Some remarks on uniqueness and regularity of Cheeger sets

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Abstract

We show that the subsets of $\mathbb{R}^N$ with finite volume have a unique Cheeger set, up to small perturbations. We also prove that Cheeger sets are $C^{1,1}$, when the ambient set is $C^{1,1}$.

Contents

1 Introduction

Given a nonempty set $\Omega \subset \mathbb{R}^N$ with finite volume, we call Cheeger constant of Ω the quantity

$$h_\Omega \ := \ \min_{F \subseteq \Omega} \frac{P(F)}{|F|}\,, \tag{1}$$

where $|F|$ denotes de N-dimensional volume of F, $P(F)$ denotes the perimeter of F [5], and the minimum is taken over all nonempty sets of finite perimeter contained in Ω. A *Cheeger set* of Ω is any set $G \subseteq \Omega$ which minimizes (1).

For any set F of finite perimeter in $\mathbb{R}^N$, let us define

$$\lambda_F := \frac{P(F)}{|F|}.$$

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Notice that for any Cheeger set G of Ω it holds $\lambda_G = h_\Omega$, as a consequence G is a Cheeger set of Ω if and only if G solves the minimum problem (whose value is zero):

$$\min_{F\subseteq\Omega} P(F) - h_\Omega|F|. \tag{2}$$

Finding the Cheeger sets of a given set Ω is, in general, a difficult task. This task is simplified if Ω is a convex set and $N = 2$. In that case, there is a unique Cheeger set and is given by $\Omega^R \oplus B_R$ where $\Omega^R := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > R\}$ and $R > 0$ is such that $|\Omega^R| = \pi R^2$ [2, 23] (we denote by $X \oplus Y$ the set $\{x + y : x \in X,\, y \in Y\}$). In particular, we observe that the Cheeger set of Ω is convex. Both features, uniqueness and convexity of the Cheeger set are due to the convexity of Ω (a simple counterexample is given in [23] when Ω is not convex).

The uniqueness of the Cheeger set inside bounded convex subsets of $\mathbb{R}^N$ was proved in [13] when the convex body is uniformly convex and of class C^2, and in [1] in the general case. In the convex case, the $C^{1,1}$ regularity of Cheeger sets is a consequence of the results in [18, 19, 28]. Moreover, a Cheeger set can be characterized in terms of the mean curvature of its boundary: the sum of the principal curvatures being bounded by the Cheeger constant (see [17, 6, 23, 2] for $N = 2$ and [3, 1] for the general case).

Let us comment on the role played by the Cheeger constant in other contexts. Given an open bounded set $\Omega \subseteq \mathbb{R}^N$ with Lipschitz boundary and $p \in (1, \infty)$, the Cheeger constant of Ω permits to give a lower bound on the first eigenvalue of the p-Laplacian on Ω with Dirichlet boundary conditions. Indeed, if we define

$$\lambda_p(\Omega) := \min_{0\neq v\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p\, dx}{\int_\Omega |v|^p\, dx}, \tag{3}$$

then

$$\lambda_p(\Omega) \geq \left(\frac{h_\Omega}{p}\right)^p. \tag{4}$$

This result was proved in [15] when $p = 2$ and extended to any $p \in (1, \infty)$ in [21]. When $p = 1$ the first eigenvalue of the 1-Laplacian is defined by

$$\lambda_1(\Omega) := \min_{0\neq v\in BV(\Omega)} \frac{\int_\Omega |Dv| + \int_{\partial\Omega} |v|\, d\mathcal{H}^{N-1}}{\int_\Omega |v|\, dx}, \tag{5}$$

where $BV(\Omega)$ denotes the space of functions of bounded variation in Ω. Then $\lambda_1(\Omega) = h_\Omega$ and both problems are equivalent in the following sense: a function $u \in BV(\Omega)$ is a minimum of (5) if and only if almost every level set is a Cheeger set (see [22]). These results have been extended in several directions, in particular, using weighted volume and perimeter [11, 7] and for anisotropic versions of the perimeter [24]. Let us also recall that Cheeger sets are related to the global behavior of solutions of the time-dependent constant-mean-curvature equation under vanishing initial condition and Dirichlet boundary data [26]. Finally, we mention an interesting interpretation of the Cheeger constant in terms of the max flow min cut theorem [27, 20].

