Crystalline curvature flow of planar networks

G. Bellettini [*] , M. Chermisi [†] , M. Novaga [‡] [§]

Abstract

We consider the evolution of a polycrystalline material with three or more phases, in presence of an even crystalline anisotropy. We analyze existence, uniqueness, regularity and stability of the flow. In particular, if the flow becomes unstable at a finite time, we prove that an additional segment (or even an arc) at the triple junction may develop in order to decrease the energy and make the flow stable at subsequent times. We discuss some examples of collapsing situations that lead to changes of topology, such as the collision of two triple junctions.

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1 Introduction

Several models in phase transitions treat phenomena in which two or more phases of the same material, or the same phase of a crystal with different orientations, can coexist without mixing. A curve or a surface Γ bounding different regions is called a surface boundary, or interface, and is moving in a non equilibrium state. In some cases the motion of Γ does not depend on the physical situation in the various phases but only on its geometry, and is described by geometric equations relating, for instance, the normal velocity of the interface to its curvatures. The crystalline curvature flow in two dimensions is the formal gradient flow of the energy functional

$$\mathcal{F}_\varphi(\Gamma) := \int_\Gamma \varphi^o(\nu)\ d\mathcal{H}^1, \tag{1.1}$$

where ν is a unit normal vector field to Γ and the energy density $\varphi^o : \mathbb{R}^2 \to [0, \infty)$, called sometimes surface tension, is a crystalline (i.e. piecewise linear) norm. When φ^o is isotropic the energy functional (1.1) is proportional to the length of the interfaces and the resulting geometric parabolic equation is the curvature flow (at least in the simplest case when Γ is the boundary of an open set). However, when dealing with crystalline and polycrystalline materials, φ^o is anisotropic and neither smooth nor strictly elliptic; in addition, multi-phase boundaries with more than two phases occur.
To our knowledge, J. E. Taylor [31], [32], [33], [35] (see also [5] and [9]) was the first to introduce the

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notion of crystalline geometry and to determine the crystalline flow of curves with triple junctions, in particular to compute the motion of the triple junction. The analysis of the evolution of grain boundaries has been pursued also by other authors, see for instance [5], [6], [7], [8], [11], [12], [23], [25], [26], [28]. See also [4], [6], [11], [22] for related physical models of crystal growth, and [14], [16], [17], [19], [20], [21], [10], [16], [17], [13], [14], [24], [27], [1], [2], [29], [30] for related results.

In the present paper we consider the evolution of a polycrystalline material with three or more phases for a crystalline φ^o whose one-sublevel set $F_\varphi := \{\varphi^o \leq 1\}$ (the Frank diagram) is a regular polygon of n sides. The dual function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(\xi) := \sup\{\xi \cdot \eta : \varphi^o(\eta) \leq 1\}$ is crystalline too and $\mathcal{W}_\varphi := \{\varphi \leq 1\}$ is called the Wulff shape. We are particularly interested in the motion by crystalline curvature of special planar networks called elementary triods, namely a regular three-phase boundary given by the union of three Lipschitz curves, the interfaces, intersecting at a point called triple junction. Each interface is the union of a segment of finite length and a half-line, reproducing two consecutive sides of $\mathcal{W}_\varphi$.

We analyze local and global existence and stability of the flow. In general, the flow may become unstable at a finite time. If this occurs, we prove that at subsequent times a regular flow can be constructed, by adding a new segment (or even an arc with zero crystalline curvature) at the triple junction. In all flows we present the crystalline curvature remains bounded (even if a segment appears or disappears) and have a jump discontinuity at the time of instability only in the case of the disappearance of a segment. We also discuss some examples of collision of two triple junctions. These examples (as well as the local in time existence result) show one of the advantages of crystalline flows with respect, for instance, to the usual mean curvature flow: explicit computations can be performed to some extent, and in case of nonuniqueness, a comparison between the energies of different evolutions (difficult in the euclidean case) can be made.

The rigorous definition of crystalline curvature for networks has been introduced in [3]: we will see that the corresponding flow essentially agrees with the one suggested in [33]. Finally, we stress that Taylor already predicted the appearance of new edges from a triple junction.

The plan of the paper is the following. In Section 2.1 we present some basic definitions and results from [3], where the crystalline curvature of partitions is computed through the first variation of $\mathcal{F}_\varphi$. The crystalline curvature is the tangential divergence of a vector field $N_{\min} : \Pi \to \mathbb{R}^2$ which minimizes the functional

$$\int_\Pi (\mathrm{div}_\tau N)^2 \, \varphi^o(\nu) \; d\mathcal{H}^1, \tag{1.2}$$

among all Cahn-Hoffman vector fields N on the elementary triod Π which satisfy the so-called balance condition at the triple junction q

$$N_{|\Sigma_1}(q) + N_{|\Sigma_2}(q) + N_{|\Sigma_3}(q) = 0. \tag{1.3}$$

Such a minimizer $N_{\min}$ is *unique* (this is true, in general, only in two dimensions) and identifies the direction along which the functional $\mathcal{F}_\varphi$ decreases most quickly. The balance condition (1.3) is the analog of the Herring condition (120 degrees condition) in the euclidean case. By definition of Cahn-Hoffman vector field, $N_{|\Sigma_j}(q) \in \partial\mathcal{W}_\varphi$. Any triplet of vectors $(X, Y, Z) \in (\partial\mathcal{W}_\varphi)^3$ satisfying $X + Y + Z = 0$ is called an admissible triplet. In Section 2.2 we introduce the notions of elementary, quasi-elementary non-polygonal and degenerate triod, and of configuration of an elementary triod. The regularity of an elementary triod is related to the regularity of each interface and to the balance condition (1.3). We recall also the notion of stability [3] and introduce the concept of stability region of a configuration. In Section 2.3 we give the definition of flow by crystalline curvature starting from an elementary triod which allows to consider also initial data which may develop a new segment or an arc from the triple junction. In Section 2.4 we determine the geometry of an elementary triod, that is the three angles at the triple junction q between the interfaces. These angles are determined by the balance condition (1.3) at q, that in turn is related to the existence of admissible triplets. We prove that any regular polygon P_n, n even ≥ 6, has a unique admissible triplet (X, Y, Z) once we fix one of the vectors of the triplet, for instance X, in ∂P_n. We also determine the range of all admissible triplets of $N_{\min}$ at q and using this result we compute in Section 2.5 the crystalline curvature of the triod. Since $N_{\min}$ is unique and its values are fixed (up to a sign change) at the three vertices of the partitions, it follows that $N_{\min}$ is given on all the interfaces by linear interpolation. Thus, as shown in [3] in the case $n = 8$, it is possible to reduce the minimum problem (1.2) to a one dimensional minimum problem. In the case of a partition consisting of two adjacent triple junctions, the solution $N_{\min}$ of (1.2) is completely determined by the values of two independent real variables. Since the Cahn-Hoffman vector fields have constant normal component, the crystalline curvature is simply the tangential derivative of the tangential component of $N_{\min}$, that is a ratio of lengths. Finally, we produce which values of the lengths of the finite segments of Π provide stable triods (*stability region*). In Section 3 we prove that there exists, locally in time, a unique stable regular flow starting from a stable regular initial datum. In Section 3.1 we show a case of global existence. The analysis of the long time behavior requires the study of the stability region of each configuration. Stability is the ingredient that ensures that no additional segments develop at the triple junction during the flow. If the initial triod is unstable then an additional segment may develop in order to decrease the surface energy and make the evolved triods stable at positive times. In Section 4 we exhibit an example of this occurrence. In Sections 5-7 we show that the flow becomes unstable at a finite time $\mathcal{T}$ and that at the subsequent times a regular flow can be constructed: in particular, a new segment (resp. an arc with zero crystalline curvature) develops at the triple junction in the flow of Theorem 5.1 (resp. of Theorem 6.1). In Theorem 7.1 we prove that the flow has two different behaviors depending on the initial datum Π. For a suitable choice of Π, we show that at $t = \mathcal{T}$ one of the three segments vanishes, its crystalline curvature remains bounded, the Cahn-Hoffman vector field $N_{\min}$ has a jump discontinuity and the triple junction translates along the remaining adjacent half-line in $[\mathcal{T}, +\infty)$. For the other choices of stable Π we prove that a curve appears from the triple junction, as in Section 6, with the difference that the adjacent segment now has positive φ-curvature and keeps on moving at subsequent times. Each of these flows has the property that all crystalline curvatures remain bounded. In Section 8 we study the crystalline curvature flow starting from a *stable* φ-regular partition formed by *two* adjacent elementary triple junctions. We discuss some examples of collapsing situations that lead to changes of topology, such as for instance the collision of two triple junctions. We present several candidates to continue the flow after the singularity (see Example 8.4). In Section 9 we introduce the notion of homothetic flow. We classify homothetic flows when $\mathfrak{n} = 6\mathfrak{m}$ and we show that the global flows studied in Section 3.1 converge to homothetic flows as $t \to +\infty$.

2 Preliminaries

In this paper, $\cdot$, $|\cdot|$ and $\mathcal{H}^1$ are respectively the euclidean canonical inner product, the euclidean norm and the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Points and vectors of $\mathbb{R}^2$ will be identified. Given two points $p, q \in \mathbb{R}^2$ we denote by pq the vector with initial and end point respectively in p and q. Given two vectors $v, w \in \mathbb{R}^2$, we denote by $v^\perp$ the counterclockwise rotation of v of $\pi/2$ around the origin and by $\vartheta(v, w) \in [0, \pi]$ the angle between v and w. Given $f : (a, b) \to \mathbb{R}$ and $t \in (a, b)$, we denote by $f(t+)$ and $f(t-)$ respectively the right and left limit of f at t (if they exist).
Given a subset U of $\mathbb{R}^2$ we denote by $\mathrm{int}\,(U)$, $\overline{U}$ and ∂U, respectively the interior, the closure and the boundary of U. In particular, given a segment $S \subset \mathbb{R}^2$, we denote by $\mathrm{int}\,(S)$ the relative interior of S. Given two parallel (possibly infinite) segments S_1, S_2, we call the distance vector of S_2 from S_1 the vector having norm $\mathrm{dist}(S_1, S_2)$ pointing from S_1 to S_2.
By a Lipschitz curve with boundary in $\mathbb{R}^2$ we mean a 1-dimensional bounded set $\Sigma \subset \mathbb{R}^2$ which can be written locally as a Lipschitz graph on an open interval of $\mathbb{R}$. Any Lipschitz function or vector field defined on Σ will be considered as defined up to $\partial\Sigma$. We denote by $\mathrm{Lip}(\Sigma; \mathbb{R}^2)$ the set of all Lipschitz vector fields on Σ. Given a point $x \in \Sigma$ we denote by $\mathrm{T}_x\Sigma$ the tangent line of Σ at x.
We denote by $\mathfrak{n}, \mathfrak{m}$ positive integers and by $P_{\mathfrak{n}}$ the regular polygon of $\mathfrak{n}$ ($\mathfrak{n}$ even) sides of length L inscribed in the unit circle centered at the origin of $\mathbb{R}^2$. $P_{\mathfrak{n}}$ has two horizontal sides and is oriented in

clockwise sense.

2.1 Crystalline curvature of regular partitions of $\mathbb{R}^2$

In this section we present some basic notations and definitions from [3]. Let $\varphi : \mathbb{R}^2 \to [0, +\infty)$ be a crystalline anisotropy on $\mathbb{R}^2$ (i.e. an even piecewise linear convex function) satisfying $\mathcal{W}_\varphi = P_\mathfrak{n}$. We denote by T_φ and T_{φ^o} the multivalued mappings (duality mappings) defined as $T_\varphi(\xi) := \frac{1}{2}\partial(\varphi^2)(\xi)$, $T_{\varphi^o}(\xi^o) := \frac{1}{2}\partial((\varphi^o)^2)(\xi^o)$, for all $\xi, \xi^o \in \mathbb{R}^2$, where ∂ denotes the usual subdifferential for convex function. We observe that T_φ (resp. T_{φ^o}) is a maximal monotone operator which takes $\partial\mathcal{W}_\varphi$ (resp. ∂F_φ) onto ∂F_φ (resp. onto $\partial\mathcal{W}_\varphi$).

Definition 2.1. *Let $\Sigma \subset \mathbb{R}^2$ be a Lipschitz curve with boundary, $x \in \partial\Sigma$ and $z \in \mathbb{R}^2 \setminus T_x(\overline{\Sigma})$. We define the vector $z^{\partial\Sigma} \in \mathbb{R}^2$ as the rotation of angle $\pi/2$ of the vector z in such a way that $z^{\partial\Sigma}$ points out of Σ.*

Definition 2.2. *A partition of $\mathbb{R}^2$ is a finite family $\{E_i\}_i$ of open subsets of $\mathbb{R}^2$ (called phases) such that $\cup_i \overline{E}_i = \mathbb{R}^2$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\partial E_i \cap \partial E_j$, when it is nonempty, is a Lipschitz curve with boundary, called interface. By a m-multiple junction of $\{E_i\}$ ($m \geq 3$ a natural number) we mean a point q belonging to m distinct interfaces.*

Given a partition $\{E_i\}$ of $\mathbb{R}^2$, we set

$$\Sigma_{ij} := \partial E_i \cap \partial E_j, \ i \neq j, \qquad \Gamma := \bigcup_{i,j} \Sigma_{ij}, \qquad J := \bigcup_{i,j} \partial\Sigma_{ij}. \tag{2.1}$$

When we write Σ_{ij} we always assume that $i \neq j$ and $\Sigma_{ij} \neq \emptyset$. We denote by ν^{ij} a $\mathcal{H}^1$-a.e. defined euclidean unit normal to Σ_{ij} and we set ${\nu_\varphi}^{ij} := \nu^{ij}/\varphi^o(\nu^{ij})$. We denote by $\mathrm{Lip}_{\nu,\varphi}(\Gamma; \mathbb{R}^2)$ the space of vector fields $N : \Gamma \to \mathbb{R}^2$ such that $N_{|\Sigma_{ij}} \in \mathrm{Lip}(\Sigma_{ij}; \mathbb{R}^2)$ and $N_{|\Sigma_{ij}}(x) \in T_{\varphi^o}(\nu_\varphi^{ij}(x))$ for $\mathcal{H}^1$-almost every $x \in \Sigma_{ij}$. Set

$$\mathcal{N} := \left\{ N \in \mathrm{Lip}_{\nu,\varphi}(\Gamma, \mathbb{R}^2) \ : \ \sum_{i,j} (N_{|\Sigma_{ij}})^{\partial\Sigma_{ij}} = 0 \ \text{ on } J \right\}. \tag{2.2}$$

The condition on J in (2.2) is usually called balance condition.

Definition 2.3. *If $\mathcal{N} \neq \emptyset$, the partition $\{E_i\}$ is said to be φ-regular and any $N \in \mathcal{N}$ is called a Cahn-Hoffman vector field on Γ.*

The following definition of φ-curvature is based on [3, Theorem 4.8] and the crucial fact that we are considering *planar* partitions: if $\{E_i\}_i$ is a φ-regular partition then the minimum problem

$$\min \left\{ \left[\int_\Gamma (\mathrm{div}_\tau N)^2 \, \varphi^o(\nu) \ d\mathcal{H}^1 \right]^{1/2} \ : \ N \in \mathcal{N} \right\} \tag{2.3}$$

admits a *unique* solution which identifies the direction along which the functional (1.1) decreases most quickly. Let $N_{\min} : \Gamma \to \mathbb{R}^2$ be the solution of problem (2.3).

Definition 2.4. *Let $\{E_i\}$ be a φ-regular partition. We define the φ-curvature κ_φ of Γ as*

$$\kappa_\varphi := \mathrm{div}_\tau N_{\min}, \qquad \textit{a.e. on } \Gamma.$$

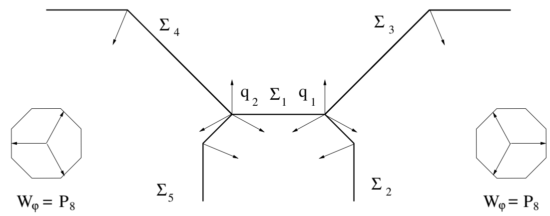

Figure 1: $\big(N_{|\Sigma_1}(q_1)^{\partial\Sigma_1}, N_{|\Sigma_2}(q_1)^{\partial\Sigma_2}, N_{|\Sigma_3}(q_1)^{\partial\Sigma_3}\big)$ and $\big(N_{|\Sigma_1}(q_2)^{\partial\Sigma_1}, N_{|\Sigma_4}(q_2)^{\partial\Sigma_4}, N_{|\Sigma_5}(q_2)^{\partial\Sigma_5}\big)$.

Remark 2.5. Let $\Sigma = \partial E$ be a simple Lipschitz curve which admits a Lipschitz Cahn-Hoffman vector field, i.e. $N \in \mathrm{Lip}(\Sigma;\mathbb{R}^2)$ with $N(x) \in T_{\varphi^o}(\nu_\varphi(x))$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$, where $\nu_\varphi := \nu/\varphi^o(\nu)$ and ν is a $\mathcal{H}^1$-a.e. defined euclidean unit normal to Σ. It is easy to see that $\kappa_\varphi = 0$ on any nonflat arc γ contained in Σ since N on γ is constantly equal to a vertex of $\partial\mathcal{W}_\varphi$. Assume now that S is an open segment of length $L > 0$ contained in Σ. Denote by N_1, N_2 respectively the values of N at the initial and final endpoint of S according to $\tau := -\nu^\perp$. The φ-curvature of S is zero if $L = +\infty$, while if $L < +\infty$,

$$\kappa_\varphi(p) = \frac{1}{L}(N_2 - N_1)\cdot\tau, \qquad p \in S. \tag{2.4}$$

Hence S has constant φ-curvature which, setting $l := L\tau$, will be denoted by $\kappa_\varphi(l)$. Notice that κ_φ in (2.4) changes sign if we change sign to ν.

2.2 Elementary, quasi-elementary, non-polygonal triods

In this section we introduce the notions of elementary, quasi-elementary and non-polygonal triod, of configuration of an elementary triod and we fix the orientation of a triod.

Definition 2.6. *When $\{E_1, E_2, E_3\}$ is a partition of $\mathbb{R}^2$ into three sets having only one 3-multiple junction, called triple junction and denoted by q, the set Γ defined in (2.1) will be called triod, and denoted by Π. If the partition is φ-regular, the triod is said to be φ-regular. For simplicity, Σ_{12}, Σ_{23}, Σ_{13} will be denoted respectively by Σ_1, Σ_2, Σ_3 and correspondingly ν_φ^{ij} will be denoted by ν_φ^j. We call angles of Π the three angles at q between Σ_1, Σ_2, Σ_3 (see Figure 5).*

Remark 2.7. The notion of regularity in Definition 2.6 is essentially the same given by J. E. Taylor in [35] when each Σ_j is polygonal. In the case of a φ-regular partition with $\Gamma = \cup_{i=1}^5 \Sigma_i$, $J = \{q_1, q_2\}$, and $N \in \mathcal{N}$ as in Figure 1, the triplet of vectors

$$\big(N_{|\Sigma_1}(q_1)^{\partial\Sigma_1}, N_{|\Sigma_2}(q_1)^{\partial\Sigma_2}, N_{|\Sigma_3}(q_1)^{\partial\Sigma_3}\big),$$

is the counterclockwise rotation of the triplet $\big(N_{|\Sigma_1}(q_1), N_{|\Sigma_2}(q_1), N_{|\Sigma_3}(q_1)\big)$, while the triplet of vectors

$$\big(N_{|\Sigma_1}(q_2)^{\partial\Sigma_1}, N_{|\Sigma_4}(q_2)^{\partial\Sigma_4}, N_{|\Sigma_5}(q_2)^{\partial\Sigma_5}\big),$$

is the clockwise rotation of the triplet $\big(N_{|\Sigma_1}(q_2), N_{|\Sigma_4}(q_2), N_{|\Sigma_5}(q_2)\big)$.

Definition 2.8. *Let $\Pi = \cup_{j=1}^3 \Sigma_j$ be a φ-regular triod. We say that Π is elementary if*

- *($\mathcal{E}$) each interface Σ_j is the union of a segment S_j of finite length $L_j > 0$ and a half-line R_j such that S_j and R_j reproduce two consecutive sides of $\mathcal{W}_\varphi$, see Figure 2 (i).*

We say that Π is degenerate if two interfaces satisfy ($\mathcal{E}$) and the remaining one $\Sigma_{\bar{k}}$ is a half-line.
We say that Π is quasi-elementary if two interfaces satisfy ($\mathcal{E}$) and the remaining one Σ_k is the union

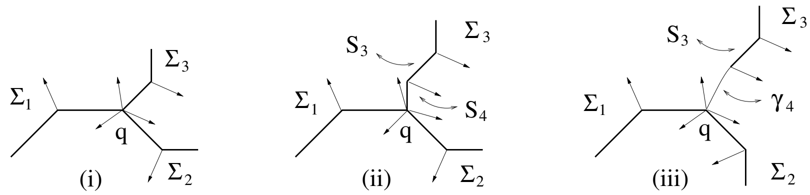

Figure 2: (i) Elementary, (ii) quasi-elementary, (iii) non-polygonal triod ($\mathcal{W}_\varphi = P_8$). Note that $\kappa_\varphi = 0$ on S_3 in (i) and (ii), $\kappa_\varphi < 0$ on S_4 in (ii), and $\kappa_\varphi = 0$ on γ_4 in (iii).

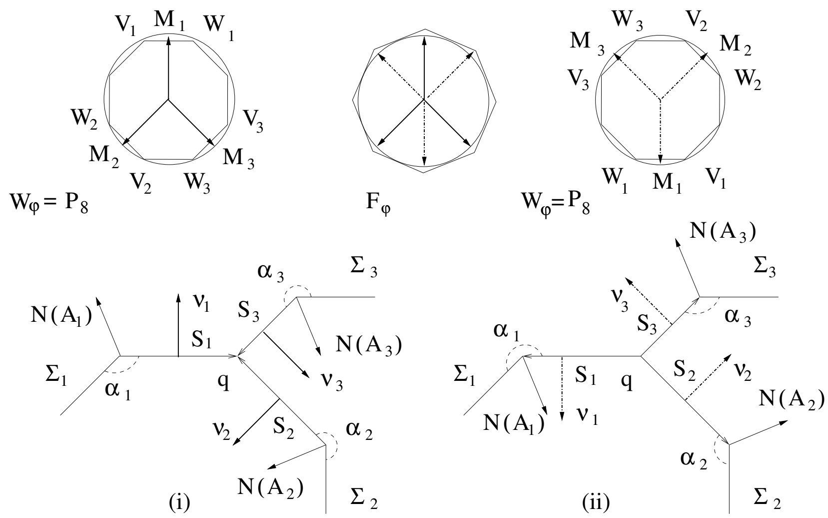

Figure 3: These triods have the same evolution according to system (2.9). Our convention is to take the orientation as in (i).

of two segments S_4 and S_k of finite lengths, $L_4 > 0$ and $L_k > 0$ respectively, and a half–line R_k such that S_4 and S_k, and S_k and R_k, reproduce two consecutive sides of $\mathcal{W}_\varphi$, see Figure 2 (ii)).
We say that Π is non-polygonal if two interfaces satisfy $(\mathcal{E})$ and the remaining one Σ_k is the union of a curve γ_4, a segment S_k of finite length $L_k > 0$ and a half–line R_k such that S_k and R_k reproduce two consecutive sides of $\mathcal{W}_\varphi$, see Figure 2 (iii).

