0<r<\overline{R}$

$$\Big(\big(W^\psi(0,R)\big)^{\phi^\circ}_{-\overline{R}}\Big)^{\phi^\circ}_r=\big(W^\psi(0,R)\big)^{\phi^\circ}_{-(\overline{R}-r)}\,.$$

Since for every $x\in\big(W^\psi(0,R)\big)^{\phi^\circ}_{-\overline{R}}$ we have $W^\phi(x,\overline{R})\subseteq W^\psi(0,R)$, from the discrete comparison principle and the analysis performed in Subsection 3.3 it follows that

$$\big(W^\psi(0,R)\big)^{\phi^\circ}_{-(\overline{R}-r^{\overline{R}}(s))}\subseteq E_h(s)$$

for all $0\le s-t\le \frac{c_1\overline{R}^2}{4\big(2(N-1)+\|g\|_\infty\big)}$ and $h\le c_1C(N)\overline{R}^2$, where $r^{\overline{R}}$ is the function defined in (3.14) (with R replaced by $\overline{R}$).

Now we return to an arbitrary discrete motion $E_h(\cdot)$. If for some $(x,t)\in\mathbb{R}^N\times[0,T_h^*)$ we have $d_h(x,t)>R$ (see (3.9)), then $W^\psi(x,R)\cap E_h(t)=\emptyset$. Thus, again by the discrete comparison principle and the results of Subsection 3.3, we infer that

$$\big(W^\psi(0,R)\big)^{\phi^\circ}_{-(\overline{R}-r^{\overline{R}}(s))}\cap E_h(s)=\emptyset$$

for all $0\le s-t\le \frac{c_1\overline{R}^2}{4\big(2(N-1)+\|g\|_\infty\big)}$ and $h\le c_1C(N)\overline{R}^2$. Taking into account also (3.11) and the definition of $r^{\overline{R}}$, it follows that

$$\begin{aligned}
d_h(x,s)&\ge d_h(x,t)-c_2\left(\overline{R}-\sqrt{\overline{R}^2-2\frac{s-t}{c_1}\big(2(N-1)+\|g\|_\infty\big)}\right)\\
&=d_h(x,t)-\frac{c_2}{c_1}\frac{\big(4(N-1)+2\|g\|_\infty\big)(s-t)}{\overline{R}+\sqrt{\overline{R}^2-2\frac{s-t}{c_1}\big(2(N-1)+\|g\|_\infty\big)}}\\
&\ge d_h(x,t)-\frac{c_2}{c_1}\frac{\big(4(N-1)+2\|g\|_\infty\big)(s-t)}{\overline{R}}\\
&=d_h(x,t)-\frac{c_2}{c_1}\frac{\big(4(N-1)+2\|g\|_\infty\big)}{(\varepsilon_0 R)\wedge 1}(s-t)
\end{aligned}$$

for all $0\le s-t\le \frac{c_1}{4}\frac{(\varepsilon_0^2R^2)\wedge 1}{2(N-1)+\|g\|_\infty}$ and $h\le c_1C(N)((\varepsilon_0^2R^2)\wedge 1)$. Letting $R\nearrow d_h(x,t)$ we obtain

$$d_h(x,s)\ge d_h(x,t)-\frac{c_2}{c_1}\frac{4(N-1)+2\|g\|_\infty}{(\varepsilon_0 d_h(x,t))\wedge 1}(s-t) \tag{4.1}$$

for all $0\le s-t\le \frac{c_1}{4}\frac{\big(\varepsilon_0^2d_h^2(x,t)\big)\wedge 1}{2(N-1)+\|g\|_\infty}$ and $h\le c_1C(N)((\varepsilon_0^2d_h^2(x,t))\wedge 1)$, whenever $d_h(x,t)>0$.

By an entirely similar argument if $d_h(x,t)<0$, then we obtain

$$d_h(x,s)\le d_h(x,t)+\frac{c_2}{c_1}\frac{4(N-1)+2\|g\|_\infty}{(\varepsilon_0|d_h(x,t)|)\wedge 1}(s-t) \tag{4.2}$$

for all $0 \le s - t \le \frac{c_1}{4}\frac{\left(\varepsilon_0^2 d_h^2(x,t)\right)\wedge 1}{2(N-1)+\|g\|_\infty}$ and $h \le c_1 C(N)((\varepsilon_0^2 d_h^2(x,t)) \wedge 1)$.

4.2. **Convergence of the ATW scheme.** For every $h > 0$ let E_h be the discrete evolution defined in (3.9). We extract a subsequence $\{E_{h_l}\}_{l\in\mathbb{N}}$ such that

$$E_{h_l} \xrightarrow{\mathcal{K}} E \qquad \text{and} \qquad \big(\mathring{E}_{h_l}\big)^c \xrightarrow{\mathcal{K}} A^c$$

for a suitable closed set E and a suitable open set $A \subset E$. We define $E(t)$ and $A(t)$ as in (3.9).

Observe that if $E(t) = \emptyset$ for some $t \ge 0$, then (4.1) implies that $E(s) = \emptyset$ for all $s \ge t$ so that we can define, as in Definition 2.2, the extinction time T^* of E, and similarly the extinction time T'^* of A^c. Notice that at least one between T^* and T'^* is $+\infty$. Possibly extracting a further subsequence, we have the following result, which can be proven arguing exactly as in [22, Proof of Proposition 4.4], using now (4.1) and (4.2).

Proposition 4.2. *There exists a countable set $\mathcal{N} \subset (0,+\infty)$ such that $d_{h_l}(\cdot,t)^+ \to \mathrm{dist}(\cdot, E(t))$ and $d_{h_l}(\cdot,t)^- \to \mathrm{dist}(\cdot, A^c)$ locally uniformly for all $t \in (0,+\infty)\setminus\mathcal{N}$. Moreover, E and A^c satisfy the continuity properties* (b) *and* (c) *of Definition 2.2. Finally, $E(0) = E^0$ and $A(0) = \mathring{E}^0$.*

Theorem 4.3. *The set E is a superflow in the sense of Definition 2.2 with initial datum E^0, while A is a subflow with initial datum E^0.*

Proof. Points *(a), (b)* and *(c)* of Definition 2.2 follow from Proposition 4.2. It remains to show *(d)*. We will use the notation in (3.9). Possibly extracting a further subsequence and setting $z_{h_l}(\cdot,t) := 0$ for $t > T^*_{h_l}$ if $T^*_{h_l} < T^*$, we may assume that z_{h_l} converges weakly-$*$ in $L^\infty(\mathbb{R}^N\times(0,T^*))$ to some vector-field z satisfying $\phi^\circ(z) \le 1$ almost everywhere. Recall that by (3.8) we have $u^{k+1}_{h_l} \le (1+Lh_l)d^{\psi^\circ}_{E^{k+1}_{h_l}}$, whenever $d_{E^{k+1}_{h_l}} \ge 0$. In turn, it follows from (3.6) that

$$\mathrm{div} z^{k+1}_{h_l} + \frac{1}{h_l}\int_{kh_l}^{(k+1)h_l} g(\cdot,s)\,ds \le \frac{(1+Lh_l)d^{\psi^\circ}_{E^{k+1}_{h_l}} - d^{\psi^\circ}_{E^k_{h_l}}}{h_l}. \tag{4.3}$$

Consider a nonnegative test function $\eta \in C^\infty_c((\mathbb{R}^N\times(0,T^*))\setminus E)$. If l is large enough, then the distance of the support of η from E_{h_l} is bounded away from zero. In particular, d_{h_l} is finite (as a consequence of (4.1)) and positive on $\mathrm{Supp}\,\eta$. We deduce from (4.3) that

$$\begin{aligned}
0 \le \iint \eta(x,t)\Bigg(&\frac{d_{h_l}(x,t+h_l)-d_{h_l}(x,t)}{h_l} - \mathrm{div} z_{h_l}(x,t+h_l)\\
&- \frac{1}{h_l}\int_{[\frac{t}{h_l}]h_l}^{([\frac{t}{h_l}]+1)h_l} g(x,s)\,ds + Ld_{h_l}(x,t+h_l)\Bigg)dtdx\\
= -\iint\Bigg(&\frac{\eta(x,t)-\eta(x,t-h_l)}{h_l}d_{h_l}(x,t) - z_{h_l}(x,t+h_l)\cdot\nabla\eta(x,t)\\
&- \eta(x,t)\Big(\frac{1}{h_l}\int_{[\frac{t}{h_l}]h_l}^{([\frac{t}{h_l}]+1)h_l} g(x,s)\,ds + Ld_{h_l}(x,t+h_l)\Big)\Bigg)\,dtdx.
\end{aligned}$$

Passing to the limit $l\to\infty$ we obtain (2.5) with $M = L$.

Next, we establish an upper bound for $\mathrm{div} z_h$ away from E^k_h. To this aim let $x \in \mathbb{R}^N \setminus E^k_h$ be such that $d^{\psi^\circ}_{E^k_h}(x) =: R > 0$.

