# Uniqueness of the Cheeger set of a convex body

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**Abstract**

We prove that if $C \subset \mathbb{R}^N$ is of class $C^2$ and uniformly convex, then the Cheeger set of $C$ is unique. The Cheeger set of $C$ is the set which minimizes, inside $C$, the ratio perimeter over volume.

## 1 Introduction

Given an nonempty open bounded subset $\Omega$ of $\mathbb{R}^N$, we call Cheeger constant of $\Omega$ the quantity

$$h_\Omega = \min_{K \subseteq \Omega} \frac{P(K)}{|K|} \tag{1}$$

where $|K|$ denotes de $N$-dimensional volume of $K$ and $P(K)$ denotes the perimeter of $K$. The minimum in (1) is taken over all nonempty sets of finite perimeter contained in $\Omega$. It is well-known that the minimum in (1) is attained at a subset $G$ of $\Omega$ such that $\partial G$ touches $\partial\Omega$ (otherwise we would diminish the quotient $P(G)/|G|$ by dilating $G$). A Cheeger set of $\Omega$ is any set $G \subseteq \Omega$ which minimizes (1). We say that $\Omega$ is Cheeger in itself if $\Omega$ minimizes (1).

For any set of finite perimeter $K$ in $\mathbb{R}^N$, let us denote

$$\lambda_K := \frac{P(K)}{|K|}.$$

Notice that for any Cheeger set $G$ of $\Omega$, $\lambda_G = h_G$. Observe also that $G$ is a Cheeger set of $\Omega$ if and only if $G$ minimizes

$$\min_{K \subseteq \Omega} P(K) - \lambda_G |K|. \tag{2}$$

We say that a set $\Omega \subset \mathbb{R}^N$ is calibrable if $\Omega$ minimizes the problem

$$\min_{K \subseteq \Omega} P(K) - \lambda_\Omega |K|. \tag{3}$$

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In particular, if $G$ is a Cheeger set of $\Omega$, then $G$ is calibrable. Thus, $\Omega$ is a Cheeger set of itself if and only if is calibrable.

Finding the Cheeger sets of a given $\Omega$ is a difficult task. This task is simplified if $\Omega$ is a convex set and $N = 2$. In that case, the Cheeger set in $\Omega$ is unique and is identified with the set $\Omega^R \oplus B(0, R)$ where $\Omega^R := \{x \in \Omega : \text{dist}(x, \partial\Omega) > R\}$ is such that $|\Omega^R| = \pi R^2$ and $A \oplus B := \{a + b : a \in A,\, b \in B\}$, $A, B \subset \mathbb{R}^2$ [1, 17]. In this case, we see that the Cheeger set of $\Omega$ is convex. Moreover, a convex set $\Omega \subseteq \mathbb{R}^2$ is Cheeger in itself if and only if $\max_{x\in\partial\Omega} \kappa_\Omega(x) \leq \lambda_\Omega$ where $\kappa_\Omega(x)$ denotes the curvature of $\partial\Omega$ at the the point $x$. This has been proved in [13, 8, 17] (see also [1]) though it was stated in terms of calibrability in [8, 1]. The proof in [13] had also a complement result: if $\Omega$ is Cheeger in itself then $\Omega$ is strictly calibrable, that is, for any set $K \subset \Omega$, $K \neq \Omega$, then

$$0 = P(\Omega) - \lambda_\Omega|\Omega| < P(K) - \lambda_\Omega|K|,$$

and this implies that the capillary problem in absence of gravity (with vertical contact angle at the boundary)

$$\begin{aligned} -\text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) &= \lambda_\Omega \quad \text{in } \Omega \\ -\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu^\Omega &= 1 \qquad \text{in } \partial\Omega \end{aligned} \tag{4}$$

has a solution. Indeed, both problems are equivalent [13].

Our purpose in this paper is to extend the above result to $\mathbb{R}^N$, that is, to prove the uniqueness and convexity of the Cheeger set contained in a set $\Omega \subset \mathbb{R}^N$ which is uniformly convex and of class $C^2$. The characterization of a convex set $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$ which is Cheeger in itself (also called calibrable) in terms of the mean curvature of its boundary was proved in [2]. The precise result states that such a set $\Omega$ is Cheeger in itself if and only if $\kappa_\Omega(x) \leq \lambda_\Omega$ for any $x \in \partial\Omega$ where $\kappa_\Omega(x)$ denotes the sum of the principal curvatures (or total curvature) of the boundary of $\Omega$. Moreover, in [2], the authors also proved that for any convex set $\Omega \subset \mathbb{R}^N$ there exists a maximal Cheeger set contained in $\Omega$ which is convex. These results were extended to convex sets $\Omega$ satisfying a regularity condition and anisotropic norms in $\mathbb{R}^N$ (including the crystalline case) in [11].