The plan of the paper is the following: in Section 2 we show the existence of the maximal and the minimal Cheeger sets, inside any set $\Omega \subset \mathbb{R}^N$ of finite volume. In Section 3 we prove that there exists a unique Cheeger set, up to arbitrarily small perturbations of Ω. Finally, in Section 4 we show that Cheeger sets are always of class $C^{1,1}$, out of a singular set of dimension at most $N-8$, when Ω is also of class $C^{1,1}$. In Remark 4.2, we point out that the uniqueness and regularity results can be extended to minimizers of (2), with h_Ω replaced by a generic $\lambda > h_\Omega$.

2 Maximal and minimal Cheeger sets

Definition 2.1. *Let Ω be a measurable set in $\mathbb{R}^N$ of finite volume. We say that a Cheeger set $X \subseteq \Omega$ is a maximal Cheeger set if $Y \subseteq X$ for all Cheeger sets $Y \subseteq \Omega$. We say that X is a minimal Cheeger set if either $Y \supseteq X$ or $Y \cap X = \emptyset$ for all Cheeger sets $Y \subseteq \Omega$.*

Lemma 2.2. *Let X, Y be two Cheeger sets in Ω. Then $X \cup Y$ and $X \cap Y$ (if non-empty) are also Cheeger sets in Ω.*

Proof. Since X, Y are Cheeger sets, we have

$$P(X \cup Y) + P(X \cap Y) \leq P(X) + P(Y) = h_\Omega(|X| + |Y|) = h_\Omega(|X \cup Y| + |X \cap Y|).$$

Now, using that

$$\frac{P(X \cap Y)}{|X \cap Y|} \geq h_\Omega$$

we have that

$$P(X \cup Y) \leq h_\Omega |X \cup Y|.$$

As a consequence $X \cup Y$ is Cheeger, hence $P(X \cup Y) = h_\Omega |X \cup Y|$. Then, we deduce that $P(X \cap Y) = h_\Omega |X \cap Y|$, that is $X \cap Y$ is also a Cheeger set. □

As a consequence of Lemma 2.2, we obtain:

Lemma 2.3. *There exists a maximal Cheeger set $C_{\max} \subseteq \Omega$. Moreover $C_{\max}$ is a bounded set.*

The second assertion easily follows from standard density estimates for solutions of (2): there exists $\rho_0 > 0$ and $\delta > 0$ such that if $\rho < \rho_0$, either $|B_\rho(x) \cap C_{\max}| > \delta$, or there exists $\rho' < \rho$ with $|B_{\rho'}(x) \cap C_{\max}| = 0$, see [4]. In particular, it shows that the set of points where $C_{\max}$ (or any other Cheeger set of Ω) has Lebesgue density zero is an open set. This is not true for the points of density one, at least if Ω is not open, as shown by the example of a set Ω with empty interior.

Lemma 2.4. *Let X, Y be two Cheeger sets in Ω. Assume that X is minimal, that is, it contains no other Cheeger set inside. Then either $X \subseteq Y$ or $X \cap Y = \emptyset$. In particular, two minimal Cheeger sets are disjoint.*

Proof. If $X \cap Y$ is nonempty, then it is also a Cheeger set contained in X. Since X is minimal, we have $X \cap Y = X$, that is $X \subseteq Y$. □

Recall that, by the isoperimetric inequality, there exists a constant $\alpha = \alpha(\Omega) > 0$ such that any Cheeger set in Ω has volume greater or equal to α.