There exists $\xi \in E_h^k$ with $x \in \partial W^\psi(\xi, R)$, and recalling Definition 4.1 there is $\bar{x}$ such that, setting $\bar{R} := (\varepsilon_0 R) \wedge 1$, $W^\phi(\bar{x}, \bar{R}) \subset W^\psi(\xi, R)$ and $x \in \partial W^\phi(\bar{x}, \bar{R})$. In particular, $d^{\psi^\circ}_{E_h^k} - R \leq \psi^\circ(\cdot - \xi) - R = d^{\psi^\circ}_{W^\psi(\xi,R)} \leq d^{\psi^\circ}_{W^\phi(\bar{x},\bar{R})}$. Using (3.12), one has for all $y \in \mathbb{R}^N$

$$d^{\psi^\circ}_{E_h^k}(y) \leq R + c_2(\phi^\circ(y - \bar{x}) - \bar{R}) \vee c_1(\phi^\circ(y - \bar{x}) - \bar{R}).$$

Thanks to (3.11),

$$\int_{kh}^{(k+1)h} g(y,s)\, ds \leq \int_{kh}^{(k+1)h} g(x,s)\, ds + Lh\psi^\circ(y-x) \leq \int_{kh}^{(k+1)h} g(x,s)\, ds + Lhc_2\phi^\circ(y-x),$$

hence, since $\phi^\circ(y-x) \leq \phi^\circ(y-\bar{x}) + \bar{R}$, summing the two previous inequalities we obtain:

$$\begin{aligned} &d^{\psi^\circ}_{E_h^k}(y) + \int_{kh}^{(k+1)h} g(y,s)\, ds \\ &\leq R + 2c_2Lh\bar{R} + \big(c_2(1+Lh)(\phi^\circ(y-\bar{x}) - \bar{R})\big) \vee \big((c_1 + c_2Lh)(\phi^\circ(y-\bar{x}) - \bar{R})\big) + \int_{kh}^{(k+1)h} g(x,s)\, ds. \end{aligned}$$

As a consequence (*cf* Lemma 3.4),

$$\begin{aligned} u_h^{k+1}(y) \leq R + 2c_2Lh\bar{R} + &\int_{kh}^{(k+1)h} g(x,s)\, ds \\ &+ c_2(1+Lh)(\phi^\circ(y-\bar{x}) - \bar{R}) \vee (c_1 + c_2Lh)(\phi^\circ(y-\bar{x}) - \bar{R}) + \frac{h(N-1)}{\phi^\circ(y-\bar{x})} \end{aligned}$$

if $\phi^\circ(y-\bar{x}) \geq \sqrt{h(N+1)/(c_1+c_2Lh)}$ and as long as this quantity is less than $\frac{\bar{R}}{\sqrt{N+1}}$. Evaluating this inequality at $y = x$, we deduce that if $h \leq C(N)\bar{R}^2$,

$$\begin{aligned} u_h^{k+1}(x) &\leq R + 2c_2Lh\bar{R} + \int_{kh}^{(k+1)h} g(x,s)\, ds + \frac{h(N-1)}{\bar{R}} \\ &\leq d^{\psi^\circ}_{E_h^k}(x) + 2c_2Lh + \int_{kh}^{(k+1)h} g(x,s)\, ds + \frac{h(N-1)}{(\varepsilon_0 R) \wedge 1}, \end{aligned}$$

as $\bar{R} \leq 1$. Thanks to (3.6), it follows

$$\mathrm{div} z_h^{k+1}(x) \leq 2c_2L + \frac{N-1}{(\varepsilon_0 d^{\psi^\circ}_{E_h^k}(x)) \wedge 1}. \tag{4.4}$$

In the limit $h_l \to 0$, we deduce that $\mathrm{div} z$ is a Radon measure in $\mathbb{R}^N \times (0,T^*) \setminus E$, and $(\mathrm{div} z)^+ \in L^\infty(\{(x,t) \in \mathbb{R}^N \times (0,T^*) : d(x,t) \geq \delta\})$ for every $\delta > 0$.

We now provide a lower (h-dependent) bound for $\mathrm{div} z_h$. To this aim, note that if $d^{\psi^\circ}_{E_h^k}(x) =: R > 0$, then $d^{\psi^\circ}_{E_h^k} \geq R - \psi^\circ(\cdot - x) \geq R - c_2\phi^\circ(\cdot - x)$. Thus, by comparison with the explicit solution given by (3.10) (with a change of sign), and using that $(c_2\phi^\circ)_h = c_2\phi^\circ_{h/c_2}$, we get

$$u_h^{k+1}(x) \geq R - c_2\phi^\circ_{\frac{h}{c_2}}(0) - h\|g\|_\infty \geq R - \frac{3N\sqrt{c_2h}}{\sqrt{N+1}},$$

for $h = h_l$ small enough (so that $h\|g\|_\infty \leq N\sqrt{c_2h}/\sqrt{N+1}$). In turn, by (3.6), we deduce both

$$\mathrm{div} z_h^{k+1} \geq -\frac{1}{\sqrt{h}} \frac{4N\sqrt{c_2}}{\sqrt{N+1}} \qquad \text{a.e. in } \{x : d_{E_h^k}(x) > 0\}.$$

and, using also (4.4), that for any $\delta > 0$

$$\|u_{h_l}(\cdot,t) - d_{h_l}(\cdot, t-h_l)\|_{L^\infty(\{x: d_{h_l}(x,t-h_l)\geq\delta\})} \leq \sqrt{h_l}\frac{3N\sqrt{c_2}}{\sqrt{N+1}} + o(\sqrt{h_l}),$$

provided that l is large enough. In particular, recalling the convergence properties of E_{h_l} and d_{h_l} (see also [22, Equation (4.9)]), we deduce that for all $t \in (0,T^*)\setminus\mathcal{N}$ (where recall that $\mathcal{N}$ is introduced in Proposition 4.2),

$$u_{h_l} \to d \qquad \text{a.e. in } \mathbb{R}^N \times (0,T^*)\setminus E, \tag{4.5}$$

with the sequence $\{u_{h_l}\}$ locally (in space and time) uniformly bounded.

Now, with (4.4) and (4.5) at hand, we proceed as in the final part of the proof of [22, Theorem 4.5] to show that $\phi(\nabla d) = z\cdot\nabla d$, which will imply that $z \in \partial\phi(\nabla d)$ (as clearly, $\phi^\circ(z)\leq 1$ a.e.). For this, it is enough to show that $\phi(\nabla d) \leq z\cdot\nabla d$. On the one hand, since $z_{h_l} \in \partial\phi(\nabla u_{h_l})$, one has for a nonnegative test function $\eta \in C_c^\infty(E^c;\mathbb{R}_+)$, using $u_{h_l}\to d$, that

$$\int\!\!\int \eta\,\phi(\nabla d)dxdt \leq \liminf_l \int\!\!\int \eta\,\phi(\nabla u_{h_l})dxdt = \liminf_l \int\!\!\int \eta\, z_{h_l}\cdot\nabla u_{h_l}dxdt.$$

On the other hand,

$$\int\!\!\int \eta\, z_{h_l}\cdot\nabla u_{h_l}dxdt = \int\!\!\int \eta\, z_{h_l}\cdot\nabla d\,dxdt + \int\!\!\int \eta\, z_{h_l}\cdot\nabla(u_{h_l}-d)dxdt$$

and as $z_{h_l} \overset{*}{\rightharpoonup} z$, we obtain that $\phi(\nabla d) \leq z\cdot\nabla d$ a.e., provided we can show that

$$\lim_l \int\!\!\int \eta\, z_{h_l}\cdot\nabla(u_{h_l}-d)dxdt = 0.$$

The proof is as in [22]: we introduce $m_l(t) := \min_{x\in\operatorname{Supp}\eta(\cdot,t)}(u_{h_l}(x,t)-d(x,t))$ which is bounded and goes to zero for almost all t, and write

$$\begin{aligned}\int\!\!\int \eta\,(z_{h_l}\cdot\nabla(u_{h_l}-d))dxdt &= \int\!\!\int \eta\,(z_{h_l}\cdot\nabla(u_{h_l}-d-m_l))dxdt\\ &= -\int\!\!\int z_{h_l}\cdot\nabla\eta\,(u_{h_l}-d-m_l)dxdt - \int\!\!\int \eta\,(u_{h_l}-d-m_l)\mathrm{div} z_{h_l}dxdt.\end{aligned}$$

The first integral in the right-hand side clearly goes to zero, while, using (4.4) and $u_{h_l}-d-m_l\geq 0$, the second is bounded from above by a quantity which vanishes as $l\to\infty$. We deduce that $\liminf_l \int\!\int \eta\, z_{h_l}\cdot\nabla(u_{h_l}-d)dxdt \geq 0$, and the proof of the reverse inequality is identical.

It follows that $z \in \partial\phi(\nabla d)$ a.e. in $\mathbb{R}^N\times(0,T^*)\setminus E$. This concludes the proof that E is a superflow. The proof that A is a subflow is identical. □

Remark 4.4 (Stability of sub- and superflows). From the proof of Theorem 4.3, we note that the minimizing movement scheme provides a superflow such that the corresponding field z satisfies (see (4.4))

$$\mathrm{div} z \leq 2c_2L + \frac{(N-1)}{(\varepsilon_0 R)\wedge 1} \qquad \text{a.e. in } \{d\geq R\}, \tag{4.6}$$

where c_2 is the constant appearing in (3.11) , L is the Lipschitz constant of g, and ε_0 is given in Definition 4.1. Moreover, from (4.1) we deduce that

$$d(x,s) \geq d(x,t) - \frac{c_2}{c_1}\frac{4(N-1)+2\|g\|_\infty}{(\varepsilon_0 d(x,t))\wedge 1}(s-t) \tag{4.7}$$

for all $0 \le s - t \le \frac{c_1}{4}\frac{\left(\varepsilon_0^2 d^2(x,t)\right)\wedge 1}{2(N-1)+\|g\|_\infty}$, whenever $d(x,t) > 0$. An analogous statement clearly holds also for the subflow provided by the ATW scheme.

We remark that thanks to these estimates, the following stability property holds: Let $\phi_n \to \phi$ and $\psi_n \to \psi$ and assume that ψ_n is ϕ_n-regular uniformly in n: there exists $\varepsilon_0 > 0$ such that the ψ_n-Wulff shape satisfies a uniform inner ϕ_n-Wulff shape condition with radius ε_0 for all n (see Definition 4.1)

For every n let E^n be a superflow as in Definition 2.2, with ϕ and ψ replaced by ϕ_n and ψ_n, respectively, and with initial datum E_0^n. Denote by d_n the corresponding distance function, that is, $d_n(\cdot,t) := \mathrm{dist}^{\psi_n^\circ}(\cdot, E^n(t))$ in $\mathbb{R}^N \setminus E^n(t)$, and by z_n the corresponding Cahn-Hoffmann field given by Definition 2.2, and assume that (4.6) and (4.7) hold with z and d replaced by z_n and d_n (and again with c_1, c_2, L, and ε_0 independent of n). Finally, assume that $E^n \xrightarrow{\mathcal{K}} E$ and $E_0^n \xrightarrow{\mathcal{K}} E_0$. Then, E is a superflow with respect to the anisotropy ϕ and the mobility ψ satisfying $E(0) = E_0$. This follows by the same arguments employed in the proofs of Proposition 4.2 (see also [22, Proof of Proposition 4.4]) and Theorem 4.3. Analogous stability properties hold also for subflows.