In particular, we obtain that $\Omega \subset \mathbb{R}^N$ is the unique Cheeger set of itself, whenever $\Omega$ is a $C^2$, uniformly convex calibrable set. We point out that, by Theorems 1.1 and 4.2 in [13], this uniqueness result is equivalent to the existence of a solution $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ of the capillary problem (4).

Let us explain the plan of the paper. In Section 2 we collect some definitions and recall some results about the mean curvature operator in (4) and the subdifferential of the total variation. In Section 3 we state and prove the uniqueness result.

## 2 Preliminaries

### 2.1 $BV$ functions

Let $\Omega$ be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient $Du$ in the sense of distributions is a (vector valued) Radon measure with finite total variation in $\Omega$ is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of $Du$ on $\Omega$ turns out to be

$$\sup\left\{\int_\Omega u \operatorname{div} z \, dx : z \in C_0^\infty(\Omega;\mathbb{R}^N), \|z\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |z(x)| \le 1\right\}, \tag{5}$$

(where for a vector $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. The map $u \to |Du|(\Omega)$ is $L^1_{\rm loc}(\Omega)$-lower semicontinuous. $BV(\Omega)$ is a Banach space when endowed with the norm $\int_\Omega |u| \, dx + |Du|(\Omega)$. We recall that $BV(\mathbb{R}^N) \subseteq L^{N/(N-1)}(\mathbb{R}^N)$.

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in $\mathbb{R}^N$ if (5) is finite when $u$ is substituted with the characteristic function $\chi_E$ of $E$ and $\Omega = \mathbb{R}^N$. The perimeter of $E$ is defined as $P(E) := |D\chi_E|(\mathbb{R}^N)$. For results and informations on functions of bounded variation we refer to [4].

Finally, let us denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. We recall that when $E$ is a finite-perimeter set with regular boundary (for instance, Lipschitz), its perimeter $P(E)$ also coincides with the more standard definition $\mathcal{H}^{N-1}(\partial E)$.

### 2.2 A generalized Green's formula

Let $\Omega$ be an open subset of $\mathbb{R}^N$. Following [6], let

$$X_2(\Omega) := \{z \in L^\infty(\Omega;\mathbb{R}^N) : \operatorname{div}\ z \in L^2(\Omega)\}.$$

If $z \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap BV(\Omega)$ we define the functional $(z \cdot Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$< (z \cdot Dw), \varphi >:= -\int_\Omega w\,\varphi \operatorname{div}\ z\,dx - \int_\Omega w\, z \cdot \nabla\varphi\, dx.$$

Then $(z \cdot Dw)$ is a Radon measure in $\Omega$,

$$\int_\Omega (z \cdot Dw) = \int_\Omega z \cdot \nabla w\, dx \qquad \forall w \in L^2(\Omega) \cap W^{1,1}(\Omega).$$

Recall that the outer unit normal to a point $x \in \partial\Omega$ is denoted by $\nu^\Omega(x)$. We recall the following result proved in [6].

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $z \in L^\infty(\Omega;\mathbb{R}^N)$ with $\operatorname{div} z \in L^2(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ satisfying $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and such that for any $u \in BV(\Omega) \cap L^2(\Omega)$ we have*

$$\int_\Omega u \operatorname{div} z \, dx + \int_\Omega (z \cdot Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u \, d\mathcal{H}^{N-1}.$$

*Moreover, if $\varphi \in C^1(\overline{\Omega})$ then $[(\varphi z) \cdot \nu^\Omega] = \varphi[z \cdot \nu^\Omega]$.*

This result is complemented with the following result proved by Anzellotti in [7].

**Theorem 2.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary of class $C^1$. Let $z \in C(\overline{\Omega}; \mathbb{R}^N)$ with* $\operatorname{div} z \in L^2(\Omega)$. *Then*

$$[z \cdot \nu^{\Omega}](x) = z(x) \cdot \nu^{\Omega}(x) \qquad \mathcal{H}^{N-1} \text{ a.e. on } \partial\Omega.$$

### 2.3 Some auxiliary results

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and let $\varphi \in L^1(\Omega)$. For all $\epsilon > 0$, we let $\Psi_{\varphi}^{\varepsilon} : L^2(\Omega) \to (-\infty, +\infty]$ be the functional defined by

$$\Psi_{\varphi}^{\epsilon}(u) := \begin{cases} \displaystyle\int_{\Omega} \sqrt{\epsilon^2 + |Du|} + \int_{\partial\Omega} |u - \varphi| & \text{if } \ u \in L^2(\Omega) \cap BV(\Omega) \\ +\infty & \text{if } \ u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{6}$$