Lemma 2.5. *There are minimal Cheeger sets in Ω and they are finite in number. In particular, Cheeger sets of minimal volume are minimal Cheeger sets, and any Cheeger set contains a minimal Cheeger set.*

Proof. Consider the problem $\min\{|X| : X \text{ is a Cheeger set of } \Omega\}$. Then any minimizing sequence has a subsequence converging to a set, say X, such that X is a Cheeger set of minimal volume. By Lemma 2.2, the set X does not intersect any other Cheeger set, therefore is minimal. Since any of such sets has a volume $\geq \alpha$, there are only finitely many of them. To prove the last assertion, we just take a minimal volume Cheeger set between the ones contained in the given Cheeger set. □

Remark 2.6. If Ω is an open set and C is a minimizer of (2), by classical regularity results [25] we know that $(\partial C \setminus \Sigma) \cap \Omega$ is analytic, where Σ is a closed singular set of dimension at most $N-8$. Moreover, if Ω is of class $C^{1,1}$, then C is a minimizer of a prescribed curvature problem with curvature in L^∞ [8], hence $\partial C \setminus \Sigma$ is of class $W^{2,p}$ for all $p < \infty$ (see also [29] for the case $N = 3$).

Remark 2.7. By a result of Giusti [17], an open set $X \subset \Omega$ is a minimal Cheeger set iff X has finite perimeter and there is a solution of the capillary problem in X (with vertical contact angle), i.e. there exists a vector field $z : X \to \mathbb{R}^N$ such that $|z| < 1$ and $-\mathrm{div}\, z = h_\Omega$.

Remark 2.8. The computation of the maximal Cheeger set has been the object of recent interest [12]. By adapting the proof of Proposition 4 in [3] one can prove the following result. Let Ω be a bounded subset of $\mathbb{R}^N$ with Lipschitz continuous boundary, and let $u \in BV(\Omega) \cap L^2(\Omega)$ be the solution of the variational problem

$$(Q)_\lambda : \quad \min_{u \in BV(\Omega)\cap L^2(\Omega)} \left\{ \int_\Omega |Du| + \frac{\lambda}{2} \int_\Omega (u-1)^2\, dx + \int_{\partial\Omega} |u|\, d\mathcal{H}^{N-1} \right\}. \tag{6}$$

Then $0 \leq u \leq 1$. Let $E_s := \{u \geq s\}$, $s \in (0,1]$. Then for any $s \in (0,1]$ we have

$$P(E_s) - \lambda(1-s)|E_s| \leq P(F) - \lambda(1-s)|F| \tag{7}$$

for any $F \subseteq \Omega$. If $\lambda > 0$ is big enough, indeed greater than $1/\|\chi_\Omega\|_*$ where

$$\|\chi_\Omega\|_* := \max\left\{ \int_{\mathbb{R}^N} u\, \chi_\Omega\, dx : \ u \in BV(\mathbb{R}^N), \ \int_\Omega |Du| \leq 1 \right\},$$

then the level set $\{u = \|u\|_\infty\}$ is the maximal Cheeger set of Ω. In particular, the maximal Cheeger set can be computed by solving (6), and for that we can use the algorithm in [14].

3 Uniqueness of Cheeger sets up to small perturbations

We prove that the Cheeger is unique, up to arbitrarily small perturbations of the ambient set Ω.

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite volume. Then, for any compact set $K \subset \Omega$ there exists a bounded open set $\Omega_K \subseteq \Omega$ such that $K \subset \Omega_K$ and Ω_K has a unique Cheeger set.*

