Eventual regularity for the parabolic minimal surface equation

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Abstract

We show that the parabolic minimal surface equation has an eventual regularization effect, that is, the solution becomes smooth after a strictly positive finite time.

1 Introduction

In this paper we consider the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & \text{in } (0,\infty)\times\Omega, \\ \\ \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot \nu^\Omega = 0 & \text{in } (0,\infty)\times\partial\Omega, \\ \\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open subset with smooth boundary, ν^Ω denotes the exterior unit normal to $\partial\Omega$, and $\cdot$ is the scalar product in $\mathbb{R}^N$.

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Problem (1.1) corresponds to the L^2-gradient flow [9] of the convex lower semicontinuous functional $F : L^2(\Omega) \to [0, +\infty]$, defined as

$$F(u) := \begin{cases} \displaystyle\int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + |D^s u|(\Omega) & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where, for $u \in BV(\Omega)$, we write its distributional derivative Du as

$$Du = \nabla u\, dx + D^s u.$$

Here, $D^s u$ is the singular part of the measure Du, with respect to the Lebesgue measure (see [20]), and $|D^s u|$ stands for its total variation.

Equation (1.1) arises in several models of physical systems describing, for instance, the motion of capillary surfaces and the motion of grain boundaries in annealing metals (see [8]). For such reasons, this evolution problem has been already considered in the mathematical literature. In particular, Lichnewski and Temam [21] showed existence of generalized solutions, while Gerhardt [19] and Ecker [15] proved esimates on u, u_t, $|Du|$ similar to the ones we present in this paper (see Lemma 3.1).

We point out that equation (1.1) should not be confused with the mean curvature flow for graphs, which has been deeply studied in [16, 17], and reads as

$$\frac{\partial u}{\partial t} = \sqrt{1 + |\nabla u|^2}\, \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

Let us state our main result.

Theorem 1.1. *Let $u_0 \in L^2(\Omega)$ and let u be the unique solution of (1.1). Suppose that one of the two following cases hold:*

1) $N = 1$,

2) $N > 1$, and the generalized graph of u_0[1],

$$\operatorname{graph}(u_0) := \overline{\partial\{(x, y) : x \in \Omega,\ y < u_0(x)\} \cap (\Omega \times \mathbb{R})},$$

is a compact hypersurface of class $\mathcal{C}^{1,1}$, meeting orthogonally $(\partial\Omega) \times \mathbb{R}$.

[1] Roughly, the graph of u_0 with the addition of the "vertical" parts.

Then there exists $T > 0$ such that

$$u \in \mathcal{C}^{\omega}((T, +\infty) \times \Omega).$$

Namely, $u(t)$ is analytic in Ω, for any $t \in (T, +\infty)$. Moreover, $u(t)$ converges to the mean value $\overline{u}_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0 \, dx$ of u_0 in $\mathcal{C}^{\infty}(\Omega)$ as $t \to +\infty$.

The assumption on graph(u_0) when $N > 1$ is technical, but we are presently not able to remove it. Notice that from this condition it follows that

$$u_0 \in L^{\infty}(\Omega) \cap BV(\Omega),$$

and u_0 is of class $\mathcal{C}^{1,1}$ in a neighbourhood of $\partial\Omega$, with Neumann boundary condition on $\partial\Omega$.

Theorem 1.1 states that the solution $u(t)$ to (1.1) becomes smooth after some time T, which in general is strictly positive. This behaviour is somewhat different from the usual regularity results for parabolic partial differential equations. Indeed, for this problem there is no instantaneous regularization of the solution, which holds for uniformly parabolic equations but does *not* hold in general for (1.1).

Another well-known degenerate parabolic problem which shares this property is the so-called total variation flow, which has relevant applications in image analysis and denoising (see, *e.g.*, [24, 6, 3, 12, 14])[2].

Example 1.1. As an example of eventual but not instantaneous regularization of u, we consider the following situation: $N = 1$, $\Omega = (0, 2)$, c a positive constant, and

$$u_0(x) := \begin{cases} \sqrt{1 - x^2} + c & \text{if } x \in (0, 1), \\ -\sqrt{1 - (2 - x)^2} & \text{if } x \in (1, 2). \end{cases}$$

Then u_0 is discontinuous at $x = 1$, and graph(u_0) is a curve of class $\mathcal{C}^{1,1}$ consisting of two quarters of unit circles (hence with constant curvature equal to 1) and a vertical segment of length c, with the correct boundary condition. Then

$$u(t, x) = \begin{cases} -t + \sqrt{1 - x^2} + c & \text{if } (t, x) \in \left(0, \frac{c}{2}\right) \times (0, 1), \\ t - \sqrt{1 - (2 - x)^2} & \text{if } (t, x) \in \left(0, \frac{c}{2}\right) \times (1, 2), \end{cases}$$

[2] As shown in [2], solutions to the total variation flow in the whole of $\mathbb{R}^N$ become extinct (hence in particular smooth) in finite time, even if the discontinuity set does not disappear immediately (see, *e.g.*, [7, 11]).

which is still discontinuous at $x = 1$, because the upper quarter of circle as unit negative vertical velocity, while the lower quarter of circle as unit positive vertical velocity. Hence the time necessary to let the jump disappear is $T := c/2$, and one checks that the solution becomes smooth in $(T, +\infty) \times (0, 2)$.