As it is proved in [14], if $f \in W^{1,\infty}(\Omega)$, then the minimum $u \in BV(\Omega)$ of the functional

$$\Psi_{\varphi}^{\epsilon}(u) + \int_{\Omega} |u(x) - f(x)|^2 \, dx \tag{7}$$

belongs to $u \in C^{2+\alpha}(\Omega)$, for every $\alpha < 1$. The mimimum $u$ of (7) is a solution of

$$\begin{cases} u - \dfrac{1}{\lambda} \operatorname{div} \dfrac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} = f(x) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{8}$$

where the boundary condition is taken in a generalized sense [18], i.e.,

$$\left[ \frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \cdot \nu^{\Omega} \right] \in \operatorname{sign}(\varphi - u) \qquad \mathcal{H}^{N-1} \text{ a.e. on } \partial\Omega.$$

Observe that (8) can be written as

$$u + \frac{1}{\lambda} \partial \Psi_{\varphi}^{\epsilon}(u) \ni f. \tag{9}$$

We are particularly interested in the case where $\varphi = 0$. As we shall show below (see also [2]) in the case of interest to us we have $u > 0$ on $\partial\Omega$ and, thus, $\left[ \frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \cdot \nu^{\Omega} \right] = -1$ $\mathcal{H}^{N-1}$ a.e. on $\partial\Omega$. It follows that $u$ is a solution of the first equation in (8) with vertical contact angle at the boundary.

As $\epsilon \to 0^+$, the solution $u_\epsilon$ of (8) converges to the solution of

$$\begin{cases} u + \dfrac{1}{\lambda} \partial \Psi_{\varphi}(u) = f(x) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{10}$$

where $\Psi : L^2(\Omega) \to (-\infty, +\infty]$ is given by

$$\Psi_\varphi(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} |Du| + \int_{\partial\Omega} |u - \varphi| & \text{if} \quad u \in L^2(\Omega) \cap BV(\Omega) \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{11}$$

In this case $\partial\Psi_\varphi$ represents the operator $-\mathrm{div}\,(Du/|Du|)$ with the boundary condition $u = \varphi$ in $\partial\Omega$, and this connection is precisely given by the following Lemma (see [5]).

**Lemma 2.1.** *The following assertions are equivalent:*

*(a)* $v \in \partial\Psi_\varphi(u)$*;*

*(b)* $u \in L^2(\Omega) \cap BV(\Omega)$*,* $v \in L^2(\Omega)$*, and there exists* $z \in X_2(\Omega)$ *with* $\|z\|_\infty \leq 1$*, such that*

$$v = -\mathrm{div}\, z \quad \text{in} \quad \mathcal{D}'(\Omega),$$

$$(z \cdot Du) = |Du|,$$

*and*

$$[z \cdot \nu^\Omega] \in \mathrm{sign}(\varphi - u) \qquad \mathcal{H}^{N-1} \text{ a.e. on } \partial\Omega.$$

Notice that the solution $u \in L^2(\Omega)$ of (10) minimizes the problem

$$\min_{u \in BV(\Omega)} \int_\Omega |Du| \, + \, \int_{\partial\Omega} |u(x) - \varphi(x)| \, d\mathcal{H}^{N-1}(x) \, + \, \frac{\lambda}{2} \int_\Omega |u(x) - f(x)|^2 \, dx, \tag{12}$$

and the two problems are equivalent.

# 3 The uniqueness theorem

We now state our main result.

**Theorem 3.** *Let $C$ be a convex body in $\mathbb{R}^N$. Assume that $C$ is uniformly convex, with boundary of class $C^2$. Then the Cheeger set of $C$ is convex and unique.*

We do not believe that the $C^2$ assumption is essential for this result, although we could not remove it. Removing the assumption of uniform convexity is probably more tricky. Let us recall the following result proved in [2] (Theorems 6 and 8 and Proposition 4).