4.3. **Existence and uniqueness of the level set flow.** The convergence theorem proved in the previous subsection combined with the comparison principles established in Subsection 2.3 yields existence and uniqueness of the level set formulation of the crystalline curvature flow, when ψ is ϕ-regular. In the following we briefly set up the discrete version of such formulation and we give the precise statements.

Let $u^0 : \mathbb{R}^N \to \mathbb{R}$ be a uniformly continuous function. Let $E^{0,\lambda} := \{u^0 \le \lambda\}$ and let $E_{\lambda,h}$ be the corresponding time discrete evolutions, defined according with (3.9) with E^0 replaced by $E^{0,\lambda}$.

We introduce the level set discrete evolution $u_h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$u_h(x,t) := \inf\{\lambda \in \mathbb{R} \,:\, x \in E_{\lambda,h}(t)\}. \tag{4.8}$$

(We warn the reader that the discrete level set function u_h defined above does not coincide with the discrete total variation flow function (also denoted by u_h) defined in (3.9).) Note that by construction

$$\{u_h(\cdot,t) < \lambda\} \subseteq E_{\lambda,h}(t) \subseteq \{u_h(\cdot,t) \le \lambda\}\,. \tag{4.9}$$

Let ω denote an increasing modulus of continuity for u^0, with respect to the metric induced by ψ°. Thus, in particular, if $\lambda_1 < \lambda_2$ we have

$$\mathrm{dist}^{\psi^\circ}(E^{0,\lambda_1}, \mathbb{R}^N \setminus E^{0,\lambda_2}) \ge \omega^{-1}(\lambda_2 - \lambda_1)\,.$$

Let $L > 0$ be the spatial Lipschitz constant of g with respect to ψ° and choose $\bar h > 0$ so small that $(1 - Lh)^{-\frac{1}{Lh}} \le 2e$ for all $h \in (0, \bar h)$. By Lemma 5.2 below (with $\eta = \psi$, $\beta = 1$, $M = L$, $g_1 = g_2 = g$ and $c = 0$) we deduce that

$$\mathrm{dist}^{\psi^\circ}(E_{\lambda_1,h}(t), \mathbb{R}^N \setminus E_{\lambda_2,h}(t)) \ge \omega^{-1}(\lambda_2 - \lambda_1)(1 - Lh)^{[\frac{t}{h}]} \ge \omega^{-1}(\lambda_2 - \lambda_1)(2e)^{-Lt}$$

for all $t > 0$ and $h \in (0, \bar h)$. In turn, it easily follows that $\widehat{\omega} : \lambda \mapsto \omega\big((2e)^{Lt}\lambda\big)$ is a spatial modulus of continuity for $u_h(\cdot,t)$. Hence:

Lemma 4.5. *For any $x, x' \in \mathbb{R}^N$, $t \ge 0$, $|u_h(x,t) - u_h(x',t)| \le \omega(2e^{Lt}|x - x'|)$.*

As for the continuity in time, we have:

Lemma 4.6. *For any $\varepsilon > 0$, $T > 0$, there exists $\tau > 0$ and $h_0 > 0$ (depending on ε) such that for all $t, t' \le T$ with $|t - t'| \le \tau$ and $h \le h_0$ we have $|u_h(\cdot, t) - u_h(\cdot, t')| < \varepsilon$.*

(We cannot expect more, as u_h is discontinuous at times kh, k integer.)

The proof of the lemma follows by standard comparison arguments with the evolution of the ϕ-Wulff shape, whose extinction time can be estimated (see Remark 4.7 below). We refer to [21, Lemma 6.13] for the details.

Remark 4.7. Let us remark that the extinction time of a ϕ-Wulff shape of radius R, evolving according to the forced mean crystalline curvature flow with forcing term g and mobility ψ, is bounded away from zero by a constant which depends only on R, the infinity norm of g and the constant c_1 in (3.11) (see section 3.3). In turn, h_0 and τ depend only on ω, ε, $\|g\|_\infty$ and c_1.

We are ready to prove the main result of this section. In the following, together with $\mathring{F}$, we will make use of the notation $\operatorname{Int} F$ to denote the interior of any set $F \in \mathbb{R}^m$, $m \in \mathbb{N}$.

Theorem 4.8. *Let ψ, g, and u^0 be a ϕ-regular mobility, an admissible forcing term, and a uniformly continuous function on $\mathbb{R}^N$, respectively. Then the following holds:*

(i) (Existence and uniqueness) *There exists a unique solution u to the level set flow with initial datum u^0, in the sense of Definition 2.5.*

(ii) (Approximation via minimizing movements) *The solution u is the locally uniform limit in $\mathbb{R}^N \times [0, +\infty)$, as $h \to 0^+$, of the level set minimizing movements u_h defined in* (4.8)*.*

(iii) (Properties of the level set flow) *For all but countably many $\lambda \in \mathbb{R}$, the fattening phenomenon does not occur and, that is: $\partial\{x : u^0(x) < \lambda\} = \partial\{x : u^0(x) \le \lambda\} = \{x : u^0(x) = \lambda\}$,* i.e.*:*

$$\begin{aligned} \{x : u^0(x) < \lambda\} &= \operatorname{Int}\left(\{x : u^0(x) \le \lambda\}\right), \\ \overline{\{x : u^0(x) < \lambda\}} &= \{x : u^0(x) \le \lambda\}. \end{aligned} \tag{4.10}$$

and $\partial\{(x,t) : u(x,t) < \lambda\} = \partial\{(x,t) : u(x,t) \le \lambda\} = \{(x,t) : u(x,t) = \lambda\}$, i.e.*:*

$$\begin{aligned} \{(x,t) : u(x,t) < \lambda\} &= \operatorname{Int}\left(\{(x,t) : u(x,t) \le \lambda\}\right), \\ \overline{\{(x,t) : u(x,t) < \lambda\}} &= \{(x,t) : u(x,t) \le \lambda\}. \end{aligned} \tag{4.11}$$

Moreover, for every λ such that (4.10), (4.11) *hold true the sublevel set $\{(x,t) : u(x,t) \le \lambda\}$ is the unique solution to* (1.1) *in the sense of Definition 2.2, with initial datum $E^{0,\lambda}$, and*

$$E_{\lambda,h} \xrightarrow{\mathcal{K}} \{(x,t) : u(x,t) \le \lambda\} \quad \text{and} \quad (\mathring{E}_{\lambda,h})^c \xrightarrow{\mathcal{K}} \{(x,t) : u(x,t) \ge \lambda\}. \tag{4.12}$$

Finally, for all $\lambda \in \mathbb{R}$ the sets $\{(x,t) : u(x,t) \le \lambda\}$ and $\{(x,t) : u(x,t) < \lambda\}$ are respectively the maximal superflow and minimal sublow with initial datum $E^{0,\lambda}$.

Proof. The arguments rely on Theorems 2.7 and 4.3, and are somewhat standard (see for instance [24, 21]). For the reader's convenience we outline below a self-contained proof.

Step 1. (Convergence) Thanks to Lemmas 4.5 and 4.6, the family $\{u_h\}$ is relatively compact with respect to the local uniform convergence in $\mathbb{R}^N \times [0, +\infty)$. Indeed, the classical proof of Ascoli-Arzelà's theorem, which consists in first extracting a subsequence which converges in all points of a countable dense subset (such as $\mathbb{Q}^N \times \mathbb{Q}_+$), and then observing that this subsequence is a Cauchy sequence in $L^\infty(K)$ for all compacts $K \subset \mathbb{R}^N \times [0, +\infty)$, can be reproduced without modification.

Observe now that if u is a cluster point for $\{u_h\}$, then by (4.9) and by Theorem 4.3 for all $\lambda \in \mathbb{R}$ there exist a superflow E_λ and a subflow A_λ, with initial datum $E^{0,\lambda}$, such that

$$\{(x,t) : u(x,t) < \lambda\} \subseteq A_\lambda \subseteq E_\lambda \subseteq \{(x,t) : u(x,t) \le \lambda\}\,. \tag{4.13}$$

Let u_1, u_2 be two cluster points for $\{u_h\}$ and for any $\lambda \in \mathbb{R}$ let A^i_λ, E^i_λ be as in (4.13), with u replaced by u_i, $i = 1, 2$. Fix $\lambda < \lambda'$. Since

$$\operatorname{dist}^{\psi^\circ}(E^1_\lambda(0), \mathbb{R}^N \setminus A^2_{\lambda'}(0)) \ge \operatorname{dist}^{\psi^\circ}(\{u^0 \le \lambda\}, \mathbb{R}^N \setminus \{u^0 < \lambda'\}) > 0\,,$$

where the last inequality follows from the uniform continuity of u^0, it follows from Theorem 2.7 that $E^1_\lambda(t) \subseteq A^2_{\lambda'}(t)$ and, in turn, from (4.13)

$$\{u_1(\cdot,t) < \lambda\} \subseteq E^1_\lambda(t) \subseteq A^2_{\lambda'}(t) \subseteq \{u_2(\cdot,t) \le \lambda'\}$$

for all $t > 0$. The arbitrariness of $\lambda < \lambda'$ in the above chain of inequalities clearly implies that $u_2 \le u_1$. Exchanging the role of u_1 and u_2, we get in fact $u_1 = u_2$. Thus, there exists a unique cluster point u and $u_h \to u$ locally uniformly in $\mathbb{R}^N \times [0, +\infty)$ as $h \to 0^+$.