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2 Notation and preliminary results

We denote by ∂F the subdifferential of F in the sense of convex analysis, which defines a maximal monotone operator in $L^2(\Omega)$. A characterization of ∂F is given in Remark 2.2.

Definition 2.1. *Let $u_0 \in L^2(\Omega)$. We say that a function $u : [0, +\infty) \to \Omega$ is a strong solution (briefly, a solution) of* (1.1)*, if*

(i) $u \in H^1((0,T); L^2(\Omega)) \cap L^\infty((\tau, +\infty); BV(\Omega))$ *for any* $T > 0$ *and* $\tau > 0$,

(iii) the following inclusion holds:

$$u_t(t) + \partial F(u(t)) \ni 0 \qquad \text{for a.e. } t \in (0, +\infty), \tag{2.1}$$

(ii) $\lim_{t \to 0^+} u(t) = u_0$ *in* $L^2(\Omega)$.

We let

$$X(\Omega) := \{ z \in L^2(\Omega; \mathbb{R}^N) : \operatorname{div} z \in L^2(\Omega) \}.$$

It is known that, for $z \in X(\Omega)$, the normal trace $[z, \nu^\Omega]$ of z on $\partial\Omega$ is well defined (see [4, 3]).

Remark 2.2. Following [3], inclusion (2.1) can be equivalently written as

$$\begin{cases} u_t = \operatorname{div} z & \text{in } \mathcal{D}'(\Omega), \text{ for a.e. } t \in (0, +\infty), \\ z = \dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} & \text{a.e. in } (0, +\infty) \times \Omega, \\ z(t) \in X(\Omega) & \text{for a.e. } t \in (0, +\infty), \\ [z(t, \cdot), \nu^\Omega] = 0 & \text{for a.e. } t \in (0, +\infty). \end{cases} \tag{2.2}$$

Note that, in the expression of z, only the absolutely continuous part of the spatial gradient of u is involved.

Let us recall the following results, proved in [9, Theorems 3.2, 3.7, 3.11] (see also [3, Chapter 6]).

Theorem 2.1. *Let $u_0 \in L^2(\Omega)$. Then there exists a unique solution u of* (1.1). *Moreover:*

(i) the function $t \in (0, +\infty) \mapsto F(u(t))$ is nonincreasing, and

$$\frac{d}{dt} F(u(t)) = -\int_\Omega u_t^2 \, dx \qquad \textit{for a.e. } t \in (0, +\infty);$$

(ii) $\displaystyle\lim_{t \to +\infty} u(t) = \overline{u}_0 := \frac{1}{|\Omega|} \int_\Omega u_0 \, dx$ *in* $L^2(\Omega)$;

(iii) $\displaystyle\|u_t\|_{L^2(\Omega)} \leq \frac{\|u_0\|_{L^2(\Omega)}}{t}$ *for almost any $t > 0$.*

Remark 2.3. Regarding point *(ii)* of Theorem 2.1, from [9, Theorem 3.11] it follows that $u(t)$ converges in $L^2(\Omega)$, as $t \to +\infty$, to a minimizer of F, that is, to a constant in Ω. The value of this constant is fixed from

$$\frac{d}{dt} \int_\Omega u(t, x) \, dx = \int_\Omega \operatorname{div} z(t, x) \, dx = 0,$$

where the last equality follows from the Gauss-Green Theorem [4].

The maximum and minimum principles ensure the following result.

Proposition 2.4. *Let $u_0 \in L^\infty(\Omega)$. Then the function*

$$t \in (0, +\infty) \to \|u(t)\|_{L^\infty(\Omega)}$$

is nonincreasing.

The following approximation result is proved in [9, Theorem 3.16].

Proposition 2.5. *Let u_0, $u_{0n} \in L^2(\Omega)$ be such that*

$$\lim_{n \to +\infty} \|u_0 - u_{0n}\|_{L^2(\Omega)} = 0.$$

Let u be the solution to (1.1), *and let u_n be the solution to the first two equations of* (1.1), *and with $u_n(0, \cdot) = u_{0n}(\cdot)$. Then, for all $T > 0$ we have*

$$\lim_{n \to +\infty} \|u(t) - u_n(t)\|_{L^2(\Omega)} = 0 \qquad \textit{uniformly in } [0, T]. \tag{2.3}$$

3 Proof of the main result

We start with the following estimates, which have been shown in [19, 15]. It can be useful to have a detailed proof, which we include here for completeness.

Lemma 3.1. *Let $u_0 \in \mathcal{C}^\infty(\overline{\Omega})$, and let u be the solution to* (1.1) *given by Theorem 2.1. For all $t > 0$ we have*

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \leq \ \|\nabla u_0\|_{L^\infty(\Omega)} \tag{3.1}$$

$$\|u_t(t)\|_{L^\infty(\Omega)} \leq \ \|u_t(0)\|_{L^\infty(\Omega)} \,. \tag{3.2}$$

In particular, by parabolic regularity theory, from (3.1) *it follows that $u \in C^\infty([0,+\infty)\times\overline{\Omega})$.*

Proof. Both estimates follow by a direct computation and by the maximum principle. By [22] we have that there exists $\tau > 0$ such that $u \in \mathcal{C}^{(2+\alpha)/2,2+\alpha}([0,\tau)\times\Omega)$ for all $\alpha \in (0,1)$, and therefore, by parabolic regularity, we have $u \in \mathcal{C}^\infty([0,\tau)\times\overline{\Omega})$.