**Theorem 4.** *Let $C$ be a convex body in $\mathbb{R}^N$ with boundary of class $C^{1,1}$. For any $\lambda, \varepsilon > 0$, there is a unique solution $u_\varepsilon$ of the equation:*

$$\begin{cases} u_\varepsilon \; - \; \dfrac{1}{\lambda}\mathrm{div}\, \dfrac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \; = \; 1 & \text{in } C \\ u_\varepsilon \; = \; 0 & \text{on } \partial C, \end{cases} \tag{13}$$

*such that $0 \le u_\varepsilon \le 1$. Moreover, there exist $\lambda_0$ and $\varepsilon_0$, depending only on $\partial C$, such that if $\lambda \ge \lambda_0$ and $\varepsilon \le \varepsilon_0$, then $u_\varepsilon$ is a concave function such that $u_\varepsilon \ge \alpha > 0$ on $\partial C$ for some $\alpha > 0$. Hence, $u_\varepsilon$ satisfies*

$$\left[\frac{Du^\epsilon}{\sqrt{\epsilon^2 + |Du^\epsilon|^2}} \cdot \nu^C\right] = \mathrm{sign}(0 - u^\epsilon) = -1 \quad \text{on } \partial C. \tag{14}$$

*As $\varepsilon \to 0$, the functions $u_\varepsilon$ converge to the concave function $u$ which minimizes the problem*

$$\min_{u \in BV(C)} \int_C |Du| \;+\; \int_{\partial C} |u(x)|\, d\mathcal{H}^{N-1}(x) \;+\; \frac{\lambda}{2}\int_C |u(x) - 1|^2\, dx \tag{15}$$

*or, equivalently, if $u$ is extended with zero out of $C$, $u$ minimizes*

$$\int_{\mathbb{R}^N} |Du| + \frac{\lambda}{2}\int_{\mathbb{R}^N} |u - \chi_C|^2\, dx.$$

*The function $u$ satisfies $0 \le u < 1$. Moreover, the level set $\{u \ge t\}$, $t \in (0,1]$, is contained in $C$ and minimizes the problem*

$$\min_{F \subset C} P(F) - \lambda(1-t)|F|. \tag{16}$$

It was proved in [2] (see also [11]) that the set $C^* = \{u = \max_C u\}$ is the maximal Cheeger set contained in $C$, that is, the maximal set that solves (1). Moreover, one has $u = 1 - h_C/\lambda > 0$ in $C^*$ and $h_C = \lambda_{C^*}$.

If we want to consider what happens inside $C^*$ and, in particular, if there are other Cheeger sets, we have to analize the level sets of $u_\varepsilon$ before passing to the limit as $\epsilon \to 0^+$. In order to do this, let us introduce the following rescaling of $u_\varepsilon$:

$$v_\varepsilon \;=\; \frac{u_\varepsilon - m_\varepsilon}{\varepsilon} \;\le\; 0,$$

where $m_\varepsilon = \max_C u_\varepsilon \to 1 - h_C/\lambda$ as $\varepsilon \to 0$. The function $v_\varepsilon$ is a generalized solution of the equation:

$$\begin{cases} \varepsilon v_\varepsilon \;-\; \dfrac{1}{\lambda}\mathrm{div}\,\dfrac{Dv_\varepsilon}{\sqrt{1+|Dv_\varepsilon|^2}} \;=\; 1 - m_\varepsilon & \text{in } C \\ v_\varepsilon \;=\; -m_\varepsilon/\varepsilon & \text{on } \partial C. \end{cases} \tag{17}$$

We let $z_\varepsilon = Du_\varepsilon/\sqrt{\varepsilon^2 + |Du_\varepsilon|^2} = Dv_\varepsilon/\sqrt{1+|Dv_\varepsilon|^2}$. Notice that $z_\varepsilon$ is a vector field in $L^\infty(C)$, with uniformly bounded divergence, such that $|z_\varepsilon| \le 1$ a.e. in $C$ and, by (14), $[z_\varepsilon \cdot \nu_C] = -1$ on $\partial C$.

Let us study the limit of $v_\varepsilon$ and $z_\varepsilon$ as $\varepsilon \to 0$. Let us observe that, for each $\varepsilon > 0$ small enough and each $s \in (0, |C|)$, there is a (convex) superlevel set $C_s^\varepsilon$ of $v_\varepsilon$ such that $|C_s^\varepsilon| = s$ for $s \in (0, |C|)$. First we observe that $\{v_\varepsilon = 0\}$ is a null set. Otherwise, since $v_\varepsilon$ is concave, it would be a convex set of positive measure, and it would have a nonempty interior. We would have that $v_\varepsilon = \mathrm{div}\, z_\varepsilon = 0$, hence $1 - m_\varepsilon = 0$ in the interior of $\{v_\varepsilon = 0\}$. This is a contradiction with Theorem 4 for $\varepsilon > 0$ small enough. Hence we may take $C_0^\varepsilon := \{v_\varepsilon = 0\}$.

Now, the concavity of $v^\varepsilon$ guarantees the existence of the foliation $C_s^\varepsilon$ made of superlevel sets of $v^\varepsilon$ such that $|C_s^\varepsilon| = s$ for $s \in (0, |C|)$.