Given either an elementary or a degenerate or a quasi-elementary or a non-polygonal triod Π and $N \in \mathcal{N}$, we set $A_j := \overline{S_j} \cap \overline{R_j}$ for any $j = 1,2,3$ such that $R_j \neq \emptyset$, $A_4 := \overline{S_4} \cap \overline{S_k}$ if Π is quasi-elementary, and $A_4 := \overline{\gamma_4} \cap \overline{S_k}$ if Π is non-polygonal. We denote by α_j the angle of Σ_j at A_j opposite to the region where $N(A_j)$ lies, see Figure 3. Notice that $\alpha_j \in \{\pi - \pi/2\mathfrak{n}, \pi + \pi/2\mathfrak{n}\}$.
Let ν be the $\mathcal{H}^1$-almost everywhere defined euclidean unit normal to Π oriented in such a way that $\nu_{|\mathrm{int}\,(S_j)} \cdot N(A_j) > 0$. We set $\nu_j := \nu_{|\mathrm{int}\,(S_j)}$, $\tau_j := -\nu_j^\perp$ and $l_j := L_j \tau_j$, for any $j = 1,2,3$, and also $j = 4$ if Π is quasi-elementary. Thus $\{\tau_j, \nu_j\}$ is a positively oriented basis of $\mathbb{R}^2$ and, without loss of generality, we assume that each l_j points towards q. We denote by $\kappa_\varphi(l_j)$ the φ-curvature of S_j.
For an elementary triod, we always assume that S_1 is horizontal and Σ_2 and Σ_3 are given in counter-clockwise sense as in Figure 3. We denote by V_j, W_j the vertices of the side of $P_\mathfrak{n}$ (in clockwise sense) having ν_j as outer normal and by M_j the middle point of the segment $[V_j, W_j]$. Note that

$$\tau_1 \cdot \nu_3 = -\tau_1 \cdot \nu_2, \quad \nu_1 \cdot \tau_3 = -\nu_1 \cdot \tau_2, \quad \tau_1 \cdot \nu_3 = -\nu_1 \cdot \tau_3. \tag{2.5}$$

Definition 2.9. *Let Π, Π' be two elementary triods. We say that Π and Π' are equivalent (or that belong to the same configuration) if they coincide, after possible rescalings of their bounded edges and after a rotation. We denote by $[\Pi]$ the configuration of Π, i.e. the equivalence class of Π, and by $\mathfrak{C}$ the set of all possible configurations for elementary triods.*

We recall also the notion of stability [3] and introduce the concept of stability region of a configuration.

Definition 2.10. *Let Π be a φ-regular triod. We say that Π is stable if $(N_{\min})_{|\Sigma_j}(q)$ is not a vertex of $\mathcal{W}_\varphi$ for any $j = 1, 2, 3$. We say that Π is unstable if it is not stable.*

It follows that non-polygonal triods are always unstable (see Remark 2.5). Elementary, degenerate and quasi-elementary triods can be either stable or unstable.

Definition 2.11. *Given a configuration $(\mathsf{e}) \in \mathfrak{C}$, the stability region of (e), denoted by $\mathcal{S}_\mathsf{e}$, is the set of all $(\Lambda_1, \Lambda_2, \Lambda_3) \in (0, +\infty)^3$ such that, if $\Pi \in (\mathsf{e})$ is an elementary triod with $|S_j| = \Lambda_j$ for any $j = 1, 2, 3$, then Π is stable. For $j_1, j_2, j_3 \in \{1, 2, 3\}$, $j_1 \neq j_2 \neq j_3 \neq j_1$, we let*

$$\mathcal{S}_\mathsf{e}(j_2, j_3) := \left\{ \left(\frac{\Lambda_{j_1}}{\Lambda_{j_2}}, \frac{\Lambda_{j_1}}{\Lambda_{j_3}} \right) \; : \; (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathcal{S}_\mathsf{e} \right\}.$$

2.3 Definition of crystalline flows of triods

Our object is to provide a definition of φ-curvature flow allowing to consider also initial data for which a new segment or a curve (with zero φ-curvature) can develop from the triple junction at time zero.

Definition 2.12. *Let $T > 0$ and Π be an elementary triod (resp. degenerate). For any $t \in [0, T)$, let $\Pi(t)$ be a φ-regular triod and $q(t)$ its triple junction. We say that $t \in [0, T) \mapsto \Pi(t)$ is a φ-curvature flow starting from $\Pi = \Pi(0)$ if for any $t \in (0, T)$*

(i) *$\Pi(t)$ is either elementary or quasi-elementary or non-polygonal (resp. degenerate);*

(ii) *for any $j = 1, 2, 3$, each $R_j(t)$ has zero normal velocity and each $S_j(t)$ is parallel to $S_j(0) = S_j$;*

(iii) *for each $j = 1, 2, 3$, and also $j = 4$ if $\Pi(t)$ is quasi-elementary, denoting by $h_j(t)$ the distance vector of the segment $S_j(t)$ from $S_j(0) = S_j$, then $h_j \in C^1((0, T); \nu_j \mathbb{R})$ and*

$$\begin{cases} \dfrac{\dot{h}_j(t)}{\varphi^o(\nu_j)} = -\kappa_\varphi(l_j(t)) \, \nu_j \\ h_j(0) = 0. \end{cases} \tag{2.6}$$

The flow is said to be stable if $\Pi(t)$ is stable for any $t \in \; (0, T)$.

Remark 2.13. Since $\varphi^o(\nu_j)$ is a constant independent of $j \in \{1, 2, 3, 4\}$, the system in (2.7) is equivalent, up to a rescaling in time, to

$$\dot{h}_j(t) = -\kappa_\varphi(l_j(t)) \, \nu_j. \tag{2.7}$$

For simplicity, we will consider (2.7) in place of (2.6).

Note that, in Definition 2.12, Π is not required to be stable (even in the definition of stable flow). Let

$$h_j^\nu(t) := h_j(t) \cdot \nu_j, \quad \text{for} \quad j = 1, 2, 3, 4. \tag{2.8}$$

Then $h_j(t) = h_j^\nu(t) \, \nu_j$ and, with this notation, system (2.7) becomes

$$\begin{cases} \dot{h}_j^\nu(t) = -\,\kappa_\varphi(l_j(t)) = -\,\dfrac{1}{L_j(t)} \left[\, {N_{\min}}_{|\Sigma_j(t)}(\, q(t) \,) - N_{\min}(\, A_j(t) \,) \, \right] \cdot \, \tau_j \\ h_j^\nu(0) = 0. \end{cases} \tag{2.9}$$

Remark 2.14. We observe that $S_j(t)$ moves in the same direction of ν_j if and only if $\kappa_\varphi(l_j(t)) < 0$. Furthermore, system (2.9) is invariant under the change of the orientation of $\Pi(t)$ (see Figure 3).

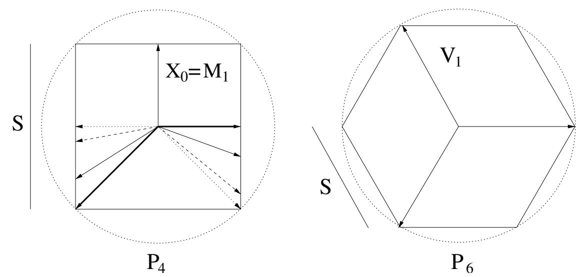

Figure 4: P_4 admits infinitely many pairs $\{Y, Z\}$ satisfying $X_0 + Y + Z = 0$ in correspondence of $X_0 = M_1$. P_6 has a unique pair in correspondence of all $X \in \partial P_6$.

2.4 Geometry of elementary triods

The angles of an elementary triod are given by the angles between the vectors ν_j's and are determined by the balance condition at q (see (2.2)) that, in turn, is related to the existence of admissible triplets.

Definition 2.15. *We call admissible triplet any triplet of vectors* $(X, Y, Z) \in (\partial \mathcal{W}_\varphi)^3$ *satisfying*

$$X + Y + Z = 0. \tag{2.10}$$

Lemma 2.16. *Let* $\psi : \mathbb{R}^2 \to [0, +\infty)$ *be a Finsler norm on* $\mathbb{R}^2$, *i.e. an even one-homogeneous convex function for which there exists* $c > 0$ *such that* $\psi(\xi) \geq c|\xi|$ *for any* $\xi \in \mathbb{R}^2$, *and define* $\mathcal{W}_\psi := \{\xi \in \mathbb{R}^2 : \ \psi(\xi) \leq 1\}$. *Let* $X \in \partial\mathcal{W}_\psi$. *Then there exist two distinct vectors* Y, Z *in* $\partial\mathcal{W}_\psi$ *such that* (X, Y, Z) *is an admissible triplet. Moreover, if either* $\mathcal{W}_\psi$ *is strictly convex or for any segment* $S \subset \partial\mathcal{W}_\psi$ *parallel to* $X \in \partial\mathcal{W}_\psi$ *we have* $|S| \leq |X|$, *then the unordered pair* $\{\, Y,\, Z \,\}$ *is unique. Finally, if there exist* $X_0 \in \partial\mathcal{W}_\psi$ *and a segment* $S \subset \partial\mathcal{W}_\psi$ *parallel to* X_0 *with* $|S| > |X_0|$, *then there are infinitely many unordered pairs* $\{\, Y,\, Z \,\}$ *of distinct vectors in* $\partial\mathcal{W}_\psi$ *such that* (X_0, Y, Z) *is an admissible triplet.*

Proof. Let $2\, h_M$ be the length of the orthogonal projection of $\mathcal{W}_\psi$ on $X^\perp\mathbb{R}$ and set $\hat{X} := X/|X|$. Define the multifunctions α_r and α_l as $\alpha_r(h) := (-h\hat{X}^\perp + X\mathbb{R}) \cap \partial\mathcal{W}_\psi$ and $\alpha_l(-h) := -\alpha_r(h)$ for any $h \in [0, h_M]$. It is easy to see that $\alpha_r(h)$ contains exactly two points for $h \neq h_M$ while $\alpha_r(h_M)$ can be either a point or a closed segment. Define the functions $\alpha_r^-, \alpha_l^- : [0, h_M] \to \mathbb{R}^2$ as $\alpha_l^-(-h) := \{Z \in \alpha_l(-h) \ : \ Z \cdot X \leq Y \cdot X,\, Y \in \alpha_l(-h)\}$ and $\alpha_r^-(h) := \{Z \in \alpha_r(h) \ : \ Z \cdot X \leq Y \cdot X,\, Y \in \alpha_r(h)\}$. Note that α_l^- and α_r^- are local parametrizations of $\partial\mathcal{W}_\psi$ which can be written with respect to the basis $(-\hat{X}^\perp, \hat{X})$ as $\alpha_l^-(-h) = (-h, \alpha_l^-(-h) \cdot \hat{X})$ and $\alpha_r^-(h) = (h, \alpha_r^-(h) \cdot \hat{X})$. Define now the function $\Phi : [0, h_M] \to \mathbb{R}$ as $\Phi(h) := \frac{1}{|X|}[\alpha_l^-(-h) + \alpha_r^+(h)] \cdot \hat{X}$. Then Φ is convex, since so are $h \mapsto \alpha_l^-(-h) \cdot \hat{X}$, $h \mapsto \alpha_r^-(h) \cdot \hat{X}$. Furthermore, $\Phi(0) = -2$, $\Phi(h_M) = 0$ if and only if $\alpha_r(h_M)$ is a singleton, while $\Phi(h_M) < 0$ if $\alpha_r(h_M)$ is a proper segment.
We divide the proof into two cases. First we observe that the existence of $h_* \in (0, h_M]$ with $\Phi(h_*) = -1$ implies that (2.10) is satisfied by choosing $Y := \alpha_r(h_*)$ and $Z := \alpha_l(-h_*)$ and, conversely, the existence of X, $Y \in \partial\mathcal{W}_\psi$ satisfying (2.10) implies that $\Phi(h_*) = -1$, where $h_* := \max\{Y \cdot \hat{X}, Z \cdot \hat{X}\}$.
Case 1: if either $\mathcal{W}_\psi$ is strictly convex (i.e. $\alpha_r(h_M)$ is a singleton) or $\alpha_r(h_M) \subset \partial\mathcal{W}_\psi$ is a segment parallel to X with length $|\alpha_r(h_M)| \leq |X|$, then $\Phi(h_M) \geq -1$ with the equality holding if and only if $|S| = |X|$ (for instance if $\mathcal{W}_\psi = P_6$, see Figure 4). From the convexity of Φ, the existence of $h_* \in (0, h_M]$ with $\Phi(h_*) = -1$ follows. Assume now that there exists $h^* \in (h_*, h_M]$ satisfying $\Phi(h^*) = \Phi(h_*) = -1$. Then, by the convexity of Φ, for every $\lambda \in (0, 1)$ we must have $\Phi\left((1 - \lambda)h_* + \lambda h^*\right) = -1$, that is $\partial\mathcal{W}_\psi$ should be flat along the direction $X^\perp$, but this contradicts the convexity of $\mathcal{W}_\psi$.

Case 2: if $\alpha_r(h_M) \subset \partial\mathcal{W}_\psi$ is a segment parallel to X with length strictly greater than $|X|$, then $\Phi(h_M) < -1$ (for instance if $\mathcal{W}_\psi = P_4$, see Figure 4). Thus, we can find infinitely many pairs $\{Y, Z\}$ (as many as the points of a segment of length $|X|$) of distinct vectors in $\partial\mathcal{W}_\psi$ satisfying (2.10). □

Remark 2.17. If $\mathcal{W}_\psi = P_4$ and $X_0 = M_1$ (see Figure 4), then $|S| = 2|X_0|$; hence there are infinitely many pairs $\{Y, Z\}$ of distinct vectors in ∂P_4 satisfying $X_0 + Y + Z = 0$. Moreover, any elementary triod has always two angles of $\pi/2$. If $\mathcal{W}_\psi = P_6$ and $X = V_1$ (see Figure 4), then $|S| = |V_1|$; hence for any $X \in \mathcal{W}_\psi$ there exists a unique unordered pair $\{Y, Z\}$ satisfying (2.10).

Corollary 2.18. *Let* $\mathtt{n} \geq 6$. *For any* $X \in [V_1, W_1]$ *there exist unique* $Y = Y(X) \in [V_2, W_2]$ *and* $Z = Z(X) \in [V_3, W_3]$ *such that* (X, Y, Z) *is an admissible triplet.*

A direct computation yields the following result.

Proposition 2.19. *Let* $\mathtt{n} \in \mathbb{N}, \mathtt{n} \geq 6$, $j = 2, 3$ *and* Π *be elementary. Then*

$$\vartheta(\nu_1, \nu_j) = \vartheta_{\mathtt{n}} := \begin{cases} 2\pi/3 & \mathtt{n} = 6\mathtt{m},\ \mathtt{m} \geq 1 \\ 2\pi/3\,(1 + 1/\mathtt{n}) & \mathtt{n} = 6\mathtt{m} - 4,\ \mathtt{m} \geq 2 \\ 2\pi/3\,(1 - 1/\mathtt{n}) & \mathtt{n} = 6\mathtt{m} - 2,\ \mathtt{m} \geq 2. \end{cases} \tag{2.11}$$

Moreover, the cardinality of $\mathfrak{C}$ *in Definition 2.9 is* 4 *if* $\mathtt{n} = 6\mathtt{m}$ *and* 8 *if* $\mathtt{n} \in \{6\mathtt{m} - 4, 6\mathtt{m} - 2\}$.

The angles of Π are strictly greater than $\pi/2$ and strictly less than π when $\mathtt{n} \geq 6$ and $\mathtt{n} \neq 8$. If $\mathtt{n} = 8$ then $\vartheta(\nu_2, \nu_3) = \pi/2$. From Proposition 2.19, when $\mathtt{n} \in \{6\mathtt{m} - 4, 6\mathtt{m} - 2\}$, there are eight different configurations which will be denoted by (a), (b), (c), (d), (a′), (b′), (c′), (d′), see Figure 5; when $\mathtt{n} = 6\mathtt{m}$, the four different configurations are the one corresponding to (a), (d), (a′), (d′).
From Proposition 2.19 we deduce the following formulas which are used throughout the paper:

$$\tau_1 \cdot \tau_j = \nu_1 \cdot \nu_j = \cos\vartheta_{\mathtt{n}}, \quad j = 2, 3, \tag{2.12}$$
$$\nu_1 \cdot \tau_2 = \tau_1 \cdot \nu_3 = \cos(\vartheta_{\mathtt{n}} - \pi/2) = \sin\vartheta_{\mathtt{n}}, \tag{2.13}$$
$$\tau_1 \cdot \nu_2 = \nu_1 \cdot \tau_3 = \cos(\vartheta_{\mathtt{n}} + \pi/2) = -\sin\vartheta_{\mathtt{n}}. \tag{2.14}$$

Remark 2.20 (quasi-elementary and non-polygonal triods). The angles of a quasi-elementary triod $\widetilde{\Pi}$ are still determined by the balance condition at q (see (2.2)) and are exactly equal to ϑ_n, ϑ_n and $2\pi - 2\vartheta_n$, as in the case of an elementary triod. The notion of local configuration of $\widetilde{\Pi}$ at q can be introduced by considering the equivalence relation introduced in Definition 2.9 on

$$(S_4 \cup S_k) \cup \Sigma_{j_1} \cup \Sigma_{j_2}, \qquad j_1, j_2 \in \{1, 2, 3\} \setminus \{k\},\ j_1 \neq j_2,$$

with k as in Definition 2.8 of quasi-elementary triod. The different local configurations of $\widetilde{\Pi}$ in q will be denoted by (a*), (b*), (c*), (d*), (a′*), (b′*), (c′*), (d′*). For non-polygonal triods, only the angle between the interfaces Σ_{j_1} and Σ_{j_2}, $j_1, j_2 \in \{1, 2, 3\} \setminus \{k\}$, $j_1 \neq j_2$, with k as in Definition 2.8 of non-polygonal triod, is known and equal to ϑ_n.

We set $\delta := |V_1 - X(V_3)|$ if $\mathtt{n} = 6\mathtt{m} - 4$, $\delta := |W_1 - X(W_3)|$ if $\mathtt{n} = 6\mathtt{m} - 2$ (see Figure 6),

$$[a, b] := \begin{cases} [0, L] & \mathtt{n} = 6\mathtt{m} \\ [\delta, L - \delta] & \mathtt{n} = 6\mathtt{m} - 4, 6\mathtt{m} - 2, \end{cases} \qquad m := \begin{cases} 1 & \mathtt{n} = 6\mathtt{m} \\ \delta/(L - 2\delta) & \mathtt{n} = 6\mathtt{m} - 4, 6\mathtt{m} - 2, \end{cases} \tag{2.15}$$

and

$$q_y := \begin{cases} L & \mathtt{n} = 6\mathtt{m} \\ m(L - \delta) & \mathtt{n} = 6\mathtt{m} - 4 \\ L + m\delta & \mathtt{n} = 6\mathtt{m} - 2 \end{cases} \qquad q_z := \begin{cases} 0 & \mathtt{n} = 6\mathtt{m} \\ -m\delta & \mathtt{n} = 6\mathtt{m} - 4 \\ L - m(L - \delta) & \mathtt{n} = 6\mathtt{m} - 2. \end{cases} \tag{2.16}$$

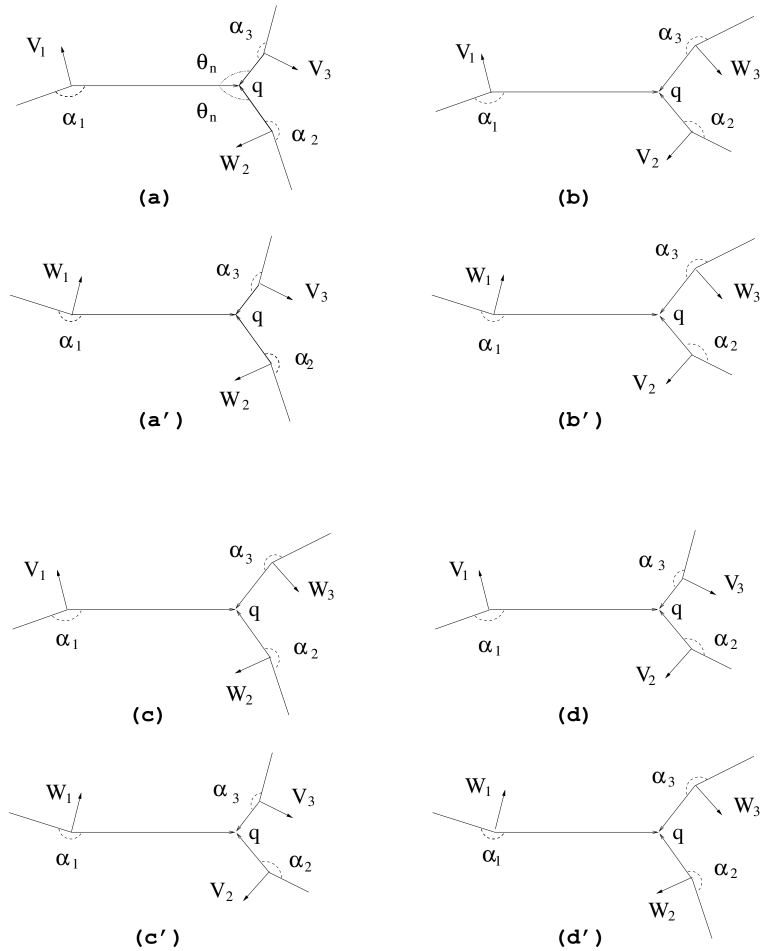

Figure 5: Eight different configurations (up to rotations of $2\pi/\mathfrak{n}$) when $\mathfrak{n} \in \{6\mathfrak{m}-4, 6\mathfrak{m}-2\}$.

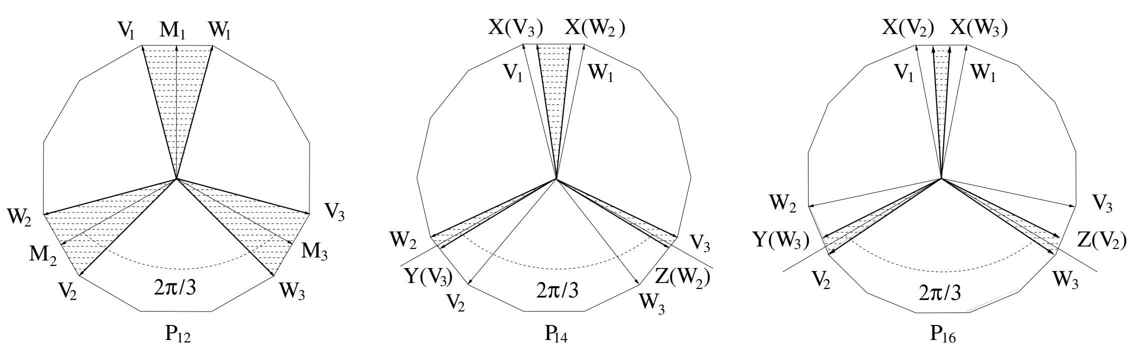

Figure 6: Wulff shapes $\mathcal{W}_\varphi = P_{12}$, $\mathcal{W}_\varphi = P_{14}$ ($\mathfrak{n} = 6\mathfrak{m}-4$) and $\mathcal{W}_\varphi = P_{16}$ ($\mathfrak{n} = 6\mathfrak{m}-2$) and relative regions of ranging of all admissible triplets (filled regions).