Step 2. (Proof of (4.10), (4.11)) For $\lambda \in \mathbb{R}$ set $K_\lambda := \{(x,t) : u(x,t) \le \lambda\}$. Since for any (x,t) the map $\lambda \mapsto \operatorname{dist}((x,t), K_\lambda)$ is non-increasing (here dist denotes the Euclidean distance in $\mathbb{R}^N \times [0, +\infty)$) and since $\operatorname{dist}(\cdot, K_\lambda)$ is (Lipschitz) continuous, we easily deduce the existence of a countable set $N_1 \subset \mathbb{R}$ such that for all $\lambda_0 \in \mathbb{R} \setminus N_1$ the map $\lambda \mapsto \operatorname{dist}((x,t), K_\lambda)$ is continuous at λ_0 for all $(x,t) \in \mathbb{R}^N \times [0, +\infty)$. In turn, by equicontinuity, it follows that $\operatorname{dist}(\cdot, K_\lambda) \to \operatorname{dist}(\cdot, K_{\lambda_0})$ locally uniformly in $\mathbb{R}^N \times [0, +\infty)$, or equivalently, $K_\lambda \xrightarrow{\mathcal{K}} K_{\lambda_0}$ as $\lambda \to \lambda_0$. In particular, by taking $\lambda_n \nearrow \lambda_0$ and using that $\bigcup_{n\ge1} K_{\lambda_n} = \{u < \lambda_0\}$, we deduce $\overline{\{(x,t) : u(x,t) < \lambda_0\}} = \{(x,t) : u(x,t) \le \lambda_0\}$. Analogously, one can show that there exists a countable set $N_2 \subset \mathbb{R}$ such that for all $\lambda_0 \notin N_2$ we have $\{(x,t) : u(x,t) \ge \lambda\} \xrightarrow{\mathcal{K}} \{(x,t) : u(x,t) \ge \lambda_0\}$ as $\lambda \to \lambda_0$, so that $\overline{\{(x,t) : u(x,t) > \lambda_0\}} = \{(x,t) : u(x,t) \ge \lambda_0\}$. We conclude that for all $\lambda \notin N_1 \cup N_2$, (4.11) holds.

A similar proof applied, this time, to the level sets of u^0 in $\mathbb{R}^N$ will ensure that (4.10) holds for all λ up to an at most countable set of levels N_3. We conclude that (4.10) and (4.11) hold for all $\lambda \notin N_0 := N_1 \cup N_2 \cup N_3$.

Step 3. (Conclusion) Fix $\lambda \in \mathbb{R} \setminus N_0$ and let E_λ and $(A_\lambda)^c$ be Kuratowski limits along a common subsequence of $E_{\lambda,h}$ and $(\mathring{E}_{\lambda,h})^c$, respectively. Then, by Theorem 4.3, E_λ and A_λ are a superflow and subflow, respectively, with initial datum $E^{0,\lambda}$. Moreover, (4.13) holds. Thus, recalling (4.11),

$$A_\lambda = \{(x,t) : u(x,t) < \lambda\}\,, \quad E_\lambda = \{(x,t) : u(x,t) \le \lambda\} \quad \text{and} \quad A_\lambda = \mathring{E}_\lambda\,.$$

This, together with (4.10), shows that E_λ is a solution to the curvature flow with initial datum $E^{0,\lambda}$. Let now E' be any superflow with initial datum $E^{0,\lambda}$. Then, for all $\lambda' > \lambda$, with $\lambda' \notin N_0$, thanks to Theorem 2.7 we easily deduce that $E' \subset \{(x,t) : u(x,t) < \lambda'\}$. Thus, $E' \subset E_\lambda$. Analogously, if A' is a subflow with initial datum $E^{0,\lambda}$ one has $\{(x,t) : u(x,t) \le \lambda'\} \subset A'$ for all $\lambda' < \lambda$, with $\lambda' \notin N_0$, and thus $A_\lambda \subset A'$. Therefore, we have

$$E' \subseteq E_\lambda = \bar{A}_\lambda \subseteq \bar{A}',$$
$$\mathring{E}' \subseteq \mathring{E}_\lambda = A_\lambda \subseteq A'.$$

In particular in case E' and A' define another solution (in the sense of Definition 2.2), that is, $\bar{A}' = E'$, $\mathring{E}' = A'$, we deduce that $E' = E_\lambda$, $A' = A_\lambda$, i.e., the uniqueness of the solution to (1.1), starting from $E^{0,\lambda}$.

Consider now the case $\lambda \in N_0$: the same argument above shows that if E' is a superflow with initial datum $E^{0,\lambda}$, then still $E' \subset \{(x,t) : u(x,t) \leq \lambda\}$. Let now $\lambda_n \searrow \lambda$, $\lambda_n \notin N_0$ for all n. Since $\{(x,t) : u(x,t) \leq \lambda_n\} \xrightarrow{\mathcal{K}} \{(x,t) : u(x,t) \leq \lambda\}$ it easily follows from the stability property stated in Remark 4.4 that $\{(x,t) : u(x,t) \leq \lambda\}$ is itself a superflow, thus the maximal superflow with initial datum $E^{0,\lambda}$. Analogously, one can show that $\{(x,t) : u(x,t) < \lambda\}$ is the minimal subflow with initial datum $E^{0,\lambda}$.

Finally, the uniqueness of the level set flow follows from the comparison principle proved in Theorem 2.8. $\square$

We conclude this section with the following remarks.

Remark 4.9 (Independence of the initial level set function). From the minimality and the maximality properties stated at the end of Theorem 4.8, we immediately deduce that if $\{u^0 < \lambda\} = \{v^0 < \lambda\}$, then $\{u(\cdot,t) < \lambda\} = \{v(\cdot,t) < \lambda\}$ for all $t > 0$. Analogously, if $\{u^0 \leq \lambda\} = \{v^0 \leq \lambda\}$, then $\{u(\cdot,t) \leq \lambda\} = \{v(\cdot,t) \leq \lambda\}$ for all $t > 0$.

Remark 4.10 (Stability of level set flows with respect to varying anisotropies and mobilities). Let $\{\phi_n\}$ and $\{\psi_n\}$ be sequences of anisotropies and mobilities, respectively, such that ψ_n is ϕ_n-regular uniformly in n (*cf* Remark 4.4). Assume also that $\phi_n \to \phi$ and $\psi_n \to \psi$. Let u_n be the unique level set solution in the sense of Definition 2.5, with ϕ and ψ replaced by ϕ_n and ψ_n, respectively, and with initial datum u^0. Then $u_n \to u$ locally uniformly, where u is the unique level set solution in the sense of Definition 2.5, with anisotropy ϕ, mobility ψ and initial datum u^0.

To see this, we start by observing that the sequence $\{u_n\}$ is equicontinuous in $\mathbb{R}^N \times [0,T]$ for all $T > 0$ (see the discussion at the beginning of Subsection 4.3 and before Definition 2.5). Thus, up to a not relabeled subsequence, we may assume that $u_n \to u$ locally uniformly in $\mathbb{R}^N \times [0,+\infty)$. It is enough to show that u is a solution in the sense of Definition 2.5, since then we conclude by uniqueness. Let now N_0 be a countable set such that if $\lambda \notin N_0$, then (4.11) holds for u and for u_n for all n. Set $E^n := \{u_n \leq \lambda\}$ and $d_n(\cdot,t) := \mathrm{dist}^{\psi_n^\circ}(\cdot, E^n(t))$ in $\mathbb{R}^N \setminus E^n(t)$. By Theorem 4.8 we have that E^n is a superflow with anisotropy ϕ_n, mobility ψ_n and initial datum $E^{0,\lambda}$ for all n. Extracting a further subsequence, if needed, we may also assume that $E^n \xrightarrow{\mathcal{K}} E$ and $(\mathring{E}^n)^c \xrightarrow{\mathcal{K}} A^c$ for suitable E and A such that

$$\{u < \lambda\} \subseteq A \subseteq E \subseteq \{u \leq \lambda\}\,.$$

But then, recalling (4.11), $\{u < \lambda\} = A$, $E = \{u \leq \lambda\}$, and $A = \mathring{E}$. Moreover, by Remark 4.4, E is a superflow with anisotropy ϕ, mobility ψ and initial datum $E^{0,\lambda}$. Analogously, one can show that A is a subflow with anisotropy ϕ, mobility ψ and initial datum $E^{0,\lambda}$. We conclude that $E = \{u \leq \lambda\}$ is a solution in the sense of Definiton 2.2 with initial datum $E^{0,\lambda}$ for all but countably many λ's, thus showing that u is a level set solution in the sense of Definition 2.5, with initial datum u^0.

5. The case of general mobilities: Existence and uniqueness by approximation

In this section we prove one of the main results of this paper: namely the existence via approximation by ϕ-regular mobilities of a unique solution to the level set crystalline flow with a general mobility. As a byproduct of the proof we will also obtain uniqueness, up to fattening, of the flat flow, i.e. of the flow obtained by the ATW scheme. The main results are stated and proven in

Subsection 5.2. In the next subsection we collect some preliminary stability estimates on the ATW scheme.

5.1. **Stability of the ATW scheme with respect to changing mobilities.** We start with the following remark:

Remark 5.1. For any norm η and any closed set $E \subset \mathbb{R}^N$, it is easily seen that

$$d^{\eta}_{(E)^{\eta}_r} \le d^{\eta}_E - r$$

where we have used the notation in (3.20) (with ϕ° replaced by η).

We recall (see Remark 3.5) that given a closed set H, $H^{\psi,k}_{g,h}$ denotes a k-th minimizing movement starting from H, with mobility ψ, forcing term g, and time step h (and the given anisotropy ϕ). As already observed, in the previous notation the dependence on the anisotropy ϕ is omitted since we think of ϕ as fixed. We finally recall that by an admissible forcing term g we mean a function satisfying assumptions H1), H2) of Subsection 2.2.

The next lemma establishes a comparison result for minimizing movements with different forcing terms.

Lemma 5.2. *Let ψ, η be two norms such that $\psi \le \beta\eta$ for some $\beta > 0$, and let g_1, g_2 be admissible forcing terms satisfying*

$$g_2 - g_1 \le c < +\infty \text{ in } \mathbb{R}^N \times [0, +\infty)\,.$$

If $E \subset F$ are closed sets with $\mathrm{dist}^{\eta^\circ}(E, \mathbb{R}^N \setminus F) := \Delta > 0$, then, for all $k \in \mathbb{N}$ we have

$$\mathrm{dist}^{\eta^\circ}(E^{\psi,k}_{g_1,h}, \mathbb{R}^N \setminus F^{\psi,k}_{g_2,h}) \ge \left(\Delta + \frac{c}{L_{\eta^\circ}}\right)(1 - \beta L_{\eta^\circ} h)^k - \frac{c}{L_{\eta^\circ}}, \tag{5.1}$$

where L_{η° is the Lipschitz constant of g_1 and g_2 with respect to η°.