We observe that a sequence of uniformly bounded convex sets is compact both for the $L^1$ and Hausdorff topologies. Hence, up to a subsequence, we may assume that $C_s^\varepsilon$ converge to convex sets $C_s$, each of volume $s$, first for any $s \in \mathbb{Q} \cap (0, |C|)$ and then by continuity for any $s$. Possibly extracting a further subsequence, we may assume that there exists $s_* \in [0, |C|]$ such that $v_\varepsilon$ goes to a concave function $v$ in $C_s$ for any $s < s_*$, and to $-\infty$ outside $C_* := C_{s_*}$. We may also assume that $z_\varepsilon \rightharpoonup z$ weakly* in $L^\infty(C)$, for some vector field $z$, satisfying $|z| \le 1$ a.e. in $C$. From (13) we have in the limit

$$-\mathrm{div}\, z \;=\; \lambda(1-u) \qquad \text{in } \mathcal{D}'(C). \tag{18}$$

Moreover, by the results recalled in Section 2, it holds $-\mathrm{div}\, z \in \partial\Psi_0(u)$. We see from (18) that

$$-\mathrm{div}\, z = h_C \qquad \text{in } C^*, \tag{19}$$

while $-\mathrm{div}\, z > h_C$ a.e. on $C \setminus C^*$. We let $s^* := |C^*|$, so that $C^* = C_{s^*}$. By Theorem 4, for $s \ge s^*$, the set $C_s$ is a minimizer of $P(E) - \mu_s |E|$ among all $E \subseteq C$, for some $\mu_s \ge h_C$ which is equal to the constant value of $-\mathrm{div}\, z$ on $\partial C_s \cap C$, and is bounded by $P(C)/(|C| - s)$. For $s > s^*$, we have $\mu_s > h_C$ and the set $C_s$ is the unique minimizer of the variational problem. As a consequence (see [2, 11]) for any $s > s^*$ the set $C_s$ is also the unique minimizer of $P(E)$ among all $E \subseteq C$ of volume $s$.

**Lemma 3.1.** *We have $s_* > 0$ and the sets $C_s$ are Cheeger sets in $C$ for any $s \in [s_*, s^*]$.*

**Proof.** Let $s_* < s \le |C|$. If $x \in \partial C_s^\epsilon \setminus \partial C$, then

$$0 - v_\varepsilon(x) \;\le\; Dv_\varepsilon(x) \cdot (\bar{x}_\varepsilon - x)$$

where $v_\varepsilon(\bar{x}_\varepsilon) = \max_C v_\varepsilon$. Hence, $\lim_{\varepsilon \to 0} \inf_{\partial C_s^\varepsilon \setminus \partial C} |Dv_\varepsilon| = +\infty$. Since $[z_\varepsilon \cdot \nu^C] = -1$ on $\partial C$ and $P(C_s^\varepsilon) \to P(C_s)$, we deduce

$$\begin{aligned} &-\int_{\partial C_s^\varepsilon} [z_\varepsilon(x) \cdot \nu^{C_s^\varepsilon}(x)]\, d\mathcal{H}^{N-1}(x) \\ &\qquad = \int_{\partial C_s^\varepsilon \setminus \partial C} \frac{|Dv_\varepsilon(x)|}{\sqrt{1 + |Dv_\varepsilon(x)|^2}}\, d\mathcal{H}^{N-1}(x) \,+\, \mathcal{H}^{N-1}(\partial C_s^\varepsilon \cap \partial C) \,\to\, P(C_s) \end{aligned}$$

as $\varepsilon \to 0^+$. Hence,

$$\begin{aligned} \int_{\partial C_s} \left[z \cdot \nu^{C_s}\right]\, d\mathcal{H}^{N-1} \;&=\; \int_{C_s} \mathrm{div}\, z = \lim_{\varepsilon \to 0} \int_{C_s^\varepsilon} \mathrm{div}\, z_\varepsilon \\ &=\; \lim_{\varepsilon \to 0} \int_{\partial C_s^\varepsilon} [z_\varepsilon \cdot \nu_{C_s^\varepsilon}]\, d\mathcal{H}^{N-1} \;=\; -\,P(C_s). \end{aligned}$$

Since $|z| \le 1$ a.e. in $C$, we deduce that $\left[z \cdot \nu^{C_s}\right] = -1$ on $\partial C_s$ for any $s > s_*$ (in particular, we have $|z| = 1$ a.e. in $C \setminus C_*$). Using this and (19), for all $s_* < s \le s^*$ we have

$$\frac{P(C_s)}{|C_s|} = h_C. \tag{20}$$

This has two consequences. First, from the isoperimetric inequality, we obtain

$$h_C = \frac{P(C_s)}{|C_s|} \geq \frac{P(B_1)}{|B_1|^{\frac{N-1}{N}} s^{\frac{1}{N}}},$$

if $s \in (s_*, s^*]$, so that $s_* > 0$. Moreover, $C_s$ is a Cheeger set for any $s \in (s_*, s^*]$, and by continuity $C_*$ is also a Cheeger set. $\square$