Given an admissible triplet $(X, Y, Z) \in [V_1, W_1] \times [V_2, W_2] \times [V_3, W_3]$, we set

$$x := |V_1 - X|, \quad y := |W_2 - Y|, \quad z := |V_3 - Z|. \tag{2.17}$$

The proof of the next result is omitted and follows by a direct computation.

Proposition 2.21. *If* $\mathfrak{n} = 6\mathfrak{m} - 4$ *then* $\delta = |W_1 - X(W_2)| = |W_2 - Y(V_3)| = |V_3 - Z(W_2)|$. *If* $\mathfrak{n} = 6\mathfrak{m} - 2$ *then* $\delta = |V_1 - X(V_2)| = |V_2 - Y(W_3)| = |W_3 - Z(V_2)|$. *Furthermore,*

$$2\delta = \left(1 - \cos\vartheta_{\mathfrak{n}}\right)^{-1} L, \qquad m = -(2\cos\vartheta_{\mathfrak{n}})^{-1}, \qquad \mathfrak{n} \in \{6\mathfrak{m} - 4, 6\mathfrak{m} - 2\}. \tag{2.18}$$

Finally

$$y = y(x) := -mx + q_y, \quad z = z(x) := mx + q_z, \quad x \in [a, b], \quad \mathfrak{n} \geq 6. \tag{2.19}$$

2.5 Crystalline curvature of elementary triods

In this section we compute the φ-curvatures $\kappa_\varphi(l_1)$, $\kappa_\varphi(l_2)$ and $\kappa_\varphi(l_3)$ (see Definition 2.4) of an elementary triod Π. Each configuration gives rise to a different vector field $N_{\min} : \Pi \to \mathbb{R}^2$. Since in two dimensions the value of $N_{\min}$ are fixed (up to a sign change) at each vertex A_j, the value of $N_{\min|\Sigma_j}$ at q uniquely determines $N_{\min}$ on Σ_j simply by linear interpolation. Hence, we can restrict the minimum problem (2.3) to the class of vector fields $N \in \mathcal{N}$ which are given by linear interpolation on each Σ_j. From Proposition 2.21, the admissible triplet $(N_{|\Sigma_1}(q), N_{|\Sigma_2}(q), N_{|\Sigma_3}(q))$ is uniquely associated with $(x, y(x), z(x))$ satisfying (2.19). Hence, we can rewrite the functional in (2.3) as a function of x. The problem of finding $N_{\min}$ in (2.3) reduces to the problem

$$\min_{x \in [a,b]} f(x), \qquad f(x) := \int_\Pi (\mathrm{div}_\tau N)^2 \, \varphi^o(\nu)\, d\mathcal{H}^1 = \alpha x^2 + \beta x + \gamma, \tag{2.20}$$

where α, β, γ are coefficients depending on the configuration of Π.
Let $x_{\min}$ be the minimizer of (2.20), $y_{\min} := y(x_{\min})$ and $z_{\min} := z(x_{\min})$. The stability of an elementary triod is equivalent to the condition

$$x_{\min} \in (a, b).$$

Proposition 2.22. *If* $\Pi \in (\mathtt{d})$ *then* $x_{\min} = a$, *where* a *is defined as in* (2.15). *If* $\Pi \in \{(\mathtt{a}), (\mathtt{b}), (\mathtt{c})\}$ *is stable then the expression of* $x_{\min}$ *is*

$$x^{(\mathtt{a})}_{\min}(L_1, L_2, L_3) = m\left(\frac{q_y}{L_2} - \frac{q_z}{L_3}\right)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right]^{-1}, \tag{2.21}$$

$$x^{(\mathtt{b})}_{\min}(L_1, L_2, L_3) = m\left(-\frac{L - q_y}{L_2} + \frac{L - q_z}{L_3}\right)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right]^{-1}, \tag{2.22}$$

$$x^{(\mathtt{c})}_{\min}(L_1, L_2, L_3) = m\left(\frac{q_y}{L_2} + \frac{L - q_z}{L_3}\right)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right]^{-1}, \tag{2.23}$$

where m, q_y, q_z *are given by* (2.15) *and* (2.16).

Proof. Let $N \in \mathcal{N}$ be given by linear interpolation on each Σ_j, $(X, Y, Z) := (N_{|\Sigma_1}(q), N_{|\Sigma_2}(q), N_{|\Sigma_3}(q))$ and x, y, z be as in (2.17). We observe that $\mathrm{div}_\tau N_{|_{\Sigma_1}}$, $\mathrm{div}_\tau N_{|_{\Sigma_2}}$, $\mathrm{div}_\tau N_{|_{\Sigma_3}}$ are constant and given as in Table 1 after replacing $x_{\min}$, $y_{\min}$, $z_{\min}$ with x, y, z, and

$$f(x) = \left(\mathrm{div}_\tau N_{|_{\Sigma_1}}\right)^2 L_1 \varphi^o(\nu_1) + \left(\mathrm{div}_\tau N_{|_{\Sigma_2}}\right)^2 L_2 \varphi^o(\nu_2) + \left(\mathrm{div}_\tau N_{|_{\Sigma_3}}\right)^2 L_3 \varphi^o(\nu_3).$$

Furthermore, $\varphi^o(\nu_1) = \varphi^o(\nu_2) = \varphi^o(\nu_3)$. In the case of configuration (d), being $f(x)$ an increasing function of $x \in [a,b]$, it follows that the minimizer is given by $x_{\min} = a$ and the first assertion follows. In the other cases, since $\alpha = \varphi^o(\nu_1)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right] > 0$, it follows that $x_{\min} = -\dfrac{\beta}{2\alpha} \in (a,b)$. Formulas (2.21)-(2.23) follow since $\beta = -2m\varphi^o(\nu_1)\left(\frac{q_y}{L_2} - \frac{q_z}{L_3}\right)$ in configuration (a), $\beta = -2m\varphi^o(\nu_1)\left(-\frac{L-q_y}{L_2} + \frac{L-q_z}{L_3}\right)$ in configuration (b) and $\beta = -2m\varphi^o(\nu_1)\left(\frac{q_y}{L_2} + \frac{L-q_z}{L_3}\right)$ in configuration (c). □

Remark 2.23. Since (a′), (b′), (c′) and (d′) are respectively symmetric to (a), (b), (c) and (d) with respect to the l_1-axis, we can derive the expression of $x_{\min}$ for the configurations (a′), (b′) (c′) and (d′) from those of (a), (b), (c) and (d) using the mirror law:

$$x^{(\mathtt{e})}_{\min}(L_1, L_2, L_3) \in [a,b] \mapsto x^{(\mathtt{e}')}_{\min}(L_1, L_2, L_3) \equiv L - x^{(\mathtt{e})}_{\min}(L_1, L_3, L_2) \in [a,b]. \tag{2.24}$$

Since $(x_{\min}, y_{\min}, z_{\min})$ identifies $N_{\min}$ at q, $\kappa_\varphi(l_j)$ is explicitly determined for each configuration, as shown in Table 1.

	α_1/π	α_2/π	α_3/π	$\kappa_\varphi(l_1)$	$\kappa_\varphi(l_2)$	$\kappa_\varphi(l_3)$
(a)	$1-2/\mathtt{n}$	$1+2/\mathtt{n}$	$1-2/\mathtt{n}$	$x_{\min}/L_1$	$-y_{\min}/L_2$	$z_{\min}/L_3$
(b)	$1-2/\mathtt{n}$	$1-2/\mathtt{n}$	$1+2/\mathtt{n}$	$x_{\min}/L_1$	$(L-y_{\min})/L_2$	$-(L-z_{\min})/L_3$
(c)	$1-2/\mathtt{n}$	$1+2/\mathtt{n}$	$1+2/\mathtt{n}$	$x_{\min}/L_1$	$-y_{\min}/L_2$	$-(L-z_{\min})/L_3$
(d)	$1-2/\mathtt{n}$	$1-2/\mathtt{n}$	$1-2/\mathtt{n}$	$a/L_1,\ a = x_{\min}$	$(L-y_{\min})/L_2$	$z_{\min}/L_3$
(a′)	$1+2/\mathtt{n}$	$1+2/\mathtt{n}$	$1-2/\mathtt{n}$	$-(L-x_{\min})/L_1$	$-y_{\min}/L_2$	$z_{\min}/L_3$
(b′)	$1+2/\mathtt{n}$	$1-2/\mathtt{n}$	$1+2/\mathtt{n}$	$-(L-x_{\min})/L_1$	$(L-y_{\min})/L_2$	$-(L-z_{\min})/L_3$
(c′)	$1+2/\mathtt{n}$	$1-2/\mathtt{n}$	$1-2/\mathtt{n}$	$-(L-x_{\min})/L_1$	$(L-y_{\min})/L_2$	$z_{\min}/L_3$
(d′)	$1+2/\mathtt{n}$	$1+2/\mathtt{n}$	$1+2/\mathtt{n}$	$-(L-b)/L_1,\ b = x_{\min}$	$-y_{\min}/L_2$	$-(L-z_{\min})/L_3$

Table 1: Angles α_j and φ-curvatures $\kappa_\varphi(l_j)$ of an elementary triod.

Remark 2.24. When $\mathtt{n} = 6\mathtt{m}$, Π is unstable if and only if

$$\left(N_{\min|\Sigma_1}(q), N_{\min|\Sigma_2}(q), N_{\min|\Sigma_3}(q)\right) \in \{(V_1,V_2,V_3),\ (W_1,W_2,W_3)\}, \text{ i.e. } x_{\min} \in \{0, L\},$$

see Figure 6. When $\mathtt{n} = 6\mathtt{m}-4$ (resp. $\mathtt{n} = 6\mathtt{m}-2$), Π is unstable if and only if either $N_{\min|\Sigma_3}(q) = V_3$ or $N_{\min|\Sigma_2}(q) = W_2$ (resp. either $N_{\min|\Sigma_2}(q) = V_2$ or $N_{\min|\Sigma_3}(q) = W_3$), i.e. either $x_{\min} = \delta$ or $x_{\min} = L - \delta$, see Figure 6. In particular, configurations (d) and (d′) are always unstable.

Remark 2.25. If $\widetilde{\Pi}$ is a quasi-elementary triod, the solution $N_{\min}$ of the minimum problem (2.3) with Γ replaced by $\widetilde{\Pi}$ is still determined as solution of the minimum problem (2.20) with Π replaced by $\widetilde{\Pi}$. Hence, formulas in Table 1 and in Proposition 2.22 still hold with (a), (b), (c), (d) replaced by (a*), (b*), (c*), (d*) and with l_1, l_2, l_3 suitably replaced by l_{j_1}, l_{j_2}, l_4, $j_1, j_2 \in \{1,2,3\} \setminus \{k\}$, k as in Definition 2.8 of quasi-elementary triod.

3 Short time existence and uniqueness of a crystalline flow

Before proving the short time existence result (Theorem 3.3) we need to understand the relations between the distances $h_j^\nu(t)$ (see (2.8) and Definition 2.12) and the lengths $L_j(t)$ (see Definition 2.8), in order to write the left hand side of system (2.9) as function of $\dot{L}_k(t)$.

Proposition 3.1. *Let Π be an elementary triod and $T > 0$. Assume that $t \in [0,T) \mapsto \Pi(t) \in [\Pi]$ is a flow starting from Π which satisfies (ii) of Definition 2.12 and that, denoting by $h_j(t)$ the distance vector of the segment $S_j(t)$ from $S_j(0) = S_j$, $h_j \in C^0((0,T); \nu_j\mathbb{R})$. Then, defining $h_j^\nu(t)$ as in (2.8),*

$$\begin{cases} L_1(t) = L_1 + \mathrm{cotg}\alpha_1 h_1^\nu(t) + \dfrac{h_k^\nu(t) - \nu_1 \cdot \nu_k h_1^\nu(t)}{\tau_1 \cdot \nu_k}, \quad k = 2,3, \\ L_2(t) = L_2 + \mathrm{cotg}\alpha_2 h_2^\nu(t) + \dfrac{h_1^\nu(t) - \nu_1 \cdot \nu_2 h_2^\nu(t)}{\nu_1 \cdot \tau_2} \\ L_3(t) = L_3 + \mathrm{cotg}\alpha_3 h_3^\nu(t) + \dfrac{h_1^\nu(t) - \nu_1 \cdot \nu_3 h_3^\nu(t)}{\nu_1 \cdot \tau_3}, \end{cases} \tag{3.1}$$

$$h_2^\nu(t) + h_3^\nu(t) = 2\,\tau_1 \cdot \tau_3\, h_1^\nu(t), \tag{3.2}$$

and

$$\mathrm{rank} \begin{pmatrix} \mathrm{cotg}\alpha_1 - \dfrac{\nu_1 \cdot \nu_2}{\tau_1 \cdot \nu_2} & \dfrac{1}{\tau_1 \cdot \nu_2} & L_1(t) - L_1 \\ \dfrac{1}{\nu_1 \cdot \tau_2} & \mathrm{cotg}\alpha_2 - \dfrac{\nu_1 \cdot \nu_2}{\nu_1 \cdot \tau_2} & L_2(t) - L_2 \\ 2\tau_1 \cdot \tau_3 \mathrm{cotg}\alpha_3 + \dfrac{1 - 2(\nu_1 \cdot \nu_3)^2}{\nu_1 \cdot \tau_3} & -\mathrm{cotg}\alpha_3 + \dfrac{\nu_1 \cdot \nu_3}{\nu_1 \cdot \tau_3} & L_3(t) - L_3 \end{pmatrix} = 2. \tag{3.3}$$

Conversely, for any $j = 1,2,3$, let $L_j : [0,T) \to (0,+\infty)$, $h_j^\nu : [0,T) \to \mathbb{R}$ be continuous functions satisfying $L_j(0) = L_j$, $h_j^\nu(0) = 0$, (3.1), (3.2), and (3.3). If $\Pi(t) \in [\Pi]$ is the elementary triod having $|S_j(t)| = L_j(t)$ and $h_j(t) := h_j^\nu(t)\nu_j$, then $t \in [0,T) \mapsto \Pi(t)$ is a flow starting from Π which satisfies (ii) of Definition 2.12 and $h_j(t)$ is the distance vector of $S_j(t)$ from $S_j(0) = S_j$.

Proof. From

$$qq(t) \;=\; qq(t) \cdot \nu_j\, \nu_j + qq(t) \cdot \tau_j\, \tau_j \;=\; h_j^\nu(t)\nu_j + [L_j(t) - L_j - \mathrm{cotg}\alpha_j\, h_j^\nu(t)]\,\tau_j, \tag{3.4}$$

we get, for $j = 2,3$,

$$h_1^\nu(t) \;=\; \nu_1 \cdot \nu_j h_j^\nu(t) + \nu_1 \cdot \tau_j\, [L_j(t) - L_j - \mathrm{cotg}\alpha_j h_j^\nu(t)], \tag{3.5}$$

$$L_1(t) \;=\; L_1 + \mathrm{cotg}\alpha_1 h_1^\nu(t) + \tau_1 \cdot \nu_j h_j^\nu(t) + \tau_1 \cdot \tau_j\, [L_j(t) - L_j - \mathrm{cotg}\alpha_j h_j^\nu(t)], \tag{3.6}$$

Thus, by (3.5)

$$L_j(t) - L_j - \mathrm{cotg}\alpha_j h_j^\nu(t) = \frac{h_1^\nu(t) - \nu_1 \cdot \nu_j h_j^\nu(t)}{\nu_1 \cdot \tau_j}, \quad j = 2,3, \tag{3.7}$$

and the second and the third equalities in (3.1) are proved.
Inserting (3.7) in (3.6), subtracting the resulting equations and using (2.12) and (2.5) yields

$$\tau_1 \cdot \nu_3\, [h_2^\nu(t) + h_3^\nu(t)] + 2\frac{\tau_1 \cdot \tau_3}{\nu_1 \cdot \tau_3} h_1^\nu(t) - \frac{(\nu_1 \cdot \nu_3)^2}{\nu_1 \cdot \tau_3}\, [h_2^\nu(t) + h_3^\nu(t)] = 0.$$

Hence, by (2.5),

$$\left(\frac{(\tau_1 \cdot \nu_3)^2 + (\nu_1 \cdot \nu_3)^2}{\tau_1 \cdot \nu_3} \right) [h_2^\nu(t) + h_3^\nu(t)] - 2\frac{\tau_1 \cdot \tau_3}{\tau_1 \cdot \nu_3} h_1^\nu(t) = 0,$$

which proves (3.2).
Similarly, inserting (3.7) in (3.6), adding the resulting equations, and using (2.12) and (2.5) yields

$$\begin{aligned} 2L_1(t) &= 2L_1 + 2\mathrm{cotg}\alpha_1 h_1^\nu(t) + \tau_1\cdot\nu_3\,[h_3^\nu(t) - h_2^\nu(t)] - \frac{(\nu_1\cdot\nu_3)^2}{\nu_1\cdot\tau_3}\,[h_3^\nu(t) - h_2^\nu(t)] \\ &= 2L_1 + 2\mathrm{cotg}\alpha_1 h_1^\nu(t) + \frac{1}{\tau_1\cdot\nu_3}\,[h_3^\nu(t) - h_2^\nu(t)]. \end{aligned} \tag{3.8}$$

Substituting (3.2) into (3.8) gives the first equality in (3.1). Now, system (3.1) and (3.2) imply

$$\begin{pmatrix} L_1(t) - L_1 \\ L_2(t) - L_2 \\ L_3(t) - L_3 \end{pmatrix} = \begin{pmatrix} \mathrm{cotg}\alpha_1 - \dfrac{\nu_1\cdot\nu_2}{\tau_1\cdot\nu_2} & \dfrac{1}{\tau_1\cdot\nu_2} & 0 \\ \dfrac{1}{\nu_1\cdot\tau_2} & \mathrm{cotg}\alpha_2 - \dfrac{\nu_1\cdot\nu_2}{\nu_1\cdot\tau_2} & 0 \\ 2\tau_1\cdot\tau_3\mathrm{cotg}\alpha_3 + \dfrac{1-2(\nu_1\cdot\nu_3)^2}{\nu_1\cdot\tau_3} & -\mathrm{cotg}\alpha_3 + \dfrac{\nu_1\cdot\nu_3}{\nu_1\cdot\tau_3} & 0 \end{pmatrix} \begin{pmatrix} h_1^\nu(t) \\ h_2^\nu(t) \\ h_3^\nu(t) \end{pmatrix}. \tag{3.9}$$

If we show that system (3.9) has always rank 2 then, being h_j^ν solutions of (3.1)-(3.2), the condition (3.3) follows. Let $A_{1,2}$ be the (2×2) matrix given by the first two rows and the first two columns of the matrix in (3.9). If $\mathrm{cotg}\alpha_1 = \mathrm{cotg}\alpha_2$, i.e. $\Pi \in \{(\mathsf{b}),(\mathsf{d}),(\mathsf{a}'),(\mathsf{d}')\}$, then $\det A_{1,2} = \mathrm{cotg}^2\alpha_1 + 1 \neq 0$. If $\mathrm{cotg}\alpha_1 = -\mathrm{cotg}\alpha_2$, i.e. $\Pi \in \{(\mathsf{a}),(\mathsf{c}),(\mathsf{b}'),(\mathsf{c}')\}$, then $\det A_{1,2} = \dfrac{1}{\sin^2\vartheta_{\mathfrak{n}}}\left(1 - \dfrac{\sin^2(\alpha_1+\vartheta_{\mathfrak{n}})}{\sin^2\alpha_1}\right)$. Using Table 1, one checks that $\det A_{1,2} \neq 0$ if either $\Pi \in \{(\mathsf{a}),(\mathsf{c})\}$ and $\mathfrak{n} \neq 6$ or $\Pi \in \{(\mathsf{b}'),(\mathsf{c}')\}$ and $\mathfrak{n} \notin \{10,12,14\}$. In the remaining cases one can check similarly that the minors given by the first and the third row (or the second and the third row) of the matrix in (3.9) are not zero. Then (3.3) follows. The converse assertion of the proposition follows by construction. □

Remark 3.2. System (3.1) is not symmetric under permutations of the indices $1,2,3$, unless $\mathfrak{n} = 6\mathfrak{m}$. This is due to the fact that for $\mathfrak{n} \neq 6\mathfrak{m}$ only two of the angles at the triple junction are equal. Finally, notice that a flow starting from Π which satisfies (ii) of Definition 2.12 with $\Pi(t)$ elementary in $[0,T)$ has two degrees of freedom.

Theorem 3.3. *Let Π be elementary and stable. Then there exist $T>0$ and a unique stable φ-curvature flow $t\in[0,T)\mapsto\Pi(t)$ starting from Π with $[\Pi(t)] = [\Pi]$ for any $t\in[0,T)$. Moreover, $h_j \in C^\infty([0,T))$ for all $j=1,2,3$.*

Proof. We assume $\Pi \in (\mathsf{a})$, the proof in the other configurations being similar. Let $x^{(\mathsf{a})}_{\min}$ be defined as in (2.21). For any $w := (w_1,w_2,w_3) \in (0,+\infty)^3$, we set $G(w) := x^{(\mathsf{a})}_{\min}(w)$ and define the vector field $F = (F_1,F_2,F_3) : (0,+\infty)^3 \to \mathbb{R}^3$ as

$$\begin{aligned} F_1(w) &:= -\left(\mathrm{cotg}\alpha_1 - \frac{\nu_1\cdot\nu_3}{\tau_1\cdot\nu_3}\right)\frac{G(w)}{w_1} - \frac{1}{\tau_1\cdot\nu_3}\,\frac{mG(w)+q_z}{w_3}, \\ F_2(w) &:= -\frac{1}{\nu_1\cdot\tau_2}\,\frac{G(w)}{w_1} + \left(\mathrm{cotg}\alpha_2 - \frac{\nu_1\cdot\nu_2}{\nu_1\cdot\tau_2}\right)\frac{-mG(w)+q_y}{w_2} \\ F_3(w) &:= -\frac{1}{\nu_1\cdot\tau_3}\,\frac{G(w)}{w_1} - \left(\mathrm{cotg}\alpha_3 - \frac{\nu_1\cdot\nu_3}{\nu_1\cdot\tau_3}\right)\frac{mG(w)+q_z}{w_3}. \end{aligned}$$

Notice that F is obtained by differentiating with respect to t the right hand side of system (3.1) and by replacing the $\dot{h}_j^\nu$'s (in (2.9)) with the expressions in G and w_j (where we used Table 1, (2.21) and (2.19)). Consider the Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)) \\ w(0) = (L_1,L_2,L_3) \in (0,+\infty)^3. \end{cases} \tag{3.10}$$

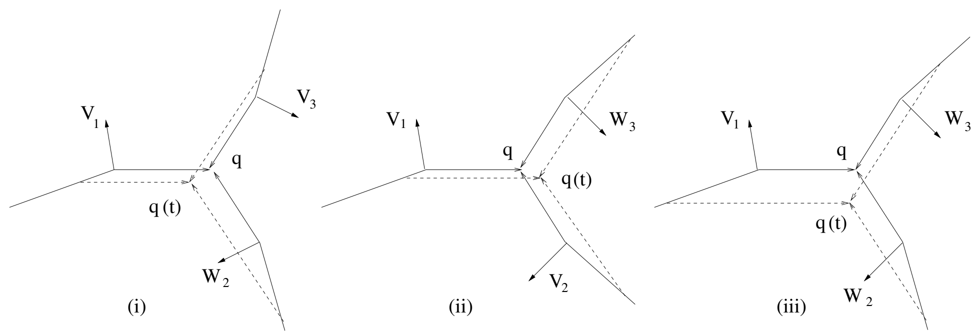

Figure 7: Stable φ-curvature flow from a stable elementary triod Π (see Theorem 3.3): (i) $\Pi \in$ (a), (ii) $\Pi \in$ (b), (iii) $\Pi \in$ (c).