Proof. We start by considering the case $k = 1$. Set $\widehat{c} := c + L_{\eta^\circ}\Delta$. Let (u, z) solve

$$-h\,\mathrm{div} z + u = d^{\psi^\circ}_E + \int_0^h g_1(\cdot, s)\, ds, \quad z \in \partial\phi(\nabla u).$$

Let $\tau \in \mathbb{R}^N$, with $\eta^\circ(\tau) \le \Delta$. By our assumptions on g_1, g_2, one has for all s, $g_1(\cdot - \tau, s) \ge g_2(\cdot, s) - c - L_{\eta^\circ}\eta^\circ(\tau) \ge g_2(\cdot, s) - \widehat{c}$, hence

$$\begin{aligned}
-h\,\mathrm{div} z(\cdot - \tau) + u(\cdot - \tau) &= d^{\psi^\circ}_{E+\tau} + \int_0^h g_1(\cdot - \tau, s)\, ds \\
&\ge d^{\psi^\circ}_{E+\tau} + \int_0^h g_2(\cdot, s)\, ds - \widehat{c}h \\
&\ge d^{\psi^\circ}_{(E)^{\psi^\circ}_{\widehat{c}h}+\tau} + \int_0^h g_2(\cdot, s)\, ds,
\end{aligned}$$

where the last inequality is Remark 5.1. Thus by comparison, and using $\psi \le \beta\eta$, we deduce that

$$E^{\psi,1}_{g_1,h} + \tau \subseteq \left((E)^{\psi^\circ}_{\widehat{c}h} + \tau\right)^{\psi,1}_{g_2,h} \subseteq \left((E)^{\eta^\circ}_{\beta\widehat{c}h} + \tau\right)^{\psi,1}_{g_2,h} \subseteq F^{\psi,1}_{g_2,h},$$

provided that $(E)^{\eta^\circ}_{\beta\widehat{c}h} + \tau \subseteq F$. The latter condition holds true if $\eta^\circ(\tau) + \beta\widehat{c}h \le \Delta$. We deduce that

$$\mathrm{dist}^{\eta^\circ}\left(E^{\psi,1}_{g_1,h}, \mathbb{R}^N \setminus F^{\psi,1}_{g_2,h}\right) \ge \Delta - \beta\widehat{c}h = \Delta(1 - \beta L_{\eta^\circ} h) - \beta c h = \left(\Delta + \frac{c}{L_{\eta^\circ}}\right)(1 - \beta L_{\eta^\circ} h) - \frac{c}{L_{\eta^\circ}}.$$

The conclusion easily follows by induction. □

In the next lemma we compare the (discrete-time) solutions corresponding to different but close mobilities and forcing terms.

Lemma 5.3. *For any G, $\Delta > 0$, $\beta \geq 1$ and $\theta \in (0,1)$, there exist positive δ_0, h_0, depending on all the previous constants, on the dimension N and on the anisotropy ϕ, and there exists $c_0 > 0$ depending on the same quantities but Δ, with the following property: Let g be an admissible forcing term, with $\|g\|_\infty \leq G$, and let ψ_1, ψ_2 be two mobilities satisfying*

$$\frac{1}{\beta}\phi \leq \psi_i \leq \beta\phi \qquad \text{for } i \in \{1, 2\} \tag{5.2}$$

and

$$\psi_2 \leq \psi_1 \leq (1+\delta)\psi_2 \tag{5.3}$$

for some $0 < \delta \leq \delta_0$. If E and F are two closed sets with $\mathrm{dist}^{\phi^\circ}(E, \mathbb{R}^N \setminus F) \geq \Delta$, then, setting $\widetilde{g} := g - c_0\frac{\delta}{\Delta}$, for all $0 < h \leq h_0$ we have

$$\mathrm{dist}^{\phi^\circ}(E^{\psi_1,k}_{g,h}, \mathbb{R}^N \setminus F^{\psi_2,k}_{\widetilde{g},h}) \geq \Delta\big(1 - \beta L_{\phi^\circ} h\big)^k \tag{5.4}$$

for all $k \in \mathbb{N}$ such that the right-hand side of the above inequality is larger than $\theta\Delta$. Here L_{ϕ° denotes the Lipschitz constant of g with respect to the metric ϕ°.

Proof. With the notation introduced in (3.20), set $H := \big((E)^{\phi^\circ}_{\Delta}\big)^{\phi^\circ}_{-\Delta}$ and note that $E \subseteq H$ and $\mathrm{dist}^{\phi^\circ}(H, \mathbb{R}^N \setminus F) \geq \Delta$. Also, it is easy to see that for $\theta \in (0,1)$ the set $\mathbb{R}^N \setminus H$ can be written as a union of closed ϕ-Wulff shapes of radius $\theta\Delta =: \Delta_0$. Thus, by Lemma 3.12 (and recalling (5.2)) there exists M_0, depending on G, β, ϕ, and the dimension N, and there exists h_0 depending on the same quantities and on Δ_0, such that

$$H^{\psi_i,1}_{\widehat{g},h} \subset (H)^{\phi^\circ}_{\frac{M_0 h}{\Delta_0}} \quad \text{for } i = 1, 2, \text{ for } 0 < h \leq h_0 \text{ and for any admissible } \widehat{g} \text{ s.t. } \|\widehat{g}\|_\infty \leq G + 1. \tag{5.5}$$

By (5.3) it follows that

$$\psi_1^\circ \leq \psi_2^\circ \leq (1+\delta)\psi_1^\circ$$

and, in turn, one has

$$\begin{cases} d_H^{\psi_1^\circ}(x) \leq d_H^{\psi_2^\circ}(x) \leq (1+\delta) d_H^{\psi_1^\circ}(x) & \text{if } x \notin \mathring{H}, \\ (1+\delta) d_H^{\psi_1^\circ}(x) \leq d_H^{\psi_2^\circ}(x) \leq d_H^{\psi_1^\circ}(x) & \text{if } x \in \overline{H} \end{cases} \tag{5.6}$$

so that

$$d_H^{\psi_2^\circ} - d_H^{\psi_1^\circ} \leq \delta\big(d_H^{\psi_1^\circ}\big)^+ .$$

In particular,

$$d_H^{\psi_2^\circ} \leq d_H^{\psi_1^\circ} + h\frac{\beta M_0}{\theta}\frac{\delta}{\Delta} \quad \text{in } (H)^{\phi^\circ}_{\frac{M_0 h}{\Delta_0}}. \tag{5.7}$$

Set $\delta_0 := \frac{\theta\Delta}{2\beta M_0}$, $c_0 := \frac{\beta M_0}{\theta}$ and note that

$$c_0\frac{\delta}{\Delta} \leq \frac{\theta}{2} \leq \frac{1}{2} \qquad \text{for } 0 < \delta \leq \delta_0. \tag{5.8}$$

Thus, setting $\widetilde{g} := g - c_0\frac{\delta}{\Delta}$, by (5.5) we have

$$H^{\psi_1,1}_{g,h},\, H^{\psi_2,1}_{\widetilde{g},h} \subset (H)^{\phi^\circ}_{\frac{M_0 h}{\Delta_0}}$$

provided that $0 < \delta \le \delta_0$, $0 < h \le h_0$ (recall $\|g\|_\infty \le G \le G + \frac{1}{2}$). Thus, we may apply (5.7) and Lemma 3.2 to deduce that

$$E^{\psi_1,1}_{g,h} \subseteq H^{\psi_1,1}_{g,h} \subseteq H^{\psi_2,1}_{\widetilde{g},h}.$$

In turn, by Lemma 5.2 (with $g_1 = g_2 = \widetilde{g}$, $c = 0$, $\psi = \psi_2$, and $\eta = \phi$) we get

$$\mathrm{dist}^{\phi^\circ}(E^{\psi_1,1}_{g,h}, \mathbb{R}^N \setminus F^{\psi_2,1}_{\widetilde{g},h}) \ge \mathrm{dist}^{\phi^\circ}(H^{\psi_2,1}_{\widetilde{g},h}, \mathbb{R}^N \setminus F^{\psi_2,1}_{\widetilde{g},h}) \ge \Delta(1 - \beta L_{\phi^\circ} h)\,,$$

which is (5.4) for $k = 1$.

We can iterate this construction as long as this distance is larger than Δ_0, deducing that (5.4) holds as long as $(1 - \beta L_{\phi^\circ} h)^k \ge \theta$. □

Combining Lemmas 5.2 and 5.3, we obtain the following proposition.

Proposition 5.4. *For any G, $\Delta > 0$, $\beta \ge 1$, and $\theta \in (0,1)$, there exist positive δ_0, h_0, depending on all the previous constants, on the dimension N and on the anisotropy ϕ, and there exists $c_0 > 0$ depending on the same quantities but Δ, with the following property: Let g be an admissible forcing term, with $\|g\|_\infty \le G$, and let ψ_1, ψ_2 be two mobilities satisfying* (5.2) *and* (5.3) *for some $0 < \delta \le \delta_0$. If E and F are two closed sets with $\mathrm{dist}^{\phi^\circ}(E, \mathbb{R}^N \setminus F) \ge \Delta$, then for all $0 < h \le h_0$ we have*

$$\mathrm{dist}^{\phi^\circ}(E^{\psi_1,k}_{g,h}, \mathbb{R}^N \setminus F^{\psi_2,k}_{g,h}) \ge \Big(\Delta + \frac{2c_0\delta}{L_{\phi^\circ}\Delta}\Big)(1 - \beta L_{\phi^\circ} h)^k - \frac{2c_0\delta}{L_{\phi^\circ}\Delta} \tag{5.9}$$

for all $k \in \mathbb{N}$ such that $(1 - \beta L_{\phi^\circ} h)^k \ge \theta$. Here L_{ϕ° denotes the Lipschitz constant of g with respect to the metric ϕ°.