We point out that, since the sets $C_s$ are convex minimizers of $P(E) - \mu_s|E|$ among all $E \subseteq C$, for $s \geq s_*$, their boundary is of class $C^{1,1}$ [9, 19], with curvature less than or equal to $\mu_s$, and equal to $\mu_s$ in the interior of $C$ (note that $\mu_s = h_C$ for $s \in [s_*, s^*]$).

**Remark 3.2.** Observe that we have either $s_* = s^*$ and therefore $C_* = C^*$, or $s_* < s^*$, and we have $C^* = \bigcup_{s \in (s_*, s^*)} C_s$. In this case, the supremum of the total curvature of $\partial C^*$ is equal to $h_C$. Indeed, if it were not the case, by considering $C' \subset \mathrm{int}(C^*)$, with curvature strictly below $h_C$, and the smallest set $C_s$, with $s > s_*$, which contains $C'$, we would have $\kappa_{C'}(x) \geq \kappa_{C_s}(x) = h_C$ at all $x \in \partial C' \cap \partial C_s$, a contradiction. In particular, if the supremum of the total curvature of $\partial C$ is strictly less than $P(C)/|C|$ (which implies $C = C^*$ by [2]) then $C = C_*$.

From the strong convergence of $Dv_\varepsilon$ to $Dv$ (in $L^2(C_s)$ for any $s < s_*$), we deduce that $z = \frac{Dv}{\sqrt{1+|Dv|^2}}$ in $C_*$. It follows that $v$ satisfies the equation

$$-\operatorname{div} \frac{Dv}{\sqrt{1+|Dv|^2}} \;=\; h_C \quad \text{in } C_*. \tag{21}$$

Integrating both terms of (21) in $C_*$, we deduce that

$$\left[\frac{Dv}{\sqrt{1+|Dv|^2}} \cdot \nu^{C_*}\right] = -1 \quad \text{on } \partial C_*.$$

**Lemma 3.3.** *The set $C_*$ is the* minimal *Cheeger set of $C$, i.e., any other Cheeger set of $C$ must contain $C_*$.*

**Proof.** Let $K \subseteq C^*$ be a Cheeger set in $C$. We have

$$h_C|K| \;=\; -\int_K \operatorname{div} z \;=\; -\int_{\partial K} [z \cdot \nu^K]\, d\mathcal{H}^{N-1} \;=\; P(K)$$

so that $[z \cdot \nu^K] = -1$ a.e. on $\partial K$. Let $\nu^\epsilon$ and $\nu$ be the vector fields of unit normals to the sets $C_s^\epsilon$ and $C_s$, $s \in [0, |C|]$, respectively. Observe that, by the Hausdorff convergence of $C_s^\epsilon$ to $C_s$ as $\epsilon \to 0^+$ for any $s \in [0, |C|]$, we have that $\nu^\epsilon \to \nu$ a.e. in $C$. On the other hand, $|z_\epsilon + \nu^\epsilon| \to 0$ locally uniformly in $C \setminus \overline{C_*}$ because of the definition of $z^\epsilon$ and the fact that $|Dv_\epsilon| \to \infty$ outside $C_*$. Both things imply that $z = -\nu$ a.e. on $C \setminus C_*$. By modifying $z$ in a set of null measure, we may assume that $z = -\nu$ on $C \setminus C_*$. We recall that the sets $C_s$, $s \geq s_*$ are minimizers of variational problems of the form $\min_{K \subseteq C} P(K) - \mu|K|$, for some values of $\mu$ (with $\mu = h_C$ as long as $s \leq s^*$ and $\mu > h_C$ continuously increasing with $s \geq s^*$). Since these sets are convex, with boundary (locally) uniformly of class $C^{1,1}$, and the map $s \to C_s$ is continuous in the Hausdorff topology, we obtain that the normal $\nu(x)$ is a continuous function in $C \setminus \mathrm{int}(C_*)$.