Let us first show (3.1), arguing at points in $(0,\tau)\times\Omega$. Differentiating (1.1), we get

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{|\nabla u|^2}{2}\right) &= \nabla u\cdot\nabla u_t = \nabla u\cdot\nabla\left(\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right)\\
&= \nabla u\cdot\nabla\left(\frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla\left(\frac{|\nabla u|^2}{2}\right)\cdot\nabla u}{(1+|\nabla u|^2)^{3/2}}\right).
\end{aligned}$$

Letting $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$, and denoting by $\nabla^2 u$ the Hessian of u, we compute

$$\begin{aligned}
\nabla u\cdot\nabla\left(\frac{\Delta u}{\sqrt{1+|\nabla u|^2}}\right) &= \frac{\sum_{i=1}^N \partial_i u\,\Delta(\partial_i u)}{\sqrt{1+|\nabla u|^2}} - \frac{\Delta u\left(\nabla u\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right)\right)}{(1+|\nabla u|^2)^{3/2}}\\
&= \frac{\Delta\left(\frac{|\nabla u|^2}{2}\right) - |\nabla^2 u|^2}{\sqrt{1+|\nabla u|^2}} - \frac{\Delta u\left(\nabla u\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right)\right)}{(1+|\nabla u|^2)^{3/2}},
\end{aligned}$$

where $|\nabla^2 u|^2 := \sum_{i,j=1}^N (\partial_i\partial_j u)^2$. Also

$$\nabla u \cdot \nabla \left(\frac{\nabla\left(\frac{|\nabla u|^2}{2}\right)\cdot \nabla u}{(1+|\nabla u|^2)^{3/2}} \right) = \frac{\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \,,\, \nabla^2 \left(\frac{|\nabla u|^2}{2}\right) \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right)}{\sqrt{1+|\nabla u|^2}} + \frac{\left|\nabla\left(\frac{|\nabla u|^2}{2}\right)\right|^2}{(1+|\nabla u|^2)^{3/2}} - 3\,\frac{\left(\nabla\left(\frac{|\nabla u|^2}{2}\right)\cdot\nabla u\right)^2}{(1+|\nabla u|^2)^{5/2}}.$$

Summing up, we get

$$\begin{aligned} \frac{\partial}{\partial t}\frac{|\nabla u|^2}{2} \le{}& \frac{\Delta\left(\frac{|\nabla u|^2}{2}\right) - \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \,,\, \nabla^2 \left(\frac{|\nabla u|^2}{2}\right) \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right)}{\sqrt{1+|\nabla u|^2}} \\ &- \frac{\left|\nabla\left(\frac{|\nabla u|^2}{2}\right)\right|^2}{(1+|\nabla u|^2)^{3/2}} + 3\frac{\left(\nabla\left(\frac{|\nabla u|^2}{2}\right)\cdot\nabla u\right)^2}{(1+|\nabla u|^2)^{5/2}} \\ &- \frac{\Delta u\left(\nabla u\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right)\right)}{(1+|\nabla u|^2)^{3/2}}. \end{aligned} \tag{3.3}$$

The estimate (3.1) then follows from the maximum principle applied to (3.3). By standard arguments [22], it now follows that $u \in \mathcal{C}^\infty([0,+\infty)\times\overline{\Omega})$, and (3.1) holds for all $t > 0$.

Let us now show (3.2). From (1.1), we get

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right) &= u_t\left(\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right)_t \\ &= u_t \operatorname{div}\left(\frac{\nabla u_t}{\sqrt{1+|\nabla u|^2}}\right) - u_t \operatorname{div}\left(\frac{(\nabla u_t\cdot\nabla u)\nabla u}{(1+|\nabla u|^2)^{3/2}}\right). \end{aligned} \tag{3.4}$$

We have

$$\begin{aligned} u_t \operatorname{div}\left(\frac{\nabla u_t}{\sqrt{1+|\nabla u|^2}}\right) &= u_t\sum_i \partial_i\left(\frac{\partial_i u_t}{\sqrt{1+|\nabla u|^2}}\right) \\ &= \frac{u_t\,\Delta u_t}{\sqrt{1+|\nabla u|^2}} - \sum_i \frac{u_t\partial_i u_t \nabla u\cdot\nabla\partial_i u}{(1+|\nabla u|^2)^{3/2}} \\ &= \frac{u_t\Delta u_t}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla\left(\frac{u_t^2}{2}\right)\cdot\nabla\left(\frac{|\nabla u|^2}{2}\right)}{(1+|\nabla u|^2)^{3/2}} =: \mathrm{I}+\mathrm{II}, \end{aligned} \tag{3.5}$$