Since F is C^∞ in $(0,+\infty)^3$, there exists a unique solution $w \in C^\infty([0,T);(0,+\infty)^3)$ of (3.10) for some $T>0$. Denote by $\Pi(t)$ the elementary triod belonging to configuration (a) and having $|S_j(t)| := w_j(t)$ for any $j=1,2,3$. Define $x(t) := x^{(\mathrm{a})}_{\min}(w(t))$, $y(t) := -mx(t)+q_y$, $z(t) := mx(t)+q_z$. By construction, for any $t \in [0,T)$, $x(t)$ is the solution of the minimum problem (2.20) with Π replaced by $\Pi(t)$ and $N_{\min}(t)$, the solution of (2.3) with Γ replaced by $\Pi(t)$, is determined by $(x(t),y(t),z(t))$. Thus

$$\kappa_\varphi(l_1(t)) = \frac{x(t)}{w_1(t)}, \quad \kappa_\varphi(l_2(t)) = -\frac{y(t)}{w_2(t)}, \quad \kappa_\varphi(l_3(t)) = \frac{z(t)}{w_3(t)},$$

where $l_j(t) := w_j(t)\tau_j$. Since $w_j \in C^\infty([0,T))$, possibly reducing $T>0$, we have $x(t) \in (a,b)$ for any $t \in (0,T)$. Therefore, $\Pi(t)$ is elementary and stable for any $t \in [0,T)$, and $\kappa_\varphi(l_j(\cdot)) \in C^\infty([0,T))$, for any $j=1,2,3$. Defining for any $j=1,2,3$

$$h_j(t) := h_j^\nu(t)\nu_j, \qquad h_j^\nu(t) := -\int_0^t \kappa_\varphi(l_j(s))\,ds, \qquad t \in [0,T), \tag{3.11}$$

we get a flow satisfying (2.7). To prove (ii) of Definition 2.12, in view of (converse part of) Proposition 3.1 it is sufficient to show that (3.1), (3.2) and (3.3) are satisfied. Denote by f_t the function in (2.20) where Π is replaced by $\Pi(t)$. Equality (3.2) follows from

$$0 = \frac{df_t(x)}{dx} = 2\varphi^\circ(\nu_1)\left[\frac{x}{w_1(t)} + \frac{y(x)}{w_2(t)}\frac{dy}{dx} + \frac{z(x)}{w_3(t)}\frac{dz}{dx}\right] = 2\varphi^\circ(\nu_1)\left[-\dot{h}_1^\nu(t) - m\Big(\dot{h}_2^\nu(t) + \dot{h}_3^\nu(t)\Big)\right]$$

and from $-m = (2\tau_1\cdot\tau_3)^{-1}$ (see (2.15) and (2.18)). Integrating (3.10) yields (3.1). Finally, (3.3) follows from (3.2), since

$$w_2(t) - L_2 = \left(2\tau_1\cdot\tau_2 \mathrm{cotg}\alpha_2 + \frac{1-2(\nu_1\cdot\nu_2)^2}{\nu_1\cdot\tau_2}\right)h_1^\nu(t) + \left(-\mathrm{cotg}\alpha_2 + \frac{\nu_1\cdot\nu_2}{\nu_1\cdot\tau_2}\right)h_3^\nu(t).$$

Uniqueness of the flow follows by uniqueness of w_j and h_j^ν. ☐

Corollary 3.4. *If Π is degenerate and stable, then there exist $T>0$ and a unique stable φ-curvature flow $t \in [0,T) \mapsto \Pi(t)$ starting from Π with $\Pi(t) \in [\Pi]$ degenerate for any $t \in [0,T)$. Moreover, $h_{j_1}^\nu, h_{j_2}^\nu \in C^\infty([0,T))$, $j_1, j_2 \in \{1,2,3\} \setminus \{\bar{k}\}$, $\bar{k}$ as in Definition 2.8 of degenerate triod.*

Proof. As in the proof of Theorem 3.3 we obtain the (two) non-degenerate lengths w_{j_1}, w_{j_2} as solutions of a system of two ordinary differential equations and the assertion follows by the same arguments. ☐

Definition 3.5. *Let* Π *be elementary and stable. We define* $\mathcal{T} = \mathcal{T}(\Pi)$ *as the supremum over all* $T > 0$ *for which there exists a unique stable* φ*-curvature flow* $t \in [0, T) \mapsto \Pi(t)$ *starting from* Π.

Corollary 3.6. *Let* Π *be elementary and stable, and* $t \in [0, \mathcal{T}) \mapsto \Pi(t) \in [\Pi]$ *be the stable* φ*-curvature flow starting from* Π*. Then*

$$-2\tau_1 \cdot \tau_3 \kappa_\varphi(l_1(t)) + \kappa_\varphi(l_2(t)) + \kappa_\varphi(l_3(t)) = 0, \qquad \forall\, t \in [0, \mathcal{T}). \tag{3.12}$$

Proof. If follows by differentiating (3.2) with respect to t and using (2.9). □

Condition (3.12) is related to the geometry of the triod near the triple junction and is equivalent to stability of the triod. In particular, if $\mathfrak{n} = 6\mathfrak{m}$, we have $\tau_1 \cdot \tau_3 = -1/2$, hence the sum of φ-curvatures at the triple junction is zero (as in the euclidean case).
Using (3.2), (2.13) and (2.5), for any $t \in [0, \mathcal{T})$ and all $\mathfrak{n} \geq 6$, system (3.1) can be written as follows:

$$L_1(t) - L_1 = [\text{cotg}\alpha_1 + \text{cotg}\vartheta_\mathfrak{n}] h_1^\nu(t) - \frac{1}{\sin \vartheta_\mathfrak{n}} h_2^\nu(t) \tag{3.13}$$

$$= [\text{cotg}\alpha_1 - \text{cotg}\vartheta_\mathfrak{n}] h_1^\nu(t) + \frac{1}{\sin \vartheta_\mathfrak{n}} h_3^\nu(t) \tag{3.14}$$

$$L_2(t) - L_2 = \frac{1}{\sin \vartheta_\mathfrak{n}} h_1^\nu(t) + [\text{cotg}\alpha_2 - \text{cotg}\vartheta_\mathfrak{n}] h_2^\nu(t) \tag{3.15}$$

$$= 2\cos\vartheta_\mathfrak{n}[\text{cotg}\alpha_2 - \text{cotg}(2\vartheta_\mathfrak{n})] h_1^\nu(t) - [\text{cotg}\alpha_2 - \text{cotg}\vartheta_\mathfrak{n}] h_3^\nu(t) \tag{3.16}$$

$$= [\text{cotg}\alpha_2 - \text{cotg}(2\vartheta_\mathfrak{n})] h_2^\nu(t) + \frac{1}{\sin(2\vartheta_\mathfrak{n})} h_3^\nu(t) \tag{3.17}$$

$$L_3(t) - L_3 = \frac{1}{\sin \vartheta_\mathfrak{n}} h_1^\nu(t) + [\text{cotg}\alpha_3 + \text{cotg}\vartheta_\mathfrak{n}] h_3^\nu(t) \tag{3.18}$$

$$= \frac{1}{\sin(2\vartheta_\mathfrak{n})} h_2^\nu(t) + [\text{cotg}\alpha_3 + \text{cotg}(2\vartheta_\mathfrak{n})] h_3^\nu(t). \tag{3.19}$$

The following proposition describes some useful qualitative properties of the flow.

Proposition 3.7. *Let* Π *be elementary and stable, and* $t \in [0, \mathcal{T}) \mapsto \Pi(t) \in [\Pi]$ *be the stable* φ*-curvature flow starting from* Π*. Denote by* j_1 *and* j_2 *the two indices for which the half-lines respectively emanating from* A_{j_1}, A_{j_2}*, parallel to* S_{j_1}, S_{j_2} *and not containing* q*, lie in the same phase. Then*

- *(i)* L_{j_1} *and* L_{j_2} *are non decreasing in* $[0, \mathcal{T})$;
- *(ii)* $\sup_j \sup_{t \in [0,\mathcal{T})} |\kappa_\varphi(l_j(t))| < +\infty$;
- *(iii)* $\sup_j \sup_{t \in [0,\mathcal{T})} |\dot{L}_j(t)| < +\infty$;
- *(iv)* $L_{j_3} \in \text{Lip}((0, \mathcal{T}))$, $j_3 \neq j_1, j_2$;
- *(v) if* $\mathcal{T} < +\infty$ *then* $L_j(\mathcal{T}-) < +\infty$ *for any* $j = 1, 2, 3$.

Proof. We show the assertion (i) for configurations (a), (b) and (c), the cases (a′), (b′), (c′) being similar (interchanging $L_2(t)$ with $L_3(t)$). Differentiating (3.14), (3.17) and (3.18) with respect to t, using $\text{cotg}\alpha \pm \text{cotg}\beta = \frac{\sin(\beta \pm \alpha)}{\sin\alpha \sin\beta}$, (2.11) and Table 1, we see that $j_1 = 2$, $j_2 = 3$, $\dot{L}_{j_1}(t) > 0$, $\dot{L}_{j_2}(t) > 0$ in configuration (a), and $j_1 = 1$, $j_2 = 3$, $\dot{L}_{j_1}(t) > 0$, $\dot{L}_{j_2}(t) > 0$ in configuration (b) and (c). Assertion (ii) follow from (i) since, using Table 1, $\dot{h}_{j_1}^\nu(t)$ and $\dot{h}_{j_2}^\nu(t)$ are bounded and, using (3.2), $\dot{h}_{j_3}^\nu(t)$ is bounded. Using (3.1), (ii) implies (iii). Conclusions (iv) and (v) follow from (iii). □

For any $t \in (0,\mathcal{T})$, for simplicity, we denote by $x(t)$ the solution of the minimum problem (2.20) with Π replaced by $\Pi(t)$ and $y(t) := -mx(t) + q_y$, $z(t) := mx(t) + q_z$. Thanks to Proposition 3.7, $L_j(\mathcal{T}-)$, $x(\mathcal{T}-)$, $\kappa_\varphi(l_j(\mathcal{T}-))$ are well-defined. We denote by $\Pi(\mathcal{T})$ the elementary triod satisfying $[\Pi(\mathcal{T})] = [\Pi]$ and $|S_j| = L_j(\mathcal{T}-)$, for any $j = 1,2,3$. Finally, $N_{\min}(\mathcal{T})$ denotes the solution of (2.3) with Γ replaced by $\Pi(\mathcal{T})$.

Remark 3.8. We conclude from Proposition 3.7 that either $L_{j_3}(\mathcal{T}-) = 0$ or $\Pi(\mathcal{T})$ is unstable or both of the previos occurences happen at time $t = \mathcal{T}$. Finally, $L_{j_3}(\mathcal{T}-) = 0$ is not equivalent to $|\kappa_\varphi(l_{j_3}(\mathcal{T}-))| = +\infty$, but whenever S_{j_3} disappears in a stable flow, its φ-curvature remains bounded.

3.1 A case of global existence

In general, $\mathcal{T}$ is finite and the flow develops singularity at time $t = \mathcal{T}$. The following result shows that in a specific case the flow is global, i.e. $\mathcal{T} = +\infty$. For all $\mathfrak{n} = 6\mathfrak{m} \geq 12$, set

$$u_\infty := \frac{v_\infty}{r}, \qquad v_\infty := 1 + \sqrt{3}\sin\alpha_1 - \cos\alpha_1, \qquad r := \left(\frac{2}{\sqrt{3}} + \frac{1}{\sin\alpha_1}\right)\left(-\mathrm{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)^{-1}. \quad (3.20)$$

Theorem 3.9. *Let $\mathfrak{n} = 6\mathfrak{m}$ and $\Pi \in (\mathsf{a})$ be stable. Then $\mathcal{T} = +\infty$.*

(i) If $\mathfrak{n} \geq 12$ then $\lim_{t\to\infty} L_j(t) = +\infty$ and $\lim_{t\to\infty} \kappa_\varphi(l_j(t)) = 0$ for any $j = 1,2,3$. Furthermore,

$$\lim_{t\to\infty} \frac{L_2(t)}{L_3(t)} = u_\infty, \qquad \lim_{t\to\infty} \frac{L_2(t)}{L_1(t)} = v_\infty, \qquad \lim_{t\to\infty} x(t) = x(\infty)$$

where u_∞, v_∞ are given as in (3.20) and $x(\infty) := l\,(u_\infty + v_\infty + 1)^{-1} \in (0, L)$.

(ii) If $\mathfrak{n} = 6$ then $\lim_{t\to\infty} L_1(t) = 0$, $\lim_{t\to\infty} L_2(t) = L_1 + L_2$, $\lim_{t\to\infty} L_3(t) = +\infty$, and $\lim_{t\to\infty} x_{\min}(t) = 0$. Furthermore,

$$\lim_{t\to\infty} \kappa_\varphi(l_3(t)) = 0 \quad \textit{and} \quad \lim_{t\to\infty} \kappa_\varphi(l_2(t)) = -\lim_{t\to\infty} \kappa_\varphi(l_1(t)) = \frac{l}{L_2 + L_1}.$$

The analysis of the long time behavior requires the following lemma (recall Definition 2.11) which in particular shows that if $\mathfrak{n} = 6\mathfrak{m}$ the stability region $\mathcal{S}_\mathsf{a}$ is the whole of $(0,+\infty)^3$.

Lemma 3.10. *For all $\mathfrak{n} \geq 6$, $\mathcal{S}_\mathsf{d} = \mathcal{S}_{\mathsf{d}'} = \emptyset$. If $\mathfrak{n} = 6\mathfrak{m}$ we have $\mathcal{S}_\mathsf{a}(3,1) = \mathcal{S}_{\mathsf{a}'}(2,1) = (0,+\infty)^2$. Moreover, if m is defined as in (2.15), then*

$$\mathfrak{n}=6\mathfrak{m}-4 \Longrightarrow \begin{cases} \mathcal{S}_\mathsf{a}(3,1) = \mathcal{S}_{\mathsf{a}'}(2,1) = \left\{(u,v) \in (0,+\infty)^2 \,:\, v < m\right\}, \\[2mm] \mathcal{S}_\mathsf{b}(2,1) = \mathcal{S}_{\mathsf{b}'}(3,1) = \left\{(u,v) \in (0,+\infty)^2 \,:\, m\,\dfrac{L-\delta-Lu}{L-\delta} < v < \dfrac{L-(L-\delta)u}{L-2\delta}\right\}, \\[2mm] \mathcal{S}_\mathsf{c}(2,1) = \mathcal{S}_{\mathsf{c}'}(3,1) = \left\{(u,v) \in (0,+\infty)^2 \,:\, m < v < \dfrac{L+\delta u}{L-2\delta}\right\}, \end{cases}$$

$$\mathfrak{n}=6\mathfrak{m}-2 \Longrightarrow \begin{cases} \mathcal{S}_\mathsf{a}(3,1) = \mathcal{S}_{\mathsf{a}'}(2,1) = \left\{(u,v) \in (0,+\infty)^2 \,:\, m\,\dfrac{L-\delta-Lu}{L-\delta} < v < \dfrac{L-(L-\delta)u}{L-2\delta}\right\}, \\[2mm] \mathcal{S}_\mathsf{b}(2,1) = \mathcal{S}_{\mathsf{b}'}(3,1) = \left\{(u,v) \in (0,+\infty)^2 \,:\, v < m\right\}, \\[2mm] \mathcal{S}_\mathsf{c}(3,1) = \mathcal{S}_{\mathsf{c}'}(2,1) = \left\{(u,v) \in (0,+\infty)^2 \,:\, m < v < \dfrac{L+\delta u}{L-2\delta}\right\}. \end{cases}$$

Proof. Using Proposition 2.22, and (2.24) for configurations (a′), (b′), (c′), the conclusion follows by imposing $x_{\min} \in (a,b)$. □

Proof of Theorem 3.9. Let $x^{(\mathrm{a})}_{\min}$ be given as in (2.21). For any $t \in [0,\mathcal{T})$ define $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t) := x^{(\mathrm{a})}_{\min}\big(L_1(t), L_2(t), L_3(t)\big)$. Set $d(t) := \left(\sum_{i<j} L_i(t)L_j(t)\right)^{-1}$. Using Table 1, (2.15) and (2.16), system (2.9) reads as

$$\begin{cases} \dot{h}^{\nu}_1(t) = -\dfrac{x(t)}{L_1(t)} = -L\,d(t)\,L_3(t) \\ \dot{h}^{\nu}_2(t) = \dfrac{y(t)}{L_2(t)} = L\,d(t)\,\big(L_1(t)+L_3(t)\big), \\ \dot{h}^{\nu}_3(t) = -\dfrac{z(t)}{L_3(t)} = -L\,d(t)\,L_1(t). \end{cases} \tag{3.21}$$

Set $A_1 := \left(\mathrm{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)\frac{\mathrm{cotg}\alpha_1+\sqrt{3}}{\mathrm{cotg}^2\alpha_1+1}$, $A_3 := \left(\mathrm{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)\frac{\mathrm{cotg}\alpha_1-\sqrt{3}}{\mathrm{cotg}^2\alpha_1+1}$. Then condition (3.3) becomes

$$L_2(t) = A_1L_1(t) + A_3L_3(t) + (L_2 - A_1L_1 - A_3L_3)(0) \quad t \in [0,\mathcal{T}). \tag{3.22}$$

Differentiating (3.14) and (3.18) with respect to t and using (3.21) yields

$$\dot{L}_1 = L\,d(t)\,f_1, \qquad \dot{L}_3 = L\,d(t)\,f_3, \tag{3.23}$$

where $f_1 := -\frac{2}{\sqrt{3}}L_1 - \left(\mathrm{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)L_3$ and $f_3 := -\left(\mathrm{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)L_1 + \frac{2}{\sqrt{3}}L_3$. Thus $\dfrac{dL_1}{f_1} = \dfrac{dL_3}{f_3}$ and, if we substitute $L_3 = RL_1$, we obtain

$$p_1\,dL_1 + p_2\,dR = 0, \tag{3.24}$$

with $p_1 := \left(\mathrm{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)R^2 + \frac{4}{\sqrt{3}}R - \left(\mathrm{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)$ and $p_2 := \left[\left(\mathrm{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)R + \frac{2}{\sqrt{3}}\right]L_1$.

Let us show (i). Integrating (3.24) yields $\log L_1 + \frac{1}{2}\int\left(\frac{1}{R-\bar{R}} + \frac{1}{R-r}\right)dR = 0$ where $r > 0$ is defined as in (3.20) and $\bar{R} := -\dfrac{\frac{2}{\sqrt{3}} - \frac{1}{\sin\alpha_1}}{\mathrm{cotg}\alpha_1 + \frac{1}{\sqrt{3}}} < 0$. Thus

$$L_1^2(t)\left(\frac{L_3}{L_1}(t) - \bar{R}\right)\left(\frac{L_3}{L_1}(t) - r\right) = C, \qquad C := (L_3 - \bar{R}L_1)(L_3 - rL_1). \tag{3.25}$$

Step 1. There exists $t_0 \in [0,\mathcal{T})$ such that $f_1(t) > 0$ (and hence $\dot{L}_1 > 0$) for any $t \in [t_0,\mathcal{T})$.
If f_1 is positive at time $t = 0$ then so is at the subsequent times since, using (3.14) and (3.18),

$$\dot{f}_1(t) = -\dot{h}^{\nu}_3(\mathrm{cotg}^2\alpha_1 + 1) > 0, \qquad t \in [0,\mathcal{T}). \tag{3.26}$$

We proceed by contradiction: assume that $f_1 \le 0$ in $[0,\mathcal{T})$, i.e.

$$L_1(t) \ge M_0L_3(t) \quad t \in [0,\mathcal{T}), \quad M_0 := -\frac{\sqrt{3}}{2}(\mathrm{cotg}\alpha_1 + \frac{1}{\sqrt{3}}) \ge 1 \tag{3.27}$$

Then, from (3.23), L_1 is decreasing in $[0,\mathcal{T})$ but bounded below from M_0L_3. From Proposition 3.10 and Proposition 3.7 (v), we conclude that $\mathcal{T} = +\infty$. Note that $A_1 \ge 0$ and $A_3 > 0$ if $\mathfrak{n} \ge 12$. Set

$$M_1 := \frac{LL_3(0)}{[(A_3/M_0)L_1(0) + L_2(0) - A_3L_3(0)]\frac{M_0+1}{M_0}L_1(0) + L_1(0)^2/M_0} > 0.$$

From (3.21), (3.22) and (3.27) we get $-\dot{h}_3^\nu \geq M_1$ and, from (3.26), $\dot{f}_1 \geq M_1(\text{cotg}^2\alpha_1 + 1)$ in $[0, \infty)$. Thus $-\left(\text{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right) L_3(t) \geq \frac{2}{\sqrt{3}} L_1(t) + M_1(\text{cotg}^2\alpha_1 + 1)t + f_1(0)$, for any $t \in [0, \infty)$. Hence, letting $t \to +\infty$ we get $L_3 \to +\infty$ and, from (3.27), $L_1 \to +\infty$. This gives a contradiction since, by our assumption, L_1 is decreasing on $[0, \infty)$. Step 1 is proved.