Since $|z| < 1$ inside $C_*$ and $[z \cdot \nu^K] = -1$ a.e. on $\partial K$, by [6, Theorem 1]) we have that the boundary of $K$ must be outside the interior of $C_*$, hence either $K \supseteq C_*$ or $K \cap C_* = \emptyset$ (modulo a null set). Let us prove that the last situation is impossible. Indeed, assume that $K \cap C_* = \emptyset$ (modulo a null set). Since $\partial K$ is of class $C^1$ out of a closed set of zero $\mathcal{H}^{N-1}$-measure (see [15]) and $z$ is continuous in $C \setminus \mathrm{int}(C_*)$, by Theorem 2 we have

$$z(x) \cdot \nu^K(x) = -1 \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial K. \tag{22}$$

Now, since $K \cap C_* = \emptyset$ (modulo a null set), then there is some $s \geq s_*$ and some $x \in \partial C_s \cap \partial K$ such that $\nu^K(x) + \nu(x) = 0$. Fix $0 < \epsilon < 2$. By a slight perturbation, if necessary, we may assume that $x \in \partial C_s \cap \partial K$ with $s > s_*$, (22) holds at $x$ and

$$|\nu^K(x) + \nu(x)| < \epsilon. \tag{23}$$

Since by (22) we have $\nu(x) = -z(x) = \nu^K(x)$ we obtain a contradiction with (23). We deduce that $K \supseteq C_*$. ☐

Therefore, in order to prove uniqueness of the Cheeger sets of $C$, it is enough to show that

$$C_* = C^*. \tag{24}$$

Recall that the boundary of both $C_*$ and $C^*$ is of class $C^{1,1}$, and the sum of its principal curvatures is less than or equal $h_C$, and constantly equal to $h_C$ in the interior of $C$. We now show that if $C_* \neq C^*$ and under additional assumptions, the sum of the principal curvatures of the boundary of $C^*$ (or of any $C_s$ for $s \in (s_*, s^*]$) must be $h_C$ out of $C_*$.

**Lemma 3.4.** *Assume that $C$ has $C^2$ boundary. Let $s \in (s_*, s^*]$ and $x \in \partial C_s \setminus \partial C_*$. If the sum of the principal curvatures of $\partial C_s$ at $x$ is strictly below $h_C$, then the Gaussian curvature of $\partial C$ at $x$ is 0.*

*Proof.* Let $x \in \partial C_s \setminus \partial C_*$ and assume the sum of the principal curvatures of $\partial C_s$ at $x$ is strictly below $h_C$ (assuming $x$ is a Lebesgue point for the curvature on $\partial C_s$). Necessarily, this implies that $x \in \partial C$. Assume then that the Gauss curvature of $\partial C$ at $x$ is positive: by continuity, in a neigborhood of $x$, $C$ is uniformly convex and the sum of the principal curvatures is less than $h_C$. We may assume that near $x$, $\partial C$ is the graph of a non-negative, $C^2$ and convex function $f : B \to \mathbb{R}$ where $B$ is an $(N-1)$-dimensional ball centered at $x$, while $\partial C_s$ is the graph of $f_s : B \to \mathbb{R}$, which is $C^{1,1}$ [9, 19], and also nonnegative and convex. In $B$, we have $f_s \geq f \geq 0$, and

$$D^2 f \geq \alpha I \quad \text{and} \quad \mathrm{div} \frac{Df}{\sqrt{1+|Df|^2}} = h$$

with $h \in C^0(\overline{B})$, $h < h_C$, $\alpha > 0$, while

$$\mathrm{div} \frac{Df_s}{\sqrt{1+|Df_s|^2}} = h\chi_{\{f=f_s\}} + h_C\chi_{\{f_s>f\}}$$

(where $\chi_{\{f=f_s\}}$ has positive density at $x$).

We let $g = f_s - f \geq 0$. Introducing now the Lagrangian $\Psi : \mathbb{R}^{N-1} \to [0, +\infty)$ given by $\Psi(p) = \sqrt{1+|p|^2}$, we have that for a.e. $y \in B$

$$\begin{aligned}(h_C - h(y))\chi_{\{g>0\}}(y) &= \operatorname{div}\,(D\Psi(Df_s(y)) - D\Psi(Df(y))) \\ &= \operatorname{div}\left(\left(\int_0^1 D^2\Psi(Df(y) + t(Df_s(y) - Df(y)))\,dt\right) Dg(y)\right)\end{aligned}$$

so that, letting $A(y) := \int_0^1 D^2\Psi(Df(y) + tDg(y))\,dt$ (which is a positive definite matrix and Lipschitz continuous inside $B$), we see that $g$ is the minimizer of

$$\int_B A(y)Dg(y)\cdot Dg(y) + (h_C - h(y))g(y)\,dy$$

under the constraint $g \geq 0$ and with boundary condition $g = f_s - f$ on $\partial B$. Adapting the results in [10] we get that $\{f = f_s\} = \{g = 0\}$ is the closure of a nonempty open set with boundary of zero $\mathcal{H}^{N-1}$-measure, unless the problem is unconstrained, which would yield $h = h_C$ a.e., but we have assumed this is not the case.