and

$$
\begin{aligned}
u_t \operatorname{div}\left(\frac{(\nabla u_t\cdot\nabla u)\nabla u}{(1+|\nabla u|^2)^{3/2}}\right) =& \Delta u\frac{\nabla u\cdot u_t\nabla u_t}{(1+|\nabla u|^2)^{3/2}} + \frac{u_t\nabla u\cdot\nabla^2 u_t\nabla u}{(1+|\nabla u|^2)^{3/2}}\\
&+\frac{\nabla^2 u\nabla u\cdot u_t\nabla u_t}{(1+|\nabla u|^2)^{3/2}}\\
&-3\sum_i \partial_i u\frac{u_t(\nabla u_t\cdot\nabla u)(\nabla\partial_i u\cdot\nabla u)}{(1+|\nabla u|^2)^{5/2}} \qquad (3.6)\\
=& \Delta u\frac{\nabla u\cdot\nabla(\frac{u_t^2}{2})}{(1+|\nabla u|^2)^{3/2}} + \frac{u_t\nabla u\cdot\nabla^2 u_t\nabla u}{(1+|\nabla u|^2)^{3/2}}\\
&+\frac{\nabla(\frac{|\nabla u|^2}{2})\cdot\nabla(\frac{u_t^2}{2})}{(1+|\nabla u|^2)^{3/2}}\\
&-3\frac{\left(\nabla(\frac{u_t^2}{2})\cdot\nabla u\right)\left(\nabla(\frac{|\nabla u|^2}{2})\cdot\nabla u\right)}{(1+|\nabla u|^2)^{5/2}}\\
=:& \mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI},
\end{aligned}
$$

with $\mathrm{V} = -\mathrm{II}$, hence

$$
\frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right) = \mathrm{I}-\mathrm{III}-\mathrm{IV}-2\mathrm{V}-\mathrm{VI}. \qquad (3.7)
$$

Now, we write $u_t\Delta u = \Delta(u_t^2/2) - |\nabla u_t|^2$, and

$$
u_t\nabla u\cdot\nabla^2 u_t\nabla u = \nabla u\cdot\nabla^2(u_t^2/2)\nabla u - (\nabla u\cdot\nabla u_t)^2,
$$

whence

$$
\begin{aligned}
\mathrm{I}-\mathrm{IV} =& \frac{\Delta\left(\frac{u_t^2}{2}\right) - \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\,,\,\nabla^2\left(\frac{u_t^2}{2}\right)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)}{\sqrt{1+|\nabla u|^2}}\\
&-\frac{|\nabla u_t|^2 - \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot\nabla u_t\right)^2}{\sqrt{1+|\nabla u|^2}}.
\end{aligned}
$$

Substituting into (3.7) we then get

$$\frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right) = \frac{\Delta\left(\frac{u_t^2}{2}\right) - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla^2\left(\frac{u_t^2}{2}\right)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}}{\sqrt{1+|\nabla u|^2}}$$

$$- \frac{|\nabla u_t|^2 - \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla u_t\right)^2}{\sqrt{1+|\nabla u|^2}} - \Delta u \frac{\nabla u \cdot \nabla(\frac{u_t^2}{2})}{(1+|\nabla u|^2)^{3/2}}$$

$$+ 3\frac{\left(\nabla(\frac{u_t^2}{2}) \cdot \nabla u\right)\left(\nabla(\frac{|\nabla u|^2}{2}) \cdot \nabla u\right)}{(1+|\nabla u|^2)^{5/2}} - 2\frac{\nabla\left(\frac{u_t^2}{2}\right) \cdot \nabla\left(\frac{|\nabla u|^2}{2}\right)}{(1+|\nabla u|^2)^{3/2}}.$$

Observing that $|\nabla u_t|^2 \leq \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla u_t\right)^2$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right) \leq & \frac{\Delta\left(\frac{u_t^2}{2}\right) - \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla^2\left(\frac{u_t^2}{2}\right)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}}{\sqrt{1+|\nabla u|^2}} \\ & - \Delta u \frac{\nabla u \cdot \nabla(\frac{u_t^2}{2})}{(1+|\nabla u|^2)^{3/2}} \\ & + 3\frac{\left(\nabla(\frac{u_t^2}{2}) \cdot \nabla u\right)\left(\nabla\left(\frac{|\nabla u|^2}{2}\right) \cdot \nabla u\right)}{(1+|\nabla u|^2)^{5/2}} \\ & - 2\frac{\nabla\left(\frac{u_t^2}{2}\right) \cdot \nabla\left(\frac{|\nabla u|^2}{2}\right)}{(1+|\nabla u|^2)^{3/2}}.\end{aligned} \tag{3.8}$$

The estimate (3.2) follows as above from the maximum principle applied to (3.8). □

Proof of Theorem 1.1.

Case 1. Assume that $N = 1$. Note that $X(\Omega) = H^1(\Omega)$. Let $L \subset (0, +\infty)$ be a set of zero Lebesgue measure such that the partial differential equation in (2.2) and Theorem 2.1 (*iii*) hold for any $t \in J := (0, +\infty) \setminus L$. We deduce that for all $\tau > 0$ there exists a constant $C = C(\tau) > 0$ such that

$$\|z(t, \cdot)\|_{H^1(\Omega)} \leq C, \qquad t \geq \tau,\ t \in J.$$

In particular, by Sobolev embedding, $z(t, \cdot)$ is 1/2–Hölder continuous, uniformly for $t \geq \tau$, $t \in J$. Therefore, for all $\eta \in (0, 1)$ there exists $\varepsilon = \varepsilon(\eta) > 0$

independent of $t \geq \tau$, $t \in J$, such that, if

$$|z(t, x_0)| \geq \eta \qquad \text{for some } (t, x_0) \in J \times \Omega, \tag{3.9}$$

then

$$|z(t, x)| \geq \frac{\eta}{2} \qquad \text{for all } x \in (x_0 - \varepsilon, x_0 + \varepsilon) \cap \Omega.$$