Since (3.27) holds when L_1 is decreasing, $\mathcal{T} = +\infty$ follows from Proposition 3.7 and formula (2.21). Let us show $\lim_{t\to\infty} L_1(t) = +\infty$. Assume that $\lim_{t\to\infty} L_1(t) < +\infty$. Then from (3.25) and (3.22) we get respectively $\lim_{t\to\infty} L_3(t) < \infty$ and $\lim_{t\to\infty} L_2(t) < \infty$. But this implies $\lim_{t\to\infty} \dot{L}_j(t) = 0$ for any $j = 1, 2, 3$. Differentiating (3.16) with respect to t and recalling that $\dot{h}_1^\nu \geq 0$ and $\dot{h}_3^\nu \geq 0$, we get $\lim_{t\to\infty} \dot{h}_1^\nu(t) = \lim_{t\to\infty} \dot{h}_3^\nu(t) = 0$, and consequently $\lim_{t\to\infty} \dot{h}_2^\nu(t) = 0$ that gives a contradiction with system (3.21).

Finally we get also $\lim_{t\to\infty} L_3(t) = +\infty$ and $\lim_{t\to\infty} L_2(t) = +\infty$ respectively from $f_1 > 0$ on $[t_0, +\infty)$ and (3.22). Using again (3.22), (3.25) and (2.21) the conclusion follows since $A_1 + A_3 r = v_\infty$.

Let us show (ii). If $\mathfrak{n} = 6$ then $\alpha_1 = \vartheta_\mathfrak{n} = 2\pi/3$. Thus, (3.22) reduces to

$$L_2(t) = -L_1(t) + S_0 \quad t \in [0, \mathcal{T}), \quad S_0 := L_1 + L_2, \tag{3.28}$$

and (3.24) to $(2R+1)\, dL_1 + L_1\, dR = 0$ so that integrating yields $\log\left(L_1\sqrt{2R+1}\right) = C$. Hence

$$L_3(t) = \frac{1}{2}\left(\frac{C_0}{L_1(t)} - L_1(t)\right) \quad t \in [0, \mathcal{T}), \quad C_0 := L_1(L_1 + 2L_3). \tag{3.29}$$

Using (3.21), (3.29), (3.28), we can rewrite the first equation in 3.23 as $\dot{L}_1 = 4LL_1^2/\sqrt{3}\, g(L_1)$, where $g(s) := 2s^3 - S_0 s^2 - C_0 S_0$. Given $\Lambda \in (0, L_1)$, let $T \in (0, \mathcal{T})$ be the time that the solution needs to achieve the value Λ. Then we get

$$\left[s^2 - S_0 s + C_0 S_0 \frac{1}{s}\right]_{L_1}^{\Lambda} = \int_{L_1}^{\Lambda} \left(2s - S_0 - \frac{C_0 S_0}{s^2}\right) ds = \frac{4}{\sqrt{3}} LT. \tag{3.30}$$

From (3.30) we discover that $T < \mathcal{T}$ for any $\Lambda \in (0, L_1)$. Hence $\Lambda = 0$ and $\mathcal{T} = +\infty$. Further, being $L_1(t)$ strictly decreasing in $[0, +\infty)$ (from (3.23)), we have $L_1 > 0$ in $[0, +\infty)$ and from (3.30) we get $\lim_{t\uparrow+\infty} L_1(t) = 0$. □

4 Configurations (d) and (d′): development of a new segment.

In this section we assume $\mathfrak{n} = 8$ and $\Pi \in$ (d). From Proposition 2.22, Π is unstable with $x_{\min} = \delta$, i.e. $N_{\min} = (X(V_3), Y(V_3), V_3)$ (see Figure 8), and from Table 1, $\kappa_\varphi(l_1) = \frac{\delta}{L_1}$, $\kappa_\varphi(l_2) = \frac{L-\delta}{L_2}$, $\kappa_\varphi(l_3) = 0$. Since $x_{\min}$ tends to be smaller than δ and the constraint $N_{\min|_{\Sigma_j}}(q) \in T_{\varphi^o}(\nu_\varphi^j)$ cannot be violated, the appearance of a vertical segment at q is forced during the flow, as explained in the following result.

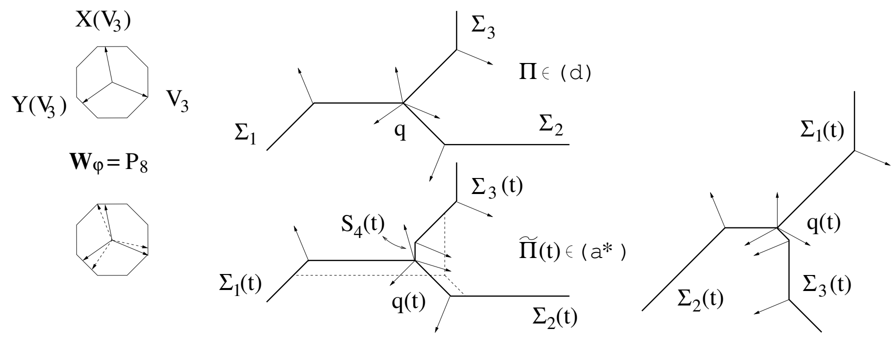

Figure 8: Development of a vertical segment $S_4(t)$. The rotated quasi-elementary triod on the right will be used in the proof of Theorem 4.1.

Theorem 4.1. *Let* $\mathfrak{n} = 8$ *and* $\Pi \in (\mathsf{d})$. *Then there exist* $T > 0$ *and a stable* φ*-curvature flow* $t \in [0,T) \mapsto \widetilde{\Pi}(t)$ *starting from* Π. *More precisely*

$$\widetilde{\Pi}(t) = \Sigma_1(t) \cup \Sigma_2(t) \cup (S_4(t) \cup \Sigma_3(t)) \in (\mathsf{a}^*), \qquad \forall t \in (0,T), \tag{4.1}$$

and, defining

$$\widetilde{x}(t) := x^{(\mathsf{a})}_{\min}\big(L_2(t), L_4(t), L_1(t)\big), \quad \widetilde{y}(t) := -m\widetilde{x}(t) + q_y, \quad \widetilde{z}(t) := m\widetilde{x}(t) + q_z, \tag{4.2}$$

with $x^{(\mathsf{a})}_{\min}$ *as in* (2.21) *and* m, q_y, q_z *as in* (2.15), (2.16), *then* $\kappa_\varphi(l_3(t)) = 0$,

$$\kappa_\varphi(l_1(t)) = \frac{\widetilde{z}(t)}{L_1(t)}, \quad \kappa_\varphi(l_2(t)) = \frac{\widetilde{x}(t)}{L_2(t)}, \quad \kappa_\varphi(l_4(t)) = -\frac{\widetilde{y}(t)}{L_4(t)}. \tag{4.3}$$

Finally, $\kappa_\varphi(l_j(\cdot)) \in C^\infty((0,T))$ *for any* $j = 1,2,3,4$, *and*

$$\lim_{t\downarrow 0} \kappa_\varphi(l_1(t)) = \frac{\delta}{L_1}, \quad \lim_{t\downarrow 0} \kappa_\varphi(l_2(t)) = \frac{L-\delta}{L_2}, \quad \lim_{t\downarrow 0} \kappa_\varphi(l_4(t)) = -\frac{\delta}{L_1} + 2\tau_1 \cdot \tau_2 \frac{L-\delta}{L_2} < 0\,. \tag{4.4}$$

The idea of the proof is to consider the φ-curvature flow starting from the rotated quasi-elementary triod $\widetilde{\Pi}$ in Figure 8, with singular initial datum $(L_2, 0, L_1)$. Notice that, from Lemma 3.10, $(L_2, 0, L_1)$ belongs to the boundary of the stability region $\mathcal{S}_{\mathsf{a}}$.

Proof. Set $G(w) := x^{(\mathsf{a})}_{\min}(w)$ for $w := (w_1, w_2, w_3) \in (0,+\infty)^3$ with $x^{(\mathsf{a})}_{\min}$ as in (2.21), and define the vector field $F = (F_1, F_2, F_3) \in C^\infty((0,+\infty)^3; \mathbb{R}^3)$ as

$$F_1(w) := \frac{1}{\sin\vartheta_{\mathfrak{n}}} \frac{mG(w) + q_z}{w_3}, \qquad F_2(w) = F_3(w) := \frac{-mG(w) + q_y}{w_2} + \frac{mG(w) + q_z}{w_3}. \tag{4.5}$$

where m, q_y, q_z are defined as in (2.15) and (2.16). Notice that F is obtained by differentiating with respect to t the right hand side of (3.14), (3.17), (3.19) and by replacing the $\dot{h}^\nu_j$'s (in (2.9)) with the expressions in G and w_j (where we used Table 1, (2.21) and (2.19)). Despite the appearance of w_2 in the denominators of F_2, the presence of G ensures that F and all its partial derivatives are bounded in $(0,+\infty) \times \{w_2 = 0\} \times (0,+\infty)$. Thus the Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)) \\ w(0) = (L_2, 0, L_1), \end{cases} \tag{4.6}$$

admits a unique solution $(\Lambda_1, \Lambda_2, \Lambda_3) \in ([0,T); (0,+\infty)^3)$ of (4.6) for some $T > 0$. Set $L_1(t) := \Lambda_3(t)$, $L_2(t) := \Lambda_1(t)$, $L_4(t) := \Lambda_2(t)$ for any $t \in [0,T)$. Let us show that $L_4 > 0$ in $[0,T)$. In order to do that, one checks that $\ddot{L}_4(0) > 0$ as a consequence of

$$\dot{L}_4(t) = \frac{q_y - mG(L_2(t), L_4(t), L_1(t))}{L_4(t)} + \frac{mG(L_2(t), L_4(t), L_1(t)) + q_z}{L_1(t)}$$

and

$$\lim_{w_2 \to 0} G(w_1, w_2, w_3) = L - \delta, \quad \lim_{w_2 \to 0} F_2(w_1, w_2, w_3) = 0 \quad \left(\lim_{t \to 0} \dot{L}_4(t) = 0 \right), \quad w_1, w_3 \in (0, +\infty).$$

Hence, we conclude that L_4 is increasing in a neighborhood of 0, say $[0,T)$.
Let $\widetilde{\Pi}(t)$ be the quasi-elementary triod defined in (4.1) and having $|S_j(t)| = L_j(t)$, $j = 1,2,3,4$. Define $\widetilde{x}(t) := G(L_2(t), L_4(t), L_1(t))$ and $\widetilde{y}(t), \widetilde{z}(t)$ as in (4.2). By construction, for any $t \in (0,T)$, the solution $N_{\min}(t)$ of (2.3) with Γ replaced by $\widetilde{\Pi}(t)$ is determined by $(\widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t))$. Since $L_j \in C^\infty([0,T))$, possibly reducing $T > 0$, we have $\widetilde{x} \in (\delta, L - \delta)$ in $(0,T)$. Therefore, $\widetilde{\Pi}$ is stable in $(0,T)$, $\kappa_\varphi(l_j(\cdot)) \in C^\infty([0,T))$ for any $j = 1,2,3$, and (4.3) holds. Defining, for any $j = 1,2,3$, $h_j^\nu : [0,T) \to \mathbb{R}$ as in (3.11) and reasoning as in the last part of Theorem 3.3 we get a stable φ-curvature flow starting from Π. Since L_1, L_2, L_4 are monotone functions in $(0,T)$, the limits in (4.4) exist and the computations follow using $\lim_{t\to 0} \widetilde{x}(t) = L - \delta$ and $\lim_{t \to 0} L_4(t) = 0$. □

Remark 4.2. The flow $t \to \widetilde{\Pi}(t)$ of Theorem 4.1 is the unique stable flow starting from Π. Indeed, if $t \mapsto \Pi'(t)$ is a stable flow starting from Π then, from Proposition 2.22, $\Pi'(t) \notin (\mathtt{d})$ and, being $x_{\min} = \delta$, i.e. $N_{\min|\Sigma_3} = V_3$ (see Figure 8), $\Pi'(t)$ must be quasi-elementary with $\Pi'(t) \in (\mathtt{a}^*)$.
We expect that $t \to \widetilde{\Pi}(t)$ is also unique among all regular flows starting from Π.

5 The case $\mathfrak{n} = 8$ and $\Pi \in (\mathtt{b})$: development of a new segment

In this section we prove that at time $t = \mathcal{T} \in (0, +\infty)$ the flow starting from a stable triod $\Pi \in (\mathtt{b})$ becomes unstable and a vertical segment develops in order to decrease the energy functional and make stable the flow at subsequent times.

Theorem 5.1. *Let $\mathfrak{n} = 8$ and $\Pi \in (\mathtt{b})$ be stable. Then $\mathcal{T} < +\infty$ and $N_{\min}(\mathcal{T}) = (X(V_3), Y(V_3), V_3)$. Furthermore, there exist $\mathcal{T}_1 \in (\mathcal{T}, +\infty]$ and a stable φ-curvature flow $t \in [\mathcal{T}, \mathcal{T}_1) \to \widetilde{\Pi}(t)$ starting from $\Pi(\mathcal{T})$. More precisely, for any $t \in (\mathcal{T}, \mathcal{T}_1)$, $\widetilde{\Pi}(t)$ is the quasi-elementary triod defined as in (4.1), $\kappa_\varphi(l_3(t)) = -L/L_3(t)$ and (4.3) holds. Finally, $\kappa_\varphi(l_j(\cdot)) \in C^\infty((\mathcal{T}, \mathcal{T}_1))$ for any $j = 1,2,3,4$, and (4.4) holds with $t \downarrow 0$ replaced by $t \downarrow \mathcal{T}$.*

The idea of the proof is that at the finite time $t = \mathcal{T}$ the solution reaches the boundary of the stability region and, at the same time an infinitesimal segment appears; then the flow is continued arguing as in Theorem 4.1.

Proof. For any $t \in [0, \mathcal{T})$ define $x(t) := x_{\min}^{(\mathtt{b})}\big(L_1(t), L_2(t), L_3(t)\big)$, $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t)$. Then system (2.9) reads as

$$\dot{h}_1^\nu(t) = -\frac{x(t)}{L_1(t)}, \qquad \dot{h}_2^\nu(t) = -\frac{L - y(t)}{L_2(t)}, \qquad \dot{h}_3^\nu(t) = \frac{L - z(t)}{L_3(t)}, \tag{5.1}$$

system (3.1) reads as

$$L_1(t) = L_1 + \sqrt{2} h_3^\nu(t), \qquad L_2(t) = L_2 - h_2^\nu(t) - h_3^\nu(t), \qquad L_3(t) = L_3 + h_2^\nu(t) + h_3^\nu(t), \tag{5.2}$$

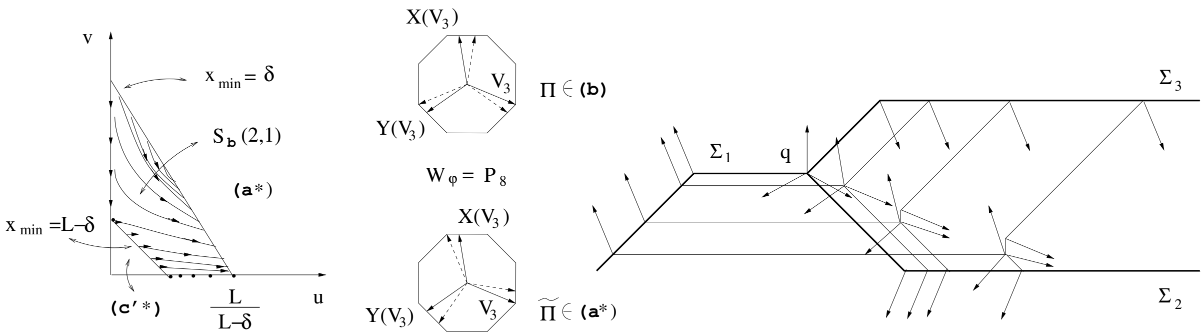

Figure 9: φ-curvature flow starting from a stable $\Pi \in \{(\mathtt{b})\}$ (see Theorem 5.1): at time $t = \mathcal{T}$ the flow becomes unstable and a vertical segment develops in order to make the triod stable.

while (3.2) and (3.3) become

$$-\sqrt{2}h_1^\nu(t) = h_2^\nu(t) + h_3^\nu(t) \qquad \text{and} \qquad L_3(t) - L_3 = L_2 - L_2(t). \tag{5.3}$$

Define $u(t) := \dfrac{L_3(t)}{L_2(t)}$ and $v(t) := \dfrac{L_3(t)}{L_1(t)}$. Differentiating (5.2) with respect to $t \in (0, \mathcal{T})$ and using (2.22), (2.19) and $m = \sqrt{2}/2$, we obtain

$$\dot{u} = \frac{L-2\delta}{L_3^2}\,\frac{u\,v\,(1+u)\,(D-Eu)}{\delta^2+\delta^2 u+(L-2\delta)^2 v}, \qquad \dot{v} = \frac{(L-2\delta)(L-\delta)}{\delta L_3^2}\,\frac{v^2(A-Bu-Cv)}{\delta^2+\delta^2 u+(L-2\delta)^2 v}, \tag{5.4}$$

where $A := \delta(L-\delta)$, $B =: \delta L$, $C := (L-\delta)(L-2\delta)$, $D := (L-\delta)^2$, $E := L^2 - 3\delta L + \delta^2$. Recall that the stability region $\mathcal{S}_{\mathtt{b}}(2,1)$ is given by Lemma 3.10, see Figure 9. Notice that $A - Bu - Cv \le 0$ is equivalent to $x_{\min} \le L - \delta$ and $D - Eu > 0$ for any $(u,v) \in \mathcal{S}_{\mathtt{b}}(2,1)$. Thus $\dot{u} > 0$ and $\dot{v} < 0$ in $\mathcal{S}_{\mathtt{b}}(2,1)$. From (5.4) we get

$$\frac{dv}{du} = \frac{L-\delta}{\delta}\,\frac{v\,(A-Bu-Cv)}{u\,(u+1)\,(D-Eu)} > -\frac{L-\delta}{\delta}\,\frac{v}{u\,(u+1)}, \tag{5.5}$$

since $-\frac{A-Bu-Cv)}{D-Eu} \le 1$ for any $(u,v) \in \mathcal{S}_{\mathtt{b}}(2,1)$. For any $(u_0,v_0) \in \mathcal{S}_{\mathtt{b}}(2,1)$ we have $v(u) \ge v_0 \left(\dfrac{u_0}{1+u_0}\,\dfrac{u+1}{u}\right)^{\frac{L-\delta}{\delta}} \ge v_0\left(\frac{u_0}{1+u_0}\right)^{\frac{L-\delta}{\delta}}$. From Proposition 3.7 we have that $L_3(\cdot)$ is non decreasing; hence, from (5.3) it follows that $L_2(\cdot)$ is non increasing, so that $0 < L_3(0) \le L_3(t)$, $L_2(t) \le L_2(0)$ for any $t \in (0,\mathcal{T})$. Since $L_3(t)/L_2(t) \le L/(l-\delta)$ it follows that $(L-\delta)L_3(t) \le LL_2(0)$ and $LL_2(t) \ge (L-\delta)L_3(0)$ for any $t \in (0,\mathcal{T})$. It follows $L_1(\mathcal{T}-) - L_1 < +\infty$, since $L_3(t)/L_1(t)$ is bounded from below. Furthermore, since we have, using (5.2) and (5.1),

$$\dot{L_2}(t) \le -\dot{h_2^\nu}(t) \le \frac{L}{L_2(t)}, \qquad \dot{L_1}(t) \ge -\sqrt{2}\dot{h_2^\nu}(t) \ge \sqrt{2}\frac{L-\delta}{L_2(t)},$$

it follows $\mathcal{T} < +\infty$, $x(\mathcal{T}-) = \delta$ and

$$\kappa_\varphi(l_1(\mathcal{T}-)) = -\frac{\delta}{L_1(\mathcal{T}-)}, \qquad \kappa_\varphi(l_2(\mathcal{T}-)) = -\frac{L-\delta}{L_2(\mathcal{T}-)}, \qquad \kappa_\varphi(l_3(\mathcal{T}-)) = \frac{L}{L_3(\mathcal{T}-)}.$$

We proceed as in proof of Proposition 4.1, with the difference that now $\kappa_\varphi(l_3(\mathcal{T}-)) > 0$, so that a system of four ODEs is required. For any $\widehat{w} := (w_1, w_2, w_3) \in (0,+\infty)^3$ we set $G(\widehat{w}) := x_{\min}^{(\mathtt{a})}(\widehat{w})$ with

$x_{\min}^{(\mathsf{a})}$ as in (2.21). For any $w := (\widehat{w}, w_4) \in (0,\infty)^4$ we define the vector field $F \in C^\infty((0,+\infty)^4;\mathbb{R}^4)$ as

$$\begin{aligned} F_1(w) &:= -\sqrt{2}\,\frac{mG(\widehat{w})+q_z}{w_3}, & F_2(w) &:= F_3(w) - \sqrt{2}\frac{L}{w_4} \\ F_3(w) &:= \frac{q_y - mG(\widehat{w})}{w_2} + \frac{mG(\widehat{w})+q_z}{w_3}, & F_4(w) &:= 2\,\frac{L}{w_4} - \sqrt{2}\,\frac{q_y - mG(\widehat{w})}{w_2}, \end{aligned} \tag{5.6}$$

where m, q_y, q_z are given by (2.15) and (2.16). Since F and all its partial derivatives are bounded in $(0,+\infty)\times\{w_2 = 0\}\times(0,+\infty)^2$, the Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)) \\ w(\mathcal{T}) = (L_2(\mathcal{T}), 0, L_1(\mathcal{T}), L_3(\mathcal{T})), \end{cases} \tag{5.7}$$

admits a unique solution $(w_1, w_2, w_3, w_4) \in C^\infty([\mathcal{T},\mathcal{T}_1);(0,+\infty)^4)$ for $\mathcal{T}_1 \in (\mathcal{T},+\infty]$ For any $t \in [\mathcal{T},\mathcal{T}_1)$, set $L_1(t) := \Lambda_3(t)$, $L_2(t) := \Lambda_1(t)$, $L_3(t) := \Lambda_4(t)$, $L_4(t) := \Lambda_2(t)$. As in the proof of Theorem 4.1, we obtain that $\ddot{L}_4(0) > 0$ as consequence of let us compute the second derivative of $L_4(t)$. Differentiating with respect to time the following expression

$$\dot{L}_4(t) = \frac{q_y - mG(L_2(t), L_4(t), L_1(t))}{L_4(t)} + \frac{mG(L_2(t), L_4(t), L_1(t)) + q_z}{L_1(t)} - \sqrt{2}\frac{L}{L_3(t)}$$

and

$$\lim_{w_2\to 0} G(w_1, w_2, w_3) = L - \delta \quad \lim_{w_2\to 0} F_2(w_1, w_2, w_3, w_4) = 0 \;\; (\; \lim_{t\to 0} \dot{L}_4(t) = 0 \;), \;\; w_1, w_3, w_4 \in (0,+\infty).$$

The conclusion follows by the same argument once we define for any $t \in [\mathcal{T},\mathcal{T}_1)$, the quasi-elementary triod $\widetilde{\Pi}(t)$ as in (4.1) with $|S_j(t)| := L_j(t)$, $\widetilde{x}(t) := G(L_2(t), L_4(t), L_1(t))$, $h_j(t) := h_j^\nu(t)\nu_j$ and $h_j^\nu(t)$ as is (3.11) with

$$\lim_{t\downarrow\mathcal{T}} h_j^\nu(t) = h_j^\nu(\mathcal{T}), \quad j = 1,2,3, \qquad \text{and} \qquad h_4^\nu(\mathcal{T}) = 0.$$

We notice that, the condition (ii) of Definition 2.12 holds since, similarly to Proposition 3.1, one can show that (ii) is satisfied for $t \in (\mathcal{T},\mathcal{T}_1) \mapsto \widetilde{\Pi}(t)$ if and only if system (5.7) with $w(t) = \big(L_2(t), L_4(t), L_1(t), L_3(t)\big)$ holds in $(\mathcal{T},\mathcal{T}_1)$ and $\widetilde{\Pi}(t)$ is stable for any $t \in (\mathcal{T},\mathcal{T}_1)$. □

Remark 5.2. The flow $t \to \widetilde{\Pi}(t)$ of Theorem 5.1 is the unique stable flow starting from $\Pi(\mathcal{T})$. Indeed, assume that $t \mapsto \Pi'(t)$ is a stable flow starting from Π. If $\Pi'(t) \in (\mathsf{b})$ and u, v are defined as in proof of Theorem 5.1 then $(u(t), v(t) \notin \mathcal{S}_{\mathsf{b}}(2,1)$, which gives a contradiction. Since any non-polygonal triod is unstable and since $x_{\min} = \delta$, i.e. $N_{\min|\Sigma_3} = V_3$ (see Figure 9), $\Pi'(t)$ must be quasi-elementary with $\Pi'(t) \in (\mathsf{a}^*)$. We expect that $t \to \widetilde{\Pi}(t)$ is also unique among all regular flows starting from $\Pi(\mathcal{T})$.