Proof. In what follows, δ_0, h_0, and c_0 are the constant provided by Lemma 5.3. With the notation introduced in (3.20), let $H := (E)^{\phi^\circ}_{\frac{\Delta}{2}}$, so that

$$\mathrm{dist}^{\phi^\circ}(E, \mathbb{R}^N \setminus H) \ge \frac{\Delta}{2}, \qquad \mathrm{dist}^{\phi^\circ}(H, \mathbb{R}^N \setminus F) \ge \frac{\Delta}{2}.$$

Set $\widetilde{g} := g - 2c_0\frac{\delta}{\Delta}$. By (5.4) (with $F^{\psi_2,k}_{\widetilde{g},h}$ and Δ replaced by $H^{\psi_2,k}_{\widetilde{g},h}$ and $\Delta/2$, respectively) we have

$$\mathrm{dist}^{\phi^\circ}(E^{\psi_1,k}_{g,h}, \mathbb{R}^N \setminus H^{\psi_2,k}_{\widetilde{g},h}) \ge \frac{\Delta}{2}(1 - \beta L_{\phi^\circ} h)^k \tag{5.10}$$

for all $0 < h \le h_0$ and for all $k \in \mathbb{N}$ such that $(1 - \beta L_{\phi^\circ} h)^k \ge \theta$.

Moreover, by Lemma 5.2 (with $\eta = \phi$, $g_1 := \widetilde{g}$, $g_2 := g$, $c := 2c_0\frac{\delta}{\Delta}$, and E replaced by H) we have

$$\mathrm{dist}^{\phi^\circ}(H^{\psi_2,k}_{\widetilde{g},h}, \mathbb{R}^N \setminus F^{\psi_2,k}_{g,h}) \ge \Big(\frac{\Delta}{2} + \frac{2c_0\delta}{L_{\phi^\circ}\Delta}\Big)(1 - \beta L_{\phi^\circ} h)^k - \frac{2c_0\delta}{L_{\phi^\circ}\Delta} \tag{5.11}$$

for all $k \in \mathbb{N}$. Since

$$\mathrm{dist}^{\phi^\circ}(E^{\psi_1,k}_{g,h}, \mathbb{R}^N \setminus F^{\psi_2,k}_{g,h}) \ge \mathrm{dist}^{\phi^\circ}(E^{\psi_1,k}_{g,h}, \mathbb{R}^N \setminus H^{\psi_2,k}_{\widetilde{g},h}) + \mathrm{dist}^{\phi^\circ}(H^{\psi_2,k}_{\widetilde{g},h}, \mathbb{R}^N \setminus F^{\psi_2,k}_{g,h}),$$

the conclusion follows directly by (5.10) and (5.11). □

Remark 5.5 (Varying anisotropies). A careful inspection of the proof of Proposition 5.4 (and of Lemma 5.3) together with Remark 3.14 shows that the constants δ_0, h_0, c_0 can be chosen as depending on ϕ only through the ellipticity constants a_1, a_2 in (3.17). This observation implies that estimate (5.9) holds uniformly with respect to converging sequences of anisotropies.

More precisely, let $\{\phi_n\}$ be a sequence of anisotropies such that $\phi_n \to \phi$ and let us denote, temporarily, by $E^{\psi,\phi_n,k}_{g,h}$ the k-th minimizing movement starting from E, with mobility ψ, forcing term g, time-step h, *and anisotropy* ϕ_n. Set $a_1' = a_1/2$, $a_2' = 2a_2$, $\beta' = 2\beta$ and observe that for n large ϕ_n satisfies (3.17) with a_i' in place of a_i. Moreover, if ψ satisfies (5.2), then it also satisfies (5.2) with ϕ, β replaced by ϕ_n, β', respectively, provided that n is large enough. Therefore, we may find δ_0, h_0, depending on β', G, Δ, a_1' (and the dimension N), and c_0 depending on all the same quantities but Δ, such that under the assumptions of Proposition 5.4 we have for all n sufficiently large

$$\mathrm{dist}^{\phi^\circ}(E^{\psi_1,\phi_n,k}_{g,h}, \mathbb{R}^N \setminus F^{\psi_2,\phi_n,k}_{g,h}) \geq \Big(\Delta + \frac{2c_0\delta}{L_{\phi^\circ}\Delta}\Big)(1-\beta L_{\phi^\circ}h)^k - \frac{2c_0\delta}{L_{\phi^\circ}\Delta}$$

for all $k \in \mathbb{N}$ such that $(1-\beta L_{\phi^\circ}h)^k \geq \theta$.

5.2. **Existence and uniqueness by approximation.** In the following, given a uniformly continuous function u^0 on $\mathbb{R}^N$, an admissible forcing term g and a mobility ψ, we denote by u^ψ_h the corresponding level set minimizing movement, defined according to (4.8). Analogously, we use the notation $E^\psi_{\lambda,h}(t)$ (in place of $E_{\lambda,h}(t)$) to denote the discrete-in-time evolution starting from $E^{0,\lambda} := \{u^0 \leq \lambda\}$ and with mobility ψ (see Subsection 4.3). In the above notation we have highlighted only the dependence on ψ since in the following we will establish stability properties of flat flows with respect to varying mobilities.

We recall that the existence theory for level set flows (in the sense of Definition 2.5) that we have so far works only for ϕ-regular mobilities. The goal of this section is to extend the existence theory to general mobilities. To this aim, we consider the following notion of *solution via approximation*:

Definition 5.6 (Level set flows via approximation)**.** *Let ψ, g, and u^0 be a mobility, an admissible forcing term, and a uniformly continuous function on $\mathbb{R}^N$, respectively.*

We will say that a continuous function $u^\psi : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is a solution via approximation *to the level set flow corresponding to* (1.1)*, with initial datum u^0, if $u^\psi(\cdot,0) = u^0$ and if there exists a sequence $\{\psi_n\}$ of ϕ-regular mobilities such that $\psi_n \to \psi$ locally uniformly and, denoting by u^{ψ_n} the unique solution to* (1.1) *(in the sense of Definition 2.5) with mobility ψ_n and initial datum u^0, we have $u^{\psi_n} \to u^\psi$ locally uniformly in $\mathbb{R}^N \times [0,+\infty)$.*

The next theorem is the main result of this section: it shows that for any mobility ψ a solution-via-approximation u^ψ in the sense of the previous definition always exists; such a solution is also unique in that it is independent of the choice of the approximating sequence of ϕ-regular mobilities $\{\psi_n\}$ and, in fact, coincides with the (unique) limit of the level set minimizing movement scheme $\{u^\psi_h\}$. In particular, in the case of a ϕ-regular mobility the notion of solution via approximation is consistent with that of Definition 2.5.

Theorem 5.7. *Let ψ, g, and u^0 be as in Definition 5.6. Then, there exists a unique solution u^ψ in the sense of Definition 5.6 with initial datum u^0. Moreover, the following holds:*

(i) (Convergence of the level set minimizing movements scheme) *The solution u^ψ is the locally uniform limit in $\mathbb{R}^N \times [0,+\infty)$, as $h \to 0^+$, of the level set minimizing movements u^ψ_h.*

(ii) (Stability) *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of mobilities such that $\psi_n \to \psi$ locally uniformly. Then u^{ψ_n} converge to u^ψ uniformly in $\mathbb{R}^N \times [0,T]$ for all $T>0$ as $n\to\infty$.*

Proof. The strategy is the following: We first show that for any ψ the minimizing movements u_h^ψ converge to a unique function u^ψ, as $h\to 0^+$. Then we establish the stability property (ii), which shows, in particular, that u^ψ is a solution in the sense of Definition 5.6. We split the proof of theorem into three steps.

Step 1. We claim that for every ε, $T>0$ and $\beta\geq 1$, there exist positive δ_0, $h_0>0$ (depending also on g, u^0, ϕ, and the dimension N) such that if ψ_1 and ψ_2 are two mobilities satisfying (5.2) and (5.3) for some $0<\delta\leq\delta_0$, then

$$\|u_h^{\psi_2}-u_h^{\psi_1}\|_{L^\infty(\mathbb{R}^N\times[0,T])}\leq\varepsilon \qquad \text{for all } 0<h\leq h_0. \tag{5.12}$$

To this aim, let ω be an increasing modulus of continuity for u^0 with respect to ϕ° and recall that for any $\lambda\in\mathbb{R}$

$$\mathrm{dist}^{\phi^\circ}(E^{0,\lambda},\mathbb{R}^N\setminus E^{0,\lambda+\varepsilon})\geq\omega^{-1}(\varepsilon)\,.$$

Set

$$\theta(T):=(2e)^{-\beta L_{\phi^\circ}T}\,,$$

where L_{ϕ° denotes the spatial Lipschitz constant of the forcing term g with respect to ϕ°, and choose $\bar h>0$ so small that $(1-\beta L_{\phi^\circ}h)^{-\frac{1}{\beta L_{\phi^\circ}h}}\leq 2e$ for all $h\in(0,\bar h)$. Let δ_0, h_0, c_0 be the positive constants provided by Proposition 5.4 and corresponding to the given β, $G:=\|g\|_\infty$, $\Delta:=\omega^{-1}(\varepsilon)$, and $\theta:=\theta(T)$. Clearly we may assume $h_0\leq\bar h$.

By Proposition 5.4, if ψ_1 and ψ_2 satisfy (5.2) and (5.3) for some $0<\delta\leq\delta_0$, then for $h\in(0,h_0]$ we have

$$\begin{aligned}\mathrm{dist}^{\phi^\circ}\big(E^{\psi_1}_{\lambda,h},\mathbb{R}^N\setminus E^{\psi_2}_{\lambda+\varepsilon,h}(t)\big)&\geq\Big(\omega^{-1}(\varepsilon)+\frac{2c_0\delta}{L_{\phi^\circ}\omega^{-1}(\varepsilon)}\Big)(1-\beta L_{\phi^\circ}h)^{[\frac{t}{h}]}-\frac{2c_0\delta}{L_{\phi^\circ}\omega^{-1}(\varepsilon)}\\ &\geq\Big(\omega^{-1}(\varepsilon)+\frac{2c_0\delta}{L_{\phi^\circ}\omega^{-1}(\varepsilon)}\Big)(2e)^{-\beta L_{\phi^\circ}t}-\frac{2c_0\delta}{L_{\phi^\circ}\omega^{-1}(\varepsilon)},\end{aligned} \tag{5.13}$$

for all $t\in(0,T]$. Clearly, by choosing δ_0 smaller if needed, we may assume that right-hand side of (5.13) is positive for all $t\in(0,T]$. Recalling (4.9), we conclude that

$$\{u_h^{\psi_1}(\cdot,t)<\lambda\}\subseteq E^{\psi_1}_{\lambda,h}(t)\subseteq E^{\psi_2}_{\lambda+\varepsilon,h}(t)\subseteq\{u_h^{\psi_2}(\cdot,t)\leq\lambda+\varepsilon\}$$

for all $\lambda\in\mathbb{R}$, $h\in(0,h_0]$, and $t\in[0,T]$. This in turn implies that

$$u_h^{\psi_2}(\cdot,t)\leq u_h^{\psi_1}(\cdot,t)+\varepsilon \qquad \text{for all } h\in(0,h_0] \text{ and } t\in[0,T].$$

We may now repeat the same argument by considering $-u^0$ as initial function, instead of u^0. This leads to the inequality

$$-u_h^{\psi_2}(\cdot,t)\leq -u_h^{\psi_1}(\cdot,t)+\varepsilon \qquad \text{for all } h\in(0,h_0] \text{ and } t\in[0,T],$$

which together with the previous one proves (5.12).

