Deterministic equivalent for the Allen-Cahn energy of a scaling law in the Ising model

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Abstract

For the Allen-Cahn functional we study the following problem: for which prescribed amount m of volume is there the appearence of a droplet of one phase inside the other? Under a suitable assumption on the domain we show that the breaking of symmetry occurs at the same value of m as for the limit of the sharp interface energy. We also prove that there exists a threshold for m of order $\varepsilon^{\frac{n}{n+1}}$ so that either there is the appearence of the droplet or there is no breaking of symmetry.

1 Introduction

In the papers [2] and [8] the following question was posed for the Ising model in equilibrium and large finite volume: for which prescribed magnetization m does one start to see the breaking of symmetry which is the origin of the appearence of a droplet of one phase inside the other phase? We expect that this is a question of surface energy, up to first order, and that it does not depend on the particular functional (i.e. free energy) whose Γ-limit is, up to rescaling, the surface energy itself (see Theorem 2.3). In statistical mechanics we could not prove such a statement, but to support our claim we look at a particular functional, the phase field functional $\mathcal{F}_\varepsilon$ with the parameter ε corresponding to the inverse linear size of the finite volume and we show, under a suitable assumption on the domain, that the breaking of symmetry occurs (up to first order) at the same value of m as for the limit of the sharp interface energy $P_{\varepsilon,m}$ (see Theorem 2.4). We also prove in

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Theorem 2.1 that there exists a threshold for m of order $\varepsilon^{\frac{n}{n+1}}$ so that either there is the appearence of the droplet or there is no breaking of symmetry. Note that this is not a Γ-limit result, as only the rescaled free energy has a Γ-limit. It is more in the spirit of a Γ-expansion, as recently investigated in [3]. In Remark 2.5 we briefly discuss an extension of the result to arbitrary (compact) Riemannian manifolds, covering in particular the periodic setting.

We want to stress that, whereas the upper estimate is obtained by an explicit construction and requires some regularity assumptions for the domain Ω in which the problem is posed, the lower estimate (Theorem 3.2) is completely general.

After this work was completed, we learned of a similar result proved independently, in the periodic setting, in [4].

2 Statement of the main results

We denote by Ω a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. We denote by $B_r(x)$ the ball of radius $r \geq 0$ centered at $x \in \Omega$; if $r = 0$ we set $B_r(x) = \emptyset$. Given $\varepsilon > 0$ and a function $u \in H^1(\Omega)$, we define the free energy $\mathcal{F}_\varepsilon(u)$ of u as

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left(\varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon} W(u) \right) \, dx,$$

where W is a regular (i.e. of class $\mathcal{C}^2$) double–well potential. In the following, we fix for simplicity $W(t) = (1-t^2)^2/2$, noticing that all the results are independent of the particular choice of the potential.

We shall use the following notation: $\sigma_W := \int_{-1}^1 \sqrt{2W(s)} \, ds$; $C_{\rm iso}(\Omega)$ is the constant in the relative isoperimetric inequality [5], which reads as

$$P(E;\Omega) \geq C_{\rm iso}(\Omega) \left(\min\{|E|, |\Omega \setminus E|\} \right)^{\frac{n-1}{n}} \quad \text{for any } E \subseteq \Omega, \tag{2.1}$$

where $P(E;\Omega)$ is the perimeter of E in Ω.

Given $m \in [0,1]$ we denote by $\overline{u}_{\varepsilon,m} \in H^1(\Omega) \cap \mathcal{C}^\infty(\Omega)$ a solution to the following minimum problem:

$$\min \left\{ \mathcal{F}_\varepsilon(u) : u \in H^1_m(\Omega) \right\}, \tag{2.2}$$

where

$$H^1_m(\Omega) := \{ u \in H^1(\Omega) : \frac{1}{|\Omega|} \int_\Omega u = m \}$$

and $|B|$ is the Lebesgue measure of any Borel set $B \subseteq \Omega$. We let

$$\overline{m}_\varepsilon := \sup\{ m \in [0,1] : \ \exists \, \overline{u}_{\varepsilon,m} \text{ solution of (2.2) with } \mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,m}) < \mathcal{F}_\varepsilon(m) \}. \tag{2.3}$$