6 The case $\mathfrak{n} = 6\mathfrak{m} - 4$ and $\Pi \in (\mathsf{a})$: development of a curve

In this section we prove that at time $t = \mathcal{T} \in (0,+\infty)$ the flow starting from a stable triod $\Pi \in (\mathsf{a})$ becomes unstable and a curve develops from the triple junction at subsequent times.

Theorem 6.1. *Let $\mathfrak{n} = 6\mathfrak{m} - 4$ and $\Pi \in (\mathsf{a})$ be stable. Then $\mathcal{T} < +\infty$ and there exists a φ-curvature flow $t \in [\mathcal{T},+\infty) \mapsto \Pi(t)$ starting from $\Pi(\mathcal{T})$. Moreover, for any $t \in [\mathcal{T},+\infty)$, the triod $\Pi(t)$ is non-polygonal and unstable with $N_{\min}(t) = (X(V_3), Y(V_3), V_3)$. Finally, $\kappa_\varphi(l_3(t)) = 0$,*

$$\kappa_\varphi(l_1(t)) = \frac{\delta}{L_1(t)}, \quad \kappa_\varphi(l_2(t)) = -\frac{\delta}{L_2(t)}, \tag{6.1}$$

$$\lim_{t\to\infty} L_1(t) = \lim_{t\to\infty} L_2(t) = +\infty, \qquad \lim_{t\to\infty} \frac{L_2(t)}{L_1(t)} = 1. \tag{6.2}$$

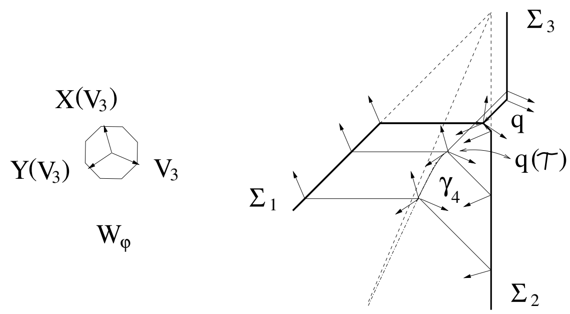

Figure 10: φ-curvature flow starting from a stable $\Pi \in (\mathsf{a})$ for $\mathfrak{n} = 8$ (see Theorem 6.1): at time $t = \mathcal{T}$ the flow becomes unstable and a curve γ_4 of zero φ-curvature develops from the triple junction.

Proof. For any $t \in [0, \mathcal{T})$ define $x(t) = x^{(\mathsf{a})}_{\min}\big(L_1(t), L_2(t), L_3(t)\big)$, $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t)$. Then system (2.9) reads as

$$\dot{h}^\nu_1(t) = -\frac{x(t)}{L_1(t)}, \qquad \dot{h}^\nu_2(t) = \frac{y(t)}{L_2(t)}, \qquad \dot{h}^\nu_3(t) = -\frac{z(t)}{L_3(t)}. \tag{6.3}$$

Step 1. L_1, L_2, L_3 are strictly positive and bounded in $[0, \mathcal{T}]$.
From Lemma 3.10, we know that $(u, v) \in \mathcal{S}_{\mathsf{a}}(3,1) = \big\{(u,v) \in (0,+\infty)^2 : v < -(2\cos\vartheta_{\mathfrak{n}})^{-1}\big\}$ in $[0, \mathcal{T})$, where $u(t) := L_2(t)/L_3(t)$ and $v(t) := L_2(t)/L_1(t)$, i.e.

$$-2\cos\vartheta_{\mathfrak{n}} L_2(t) < L_1(t), \quad t \in [0, \mathcal{T}). \tag{6.4}$$

Set $M_0 := \dfrac{\mathrm{cotg}\alpha_1 - \mathrm{cotg}\vartheta_{\mathfrak{n}}}{-2\cos\vartheta_{\mathfrak{n}}(\mathrm{cotg}\alpha_1 + \mathrm{cotg}(2\vartheta_{\mathfrak{n}}))} > 0$. From (3.14), (3.16), $\mathrm{cotg}\alpha_1 = -\mathrm{cotg}\alpha_2 < 0$, $\dot{h}^\nu_3 < 0$, $\cos\vartheta_n < 0$, we deduce $\dot{L}_1(t) \leq (\mathrm{cotg}\alpha_1 - \mathrm{cotg}\vartheta_{\mathfrak{n}})\dot{h}^\nu_1(t) \leq M_0 \dot{L}_2(t)$ for any $t \in [0, \mathcal{T})$. Hence, using (6.4), we obtain

$$L_1(t)\left(1 + \frac{M_0}{2\cos\vartheta_{\mathfrak{n}}}\right) \leq L_1 - M_0 L_2, \qquad t \in [0, \mathcal{T}).$$

Now we observe that $1 + \frac{M_0}{2\cos\vartheta_{\mathfrak{n}}} > 0$ and $L_1 - M_0 L_2 > 0$. Indeed, $L_1 - M_0 L_2 \geq (-2\cos\vartheta_{\mathfrak{n}} - M_0)L_2$ and using Table 1 and (2.11),

$$\begin{aligned} -2\cos\vartheta_{\mathfrak{n}} - M_0 &= \frac{(1 - 4\cos^2\vartheta_{\mathfrak{n}})\mathrm{cotg}\alpha_1 + \frac{-2\cos^2\vartheta_{\mathfrak{n}}}{\sin(2\vartheta_{\mathfrak{n}})}(2\cos(2\vartheta_{\mathfrak{n}}) + 1)}{2\cos\vartheta_{\mathfrak{n}}(\mathrm{cotg}\alpha_1 + \mathrm{cotg}(2\vartheta_{\mathfrak{n}}))} \\ &= \frac{(1 - 4\cos^2\vartheta_{\mathfrak{n}})(\mathrm{cotg}\alpha_1 + \mathrm{cotg}\vartheta_{\mathfrak{n}})}{2\cos\vartheta_{\mathfrak{n}}(\mathrm{cotg}\alpha_1 + \mathrm{cotg}(2\vartheta_{\mathfrak{n}}))} > 0 \qquad \forall\, \mathfrak{n} \geq 8, \end{aligned}$$

since $1 - 4\cos^2\vartheta_{\mathfrak{n}} < 0$, $\mathrm{cotg}\alpha_1 + \mathrm{cotg}\vartheta_{\mathfrak{n}} < 0$ and $\mathrm{cotg}\alpha_1 + \mathrm{cotg}(2\vartheta_{\mathfrak{n}}) < 0$. Thus L_1 is bounded in $[0, \mathcal{T})$, and from (6.4) and (3.3), so are L_2 and L_3.

Step 2. $L_3(t) - L_3 \geq C\sqrt{L_1^2 - 2(\mathrm{cotg}\alpha_1 - \mathrm{cotg}\vartheta_{\mathfrak{n}})(L - \delta)t}$, $t \in [0, \mathcal{T})$, for some constant $C > 0$.
Step 2 follows since, using (3.13), (3.18) and (6.3), we have

$$\dot{L}_1(t) \leq -(\mathrm{cotg}\alpha_1 + \mathrm{cotg}\vartheta_{\mathfrak{n}})\frac{L - \delta}{L_1(t)}, \qquad \dot{L}_3(t) \geq \frac{\delta}{\sin\vartheta_{\mathfrak{n}}\, L_1(t)}, \qquad t \in [0, \mathcal{T}).$$

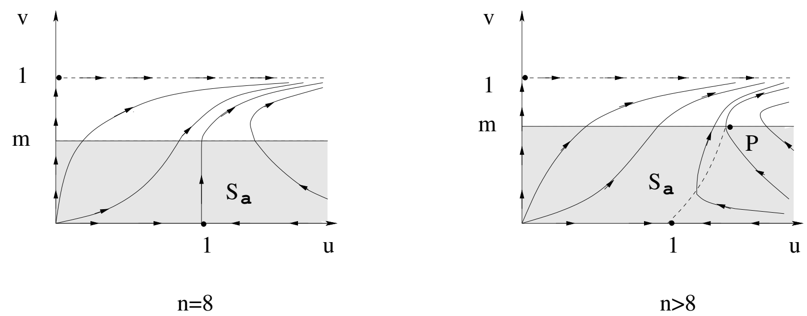

Figure 11: Flow lines diagram in the variables (u, v) corresponding to φ-curvature flows starting from stable $\Pi \in (\mathtt{a})$ (see Theorem 6.1).

From step 1 and step 2, we get $\mathcal{T} < +\infty$, and hence $x(\mathcal{T}-) = \delta$.
For any $w := (w_1, w_2) \in (0, +\infty)^2$ we define the vector field $F = (F_1, F_2) \in C^\infty((0, +\infty)^2; \mathbb{R}^2)$ as

$$F_1(w) := -\delta \frac{\mathrm{cotg}\alpha_1 + \mathrm{cotg}\vartheta_{\mathrm{n}}}{w_1} - \frac{\delta}{w_2 \sin \vartheta_{\mathrm{n}}}, \qquad F_2(w) := \frac{-\delta}{w_1 \sin \vartheta_{\mathrm{n}}} - \delta \frac{\mathrm{cotg}\alpha_1 + \mathrm{cotg}\vartheta_{\mathrm{n}}}{w_2}. \tag{6.5}$$

The Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)) \\ w(\mathcal{T}) = (L_1(\mathcal{T}), L_2(\mathcal{T})) \end{cases} \tag{6.6}$$

admits a unique solution $(w_1, w_2) \in C^\infty([\mathcal{T}, T); (0, +\infty)^2)$ for some $T \in (\mathcal{T}, +\infty]$. Notice that F is obtained differentiating with respect to t the right hand side of (3.13), (3.15) and by replacing $\dot{h}_1^\nu$ and $\dot{h}_2^\nu$ respectively by $-\delta/w_1$ and δ/w_2. For any $t \in (\mathcal{T}, T)$, $j = 1, 2$ we denote by $\Sigma_j(t)$ the interface of an elementary triod having $S_j(t)$ parallel to $S_j(\mathcal{T})$ with one of the endpoints in $R_j(\mathcal{T})$ and $L_j(t) := |S_j(t)| = w_j(t)$. It follows from Proposition 3.1 that the condition (ii) of Definition 2.12 is satisfied with $\emptyset \neq \overline{S_1(t)} \cap \overline{S_2(t)} =: q(t)$ and furthermore (6.1) holds in $[\mathcal{T}, T)$. For any $j = 1, 2$ define $h_j^\nu : (\mathcal{T}, T) \to \mathbb{R}$ as in (3.11) with $\lim_{t \downarrow \mathcal{T}} h_j^\nu(t) = h_j^\nu(\mathcal{T}-)$. Then $t \in (\mathcal{T}, T) \mapsto \Sigma_1(t) \cup \Sigma_2(t)$ is a flow starting from $\Sigma_1(\mathcal{T}) \cup \Sigma_2(\mathcal{T})$ which satisfies (i)-(iii) of Definition 2.12.
Step 3. We have $T = +\infty$. Since $L_1(t) = L_2(t)$ is the solution of the system (6.6) with initial datum $L_1(\mathcal{T}) = L_2(\mathcal{T})$, $v(t) := L_2(t)/L_1(t) < 1$ for any $t \in [\mathcal{T}, T)$. Moreover, v is increasing since

$$\dot{v}(t) = \frac{\delta(\mathrm{cotg}\alpha_1 + \mathrm{cotg}\vartheta_n)}{L_2^2} v \left(v^2 - 1\right) > 0. \tag{6.7}$$

Set $c_0 := -\dfrac{\sin \alpha_1}{\sin(\alpha_1 + \vartheta_{\mathrm{n}})} = \left(\dfrac{L - 2\delta}{\delta} + \dfrac{\sin(\alpha_1 - \vartheta_{\mathrm{n}})}{\sin \alpha_1}\right)^{-1} \leq \dfrac{\delta}{L - 2\delta} = m$. Then L_1 and L_2 are increasing in $[\mathcal{T}, T)$ since, from (6.5) it follows that $\dot{L}_1(t) > 0$ if and only if $v(t) > c_0 \in (0, m)$ while $\dot{L}_2(t) > 0$ if and only if $v(t) < \frac{1}{c_0} \in (1, +\infty)$. Substituting $w_1 = L_1$ and $w_2 = vL_1 (= L_2)$ in the second equation in (6.6) and solving in L_1, v, yields

$$(1 + v)^{a_1}(1 - v)^{a_2} = \frac{C_{\mathcal{T}}}{L_1}, \tag{6.8}$$

where $a_1 := \dfrac{1 - c_0}{2}$, $a_2 := \dfrac{1 + c_0}{2}$ and $C_{\mathcal{T}} := (L_1(\mathcal{T}) + L_2(\mathcal{T}))^{a_1}(L_1(\mathcal{T}) - L_2(\mathcal{T}))^{a_2}$. If, by contradiction, the maximal time of existence T is finite then, using the first equation in (6.6) (with $w_1 = L_1$

and $w_2 = L_2$) and $L_2 < L_1$ in $[\mathcal{T}, T)$, we get

$$\dot{L}_1(t) \le -\frac{\delta}{L_1(t)}\left(\mathrm{cotg}\alpha_1 + \mathrm{cotg}\vartheta_{\mathfrak{n}} + \frac{1}{\sin\vartheta_{\mathfrak{n}}}\right) = -\frac{\delta}{L_1(t)}\left[\mathrm{cotg}\alpha_1 + \mathrm{cotg}\left(\frac{\vartheta_{\mathfrak{n}}}{2}\right)\right], \quad t \in [\mathcal{T}, T). \quad (6.9)$$

Integrating (6.9), $L_1(T) \le \sqrt{L_1(\mathcal{T})^2 - 2\delta\left[\mathrm{cotg}\alpha_1 + \mathrm{cotg}\left(\frac{\vartheta_{\mathfrak{n}}}{2}\right)\right](T - \mathcal{T})} < +\infty$, which contradicts the maximality of T. Hence, $T = +\infty$. From (6.9) and (6.8), (6.2) follows.
Step 4. For any $t \in (\mathcal{T}, +\infty)$ let $\gamma_4(t)$ be the curve which has initial point in $q(\mathcal{T})$ and is created by the motion of $q(s) := \overline{S_1(s)} \cap \overline{S_2(s)}$ for $s \in (\mathcal{T}, t)$. Then $\gamma(t)$ is φ-regular for any $t \in (\mathcal{T}, \infty)$.
Let $(X(t), Y(t))$ be the component of $qq(t)$ with respect to the (τ_1, ν_1)-axis. Then, from (3.4), we get $\dot{X} = \dot{h}_1^{\nu}(t)\mathrm{cotg}\vartheta_{\mathfrak{n}} - \frac{\dot{h}_2^{\nu}(t)}{\sin\vartheta_{\mathfrak{n}}}$, $\dot{Y} = \dot{h}_1^{\nu}$, and the slope of the tangent to the curve with respect to the (τ_1, ν_1)-axis is given by

$$\mathcal{K}(t) = \frac{\dot{Y}}{\dot{X}} = \frac{\dot{h}_1^{\nu}(t)}{\dot{h}_1^{\nu}(t)\mathrm{cotg}\vartheta_{\mathfrak{n}} - \dot{h}_2^{\nu}(t)\dfrac{1}{\sin\vartheta_{\mathfrak{n}}}} = \left(\mathrm{cotg}\alpha_1 + \frac{1}{v(t)\sin\vartheta_{\mathfrak{n}}}\right)^{-1}.$$

Thus, γ_4 and Σ_3 join in a C^1 fashion since $\mathcal{K}(\mathcal{T}) = \mathrm{tg}(\pi - \vartheta_{\mathfrak{n}}) > 0$. Furthermore, γ_4 is concave in $[\mathcal{T}, +\infty)$ since, from (6.7) $\dot{\mathcal{K}} = \dfrac{\dot{v}\mathcal{K}^2}{v^2\sin\vartheta_{\mathfrak{n}}} > 0$, $\dot{X} = -\delta/(L_1\mathcal{K}) < 0$ and $d^2Y/dX^2 = \dot{\mathcal{K}}/\dot{X} < 0$. Finally

$$\lim_{t\to\infty}\mathcal{K}(t) = \left(\mathrm{cotg}\vartheta_{\mathfrak{n}} + \frac{1}{\sin\vartheta_{\mathfrak{n}}}\right)^{-1} = \mathrm{tg}\left(\frac{\vartheta_{\mathfrak{n}}}{2}\right) < \mathrm{tg}\left(\pi + \frac{2\pi}{n} - \vartheta_{\mathfrak{n}}\right),$$

where the right hand side gives the slope of a segment parallel to R_3.
Step 5. Conclusion of the proof.
Let $\Sigma_3(t) := \gamma_4(t) \cup \Sigma_3(\mathcal{T})$ and $\Pi(t) := \Sigma_1(t) \cup \Sigma_2(t) \cup \Sigma_3(t)$ for any $t \in (\mathcal{T}, +\infty)$. Then $\Pi(t)$ is non-polygonal, unstable, and $t \in [\mathcal{T}, +\infty) \mapsto \Pi(t)$ is a φ-curvature flow starting from $\Pi(\mathcal{T})$. □

Remark 6.2. We expect that the flow of Theorem 5.1 is the unique regular flow starting from $\Pi(\mathcal{T})$. Notice that if $x_{\min} < \delta$ in some open intervall contained in $(\mathcal{T}, \mathcal{T} + \sigma)$, for some $\sigma > 0$, then, in view of the constraint $N_{\min|\Sigma_j}(q) \in T_{\varphi^o}(\nu_{\varphi}^j)$, at time $t = \mathcal{T}$ a new segment S_4 should appear in Σ_3 in such a way that $\Sigma_1(t) \cup \Sigma_2(t) \cup (S_4(t) \cup S_3(t))] \in (\mathsf{a}')$ for any $t \in (\mathcal{T}, \mathcal{T} + \sigma)$, but this would give an unstable triod with $N_{\min} = (X(V_3), Y(V_3), V_3)$ in $(\mathcal{T}, \mathcal{T} + \sigma)$ since $L_2/L_1 < 1$ in $[\mathcal{T}, \mathcal{T} + \sigma)$ and

$$\mathcal{S}_{\mathsf{a}'}\left(\frac{L_1}{L_4}, \frac{L_1}{L_2}\right) = \left\{\left(\frac{L_1}{L_4}, \frac{L_1}{L_2}\right) \in (0, +\infty)^2 : \frac{L_1}{L_2} < \frac{\delta}{L - 2\delta}\right\},$$

a contradiction. Hence $x_{\min} = \delta$ in $[\mathcal{T}, \mathcal{T} + \sigma)$.

7 The case $\mathfrak{n} = 8$ and $\Pi \in (\mathsf{c})$: disappearance of a segment

In this section we show that the flow has two different behavior depending on the initial datum $\Pi \in (\mathsf{c})$. For a suitable choice of Π, we show that one of the three segments vanishes at $t = \mathcal{T}$, its φ-curvature remains bounded, the Cahn-Hoffman vector field $N_{\min}$ has a jump discontinuity at $q(\mathcal{T})$ on each Σ_j and the triple junction translates along the remaining adjacent half-line in $[\mathcal{T}, +\infty)$. For the other choices of stable $\Pi \in (\mathsf{c})$ we prove that at time $t = \mathcal{T} \in (0, +\infty)$ the flow becomes unstable, a curve appears from the triple junction, as in Section 6, with the difference that the adjacent segment now has positive φ-curvature and keeps on moving at subsequent times.
In the following theorem we denote by $x_{\min}^{(\mathsf{b})}(\Lambda_1, +\infty, \Lambda_3))$ the limit of $x_{\min}^{(\mathsf{b})}(\Lambda_1, \Lambda_2, \Lambda_3))$ as $\Lambda_2 \to +\infty$, where $x_{\min}^{(\mathsf{b})}$ is defined as in (2.22).

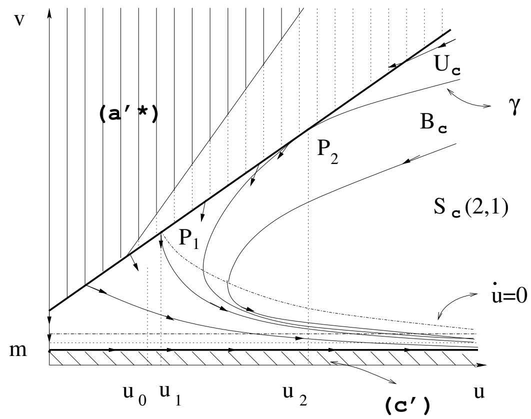

Figure 12: The white region is the stability region $\mathcal{S}_c(2,1)$. Flow lines diagram of system (7.5) corresponding to φ-curvature flows starting from stable $\Pi \in \{(\mathsf{c})\}$ for $\mathfrak{n} = 8$ (see Theorem 7.1); $x_{\min}(\mathcal{T}) \in \{\delta, L-\delta\}$.