Proof. By Lemma 2.5, we know that Ω has a finite number of disjoint minimal Cheeger sets. Let C be a minimal Cheeger set of Ω, let $\widetilde{\Omega}$ be any open set such that $K \subset \widetilde{\Omega} \subset\subset \Omega$, and let $\Omega_K := C \cup \widetilde{\Omega}$. Notice that C is also a (minimal) Cheeger set in Ω_K, and we want to show that it is the only one. Indeed, let D be a Cheeger set in Ω_K, then by Lemma 2.4 either $D \supseteq C$ or $D \cap C = \emptyset$. The latter cannot happen, since in this case we would have $D \subseteq \Omega_K \setminus C \subseteq \widetilde{\Omega} \subset\subset \Omega$, but the distance of D from the boundary of Ω cannot be positive, otherwise we could decrease the quotient $P(D)/|D|$ by rescaling D with a factor larger than one. It then follows $D \supseteq C$. By Remark 2.6 there exist singular sets $\Sigma_C \subset \partial C$ and $\Sigma_D \subset \partial D$, of dimension at most $N-8$, such that $A_C := (\partial C \setminus \Sigma_C) \cap \Omega$ and $A_D := (\partial D \setminus \Sigma_D) \cap \Omega$ are both analytic solutions of the geometric equation $(N-1)\mathbf{H} = h_\Omega$, where $\mathbf{H}$ denotes the mean curvature. As a consequence, since $\mathcal{H}^{N-1}(A_C \cap A_D) \geq \mathcal{H}^{N-1}((\partial C \setminus \widetilde{\Omega}) \cap \Omega) > 0$, by analytic continuation we get $A_D = A_C$. More precisely, assume by contradiction that we can find $\bar{x} \in A_C \cap A_D$ such that $A_C \cap B_\rho(\bar{x}) \neq A_D \cap B_\rho(\bar{x})$ for all $\rho > 0$. Letting T be the tangent hyperplane to ∂D at $\bar{x}$, we can write ∂D and ∂C as the graph of two smooth functions v^* and v_*, respectively, over $T \cap B_\rho(\bar{x})$ for $\rho > 0$ small enough. Identifying $T \cap B_\rho(\bar{x})$ with $B_\rho \subset \mathbb{R}^{N-1}$, we have that $v_*, v^* : B_\rho \to \mathbb{R}$ both solve the equation

$$-\operatorname{div} \frac{Dv}{\sqrt{1+|Dv|^2}} = h_\Omega. \tag{8}$$

Moreover, it holds $v_* \geq v^*$, $v_*(0) = v^*(0)$ and $v_*(\tilde{y}) > v^*(\tilde{y})$ for some $\tilde{y} \in B_\rho$. Let B be an open ball such that $\overline{B} \subset B_\rho$, $v_* > v^*$ on B and $v_*(y) = v^*(y)$ for some $y \in \partial B$. Notice that, since both v^* and v_* belong to $C^\infty(B) \cap C^1(\overline{B})$, the fact that $v_*(y) = v^*(y)$ also implies that $Dv_*(y) = Dv^*(y)$. In B, both functions solve (8). Letting now $w = v_* - v^*$, we have that $w(y) = 0$ and $Dw(y) = 0$, while $w > 0$ inside B. For any $x \in B$ we have

$$\begin{aligned} 0 &= \operatorname{div} \left(D\Psi(Dv_*(x)) - D\Psi(Dv^*(x)) \right) \\ &= \operatorname{div} \left(\left(\int_0^1 D^2\Psi(Dv^*(x) + t(Dv_*(x) - Dv^*(x)))\, dt \right) Dw(x) \right), \end{aligned}$$

where $\Psi(p) = \sqrt{1+|p|^2}$, so that w solves a linear, uniformly elliptic equation with smooth coefficients. Then Hopf's lemma [16] implies that $Dw(y) \cdot \nu_B(y) < 0$, a contradiction. Hence $A_C = A_D$, which is equivalent to $C = D$. □

Remark 3.1. Notice that, given any open set Ω with finite volume, for all $\epsilon > 0$, we can find a set $\Omega_\epsilon \subset \Omega$ such that $|\Omega \setminus \Omega_\epsilon| < \epsilon$ and Ω_ϵ has a unique Cheeger set. Indeed, considering as above a minimal Cheeger set $C \subset \Omega$, we can define

$$\Omega_\epsilon := \Omega \setminus \bigcup_{q \in \mathbb{Q}^N \cap \Omega \setminus C} \overline{B_{r(q)}},$$

where $r(q) > 0$ is such that $B_{r(q)} \subset \Omega \setminus C$ and

$$\sum_{q \in \mathbb{Q}^N \cap \Omega \setminus C} r(q) < \epsilon.$$

Let D be a Cheeger set in Ω_ϵ different from C, then $|D \setminus C| > 0$. By the regularity result in Remark 2.6, it follows that $D \setminus C$ has nonempty interior, which is impossible by the construction of Ω_ϵ.