For any set $E \subseteq \mathbb{R}^n$ we define the functional

$$P_{\varepsilon,m}(E) := \sigma_W P(E;\Omega) + \frac{|\Omega|}{\varepsilon} W\left(m + 2\frac{|E|}{|\Omega|}\right), \tag{2.4}$$

and we denote by $E_{\varepsilon,m}$ a solution to the minimum problem:

$$\min\{P_{\varepsilon,m}(E) : E \subseteq \Omega\}. \tag{2.5}$$

In order to study the functional $P_{\varepsilon,m}$ we introduce the following function defined for $\delta \in [0,1]$ as

$$f_\Omega(\delta) := \min\left\{P(E;\Omega) : \ E \subseteq \Omega, \ \frac{|E|}{|\Omega|} = \delta\right\}.$$

The parameter δ represents the volume fraction $|E|/|\Omega|$ letting E varying in all (measurable) subsets of Ω.

Let $\bar{r} = \bar{r}(\Omega)$ be the radius of a maximal open ball contained in Ω, and let $\beta(\Omega) \in (0,1)$ be such that $(\frac{\beta(\Omega)|\Omega|}{2\omega_n})^{1/n} = \frac{\bar{r}}{2}$, $\omega_n := |B_1(0)|$. Our first result reads as follows.

Theorem 2.1. *There exist constants $C_2 > C_1 > 0$ and $\varepsilon_0 > 0$, depending on Ω and W, such that for any $\varepsilon \in (0, \varepsilon_0)$ the following assertions hold.*

(i) If $m \in \left[1 - C_1 \varepsilon^{\frac{n}{n+1}}, 1\right]$ then any $\overline{u}_{\varepsilon,m}$ is constantly equal to m in Ω.

(ii) If $m \in \left[1 - \beta(\Omega), 1 - C_2 \varepsilon^{\frac{n}{n+1}}\right]$ then any $\overline{u}_{\varepsilon,m}$ is not constant in Ω.

Notice that, in particular, from Theorem 2.1 it follows

$$\overline{m}_\varepsilon \in \left[1 - C_2 \varepsilon^{\frac{n}{n+1}}, 1 - C_1 \varepsilon^{\frac{n}{n+1}}\right]. \tag{2.6}$$

Under the following additional assumption (H) on Ω, we can prove a stronger statement (see Theorem 2.4).

(H) There exists a point $x_0 \in \partial\Omega$ such that

(i) there exist the limit

$$\lim_{r \to 0^+} \frac{P(B_r(x_0);\Omega)}{n\omega_n r^{n-1}} = \lim_{r \to 0^+} \frac{|B_r(x_0) \cap \Omega|}{\omega_n r^n} =: c_\Omega\,; \tag{2.7}$$

(ii) there exists the limit

$$\lim_{r \to 0^+} \frac{P(B_r(x_0);\Omega)}{f_\Omega(|B_r(x_0) \cap \Omega|/|\Omega|)} = 1.$$

Remark 2.2. Since Ω has Lipschitz boundary, we have $c_\Omega \in (0,1)$ in assumption (H), and $c_\Omega = 1/2$ if $\partial\Omega$ is of class $\mathcal{C}^1$. We also observe that (H) is always satisfied whenever Ω is convex or $\partial\Omega$ is Lipschitz and piecewise of class $\mathcal{C}^1$.

Theorem 2.3. *Assume that Ω has the regularity property (H). Let $C > 0$ and $m \in [1 - C\varepsilon^{\frac{n}{n+1}}, 1]$. Then*

$$\begin{aligned}\min_{u \in H^1_m(\Omega)} \mathcal{F}_\varepsilon(u) &= \min_{E \subseteq \Omega} P_{\varepsilon,m}(E) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right) \\ &= \min_{r \geq 0} P_{\varepsilon,m}(B_r(x_0)) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right).\end{aligned} \tag{2.8}$$