Theorem 7.1. *Let $\mathfrak{n} = 8$ and $\Pi \in (\mathsf{c})$ be stable. Then $\mathcal{T} < +\infty$ and $\Pi(\mathcal{T})$ is unstable. Moreover, there exists a curve γ tangent to the line $\{x_{\min}(u,v) = \delta\}$ at $(u_2, v_2) = P_2$ (see Figure 12) which divides $\mathcal{S}_\mathsf{c}(2,1)$ in two disjoint regions $U_\mathsf{c} := \{(\bar u, \bar v) \in \mathcal{S}_\mathsf{c}(2,1) \,:\, \bar u > u_2,\, \bar v > v_\gamma := \gamma \cap \{u = \bar u\}\}$ and $B_\mathsf{c} := \mathcal{S}_\mathsf{c}(2,1) \setminus U_\mathsf{c}$ such that*

(i) if $(u(0), v(0)) \in B_\mathsf{c}$ then $N_{\min}(\mathcal{T}-) = (X(V_2), W_2, Z(W_2))$, i.e. $x(\mathcal{T}-) = L-\delta$, $L_2(\mathcal{T}-) = 0$, $\kappa_\varphi(l_2(\mathcal{T}-)) = 0$ and $\dfrac{L_3(\mathcal{T}-)}{L_1(\mathcal{T}-)} = \dfrac{\sqrt{2}}{2}$. Furthermore, there exists a stable φ-curvature flow in $(\mathcal{T}, +\infty)$ starting from $\Pi(\mathcal{T})$ with

$$\kappa_\varphi(l_1(t)) = \frac{L - \widetilde{z}(t)}{L_1(t)}, \qquad \kappa_\varphi(l_3(t)) = -\frac{L - \widetilde{x}(t)}{L_3(t)}, \qquad t \in (\mathcal{T}, +\infty) \tag{7.1}$$

where $\widetilde{x}(t) := x^{(\mathsf{b})}_{\min}\big(L_3(t), +\infty, L_1(t))\big) = \dfrac{\delta(L-\delta)^2 L_3(t)}{\delta^2 L_3(t) + (L-2\delta)^2 L_1(t)}$, $\widetilde{z}(t) = m\widetilde{x}(t) + q_z$. Finally, the triple junction translates along $R_2(\mathcal{T})$ in $(\mathcal{T}, +\infty)$ and

$$\lim_{t\uparrow\infty} \widetilde{x}(t) = L - \delta, \qquad \lim_{t\uparrow\infty} \frac{L_3(t)}{L_1(t)} = \sqrt{2}, \qquad \lim_{t\downarrow\mathcal{T}} \widetilde{x}(t) = \delta\,\frac{(1+\sqrt{2})^2}{1+2\sqrt{2}} \in (\delta, L-\delta).$$

In particular, $N_{\min}(\mathcal{T}-) \neq N_{\min}(\mathcal{T}+)$, see Figure 13 (i).

(ii) If $(u(0), v(0)) \in U_\mathsf{c}$ then $N_{\min}(\mathcal{T}) = (X(V_3), Y(V_3), V_3)$. Furthermore, there exist $\mathcal{T}_2 \in (\mathcal{T}, +\infty]$ and a φ-curvature flow in $[\mathcal{T}, \mathcal{T}_2)$ starting from $\Pi(\mathcal{T})$. Furthermore, for any $t \in [\mathcal{T}, \mathcal{T}_2)$, $\Pi(t)$ is non-polygonal, $N_{\min}(t) = (X(V_3), Y(V_3), V_3)$, $\kappa_\varphi(l_3(t)) = -L/L_3(t)$ and (6.1) hold, see Figure 13 (ii).

Proof. For any $t \in [0, \mathcal{T})$ we define $x(t) = x^{(\mathsf{c})}_{\min}(L_1(t), L_2(t), L_3(t)$, $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t)$. Then system (2.9) reads as

$$\dot{h}^\nu_1(t) = -\frac{x(t)}{L_1(t)}, \qquad \dot{h}^\nu_2(t) = \frac{y(t)}{L_2(t)}, \qquad \dot{h}^\nu_3(t) = \frac{L - z(t)}{L_3(t)}, \tag{7.2}$$

system (3.1) reads as

$$L_1(t) = L_1 + \sqrt{2}h_3^\nu(t), \qquad L_2(t) = L_2 + h_2^\nu(t) - h_3^\nu(t), \qquad L_3(t) = L_3 + h_2^\nu(t) + h_3^\nu(t), \tag{7.3}$$

while (3.2) and (3.3) become

$$-\sqrt{2}h_1^\nu(t) = h_2^\nu + h_3^\nu \qquad \text{and} \qquad \sqrt{2}(L_1(t) - L_1) = L_3(t) - L_2(t) + L_2 - L_3. \tag{7.4}$$

Define $u(t) := \dfrac{L_3(t)}{L_2(t)}$ and $v(t) := \dfrac{L_3(t)}{L_1(t)}$. Differentiating (7.3) with respect to time and using (2.23), (2.19), and $m = \frac{\sqrt{2}}{2}$, we obtain for any $t \in [0, \mathcal{T})$

$$\begin{aligned} \dot{u} &= \frac{L-\delta}{L_3^2} \, \frac{u\,[2\delta^2u^2 + v(L-2\delta)(-\delta u^2 + Lu + L - \delta)]}{\delta^2 + \delta^2 u + (L-2\delta)^2 v} \\ \dot{v} &= \frac{(L-2\delta)(L-\delta)^2}{L_3^2} \, \frac{v^2\,(1-\sqrt{2}v)}{\delta^2 + \delta^2 u + (L-2\delta)^2 v}. \end{aligned} \tag{7.5}$$

Recall that the stability region $\mathcal{S}_c(2,1)$ is given by Lemma 3.10, see Figure 12. It follows that $\dot{v} < 0$ in $\mathcal{S}_c(2,1)$ with the equality holding only if $v = 0$ or $v = m$ (i.e. $\{x_{\min}(u,v) = L - \delta\}$). Notice that $-\delta u^2 + Lu + L - \delta < 0$ for $u > u_0 := \frac{L + \sqrt{L^2 + 4\delta(L-\delta)}}{2\delta}$. Moreover, $\dot{u} \le 0$ if and only if $u > u_0$ and

$$v \geq \frac{\sqrt{2}\delta u^2}{\delta u^2 - Lu - L + \delta} \tag{7.6}$$

or, more precisely, if and only if (7.6) holds for any $u > u_1$, where $u_1 := \frac{L+\delta+\sqrt{(L+\delta)^2+4\delta L}}{2\delta}$ is the intersection point of the line $\{x_{\min}(u,v) = \delta\}$ and the curve of points satisfying $\dot{u} = 0$. Since the condition

$$\left(\frac{\dot{v}}{\dot{u}}\right)\Big|_{\{x_{\min}(u,v)=\delta\}} < \frac{\delta}{L-2\delta}, \qquad u > u_1$$

is satisfied if and only if $g(u) := -\delta^3u^3 + 2\delta^2Lu^2 + \delta L(2L-\delta)u + L^2(L-\delta) < 0$, that is, for any $u > u_2$, $u_2 > u_1$ (for $g(u_0) > 0$), it follows that the trajectories of solutions of system (7.5) intersects the line $\{x_{\min}(u,v) = \delta\}$ for $u < u_2$. Denote by P_2 the point belonging to the line $\{x_{\min}(u,v) = \delta\}$ having u-coordinate equal to u_2. Let $\gamma \subset \mathcal{S}_c$ be the flow line tangent to $\{x_{\min}(u,v) = \delta\}$ in P_2. Then γ decompose $\mathcal{S}_c(2,1)$ into B_c and U_c.
Let us first prove (ii). If $(u(0), v(0)) \in U_c$, then the trajectory of the solution of (7.5) intersects the line $\{x_{\min}(u,v) = \delta\}$ at $\mathcal{T}$. It is clear that $\mathcal{T} < +\infty$ since any $L_j(\mathcal{T})$ is bounded and

$$\dot{L_1}(t) \le -2\,\dot{h_1^\nu}(t) \le 2\,\frac{L-\delta}{L_1(t)}, \qquad \dot{L_3}(t) = -\sqrt{2}\,\dot{h_1^\nu}(t) \ge \frac{\delta}{L_1(t)}.$$

Let $(w_1, w_2) \in C^\infty([\mathcal{T}, T); (0,+\infty)^2)$ for some $T \in (\mathcal{T}, +\infty]$ be the solution of (6.6) with F defined as in (6.5). For any $t \in (\mathcal{T}, T)$, $j = 1, 2$ we define $\Sigma_j(t)$, h_j^ν and γ_4 as in Theorem 6.1. Then, following the same argument, (6.1) hold in $[\mathcal{T}, T)$, $t \in (\mathcal{T}, T) \mapsto \Sigma_1(t) \cup \Sigma_2(t)$ is a flow starting from $\Sigma_1(\mathcal{T}) \cup \Sigma_2(\mathcal{T})$ which satisfies (i)-(iii) of Definition 2.12, $T = +\infty$ and γ_4 is concave and φ-regular in $[\mathcal{T}, +\infty)$. Let $t \in [\mathcal{T}, +\infty) \to \gamma_4(\infty) \cup S_3(t) \cup R_3(t)$ be the φ-curvature flow starting from $\gamma_4(\infty) \cup (S_3 \cup R_3)(\mathcal{T})$. Since $h_3^\nu(t) = -L/L_3(t) \in C^\infty([\mathcal{T}, +\infty))$ and from (7.4) we have

$$\sqrt{2}|\dot{h_1^\nu}(\mathcal{T})| = -\sqrt{2}\dot{h_1^\nu}(\mathcal{T}) = \dot{h_2^\nu}(\mathcal{T}) + \dot{h_3^\nu}(\mathcal{T}) > \dot{h_3^\nu}(\mathcal{T}) = |\dot{h_3^\nu}(\mathcal{T})|,$$

it follows that $|\dot{h_3^\nu}(t)| \le \sqrt{2}|\dot{h_1^\nu}(t)|$ (that is $v(t) \ge \sqrt{2}/2$) for any t in a neighborhood of $\mathcal{T}$, say $(\mathcal{T}, \mathcal{T}_2)$. We conclude that the normal velocity $\dot{h_3^\nu}$ of $S_3(t)$ is smaller than the ones of $S_1(t)$ and $S_3(t)$.

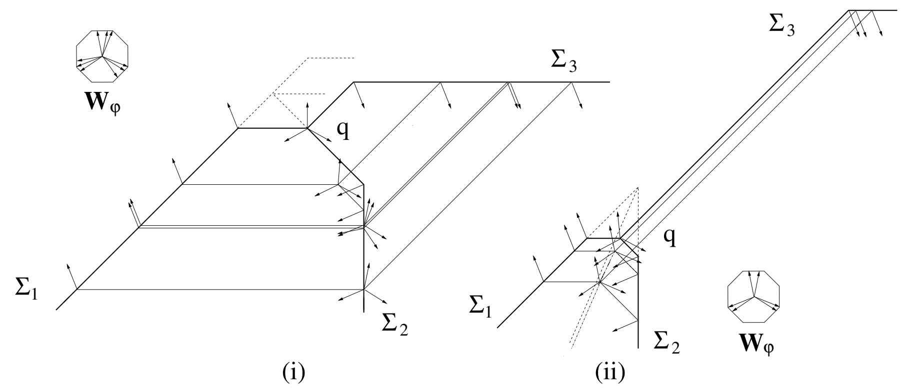

Figure 13: φ-curvature flow starting from a stable $\Pi \in \{(\mathsf{c})\}$ (see Theorem 7.1): (i) the segment S_2 has zero length at time $t = \mathcal{T}$ and $x(\mathcal{T}) = L - \delta$; (ii) $x(\mathcal{T}) = \delta$ and a curve develops from the triple junction for $t > \mathcal{T}$.

Thus, setting $\Sigma_3(t) := \gamma_4(t) \cup (S_3 \cup R_3)(t)$ and $\Pi(t) := \Sigma_1(t) \cup \Sigma_2(t) \cup \Sigma_3(t)$ for any $t \in (\mathcal{T}, \mathcal{T}_2)$, we conclude that the triod $\Pi(t)$ is φ-regular and unstable in $[\mathcal{T}, \mathcal{T}_2)$ and $t \in [\mathcal{T}, \mathcal{T}_2) \mapsto \Pi(t)$ is the unique φ-curvature flow starting from $\Pi(\mathcal{T})$.

Let us prove (i). Given $\sigma > 0$, set

$$B_{\mathsf{c}}(\sigma) := \{(u, v) \in B_{\mathsf{c}} \, : \, 2\delta^2 u^2 + v(L - 2\delta)(-\delta u^2 + Lu + L - \delta) > \sigma, \, u > u_1\}.$$

Notice that $B_{\mathsf{c}}(\sigma) = \{(u, v) \in B_{\mathsf{c}} \, : \, \dot{u}(u, v) > \sigma, \, u > u_1\}$. Without loss of generality, assume $(u(0), v(0)) \in B_{\mathsf{c}}(\sigma)$. Then $(u(t), v(t)) \in B_{\mathsf{c}}(\sigma)$ for any $t \in [0, \mathcal{T})$. Since for any $(u, v) \in B_{\mathsf{c}}(\sigma)$ we deduce the following estimates

$$-c_1 \frac{\sqrt{2}v - 1}{\sigma u} \leq \frac{\dot{v}}{\dot{u}} \leq -c_2 \frac{\sqrt{2}v - 1}{u^3}$$

with $c := (L - 2\delta)(L - \delta)v_1^2$ and $c_2 := \frac{(L-2\delta)(L-\delta)}{4\delta^2}$. Integrating yields

$$\frac{\sqrt{2}}{2}\left(1 + u^{-\frac{\sqrt{2}c_2}{\sigma}}\right) \leq v \leq \frac{\sqrt{2}}{2}\left(1 + e^{\frac{c_2}{\sqrt{2}\,u^2}}\right),$$

and hence $\lim_{t \to \mathcal{T}} u(t) = +\infty$, $\lim_{t \to \mathcal{T}} v(t) = \frac{\sqrt{2}}{2}$. Thus the first part of the assertion follows from (2.23) and (2.19).

Let us show $\mathcal{T} < +\infty$. Let $\varepsilon > 0$ and assume $v(0) \leq \sqrt{2} - \varepsilon$. Then $v(t) \leq \sqrt{2} - \varepsilon$ for any $t \in [0, \mathcal{T})$, and $\dot{L_3}(t) = \sqrt{2} \frac{x}{L_1(t)} \leq C L_3(t)$, where $C := (2 - \sqrt{2}\varepsilon)(l - \delta)$. It follows that

$$L_3(t) \leq \sqrt{L_3^2 + 2Ct}. \tag{7.7}$$

Using (7.3) and (7.2) we get

$$\dot{L_2}(t) = -\sqrt{2}\dot{h}_1^\nu(t) - 2\dot{h}_3^\nu(t) \leq \sqrt{2}\frac{L - \delta}{L_3(t)}\left(\frac{L_3(t)}{L_1(t)} - \sqrt{2}\right) \leq -\sqrt{2}\varepsilon\,\frac{L - \delta}{L_3(t)},$$

and, inserting (7.7) and integrating, yields

$$L_2(t) \leq L_2^2 - \frac{\sqrt{2}\varepsilon(L - \delta)}{C}\sqrt{L_3^2 + 2Ct}.$$

We conclude that $\mathcal{T} < +\infty$ and, from Proposition 3.7 (v), also $L_1(\mathcal{T}-)$, $L_3(\mathcal{T}-) < +\infty$. Moreover, $\Pi(\mathcal{T}-) \in (\mathtt{b}')$ is degenerate. Notice that the φ-curvature flow starting from $\Pi(\mathcal{T}-)$ can be described as φ-curvature flow starting from $\widetilde{\Pi} \in (\mathtt{b})$, obtained by a rotation and a symmetry with respect the L_3-axis from Π, i.e. $\widetilde{L_2} = +\infty$ and $\widetilde{L_3} = \sqrt{2}\widetilde{L_1}$. Recalling the mirror law (2.24), $\Pi(\mathcal{T}-)$ is stable, the φ-curvature flow starting from $\Pi(\mathcal{T}-)$ satisfies system (2.7) with φ-curvatures given as in (7.1) and

$$\lim_{t\to\infty} \frac{\widetilde{L_3}(t)}{\widetilde{L_1}(t)} = \lim_{t\to\infty} \frac{L_1(t)}{L_3(t)} = \frac{\sqrt{2}}{2}.$$

Finally, since $\delta\, \dot{h_1^\nu} = -(L-\delta)\, \dot{h_3^\nu}$ in $[\mathcal{T}, +\infty)$, the triple junction translates along $R_2(\mathcal{T})$. □

Remark 7.2. We believe that the flows of Theorem 7.1 (i) and (ii) are the unique regular flows starting from $\Pi(\mathcal{T})$.

8 Adjacent triple junctions: solutions after the collision

In this section we fix $\mathcal{W}_\varphi = P_8$ and consider the φ-curvature flow starting from a *stable* φ-regular partition, denoted by Υ, consisting of two adjacent elementary triple junctions q_1 and q_2. Given a Cahn-Hoffman vector field N on Υ as in Figure 14, we set

$$(X_1, Y_1, Z_1) := \Big(N_{|\Sigma_1}(q_1), N_{|\Sigma_2}(q_1), N_{|\Sigma_3}(q_1)\Big), \qquad (X_2, Y_2, Z_2) := \Big(N_{|\Sigma_1}(q_2), N_{|\Sigma_4}(q_2), N_{|\Sigma_5}(q_2)\Big),$$

and $x_j := |V_1 - N_{|\Sigma_1}(q_j)|$, $y_j := y(x_j)$, $z_j := z(x_j)$, $j = 1, 2$, where y and z are defined in (2.19). From Proposition 2.21 we know that the admissible triplet (X_j, Y_j, Z_j) is uniquely associated with (x_j, y_j, z_j). Noticing that we can restrict the minimum (2.3) to vector fields which are linear on each Σ_i and verify the required constraints, the problem of finding $N_{\min}$ in Definition 2.4 reduces [3] to the following minimum problem:

$$\min_{(x_1,x_2)\in[\delta, L-\delta]^2} f(x_1, x_2),$$

where

$$f(x_1, x_2) = \int_\Upsilon (\mathrm{div}_\tau N)^2\, \varphi^o(\nu)\; d\mathcal{H}^1 = \sigma_1 x_1^2 + \sigma_2 x_2^2 + \sigma_{12} x_1 x_2 + \beta_1 x_1 + \beta_2 x_2 + \gamma,$$

and σ_1, σ_2, σ_{12}, β_1, β_2, γ are coefficients depending on the configuration we are analyzing.
We say that Υ is stable if $(N_{\min})_{|\Sigma_j}(q_k)$ is not a vertex of $\mathcal{W}_\varphi$ for any $j = 1, 2, 3$ and $k = 1, 2$. We say that Υ is unstable if it is not stable. The stability of Υ is equivalent to

$$(x_{1\min}, x_{2\min}) \in \; (\delta, L-\delta)^2.$$

Notice that if Υ is stable then

$$x_{1\min} = \frac{\sigma_{12}\beta_2 - 2\sigma_2\beta_1}{4\sigma_1\sigma_2 - \sigma_{12}^2}, \qquad x_{2\min} = \frac{\sigma_{12}\beta_1 - 2\sigma_1\beta_2}{4\sigma_1\sigma_2 - \sigma_{12}^2}. \tag{8.1}$$

From now on, $x_{1\min}$, $x_{2\min}$ will be denoted simply by $\overline{x}_1$, $\overline{x}_2$ and we set $\overline{y}_j := y(\overline{x}_j)$, $\overline{z}_j := z(\overline{x}_j)$.
The discussion of finding which values of L_i, $i = 1, \dots, 5$, provide a stable Υ simplifies only in the case of adjacent triple junctions which either belong to the same symmetry classes (i.e. $L_2 = L_4$ and $L_3 = L_5$, see Figure 14 (i)) or are symmetric with respect to the axis orthogonal to Σ_1 at its middle point (see Figure 14 (ii)), leading respectively to $\overline{x}_1 = \overline{x}_2$ and $\overline{x}_1 = L - \overline{x}_2$.
Let $T > 0$ and let us introduce the orientation of Υ as in the comment after Definition 2.8. We say that $t \in [0, T) \mapsto \Upsilon(t)$ is a φ-curvature flow starting from Υ if $\Upsilon(t)$ is a φ-regular partition consisting

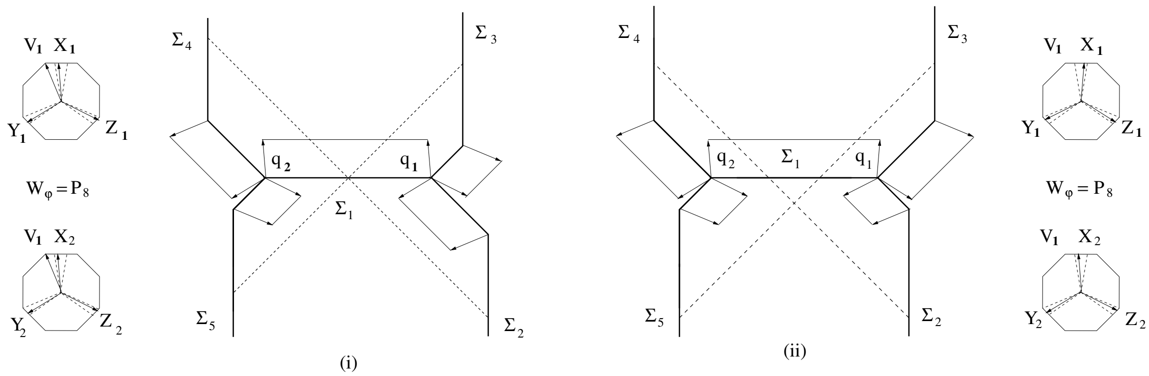

Figure 14: Example 8.1: collision of two adjacent triple junctions; in (i) we have $L_2 = L_4$, $L_3 = L_5$ and $\kappa_\varphi = 0$ on Σ_1.

of two adjacent elementary triple junctions $q_1(t)$ and $q_2(t)$, and conditions (ii)-(iii) of Definition 2.12 hold for any $j = 1, 2, 3, 4, 5$. If the adjacent triple junctions of Υ belong to the same symmetry classes or are symmetric with respect to the axis orthogonal to Σ_1 then, arguing as in Theorem 3.3, one can show that there exists a unique stable φ-curvature flow starting from a stable Υ.
When Υ is not stable, at least one of the two triple junctions is not stable. If in addition the gradient of f on $\partial[\delta, L-\delta]^2$ points inside $[\delta, L-\delta]^2$, then the appearance of a new edge from one of the two triple junctions (or from both) is forced during the subsequent crystalline flow.
The following example shows that the collision phenomenon occurs and a quadrijunction forms.