Hence

$$\text{either} \quad u_x(t, x) \geq \frac{\eta}{\sqrt{4 - \eta^2}} \quad \text{or} \quad u_x(t, x) \leq -\frac{\eta}{\sqrt{4 - \eta^2}}\,, \tag{3.10}$$

for almost every $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \cap \Omega$. We then get

$$\begin{aligned} \|u(t) - \overline{u}_0\|^2_{L^2(\Omega)} &\geq \min_{\alpha \in \mathbb{R}} \|u(t) - \alpha\|^2_{L^2(\Omega)} \\ &\geq \min_{\beta \in \mathbb{R}} f(\beta), \end{aligned} \tag{3.11}$$

where[3]

$$f(\beta) := \int_0^\varepsilon \left(\frac{\eta}{\sqrt{4 - \eta^2}}\, x - \beta \right)^2 dx, \qquad \beta \in \mathbb{R},$$

and we use that $|\Omega \cap (x_0 - \varepsilon, x_0 + \varepsilon)| \geq \varepsilon$ as soon as $\varepsilon \in (0, |\Omega|]$.

One checks that $\min_{\beta \in \mathbb{R}} f(\beta) = f\left(\frac{\varepsilon\eta}{2\sqrt{4-\eta^2}} \right) = \frac{\eta^2}{4-\eta^2}\, \frac{\varepsilon^3}{12}$. Hence

$$\|u(t) - \overline{u}_0\|^2_{L^2(\Omega)} \geq \frac{\eta^2}{4 - \eta^2}\, \frac{\varepsilon^3}{12}\,.$$

Since $u(t) \to \overline{u}_0$ in $L^2(\Omega)$ as $t \to +\infty$ by Theorem 2.1 (*ii*), in order not to have a contradiction it follows that, given $\eta \in (0, 1)$, there exists $T = T(\eta)$ such that $|z(t, x)| < \eta$ for all $(t, x) \in J \times \Omega$, $t \geq T$. Therefore $u(t, \cdot)$ is $(\eta/\sqrt{4 - \eta^2})$-Lipschitz in Ω for all $t > T$, $t \in J$, and hence for all $t > T$. Thus, by parabolic regularity theory,

$$u(t, \cdot) \in \mathcal{C}^\omega(\Omega) \qquad \text{for all } t > T\,,$$

and $u(t) \to \overline{u}_0$ in $\mathcal{C}^\infty(\Omega)$ as $t \to +\infty$.

[3] Indeed, if $c := \frac{\eta}{4-\eta^2}$, for any $\alpha \in \mathbb{R}$ and $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \cap \Omega$ we have $|u(t, x) - \alpha| = |u(t, x) - \alpha + \int_{x_0}^x u'(t, \xi)\, d\xi| \geq |u(t, x_0) - \alpha + c(x - x_0)| = |cx - \beta|$, where $\beta := \alpha - u(t, x_0) + cx_0$. Hence $\int_\Omega (u(t, x) - \alpha)^2\, dx \geq \int_\Omega (cx - \beta)^2\, dx$, and therefore $\min_{\alpha \in \mathbb{R}} \int_\Omega (u(t, x) - \alpha)^2 \geq \min_{\beta \in \mathbb{R}} \int_\Omega (cx - \beta)^2\, dx$.

Case 2. Assume that $N > 1$. Let $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$, and suppose that $\mathrm{graph}(u_0)$ is a hypersurface of class $\mathcal{C}^{1,1}$ meeting orthogonally $\partial\Omega$. We divide the proof into four steps.

Step 1. There exists a sequence $(u_0^n) \subset \mathcal{C}^\infty(\Omega) \cap \mathrm{Lip}(\Omega)$ converging to u_0 in $L^2(\Omega)$ and such that

$$\sup_{n\in\mathbb{N}} \left\| \mathrm{div}\left(\frac{\nabla u_0^n}{\sqrt{1+|\nabla u_0^n|^2}} \right) \right\|_{L^\infty(\Omega)} < +\infty. \tag{3.12}$$

Indeed, let $\Sigma(t)$ be the mean curvature evolution starting from $\mathrm{graph}(u_0) = \Sigma(0)$, with Neumann boundary conditions on $\partial\Omega \times \mathbb{R}$ (see [25, 10]). Since $\mathrm{graph}(u_0)$ is of class $\mathcal{C}^{1,1}$, there exists an evolution $t \mapsto \Sigma(t)$, with $t \in [0, \tau]$ for some $\tau > 0$, such that $\Sigma(t)$ is of class $\mathcal{C}^\infty$ (actually analytic) and $\Sigma(t)$ is of class $\mathcal{C}^{1,1}$[25][4] uniformly in $[0, \tau]$. Moreover, letting $\nu(t)$ be the unit normal to $\Sigma(t)$ pointing upward, so that $\nu_{n+1}(0) \geq 0$ on $\Sigma(0)$, by the strong maximum principle we have $\nu_{n+1}(t) > 0$ on $\Sigma(t)$ for any $t \in (0, \tau]$. That is, $\Sigma(t)$ is the graph of a function $v(t) \in \mathcal{C}^\infty(\Omega) \cap \mathrm{Lip}(\Omega)$. We conclude by letting

$$u_0^n := v(t_n),$$

where $(t_n) \subset (0, \tau)$ is a sequence converging to 0 as $n \to +\infty$, so that $\mathrm{graph}(u_0^n)$ is of class $\mathcal{C}^{1,1}$ uniformly in $n \in \mathbb{N}$, which implies (3.12).