We therefore have found an open subset $D \subset \partial C \cap \partial C_s$, disjoint from $\partial C_*$, on which $C$ is uniformly convex, with curvature less than $h_C$. Letting now $\varphi$ is a smooth, nonnegative function with compact support in $D$, one easily shows that if $\varepsilon > 0$ is small enough, $\partial C_s - \varepsilon\varphi\nu^{C_s}$ is a boundary of a set $C'_\epsilon$ which is still convex, with $P(C'_\epsilon)/|C'_\epsilon| > P(C_s)/|C_s| = h_C$ (just differentiate the map $\epsilon \to P(C'_\epsilon)/|C'_\epsilon|$), and the sum of its principal curvatures is less than $h_C$. This implies that for $\epsilon > 0$ small enough, the set $C' := C_\epsilon$ is calibrable [2], which in turn implies that $\min_{K\subset C'} P(K)/|K| = P(C')/|C'|$. But this contradicts $C_* \subset C'$, which is true if $\varepsilon$ was chosen small enough. □

**Proof of Theorem 3**. Assume that $C$ is $C^2$ and uniformly convex. Let us prove that its Cheeger set is unique. Assume by contradiction that $C^* \neq C_*$. From Lemma 3.4 we have that the sum of the principal curvatures of $\partial C^*$ is $h_C$ outside of $C_*$.

Let now $\bar{x} \in \partial C^* \cap \partial C_*$ be such that $\partial C^* \cap B_\rho(\bar{x}) \neq \partial C_* \cap B_\rho(\bar{x})$ for all $\rho > 0$ ($\partial C^* \cap \partial C_* \neq \emptyset$ since otherwise both $C^*$ and $C_*$ would be balls, which is impossible). Letting $T$ be the tangent hyperplane to $\partial C^*$ at $\bar{x}$, we can write $\partial C^*$ and $\partial C_*$ as the graph of two positive convex functions $v^*$ and $v_*$, respectively, over $T \cap B_\rho(\bar{x})$ for $\rho > 0$ small enough. Identifying $T \cap B_\rho(\bar{x})$ with $B_\rho \subset \mathbb{R}^{N-1}$, we have that $v_*, v^* : B_\rho \to \mathbb{R}$ both solve the equation

$$-\operatorname{div}\frac{Dv}{\sqrt{1+|Dv|^2}} = f, \tag{25}$$

for some function $f \in L^\infty(B_\rho)$, moreover it holds $v_* \geq v^*$, $v_*(0) = v^*(0)$ and $v_*(y) > v^*(y)$ for some $y \in B_\rho$. Notice that $f = \lambda_C$ in the (open) set where $v_* > v^*$, in particular both functions are smooth in this set. Let $D$ be an open ball such that $\overline{D} \subset B_\rho$, $v_* > v^*$ on $D$ and $v_*(y) = v^*(y)$ for some $y \in \partial D$. Notice that, since both $v^*$ and $v_*$ belong to $C^\infty(D) \cap C^1(\overline{D})$, the fact that $v_*(y) = v^*(y)$ also implies that $Dv_*(y) = Dv^*(y)$. In $D$, both functions solve

(25) with $f = \lambda_C$. Letting now $w = v_* - v^*$, we have that $w(y) = 0$ and $Dw(y) = 0$, while $w > 0$ inside $D$. Recalling the function $\Psi(p) = \sqrt{1 + |p|^2}$, we have that for any $x \in D$

$$
\begin{aligned}
0 \;=\; & \operatorname{div}\left(D\Psi(Dv_*(x)) - D\Psi(Dv^*(x))\right) \\
& = \operatorname{div}\left(\left(\int_0^1 D^2\Psi(Dv^*(x) + t(Dv_*(x) - Dv^*(x)))\, dt\right) Dw(x)\right)
\end{aligned}
$$

so that $w$ solves a linear, uniformly elliptic equation with smooth coefficients. Then Hopf's lemma [12] implies that $Dw(y) \cdot \nu_D(y) < 0$, a contradiction. Hence $C_* = C^*$. □

**Remark 3.5.** Notice that, as a consequence of Theorem 3 and the results of Giusti [13], we get that if $C$ is of class $C^2$ and uniformly convex, equation (21) has a solution on the whole of $C$, if and only if $C$ is a Cheeger set of itself, i.e. if and only if the the sum of the principal curvatures of $\partial C$ is less than or equal to $P(C)/|C|$.

**Remark 3.6.** The results of this paper can be easily extended to the anisotropic setting (see [11]) provided the anisotropy is smooth and uniformly elliptic.

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