Step 2. Here we prove that $\{u_h^\psi\}_h$ satisfies the Cauchy condition in $L^\infty(K\times[0,T])$ for all compact sets $K\subset\mathbb{R}^N$ and for all $T>0$.

To this purpose, let T, $\varepsilon>0$, let $\beta\geq 2$ satisfy $\frac{2}{\beta}\phi\leq\psi\leq\frac{\beta}{2}\phi$, and let δ_0, h_0 be the corresponding constants provided by Step 1. Clearly we may choose a ϕ-regular mobility (see Definition 4.1) $\widehat{\psi}$

such that $\psi_1 := \psi$ and $\psi_2 := \widehat{\psi}$ satisfy (5.2) and (5.3) for some $0 < \delta \le \delta_0$. Pick any sequence $h_n \searrow 0$. Then, we may write

$$\|u^{\psi}_{h_n} - u^{\psi}_{h_m}\|_{L^\infty(K\times[0,T])} \le \|u^{\psi}_{h_n} - u^{\widehat{\psi}}_{h_n}\|_{L^\infty(K\times[0,T])} + \|u^{\widehat{\psi}}_{h_n} - u^{\widehat{\psi}}_{h_m}\|_{L^\infty(K\times[0,T])} + \|u^{\widehat{\psi}}_{h_m} - u^{\psi}_{h_m}\|_{L^\infty(K\times[0,T])}\,.$$

The first and the third term on the right-hand side of the above inequality are both less than or equal to ε thanks to (5.12), provided that h_n, $h_m \le h_0$. Recall now that by Theorem 4.8-(ii) the family $\{u^{\widehat{\psi}}_h\}_h$ satisfies the Cauchy condition in $L^\infty(K \times [0,T])$; thus also the middle term on the right-hand side of the above inequality is smaller than ε for n and m large enough. We conclude that $\|u^{\psi}_{h_n} - u^{\psi}_{h_m}\|_{L^\infty(K\times[0,T])} \le 3\varepsilon$ for n, m sufficiently large. This establishes the claim and shows that u^{ψ}_h converges locally uniformly in $\mathbb{R}^N \times [0,+\infty)$. We denote its limit by u^{ψ}.

Step 3. Let $\{\psi_n\}_n$ be a sequence of mobilities such that $\psi_n \to \psi$ locally uniformly. First of all, observe that we may find $\lambda_n \to 1^-$such that $\widehat{\psi}_n := \lambda_n\psi_n \le \psi$ for all n.

Fix $\varepsilon > 0$, let $\beta \ge 2$ be as in Step 2, and let δ_0, h_0 be the corresponding constants provided by Step 1. Note that for any $0 < \delta \le \delta_0$ we have:

$$\widehat{\psi}_n \le \psi \le (1+\delta)\widehat{\psi}_n\,, \qquad \widehat{\psi}_n \le \psi_n \le (1+\delta)\widehat{\psi}_n \quad \text{and} \quad \frac{1}{\beta}\phi \le \psi, \psi_n, \widehat{\psi}_n \le \beta\phi\,, \tag{5.14}$$

provided n large enough. Thus, thanks to Step 1, for all such n's and for all $h \le h_0$ we have:

$$\|u^{\psi}_h - u^{\psi_n}_h\|_{L^\infty(\mathbb{R}^N\times[0,T])} \le \|u^{\psi}_h - u^{\widehat{\psi}_n}_h\|_{L^\infty(\mathbb{R}^N\times[0,T])} + \|u^{\widehat{\psi}_n}_h - u^{\psi_n}_h\|_{L^\infty(\mathbb{R}^N\times[0,T])} \le 2\varepsilon\,.$$

Thanks to Step 2 we may send $h \to 0$ in the above inequality to infer that $\|u^{\psi} - u^{\psi_n}\|_{L^\infty(\mathbb{R}^N\times[0,T])} \le 2\varepsilon$ for all n sufficiently large. This concludes the proof of the theorem. $\square$

In the next theorem we collect the main properties of the level set solutions introduced in Definition 5.6.

To this aim, we will say that a uniformly continuous initial function u^0 is *well-prepared* at $\lambda \in \mathbb{R}$ if the following two conditions hold:

(a) If $H \subset \mathbb{R}^N$ is a closed set such that $\mathrm{dist}(H, \{u_0 \ge \lambda\}) > 0$, then there exists $\lambda' < \lambda$ such that $H \subseteq \{u_0 < \lambda'\}$;

(b) If $A \subset \mathbb{R}^N$ is an open set such that $\mathrm{dist}(\{u_0 \le \lambda\}, \mathbb{R}^N \setminus A) > 0$, then there exists $\lambda' > \lambda$ such that $\{u_0 \le \lambda'\} \subset A$.

Remark 5.8. Note that the above assumption of well-preparedness is automatically satisfied if the set $\{u_0 \le \lambda\}$ is bounded.

Theorem 5.9 (Properties of the level set flow). *Let u^{ψ} be a solution in the sense Definition 5.6, with initial datum u^0. The following properties hold true:*

(i) (Non-fattening level sets and unique flat flows) *There exists a countable set $N \subset \mathbb{R}$ such that for all $\lambda \notin N$*

$$\begin{aligned} \{(x,t) :\, u^{\psi}(x,t) < \lambda\} &= \mathrm{Int}\left(\{(x,t) : u^{\psi}(x,t) \le \lambda\}\right), \\ \overline{\{(x,t) : u^{\psi}(x,t) < \lambda\}} &= \{(x,t) : u^{\psi}(x,t) \le \lambda\} \end{aligned} \tag{5.15}$$

and the flat flow starting from $E^{0,\lambda}$ is unique. More precisley, we have

$$E^{\psi}_{\lambda,h} \xrightarrow{\mathcal{K}} \{(x,t) : u^{\psi}(x,t) \le \lambda\} \ \textit{and}\ \left(\mathrm{Int}\, E^{\psi}_{\lambda,h}\right)^c \xrightarrow{\mathcal{K}} \{(x,t) : u^{\psi}(x,t) \ge \lambda\}$$

as $h \to 0^+$.

(ii) (Distributional formulation when ψ is ϕ-regular) *If* ψ *is* ϕ*-regular, then* u^ψ *coincides with the distributional solution in the sense of Definition 2.5.*

(iii) (Comparison) *Assume that* $u^0 \le v^0$ *and denote the corresponding level set flows by* u^ψ *and* v^ψ*, respectively. Then* $u^\psi \le v^\psi$.

(iv) (Geometricity) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be increasing and continuous. Then* u^ψ *is a solution with initial datum* u^0 *if and only if* $f \circ u^\psi$ *is a solution with initial datum* $f \circ u^0$.

(v) (Independence of the initial level set function) *Assume that* u^0 *and* v^0 *are well-prepared at* λ*. If* $\{u^0 < \lambda\} = \{v^0 < \lambda\}$*, then* $\{u^\psi(\cdot, t) < \lambda\} = \{v^\psi(\cdot, t) < \lambda\}$ *for all* $t > 0$*. Analogously, if* $\{u^0 \le \lambda\} = \{v^0 \le \lambda\}$*, then* $\{u^\psi(\cdot, t) \le \lambda\} = \{v^\psi(\cdot, t) \le \lambda\}$ *for all* $t > 0$.

Proof. Property (ii) is obvious. Properties (i) can be proven arguing as in the proof of Theorem 4.8. Property (iii) follows at once from the stability property of flat flows with respect to approximation with smooth mobilities, and from Theorem 2.8. Also property (iv) follows by approximation, since clearly it is satisfied when the mobility ψ is ϕ-regular. Let us now prove property (v): Pick $\lambda_1 < \lambda$ and note that by uniform continuity and by assumption we have

$$\mathrm{dist}(\{u^0 \le \lambda_1\}, \{v^0 \ge \lambda\}) = \mathrm{dist}(\{u^0 \le \lambda_1\}, \{u^0 \ge \lambda\}) > 0$$

and thus there exists $\lambda_2 \in (\lambda_1, \lambda)$ such that $\{u^0 \le \lambda_1\} \subset \{v^0 < \lambda_2\}$. Let now $\{\psi_n\}$ be an approximating sequence of ϕ-regular mobilities and recall that by Theorem 4.8-(iii) the set $\{(x,t) : u^{\psi_n}(x,t) \le \lambda_1\}$ is a superflow with initial datum $\{u^0 \le \lambda_1\}$, while $\{(x,t) : v^{\psi_n}(x,t) < \lambda_2\}$ is a subflow with initial datum $\{v^0 < \lambda_2\}$. Thus, from Theorem 2.7 we deduce that $\{(x,t) : u^{\psi_n}(x,t) < \lambda_1\} \subset \{(x,t) : v^{\psi_n}(x,t) < \lambda_2\}$ for all n. In turn, from the latter inclusion we easily deduce that

$$\{(x,t) : u^\psi(x,t) < \lambda_1\} \subset \{(x,t) : v^\psi(x,t) \le \lambda_2\} \subset \{(x,t) : v^\psi(x,t) < \lambda\}\,.$$

By the arbitrariness of λ_1, we conclude that $\{(x,t) : u^\psi(x,t) < \lambda\} \subseteq \{(x,t) : v^\psi(x,t) < \lambda\}$. Symmetrically, also the opposite inclusion holds. The equality between the closed sub-level sets can be proven analogously. ☐

Remark 5.10 (Generalized motion). We observe that property (v) above allows one to consider $\Gamma_t := \{u^\psi(\cdot, t) = 0\}$ as defining a *generalized motion* starting from $\Gamma_0 := \{u^0 = 0\}$.