We can require that also Ω_ϵ is open but the construction is a bit more complicated. First, we need to remove from Ω a small closed ball inside each minimal Cheeger set different from C. This ensures that any Cheeger set C' in the new set must contain C. Then, we remove from Ω a (possibly countable) union of closed balls contained in $\Omega \setminus C$, each one touching a connected component of $\partial C \cap \Omega$.

Remark 3.2. For a general open set Ω, one may also consider a different notion of Cheeger set, based on the following definition of perimeter:

$$P_\Omega(E) := \sup \left\{ \int_E \operatorname{div} \phi \, dx : \phi \in C^1(\Omega, \mathbb{R}^N),\ |\phi| \le 1,\ \operatorname{div} \phi \in L^\infty(\Omega) \right\},$$

which coincides with the lower semicontinuous relaxation of the usual perimeter restricted to the compact subsets of Ω. Notice that such notion of Cheeger set gives a higher Cheeger constant of Ω, which still verifies (4), and it coincides with the classical notion if, for instance, Ω is the subgraph of a continuous function near each point of its boundary. We observe that Theorem 1 remains true also with this definition of Cheeger set.

4 Regularity of Cheeger sets in regular domains

We now show that each Cheeger set of Ω is of class $C^{1,1}$, if Ω is also of class $C^{1,1}$.

Theorem 2. *Let Ω be a bounded open set with boundary of class $C^{1,1}$. Then any Cheeger set C of Ω has boundary of class $C^{1,1}$, out of a closed singular set $\Sigma \subset \partial C$ of dimension at most $N-8$.*

Proof. We know that any Cheeger set is a solution of the variational problem (2). Let C be a Cheeger set of Ω, and let $x_0 \in (\partial C \setminus \Sigma) \cap \partial\Omega$, where the singular set Σ is as in Remark 2.6. We may assume that near x_0, $\partial\Omega$ is the graph of a $C^{1,1}$ function $f : B_{2r} \to \mathbb{R}$ where B_{2r} is an $(N-1)$-dimensional ball centered at x_0 of radius $2r$. We may as well assume that ∂C is the graph of $u : B_{2r} \to \mathbb{R}$. We know that $u \in W^{2,p}(B_{2r})$ for any $p < \infty$, in particular $u \in C^{1,\alpha}(B_{2r})$ for any $\alpha < 1$. We observe that u is a solution of

$$\min \left\{ \int_{B_r} \left(\sqrt{1 + |\nabla v|^2} + h_\Omega v \right) dx :\ v \in BV(B_r),\ v \ge f,\ v|_{\partial B_r} = u|_{\partial B_r} \right\}. \tag{9}$$

The result follows by adapting the proof of regularity for the obstacle problem in [9]. Indeed, since $\partial\Omega$ is of class $C^{1,1}$, ∇f has modulus of continuity $\sigma(r) \leq \kappa r$, $\kappa > 0$. Letting $L(x) := f(x_0) + \nabla f(x_0) \cdot (x - x_0)$, we have

$$L(x) - \kappa r^2 \leq f(x) \leq u(x) \qquad x \in B_r.$$

We shall prove that

$$u(x) \leq L(x) + Cr^2 \qquad x \in B_{\frac{r}{2}}, \tag{10}$$

for some constant $C > 0$. We shall denote by C a positive constant that may vary from line to line. Consider $w = u - (L - \kappa r^2) \geq 0$, and observe that u satisfies the equation