Step 2. The function u satisfies

$$\sup_{t\in(0,+\infty)} \|u_t(t)\|_{L^\infty(\Omega)} < +\infty. \tag{3.13}$$

Let $u_n(t)$, $t \in [0, +\infty)$, be the solution of the first two equations of (1.1), with initial condition $u_n(0) = u_0^n$, where u_0^n is as in *step 1.* From Lemma 3.1 we have $u_n \in \mathcal{C}^\infty([0, +\infty) \times \overline{\Omega})$; from (3.12) and (3.2), it follows that

$$\sup_{n\in\mathbb{N}} \|\partial_t u_n(t)\|_{L^\infty(\Omega)} \leq C \qquad \forall t \in (0, +\infty), \tag{3.14}$$

where $C > 0$ is a constant bounding the left hand side of (3.12). Notice that (3.14) is equivalent to say that the functions $u_n(\cdot, x)$ are C-Lipschitz on $(0, +\infty)$, uniformly in $n \in \mathbb{N}$ and $x \in \Omega$. Recalling that, by Proposition

[4] The fact that $\Sigma(t)$ is of class $\mathcal{C}^{1,1}$ uniformly in $[0, \tau]$ follows from the assumption on $\mathrm{graph}(u_0)$ and the estimates in [25] (see, *e.g.*, [5, Ch. 13] for related references and a precise argument).

2.5, $\lim_{n\to+\infty} u_n = u$ in $L^2((0,T)\times\Omega)$ for all $T>0$, we can extract a (not relabelled) subsequence so that $\lim_{n\to+\infty} u_n = u$ almost everywhere in $(0,+\infty)\times\Omega$. Passing to the limit, as $n\to+\infty$, in the inequality $|u_n(t,x) - u_n(s,x)| \le C|t-s|$ for almost every $(t,s,x)\in(0,+\infty)\times(0,+\infty)\times\Omega$, we get that $u(\cdot,x)$ is also C-Lipschitz in $(0,+\infty)$, uniformly in $x\in\Omega$, which gives (3.13).

Step 3. We have that[5]

$$\text{graph}(u(t)) \text{ is } \mathcal{C}^{1,\alpha} \text{ for any } \alpha\in(0,1), \text{ uniformly in } t\in(0,+\infty). \tag{3.15}$$

To prove assertion (3.15) it is enough to closely follow [13, Proposition 4.4] and use (3.13): we repeat here the argument for completeness. Suppose by contradiction that (3.15) does not hold. Then, for any $n\in\mathbb{N}$, we can find $(t_n,x_n,y_n)\in(0,+\infty)\times\Omega\times\mathbb{R}$ such that $(x_n,y_n)\in\text{graph}(u(t_n))$ and, for all $\rho>0$, the hypersurfaces $\text{graph}(u(t_n))\cap B_\rho(x_n,y_n)$[6] are not uniformly $\mathcal{C}^{1,\alpha}$ with respect to n. Letting $\widetilde{u}_n(x) := u(t_n,x) - u(t_n,x_n)$, from (1.1) and (3.13) we have that

$$-\text{div}\left(\frac{\nabla\widetilde{u}_n(x)}{\sqrt{1+|\nabla\widetilde{u}_n(x)|^2}}\right) = \kappa_n(x), \qquad x\in\Omega. \tag{3.16}$$

with

$$\sup_{n\in\mathbb{N}} \|\kappa_n\|_{L^\infty(\Omega)} < +\infty. \tag{3.17}$$

In particular, $\widetilde{u}_n$ is a minimizer of the prescribed curvature functional

$$v\in BV(\Omega)\mapsto \int_\Omega \left(\sqrt{1+|\nabla v|^2} - \kappa_n v\right)\,dx + |D^s v|(\Omega).$$

From (3.17) and the compactness theorem for quasi minimizers of the perimeter [1], [20], the hypersurfaces $\text{graph}(\widetilde{u}_n)$ converge in $L^1(\Omega\times\mathbb{R})$, and up to a (not relabelled) subsequence, to a limit hypersurface $\Gamma_\infty\subset\overline{\Omega}\times\mathbb{R}$ of class $\mathcal{C}^{1,\alpha}$, for all $\alpha\in(0,1)$. Possibly passing to a further subsequence, we can also assume that $\lim_{n\to+\infty} x_n = x_\infty$, for some $x_\infty\in\overline{\Omega}$. Observe that $\widetilde{u}_n(x_n) = 0$, and so $(x_\infty,0)\in\Gamma_\infty$.

By [26, Theorem 1] there exists $\rho>0$ such that both $\text{graph}(\widetilde{u}_n)\cap B_\rho(x_\infty,0)$ and $\Gamma_\infty\cap B_\rho(x_\infty,0)$ can be written as graphs of functions of N

[5] α in general cannot be taken equal to one, see [1].

[6] Here $B_\rho(x,y)$ denotes the ball of radius ρ in $\mathbb{R}^{N+1}$ centered at $(x,y)\in\overline{\Omega}\times\mathbb{R}$.

variables, in the normal direction to Γ_∞ at $(x_\infty, 0)$. Therefore, by regularity of minimizers of the prescribed curvature functional [23], the hypersurfaces $\mathrm{graph}(\widetilde{u}_n) \cap B_\rho(x_\infty, 0)$ are uniformly (with respect to $n \in \mathbb{N}$) of class $\mathcal{C}^{1,\alpha}$ for all $\alpha \in (0,1)$, thus leading to a contradiction.