Remark 5.11 (Star-shaped sets, convex sets and graphs). A natural question is to understand under which circumstances fattening does not occur. To the best of our knowledge, no general results are available, even for the classical mean curvature flow. On the other hand, it is well-known [46, Sec. 9] that for the motion without forcing, strictly star-shaped sets do not develop fattening so that, in particular, their evolution is unique. The proof of this fact, given for instance in [46] for the mean curvature flow, works also for solutions in the sense of Definition 2.2 when the mobility ψ is ϕ-regular, and in turn, by approximation, also for the *generalized motion* associated to level set solutions in the sense of Definition 5.6, when ψ is general. Uniqueness also holds for motions with a time-dependent forcing $g(t)$ [14, Theorem 5] as long as the set remains strictly star-shaped. This remark obviously applies to initial convex sets, which, in addition, remain convex for all times, as was shown in [13, 18, 14] with a spatially constant forcing term.[7] The case of unbounded initial

[7]Convexity is preserved also with a spatially convex forcing term but uniqueness is not known in this case.

convex sets was not considered in these references but can be easily addressed by approximation (and uniqueness still holds with the same proof).

In the same way, if the initial set $E_0 = \{x_N \le v^0(x_1, \dots, x_{N-1})\}$ is the subgraph of a uniformly continuous functions v^0, and the forcing term does not depend on x_N, then one can show that fattening does not develop and $E(t)$ is still the subgraph of a uniformly continuous function for all $t > 0$, as in the classical case [27, 29] (see also [33] for the 2D crystalline case).

Eventually, we extend the stability property in Theorem 5.7-(ii) to varying anisotropies.

Proposition 5.12. *Let ψ, g, and u^0 be as in Theorem 5.7, let $\{\psi_n\}$ and $\{\phi_n\}$ be a sequences of mobilities and anisotropies, respectively, such that $\psi_n \to \psi$ and $\phi_n \to \phi$ locally uniformly as $n \to +\infty$. Denote by u^{ψ_n,ϕ_n} the level set solution in the sense of Definition 5.6 with ψ, ϕ replaced by ψ_n, ϕ_n, respectively, and with initial datum u^0. Then, $u^{\psi_n,\phi_n} \to u^{\psi,\phi}$ locally uniformly in $\mathbb{R}^N \times [0,+\infty)$ as $n \to \infty$.*

Proof. From Steps 1 and 3 in the proof of Theorem 5.7 combined with Remark 5.5, we have that for every ε, β, $T > 0$ there exist positive δ_0, $h_0 > 0$ (depending also on g, u^0, ϕ, the dimension N, but not on n) such that if ψ_1 and ψ_2 are two mobilities satisfying (5.2) and $\max_{|\xi|=1} |\psi_1(\xi) - \psi_2(\xi)| \le \delta_0$, then for all n large enough $\|u_h^{\psi_2,\phi_n} - u_h^{\psi_1,\phi_n}\|_{L^\infty(\mathbb{R}^N\times[0,T])} \le \varepsilon$ for all $0 < h \le h_0$. Sending $h \to 0$ we deduce $\|u^{\psi_2,\phi_n} - u^{\psi_1,\phi_n}\|_{L^\infty(\mathbb{R}^N\times[0,T])} \le \varepsilon$ for n large enough. Thus, in particular, we may choose $\delta > 0$ so small that letting $\psi_n^\delta := \psi + \delta\phi_n^\circ$, then

$$\|u^{\psi_n,\phi_n} - u^{\psi_n^\delta,\phi_n}\|_{L^\infty(\mathbb{R}^N\times[0,T])} \le \varepsilon \qquad \text{for } n \text{ large enough.} \tag{5.16}$$

Taking δ smaller, if needed, thanks to Theorem 5.7-(ii) we may also impose

$$\|u^{\psi^\delta,\phi} - u^{\psi,\phi}\|_{L^\infty(\mathbb{R}^N\times[0,T])} \le \varepsilon\,, \tag{5.17}$$

where ψ^δ is defined as ψ_n^δ, with ψ_n replaced by ψ. Note now that by construction ψ_n^δ is ϕ_n-regular with $W^{\psi_n^\delta}(0,1)$ satisfying an inner ϕ_n-Wulff shape condition of radius δ (and thus uniformly in n) and $\psi_n^\delta \to \psi^\delta$. By Remark 4.10 we then have $u^{\psi_n^\delta,\phi_n} \to u^{\psi^\delta,\phi}$ locally uniformly. Thus, for any fixed compact set $K \subseteq \mathbb{R}^N$, recalling also (5.16) and (5.17), we have

$$\limsup_n \|u^{\psi_n,\phi_n} - u^{\psi,\phi}\|_{L^\infty(K\times[0,T])} \le 2\varepsilon\,.$$

The conclusion follows by the arbitrariness of ε. □

6. Concluding remarks

We conclude the paper with the following observations.

Remark 6.1 (Comparison with the Giga-Požár solution). When ϕ is purely crystalline and $g \equiv c$, $c \in \mathbb{R}$, the unique level set solution in the sense of Definition 5.6 coincides with the viscosity solution constructed in [41, 42].

Let ϕ_n be a sequence of smooth anisotropies such that $\{\phi_n \le 1\}$ is strictly convex for every n and $\phi_n \to \phi$. By Lemma 2.9 the unique viscosity level set solution u_n corresponding to the motion

$$V = -\psi(\nu)(\kappa_{\phi_n} + c)$$

coincides with the level set solution in the sense of Definition 2.5. By Proposition 5.12, $u_n \to u$ locally uniformly with u the unique level set solution in the sense of Definition 5.6 corresponding to

$$V = -\psi(\nu)(\kappa_\phi + c)\,.$$

But thanks to [41, Theorem 8.9], it turns out that u is also the viscosity solution in the sense of Giga-Požár. This argument also holds in higher dimension, for the solutions defined in [42].

Remark 6.2 (Approximation by anisotropic Allen-Cahn equations). In [40] the authors consider the anisotropic Allen-Cahn equation

$$v_t = \psi(\nabla v)\left(\operatorname{div}\big(\phi(\nabla v)\nabla\phi(\nabla v)\big) - \frac{1}{\varepsilon^2}W'(v) + \frac{\lambda}{\varepsilon}\,g\right), \tag{6.1}$$

where ψ, ϕ are respectively a smooth mobility and anisotropy, $g \equiv c$, $c \in \mathbb{R}$, is a constant forcing term, W is a standard double-well potential with zeroes in ± 1, and λ is a suitable constant depending only on W.

Let now u_0 be a uniformly continuous function, let u be the corresponding solution to the level set flow given by Theorem 4.8, and let $\gamma : \mathbb{R} \to \mathbb{R}$ be the (unique) solution to $-\gamma'' + W'(\gamma) = 0$ with $\gamma(0) = 0$ and $\lim_{x\to\pm\infty} \gamma(x) = \pm 1$.

In [40, Theorem 2.2] it is shown that the solutions u^ε to (6.1) with initial data

$$u_0^\varepsilon(x) := \gamma\left(\frac{1}{\varepsilon} d^{\phi^0}_{\{u_0<0\}}(x)\right)$$

converge as $\varepsilon \to 0$ to a family of characteristic functions $\chi_{E(t)}$ with $\partial E(t) \subset \{x : u(x,t) = 0\}$ for all $t > 0$, where u is the level set solution corresponding to (1.1). This means that the solutions to (6.1) converge to a (generalized) solution to (1.1) which is contained in the zero-level set of u.

In [40, Theorem 2.4] the authors also show that, given two sequences ψ_n, ϕ_n of smooth mobilities and anisotropies converging to (possibly nonsmooth) limit functions ψ, ϕ, if the corresponding level set solutions u_n, with initial datum u_0, converge to a limit function u, then the corresponding solutions u_n^ε to (6.1) converge as $\varepsilon \to 0$ and $n \to \infty$ to a family of characteristic functions $\chi_{E(t)}$ with $\partial E(t) \subset \{x : u(x,t) = 0\}$ for all $t > 0$.

Thanks to Proposition 5.12 we know that the solutions u_n do indeed converge to the unique solution u given by Theorem 5.7, so that the convergence result in [40, Theorem 2.4] applies to our solutions.

Notice also that the solutions u_n^ε converge as $n \to \infty$ to the unique solution u^ε of the (nonsmooth) Allen-Cahn inclusion corresponding to (6.1) (see [15] for a precise definition), so that the convergence result also applies to such solutions u^ε, thus significantly extending the convergence result in [15].

Remark 6.3 (Non symmetric anisotropies). It is a notational simplification to have considered symmetric (i.e., norms) anisotropies and mobilities, however that there is no particular difficulty in extending the results of this paper to arbitrary convex, one-homogeneous functions ϕ and ψ. Note, though, that it requires more precision in the definitions and notation. To start with, one should replace the signed "distance" function (2.4) with:

$$d^\eta_E(x) := \inf_{y\in E} \eta(x-y) - \inf_{y\notin E} \eta(y-x).$$

Then in Definition 2.2, one should define the superflows with the differential inequality $\partial_t d \geq \mathrm{div} z + g - Md$ out of E for $d(x,t) := \inf_{y\in E(t)} \psi^\circ(x-y) = (d^{\psi^\circ}_{E(t)}(x))^+$, and the subflow with $\partial_t d \leq \mathrm{div} z + g - Md$ inside A for $d(x,t) = -\inf_{y\notin A(t)} \psi^\circ(y-x) = -(d^{\psi^\circ}_{A(t)}(x))^-$. The assumptions and statements such as (in Theorem 2.7) $\mathrm{dist}^{\psi^\circ}(E^0, {F^0}^c) = \Delta > 0$ have to be properly adapted: in that case for instance, one should ask that $E^0 + W^\psi(0,\Delta) \subset F^0$, where W^ψ is still defined by (2.3); similarly, comparison principles such as in Remark 3.7 have to be properly interpreted. Eventually, the results in [18, 1] which we use in our proofs are easily seen not to depend either on the symmetry assumption.

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