Theorem 2.4. *Suppose that Ω satisfies assumption (H). Then*

$$\lim_{\varepsilon \to 0} \frac{1 - \overline{m}_\varepsilon}{\varepsilon^{\frac{n}{n+1}}} = K_\Omega := \frac{(n+1)\omega_n^{\frac{1}{n+1}} c_\Omega^{\frac{1}{n+1}} {\sigma_W}^{\frac{n}{n+1}}}{|\Omega|^{\frac{1}{n+1}} (2W''(1))^{\frac{n}{n+1}}}. \tag{2.9}$$

Remark 2.5. Theorems 2.1 and 2.4 can be extended, with the obvious modifications, to functionals $\mathcal{F}_\varepsilon$ defined on $H^1(M)$, M a smooth compact Riemannian manifold with smooth boundary. Note that, if $\partial M \neq \emptyset$, then M satisfies assumption (H), with $c_\Omega = 1/2$. On the other hand, if $\partial M = \emptyset$, it has been shown in [1, Appendix C] that M satisfies an assumption analogous to (H), with $c_\Omega = 1$ and with x_0 any point of M. In particular, when M is the n-dimensional torus, $\mathbb{T}^n$, it covers the case of periodic boundary conditions in problem (2.2).

3 Estimate from below

We first set some notation used in the paper. If $\lambda, \beta, \sigma \in \mathbb{R}$, $\lambda < \beta$, and $u : \Omega \to \mathbb{R}$, we set $\{\lambda < u < \beta\} := \{x \in \Omega : \lambda < u(x) < \beta\}$ and $\Sigma_\sigma(u) := \{u < \sigma\}$. In the sequel C will denote a generic positive constant that may vary from line to line and even in the same line. Throughout the paper we often use the following quadratic estimates on W: there exist constants $c', c'' > 0$ such that

$$W(t) \geq c'(1-t)^2 \qquad \forall\, t \geq 0 \tag{3.1}$$

$$W(t) \leq c''(1-t)^2 \qquad \forall\, t \in [0,2]. \tag{3.2}$$

A symmetric statement holds for $t \leq 0$ by replacing 1 with -1.

We start with the following preliminary lemma.

Lemma 3.1. *There exist two constants $\bar{m} \in (0,1)$ and $\bar{c} \in (0, \frac{1}{1-\bar{m}})$, depending only on W, such that for any $m \in (\bar{m}, 1)$ and for all $u \in H^1_m(\Omega)$, with $\int_\Omega W(u)\, dx < |\Omega| W(m)$, we have*

$$|\Sigma_s(u)| \leq |\Omega|/2, \qquad \forall\, s \leq 1 - \bar{c}(1-m). \tag{3.3}$$

Proof. Set

$$\overline{s} := 1 - \overline{c}(1-m), \tag{3.4}$$

for a suitable constant $\bar{c} \in (0, \frac{1}{1-\bar{m}})$ that will be defined later, independently of m, together with $\bar{m}$. Let $u \in H^1_m(\Omega)$ with $\int_\Omega W(u)\,dx < |\Omega| W(m)$ be fixed. It is enough to show (3.3) for $s = \overline{s}$. Let us define

$$\begin{aligned} A_{\overline{s}} &:= \{x \in \Omega : \big|\, |u(x)| - 1 \big| \geq 1 - \overline{s}\}, \\ B_{\overline{s}} &:= \{x \in \Omega : |u(x) + 1| < 1 - \overline{s}\}. \end{aligned}$$

Clearly $\Sigma_{\overline{s}}(u) \subseteq A_{\overline{s}} \cup B_{\overline{s}}$, hence

$$|\Sigma_{\overline{s}}(u)| \leq |A_{\overline{s}}| + |B_{\overline{s}}|. \tag{3.5}$$

Since $W(u(x)) \geq W(\overline{s})$ for any $x \in A_{\overline{s}}$, using the assumption $\int_\Omega W(u)\,dx \leq |\Omega| W(m)$ and (3.2) we find

$$|A_{\overline{s}}| W(\overline{s}) \leq \int_{A_{\overline{s}}} W(u)\,dx \leq \int_\Omega W(u)\,dx \leq |\Omega| W(m) \leq |\Omega| C(1-m)^2. \tag{3.6}$$