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) + h_\Omega \ \geq \ 0 \qquad x \in B_r, \tag{11}$$

with equality in $D = \{x \in B_r \ : \ u(x) > f(x)\}$. Due to the regularity of u, (11) can be written

$$-a_{ij}(x)\partial_{x_i x_j} u + h_\Omega \ \geq \ 0 \qquad x \in B_r, \tag{12}$$

where $a_{ij} \in C^\alpha(B_r)$ are uniformly positive. It follows that also w satisfies (12) (and, still, with an equality in D) . Let now w_1 be the solution of

$$-a_{ij}(x)\partial_{x_i x_j} w_1 + h_\Omega \ = \ 0 \qquad x \in B_r,$$

with $w_1|_{\partial B_r} = w|_{\partial B_r} \geq 0$. Observe that $w_1 \leq w$. Without loss of generality, we assume that $x_0 = 0$. Let $\gamma = h_\Omega/(\min_{x \in B_r} Tr(A(x))$, where $A(x) = (a_{ij}(x))$, and $Q(x) = (\gamma/2)(|x|^2 - r^2)$. Then, Q is a subsolution of (12) in B_r, with $Q|_{\partial B_r} = 0$, so that $Q \leq w_1$ in B_r. In particular, we have that

$$-a_{ij}(x)\partial_{x_i x_j}(w_1 - Q) \ = \ h_\Omega \left(\frac{Tr(A(x))}{\min_{B_r} Tr(A)} - 1\right) \qquad x \in B_r,$$

and the right-hand side of this equation is bounded by Cr^α (since $A(x)$ is Hölderian of exponent α). We have

$$w_1(x_0) \leq w(x_0) = u(x_0) - (L(x_0) - \kappa r^2) = f(x_0) - (L(x_0) - \kappa r^2) = \kappa r^2,$$

while, since $w_1 - Q \geq 0$, it satisfies a Harnack inequality [16, Thms 9.20 and 9.22] in $B_{r/2}$:

$$\begin{aligned} w_1(x) - Q(x) \quad &\leq \quad C \inf_{B_r}(w_1 - Q) + Cr^2 \\ &\leq \quad Cw_1(x_0) + C\frac{\gamma}{2}r^2 + Cr^2 \ \leq Cr^2, \end{aligned}$$

hence also $w_1(x) \leq Cr^2$, for any $x \in B_{r/2}$ (for some constant $C > 0$ which does not blow-up as $r \to 0$).

Let now $w_2 := w - w_1$. The function w_2 satisfies $0 \leq w_2 \leq w - Q$, $w_2|_{\partial B_r} = 0$, and

$$-a_{ij}(x)\partial_{x_i x_j} w_2 \geq 0 \qquad x \in B_r, \tag{13}$$

again, with an equality if $x \in D$. Consider $\bar{x} \in B_r$ a point where w_2 reaches its maximum: then, either $w_2(\bar{x}) = 0$, in which case $w_2 = 0$ inside B_r, or $w_2(\bar{x}) > 0$, in which case we must have $\bar{x} \notin D$, since (13) is satisfied with an equality in D (it could be that w_2 is constant and maximal in D, in which case we may always assume $\bar{x} \in \partial D \cap B_r$).

Thus, either $w_2 = 0$ in B_r, or $u(\bar{x}) = f(\bar{x})$. In particular, in the latter case, we find that for any $x \in B_r$,

$$\begin{aligned} w_2(x) \;\le\; w_2(\bar{x}) \;\le\; w(\bar{x}) - Q(\bar{x}) \;&=\; u(\bar{x}) - (L(\bar{x}) - \kappa r^2) + \frac{\gamma}{2}(r^2 - |x|^2) \\ &\le\; f(\bar{x}) - f(0) - \nabla f(0) \cdot \bar{x} + Cr^2 \;\le\; Cr^2, \end{aligned}$$

so that $w(x) = w_1(x) + w_2(x) \;\le Cr^2$ if $x \in B_{r/2}$, which shows (10). □

Remark 4.1. Since the Cheeger sets of Ω are solutions of (2), if Ω is of class $C^{1,1}$ and C is a Cheeger set of Ω, we have $(N-1)\mathbf{H}_C(x) \le h_\Omega$ for a.e. $x \in \partial C$.

Remark 4.2. We point out that Theorems 1 and 2 extend also to minimizers of (2), with h_Ω replaced by any $\lambda > h_\Omega$ (see [3]).

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