Step 4. From (3.15) if follows that the vector field $z(t,\cdot)$ in (2.2) is α-Hölder continuous in Ω, uniformly with respect to $t \in (0,+\infty)$. We can now proceed as in case 1, with only minor changes. Indeed, letting L and $J \subset:= (0,+\infty) \setminus L$ be as in case 1, from the Hölder continuity of z we get that, for all $\eta \in (0,1)$, there exists $\varepsilon = \varepsilon(\eta) > 0$ independent of $t \in J$, such that, if for some $\nu \in \mathbb{S}^{N-1}$,

$$z(t,x_0) \cdot \nu \geq \eta \qquad \text{for some } (t,x_0) \in J \times \Omega, \tag{3.18}$$

then

$$z(t,x) \cdot \nu \geq \frac{\eta}{2} \qquad \text{for all } x \in B_\varepsilon(x_0) \cap \Omega.$$

It then follows

$$\nabla u(t,x) \cdot \nu \geq \frac{\eta}{\sqrt{4-\eta^2}} \qquad \text{for almost every } x \in B_\varepsilon(x_0) \cap \Omega, \tag{3.19}$$

which implies, as in case 1,

$$\begin{aligned} \|u(t) - \overline{u}_0\|^2_{L^2(\Omega)} &\geq \min_{\alpha \in \mathbb{R}} \|u(t) - \alpha\|^2_{L^2(\Omega)} \\ &\geq \min_{\beta \in \mathbb{R}} g(\beta), \end{aligned} \tag{3.20}$$

where

$$g(\beta) := \int_{B_\varepsilon(x_0)\cap\Omega} \left(\frac{\eta}{\sqrt{4-\eta^2}}\, x \cdot \nu - \beta \right)^2 dx, \qquad \beta \in \mathbb{R}.$$

One checks that

$$\begin{aligned} \min_{\beta \in \mathbb{R}} g(\beta) &= g\left(\frac{\eta \int_{B_\varepsilon(x_0)\cap\Omega} x \cdot \nu \; dx}{\sqrt{4-\eta^2} |B_\varepsilon(x_0) \cap \Omega|} \right) \\ &= \frac{\eta^2}{4-\eta^2} \left(\int_{B_\varepsilon(x_0)\cap\Omega} (x\cdot\nu)^2 \; dx - \frac{1}{|B_\varepsilon(x_0)\cap\Omega|} \left(\int_{B_\varepsilon(x_0)\cap\Omega} x \cdot \nu \; dx \right)^2 \right) \\ &\geq C\, \frac{\eta^2}{4-\eta^2}\, \varepsilon^{N+2}, \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0(\Omega))$, where $\varepsilon_0(\Omega) > 0$ depends only on Ω, and the constant $C > 0$ depends only on the dimension N. Hence

$$\|u(t) - \overline{u}_0\|^2_{L^2(\Omega)} \geq C \, \frac{\eta^2}{4 - \eta^2} \, \varepsilon^{N+2}.$$

Since $\|u(t) - \overline{u}_0\|^2_{L^2(\Omega)} \to 0$ as $t \to +\infty$ by Theorem 2.1, it follows that, given $\eta \in (0, 1)$, there exists $T = T(\eta)$ (in particular independent of ν) such that $|z(t, x)| \leq \eta$ for all $(t, x) \in J \times \Omega$, $t \geq T$. Therefore $u(t, \cdot)$ is $(\eta/\sqrt{4 - \eta^2})$-Lipschitz in Ω for all $t > T$, so that by parabolic regularity theory $u(t, \cdot) \in \mathcal{C}^\omega(\Omega)$ for all $t > T$, and $u(t) \to \overline{u}_0$ in $\mathcal{C}^\infty(\Omega)$ as $t \to +\infty$. $\square$

We conclude the paper with an example showing that, in contrast with the one-dimensional case, in higher dimensions there is no instantaneous regularization of graph$(u(t))$[7].

Example 3.2. Let $N \geq 3$, $\Omega = B_1$ be the unit ball of $\mathbb{R}^N$ centered at the origin, and $\sigma := \frac{1}{N-1}$. Let $u_0(x) := 1/|x|$, $x \in \Omega \setminus \{0\}$; notice that $u_0 \in L^2(\Omega) \cap BV(\Omega)$. Let u be the solution to (1.1) given by Theorem 2.1, so that $u(t, \cdot) \in L^2(\Omega) \cap BV(\Omega)$ for all $t > 0$. Let us check that the function

$$v(t, x) := \frac{a(t)}{|x|}, \qquad t > 0, \ x \in \Omega \setminus \{0\}$$

where

$$a(t) := \max\big(1 - (N-1)\, t, 0\big), \quad t > 0,$$

is a subsolution of (2.2). For all $(t, x) \in (0, \sigma) \times (\Omega \setminus \{0\})$, a direct computation gives $|\nabla v(t, x)|^2 = \frac{a(t)^2}{|x|^4}$, and

$$\begin{aligned}
&\operatorname{div}\left(\frac{\nabla v(t,x)}{\sqrt{1 + |\nabla v(t,x)|^2}}\right) \\
&= - \, a \left(\frac{N-3}{|x|} \frac{1}{\sqrt{a(t)^2 + |x|^4}} + \frac{2}{|x|} \frac{a(t)^2}{(a(t)^2 + |x|^4)^{3/2}}\right) \\
&\geq -a \frac{N-1}{|x|} \frac{1}{\sqrt{a(t)^2 + |x|^4}}.
\end{aligned}$$

[7]In dimension one, we have proven that $z(t, \cdot)$ becomes instantaneously 1/2-Hölder continuous, and this implies that the graph of $u(t, \cdot)$ becomes instantaneously of class $\mathcal{C}^{1,1/2}$.