Example 8.1. Consider the partition of Figure 14 (i) with $L_1 > 0$, $L_2 = L_4 > 0$ and $L_3 = L_5 > 0$. In this case f reads as

$$f(x_1, x_2) = \varphi^o(\nu_1)\left(\frac{(x_1 - x_2)^2}{L_1} + \frac{y_1^2}{L_2} + \frac{z_1^2}{L_3} + \frac{y_2^2}{L_4} + \frac{z_2^2}{L_5}\right), \tag{8.2}$$

so that

$$\begin{aligned} &\sigma_1 = \tfrac{1}{L_1} + m\left(\tfrac{1}{L_2} + \tfrac{1}{L_3}\right) > 0, \quad \sigma_2 = \tfrac{1}{L_1} + m\left(\tfrac{1}{L_4} + \tfrac{1}{L_5}\right) > 0, \quad \sigma_{12} = -\tfrac{2}{L_1}, \\ &\beta_1 = 2m\left(\tfrac{-q_y}{L_2} + \tfrac{q_z}{L_3}\right), \qquad \beta_2 = 2m\left(\tfrac{-q_y}{L_4} + \tfrac{q_z}{L_5}\right), \end{aligned} \tag{8.3}$$

where m, q_y, q_z are defined in (2.15) and in (2.16). Thus, the triod is *always stable* since using (8.1)

$$\overline{x}_1 = \overline{x}_2 = \frac{-\beta_1}{2\sigma_1 + \sigma_{12}} = \frac{\delta L_2 + (L-\delta)L_3}{L_2 + L_3} \in (\delta, L-\delta). \tag{8.4}$$

The evolution equations are given by

$$\dot{h_1^\nu} = 0, \quad \dot{h_2^\nu} = \frac{\overline{y}_1}{L_2}, \quad \dot{h_3^\nu} = -\frac{\overline{z}_1}{L_3}, \quad \dot{h_4^\nu} = -\frac{\overline{y}_2}{L_4}, \quad \dot{h_5^\nu} = \frac{\overline{z}_2}{L_5},$$

so that the triple junctions move along Σ_1 until they collide at the middle point at a finite time.

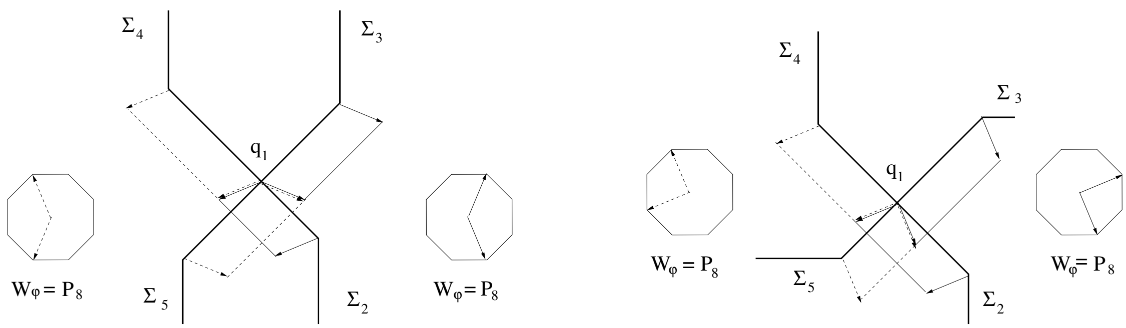

Figure 15: These quadrijunctions are φ-regular with $\kappa_\varphi = 0$ and unstable, i.e. $N_{\min|\Sigma_j}(q_1)$ is a vertex of $\mathcal{W}_\varphi$ for some $j = 2, 3, 4, 5$.

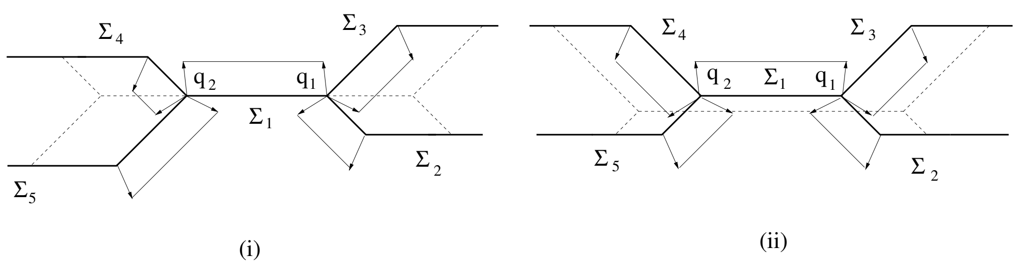

Figure 16: Example 8.2: these configurations are stable only for suitable choices of $L_j,\ j = 1, \dots, 5$.

Consider now the partition of Figure 14 (ii) with $L_1 > 0$, $L_2 = L_5 > 0$ and $L_3 = L_4 > 0$. Formulas (8.2) and (8.3) still hold and, using (8.1), we get

$$\overline{x}_1 = L - \overline{x}_2 = \frac{\dfrac{L}{m^2 L_1} + \dfrac{L-\delta}{L_2} + \dfrac{\delta}{L_3}}{\dfrac{2}{m^2 L_1} + \dfrac{1}{L_2} + \dfrac{1}{L_3}} \in (\delta, L - \delta).$$

Therefore, the triod is *always stable* and the evolution equations are given by

$$\dot{h_1^\nu} = -\frac{\overline{x}_1 - \overline{x}_2}{L_1} = \frac{L - 2\overline{x}_1}{L_1}, \quad \dot{h_2^\nu} = \frac{\overline{y}_1}{L_2}, \quad \dot{h_3^\nu} = -\frac{\overline{z}_1}{L_3}, \quad \dot{h_4^\nu} = -\frac{\overline{y}_2}{L_4}, \quad \dot{h_5^\nu} = \frac{\overline{z}_2}{L_5}.$$

If in addition $L_3 > L_2$, the triple junctions move as shown in Figure 14 (ii) until they collide at a finite time and a quadrijunction forms.

In general, it is not clear what happens after the collision. In a special case the solution can be continued in a "natural" way, see Example 8.4. A quadrijunction Ξ as in Figure 15 is φ-regular with $\kappa_\varphi = 0$ and unstable, i.e. $N_{\min|\Sigma_j}(q_1)$ is a vertex of $\mathcal{W}_\varphi$ for some $j \in \{2, 3, 4, 5\}$. Indeed, the minimizer $N_{\min}$ of (2.3) with Γ replaced by Υ which satisfies $\sum_{j=2}^{5} (N_{\min|\Sigma_j})^{\partial \Sigma_j} = 0$ is given in Figure 15. Finally, $t \mapsto \Xi$ is a stationary φ-curvature flow starting from Ξ, i.e. Ξ does not move.
The following example concerns the stability of the partitions given in Figure 16 (see (8.6)), and will be used to construct the flow after the collision of two triple junctions (see Example 8.1).

Example 8.2. Consider the partition of Figure 16 (i) with $L_1 > 0$, $L_2 = L_4 > 0$ and $L_3 = L_5 > 0$. In this case

$$f(x_1, x_2) = \varphi^o(\nu_1) \left(\frac{(x_1 - x_2)^2}{L_1} + \frac{(L - y_1)^2}{L_2} + \frac{(L - z_1)^2}{L_3} + \frac{(L - y_2)^2}{L_4} + \frac{(L - z_2)^2}{L_5} \right), \tag{8.5}$$

so that σ_1, σ_2, σ_{12} are given as in (8.3) $\beta_1 = 2m \left(\frac{L-q_y}{L_2} - \frac{L-q_z}{L_3} \right)$ and $\beta_2 = 2m \left(\frac{L-q_y}{L_4} - \frac{L-q_z}{L_5} \right)$. Furthermore, since the first two equalities in (8.4) hold, we have

$$\overline{x}_1 = \overline{x}_2 = \frac{(L-\delta)^2 L_2 - (L^2 - 3\delta L + \delta^2) L_3}{\delta(L_2 + L_3)} \in (\delta, L - \delta) \iff \frac{L_3}{L_2} \in \left(\frac{L - \delta}{L}, \frac{L}{L - \delta} \right). \tag{8.6}$$

The evolution equations are given by

$$\dot{h_1^\nu} = 0, \quad \dot{h_2^\nu} = -\frac{L - \overline{y}_1}{L_2}, \quad \dot{h_3^\nu} = \frac{L - \overline{z}_1}{L_3}, \quad \dot{h_4^\nu} = \frac{L - \overline{y}_2}{L_4}, \quad \dot{h_5^\nu} = -\frac{L - \overline{z}_2}{L_5}.$$

We observe that, because of the symmetry, $\dot{h_4^\nu} = -\dot{h_2^\nu}$, $\dot{h_5^\nu} = -\dot{h_3^\nu}$ and, by direct computations, $\dot{h_2^\nu} = -(2L - \delta)/(L_2 + L_3) = -\dot{h_3^\nu}$. Hence, assuming that the initial partition is stable, the flow is stable in the whole of $[0, +\infty)$ with the triple junctions translating in opposite directions along Σ_1.
Consider now the partition of Figure 16 (ii) with $L_1 > 0$, $L_2 = L_5 > 0$ and $L_3 = L_4 > 0$. Since (8.5) still holds, the expressions of σ_1, σ_2, σ_{12}, β_1, β_2 are the same. Using (8.1) it follows

$$\overline{x}_1 = L - \overline{x}_2 = \frac{\dfrac{L(L-2\delta)^2}{L_1} - \dfrac{\delta(L^2 - 3\delta L + \delta^2)}{L_2} + \dfrac{\delta(L-\delta)^2}{L_3}}{\dfrac{2(L-2\delta)^2}{L_1} + \dfrac{\delta^2}{L_2} + \dfrac{\delta^2}{L_3}},$$

so that

$$\overline{x}_1 > \delta \iff \frac{L_3}{L_1} > \frac{m}{L - 2\delta} \left(-L + (L - \delta) \frac{L_3}{L_2} \right),$$

$$\overline{x}_1 < L - \delta \iff \frac{L_3}{L_1} > \frac{m}{L - 2\delta} \left(L - \delta - \delta \frac{L_3}{L_2} \right).$$

Therefore, assuming that the initial partition is stable, the evolution equations are given by

$$\dot{h_1^\nu} = \frac{L - 2\overline{x}_1}{L_1}, \quad \dot{h_2^\nu} = -\frac{L - \overline{y}_1}{L_2}, \quad \dot{h_3^\nu} = \frac{L - \overline{z}_1}{L_3}, \quad \dot{h_4^\nu} = \frac{L - \overline{y}_2}{L_4}, \quad \dot{h_5^\nu} = -\frac{L - \overline{z}_2}{L_5}. \tag{8.7}$$

If in addition $L_3 > L_2$, the triple junctions move (for small times) as shown in Figure 14 (ii).

Remark 8.3. From the computations made in Example 8.2, it follows that the first partition is stable for any $L_1 > 0$ while the second one is stable provided that L_1 is small enough. In particular, if $L_1 = 0$ then $\overline{x}_1 = \overline{x}_2 = \frac{L}{2}$. Hence, in Example 8.2 we *have constructed stable φ-curvature flows starting from a quadrijunction.*

In the following example we construct flows after the collision of two triple junctions.

Example 8.4. Consider the partition of Figure 17 (i) with $L_1 > 0$, $L_2 = L_4 = L_3 = L_5 > 0$. As shown in Example 8.1 there exists a finite time $T_0 > 0$ such that $L_1(T_0) = 0$. From the considerations made in Remark 8.3, there exists at least one stable φ-curvature flow starting from the quadrijunction $\Upsilon(T_0)$ as shown in Figure 17 (i). This is not the only stable φ-curvature flow starting from $\Upsilon(T_0)$. There are other candidates to continue the flow after the singularity: the stable flows shown in Figures 17 (ii)

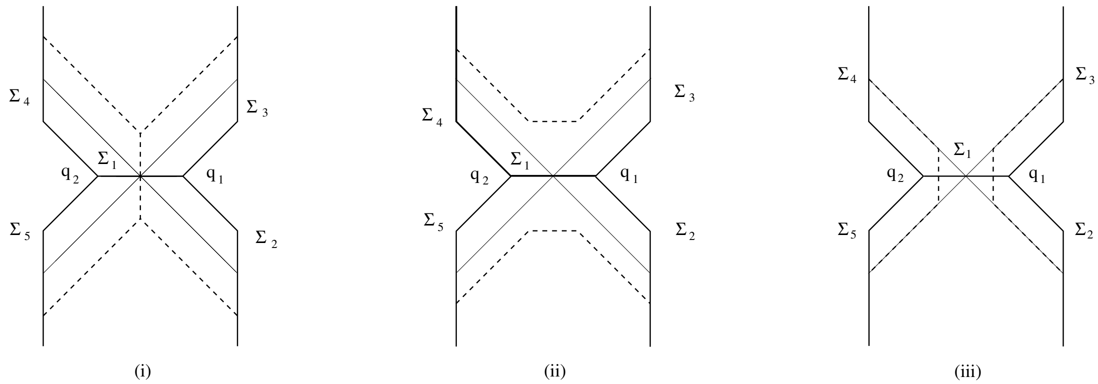

Figure 17: Example 8.4: existence of three stable flows *after* the collision of two adjacent triple junctions. Another candidate to continue the flow after the singularity is the stationary (not stable) flow $t \in [T_0, +\infty) \mapsto \Upsilon(T_0)$. The dotted partition in (i) is the rotation of 90 degrees of partition in Figure 16 (i) with $L_2 = L_3 = L_4 = L_5$.

and (iii) and the stationary flow $t \in [T_0, +\infty) \mapsto \Upsilon(T_0)$. The latter flow is not stable and has the largest energy (1.1) among the four flows.
Some explicit comparisons between the energies of the different evolutions can be made. For instance, if we denote by $\Upsilon_{(i)}(t)$ (resp. $\Upsilon_{(ii)}(t)$) the partition in Figure 17 (i) (resp. Figure 17 (ii)) at time t, then $\mathcal{F}_\varphi(\Upsilon_{(ii)}(t)) < \mathcal{F}_\varphi(\Upsilon_{(i)}(t))$ for any $t > T_0$, since $\dot{h}^\nu_{3\,(i)}(t) = \frac{2l-\delta}{2L_3(T_0)} < \frac{L}{L_3(t)} = \dot{h}^\nu_{3\,(ii)}(t)$.
Furthermore, notice that in the case of the flow in Figure 17 (i), $\kappa_\varphi(l_j(T_0-)) = \kappa_\varphi(l_j(T_0+))$ for any $j = 2, 3, 4, 5$, while in the other cases $\kappa_\varphi(l_j(T_0-)) \neq \kappa_\varphi(l_j(T_0+))$ for any $j = 2, 3, 4, 5$.
We believe that a selection of the "most natural" evolution between the three flows in Figure 17 cannot probably be done if one considers the evolution of interfaces without looking at the phases, i.e. without looking at the interfaces as the "boundaries" of their interior.

9 Homothetic flows and asymptotic convergence ($\mathfrak{n} = 6\mathfrak{m}$)

In this section we introduce the notion of homothetic flows and we assume $\mathfrak{n} = 6\mathfrak{m}$. In the case of curves the homothetic flows by crystalline curvature have been studied in [29], [15].

Definition 9.1. *Let Π be elementary. We say that $t \in [0, +\infty) \mapsto \Pi(t)$ is a homothetic flow starting from $\Pi = \Pi(0)$ if there exists $\lambda \in C^0([0, +\infty))$, $\lambda(0) = 1$, such that $\Pi(t) = \lambda(t)\Pi(0) + qq(t)$. If $\lambda \equiv 1$ we say that the flow is translating; if in addition $qq(t) = q$ the flow is stationary.*

Remark 9.2. $t \in [0, +\infty) \mapsto \Pi(t)$ is homothetic if and only if $\dfrac{L_j(t)}{L_i(t)} = \dfrac{L_j(0)}{L_i(0)}$ for any $i, j = 1, 2, 3$.
The flow is stationary whenever an elementary triod has two of the segments S_j of infinite length.

We characterize now all homothetic flows for $\mathfrak{n} = 6\mathfrak{m}$. If $\mathfrak{n} = 6$ we will consider the following limit cases of degenerate triod:

(i) $\Pi \in (\mathsf{a})$ with $L_1 = 0$ and $L_3 = +\infty$ (see Figure 18 (i));

(ii) $\Pi \in (\mathsf{a})$ with $L_2 = 0$ and $L_3 = +\infty$ (see Figure 18 (ii));

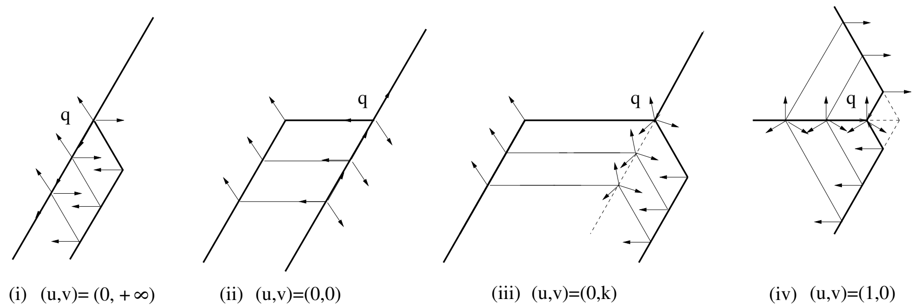

Figure 18: The case $\mathfrak{n} = 6$. The flows (i), (ii) and (iii) are translating; (iv) is homothetic.

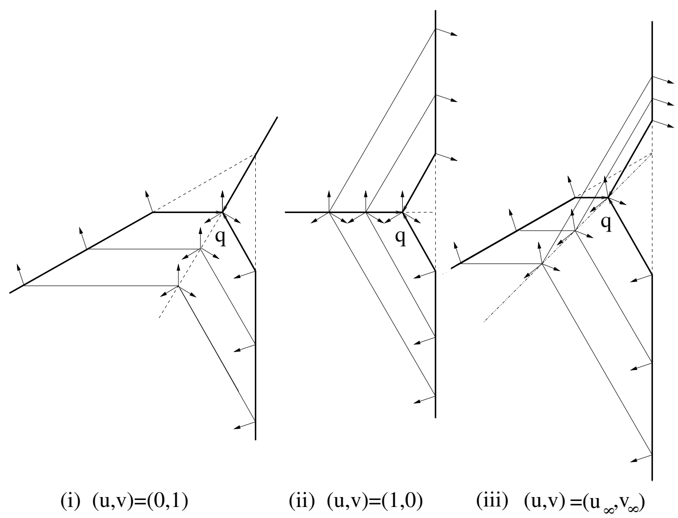

Figure 19: The case $\mathfrak{n} = 12$. The flows (i), (ii) and (iii) are homothetic.

Theorem 9.3. *Let* $\mathfrak{n} = 6\mathfrak{m}$. *If* $\Pi \in (\mathtt{d})$, *the flow is stationary for any choice of* $L_1, L_2, L_3 \in (0, +\infty]$. *If* $\Pi \in (\mathtt{a})$, *the flow is homothetic if and only if one of the following holds:*

(i) $L_2 = +\infty$ *and* $L_1, L_3 \in (0, +\infty]$. *The flow is stationary and* $\Pi(t)$ *is unstable for any* $t \in [0, +\infty)$.

(ii) $L_1 = +\infty$ *and* $L_2 = L_3$ *(see Figures 18 (iv) and 19 (ii)). The flow is stable.*

(iii) $\mathfrak{n} \geq 12$ *and* $L_3 = +\infty$, $L_1 = L_2$ *(see Figure 19 (i)). The flow is stable.*

(iv) $\mathfrak{n} = 6$ *and* $L_3 = +\infty$, $(L_1, L_2) \in [0, +\infty)^2 \setminus \{(0, 0)\}$ *(see Figure 18 (i), (ii), (iii)). The flow is translating. The triod* $\Pi(t)$ *is unstable for any* $t \in [0, +\infty)$ *if either* $L_1 = 0, L_2 \in (0, +\infty)$ *or* $L_2 = 0, L_1 \in (0, +\infty)$ *(resp. (i) and (ii) in Figure 18); in the other cases, the flow is stable.*

(v) $\mathfrak{n} \geq 12$ *and* $L_2 = u_\infty L_3$, $L_2 = v_\infty L_1$ *(see Figure 19 (iii), where* u_∞, v_∞ *are defined in* (3.20). *The flow is stable.*

Proof. If $\Pi \in (\mathtt{d})$ then $\dot{h}_j^\nu(t) = 0$ for any $t \in [0, +\infty)$, $j = 1, 2, 3$ (see Table 1).
Assume $\Pi \in (\mathtt{a})$. From (3.21), (i) follows since $x(t) = 0$ and $\dot{h}_j^\nu(t) = 0$ for any $t \in [0, +\infty)$, $j = 1, 2, 3$.

Now let $u(t) := L_2(t)/L_3(t)$ and $v := L_2(t)/L_1(t)$. Recalling that $\text{cotg}\alpha_1 = -\text{cotg}\alpha_2 = \text{cotg}\alpha_3$ and $\vartheta_{\mathfrak{n}} = 2\pi/3$, from (3.21), (3.13), (3.15) and (3.18), we obtain

$$\begin{aligned} \dot{u} &= \frac{L}{L_2^2} \frac{u\left[-\left(\text{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)u - \left(\text{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)v - \frac{2}{\sqrt{3}}uv + \left(\text{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)u^2\right]}{u+v+1} \\ \dot{v} &= \frac{L}{L_2^2} \frac{v\left[-\left(\text{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)u - \left(\text{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)v + \frac{2}{\sqrt{3}}uv + \left(\text{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)v^2\right]}{u+v+1}. \end{aligned} \tag{9.1}$$

From Remark 9.2 it follows that all homothetic flows are the constant solutions (u, v) of system (9.1). We seek for solutions of (9.1) of the form $\frac{v}{u} = K$, K constant. Imposing $\dot{v} = K\dot{u}$ we obtain three possibilities:

(1) $K = 0$, that gives (ii), since $\dot{h_1^\nu}(t) = 0$, $\dot{h_2^\nu}(t) = -\dot{h_3^\nu}(t) = L/(L_2(t) + L_3(t))$ and (3.17), (3.19) hold;

(2) $u = 0$, that gives (iii) and (iv), since $\dot{h_3^\nu}(t) = 0$, $\dot{h_2^\nu}(t) = -\dot{h_1^\nu}(t) = L/(L_1(t) + L_2(t))$ and (3.13), (3.15) hold. Notice that if $\mathfrak{n} = 6$ then $\dot{L}_1 = \dot{L}_2 = 0$ while if $\mathfrak{n} \geq 12$ then $\dot{L}_1 = \dot{L}_2 > 0$;

(3) $K = r$ and $v = v_\infty$, $u = u_\infty$ that gives (v). Indeed, equalizing the brackets on the right-hand sides of (9.1) we get

$$-\left(\text{cotg}\alpha_1 + \frac{1}{\sqrt{3}}\right)v^2 - \frac{4}{\sqrt{3}}uv + \left(\text{cotg}\alpha_1 - \frac{1}{\sqrt{3}}\right)u^2 = 0,$$

and thus $v = ru$ for $\mathfrak{n} \geq 12$ and $u = 0$ for $\mathfrak{n} = 6$ (observe that if $\mathfrak{n} \geq 12$ then $\text{cotg}\alpha_1 + \frac{1}{\sqrt{3}} \leq 0$ and the equality holds if and only if $\mathfrak{n} = 6$). Hence, if $\mathfrak{n} \geq 12$, substituting $v = ru$ in $\dot{u} = 0$ yields the conclusion, i.e. $u_\infty = (1 + \sqrt{3}\sin\alpha_1 - \cos\alpha_1)/r$.

The inverse implication follows by construction. □

Remark 9.4. The φ-curvature flows in Theorem 3.9 converge to homothetic flows, i.e. if $\mathfrak{n} \geq 12$ (resp. $\mathfrak{n} = 6$) the limit triod satisfies (v) (resp. (iv) with $L_1 = 0$) of Lemma 9.3, see Figure 19 (iii) (resp. Figure 18 (i)).

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