Hence

$$
\begin{aligned}
v_t(t,x) \;=\; -\frac{(N-1)}{|x|} \;&\leq\; -\frac{(N-1)}{|x|}\,\frac{a(t)}{\sqrt{a(t)^2+|x|^4}} \\
&\leq\; \operatorname{div}\left(\frac{\nabla v(t,x)}{\sqrt{1+|\nabla v(t,x)|^2}}\right)
\end{aligned}
$$

and therefore v is a subsolution of (2.2). By comparison principle (see [3]) it follows that $u \geq v$ almost everywhere in $(0,+\infty)\times\Omega$. As a consequence, graph$(u(t))$ is not of class $\mathcal{C}^1(\Omega)$ for $t \in [0,\sigma)$.

References

[1] L. Ambrosio, *Corso Introduttivo alla Teoria Geometrica della Misura ed alle Superfici Minime*, Edizioni della Scuola Normale, Pisa, 1997.

[2] F. Andreu, V. Caselles, J. I. Diaz and J. M. Mazón, *Some qualitative properties for the total variation flow*, J. Funct. Anal., 188 (2002), no. 2, 516-547.

[3] F. Andreu, V. Caselles, and J. M. Mazón, *Parabolic Quasilinear Equations Minimizing Linear Growth Functionals*, Oxford Mathematical Monographs, Birkhäuser, Basel, 2004.

[4] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl., 135 (1983), 293-318.

[5] G. Bellettini, *Lecture Notes on Mean Curvature Flow, Barriers and Singular Perturbations*, Edizioni della Scuola Normale, Pisa, 2013.

[6] G. Bellettini, V. Caselles and M. Novaga, *The Total Variation Flow in* $\mathbb{R}^N$, J. Differential Equations 184 (2002), 475-525.

[7] G. Bellettini, V. Caselles and M. Novaga. *Explicit solutions of the eigenvalue problem* $-\operatorname{div}\left(\frac{Du}{|Du|}\right)=u$. SIAM J. Math. Anal., 36 (2005), no. 4, 1095-1129.

[8] K. A. Brakke, *The Motion of a Surface by Its mean Curvature*, Math. Notes, Princeton Univ. Press, Princeton, N. J., 1978.

[9] H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North–Holland, 1973.

[10] J. Buckland, *Mean curvature flow with free boundary on smooth hypersurfaces*, J. Reine Angew. Math. 586 (2005), 71-90.

[11] V. Caselles, A. Chambolle and M. Novaga, *The discontinuity set of solutions of the TV denoising problem and some extensions*, Multiscale Model. Simul. 6 (2007), no. 3, 879-894.

[12] V. Caselles, A. Chambolle and M. Novaga, *Total variation in imaging*, in "Handbook of Mathematical Methods in Imaging", Springer, 1016-1057, 2011.

[13] A. Cesaroni and M. Novaga, *Long-time behavior of the mean curvature flow with periodic forcing*, Comm. Partial Differential Equations, 38 (2013), no. 5, 780-801.

[14] A. Chambolle, V. Caselles, D. Cremers, M. Novaga and T. Pock, *An introduction to Total Variation for Image Analysis*, in "Theoretical Foundations and Numerical Methods for Sparse Recovery", De Gruyter, Radon Series Comp. Appl. Math., vol. 9, 263-340, 2010.

[15] K. Ecker, *Estimates for evolutionary surfaces of prescribed mean curvature*, Math. Z. 180 (1982), no. 2, 179-192.

[16] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math., 130 (1989), no. 3, 45-471.

[17] K. Ecker and G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math., 105 (1991), no. 1, 547-569.

[18] M. Gage and R. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. 23 (1986), no. 1, 69-96.

[19] C. Gerhardt, *Evolutionary surfaces of prescribed mean curvature*, J. Differential Equations 36 (1980), 139-172.

[20] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation.* Monographs in Mathematics, 80, Boston-Basel-Stuttgart, Birkhäuser, 1984.

[21] A. Lichnewski and R. Temam, *Pseudosolutions of the time-dependent minimal surface problem*, J. Differential Equations 30 (1978), 340-364.

[22] A. Lunardi: Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Boston, 1995.

[23] U. Massari, *Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in* $\mathbb{R}^n$, Arch. Ration. Mech. Anal. 55 (1974), 357-382.

[24] I. Rudin, S. Osher and E. Fatemi, *Nonlinear total variation based noise removal argorithms*, Physica D, 60 (1992), 259-268.

[25] A. Stahl, *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differential Equations 4 (1996), no. 4, 385-407.

[26] I. Tamanini, *Boundaries of Caccioppoli sets with Hölder-continuous normal vector*, J. Reine Angew. Math., 334 (1982), 27-39.