Using (3.1), from (3.6) and (3.4) we obtain

$$|A_{\overline{s}}| \leq |\Omega| C \left(\frac{1-m}{1-\overline{s}}\right)^2 = \frac{C}{\overline{c}^2} |\Omega|.$$

We now estimate

$$\int_{A_{\overline{s}}} u dx \leq \int_{A_{\overline{s}}} (C + W(u)) dx \leq C(|A_{\overline{s}}| + |\Omega|(1-m)^2) \leq C|\Omega| \left(\frac{1}{\overline{c}^2} + (1-m)^2\right).$$

Therefore, since $u(x) \leq -\overline{s}$ for any $x \in B_{\overline{s}}$, we have

$$\begin{aligned} |\Omega| m &= \int_\Omega u\,dx = \int_{A_{\overline{s}}} u\,dx + \int_{B_{\overline{s}}} u\,dx + \int_{\Omega \setminus (A_{\overline{s}} \cup B_{\overline{s}})} u\,dx \\ &\leq C|\Omega| \left(\frac{1}{\overline{c}^2} + (1-m)^2\right) - \overline{s}|B_{\overline{s}}| + (2-\overline{s})(|\Omega| - |B_{\overline{s}}| - |A_{\overline{s}}|) \\ &= C|\Omega| \left(\frac{1}{\overline{c}^2} + (1-m)^2\right) + (2-\overline{s})|\Omega| - 2|B_{\overline{s}}| + (\overline{s}-2)|A_{\overline{s}}| \\ &\leq C|\Omega| \left(\frac{1}{\overline{c}^2} + (1-m)^2\right) + (2-\overline{s})|\Omega| - 2|B_{\overline{s}}|. \end{aligned}$$

Recalling (3.5), we deduce

$$\begin{aligned} |\Sigma_{\overline{s}}(u)| &\leq |A_{\overline{s}}| + |B_{\overline{s}}| \leq \frac{(2-\overline{s}-m)|\Omega|}{2} + C|\Omega| \left(\frac{1}{\overline{c}^2} + (1-m)^2\right) \\ &\leq \frac{(1+\overline{c})(1-\bar{m})|\Omega|}{2} + C|\Omega|(1-\bar{m})^2 + \frac{|\Omega|}{6}, \end{aligned} \tag{3.7}$$

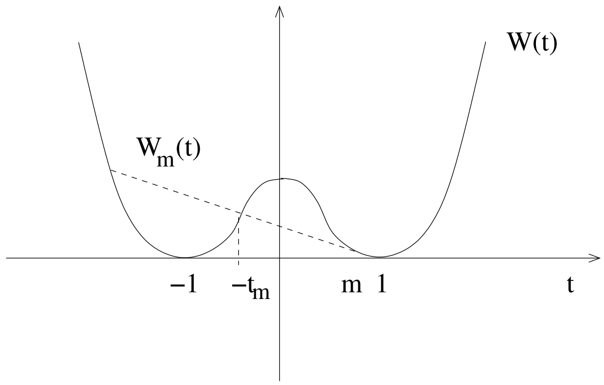

Figure 1: The double–well potential W and its convex modification W_m in Theorem 3.2.

where we have chosen $\overline{c}$ such that $\overline{c}^2 \geq 6C$. In order to conclude the proof of (3.3) it is enough to choose $\bar{m}$ in such a way that $1 - \bar{m} \leq \overline{c}^{-1}$ and

$$\frac{(1+\overline{c})(1-\bar{m})}{2} + C(1-\bar{m})^2 \leq \frac{1}{3}.$$

□

Lemma 3.1, together with the isoperimetric inequality, allows us to estimate the volume of sublevel sets of a given $u \in H^1_m(\Omega)$ the energy of which is lower than the energy of the constant m.

The following estimate from below is the crucial result of this section.

Theorem 3.2. *There exist two constants $\tilde{m}, \tilde{C} > 0$, depending only on W, such that for any $m \in (\tilde{m}, 1)$ and any $u \in H^1_m(\Omega)$ satisfying $\mathcal{F}_\varepsilon(u) < \mathcal{F}_\varepsilon(m)$ it is possible to find $t^*_m \in [-1+\tilde{C}\sqrt{1-m}, 1-\tilde{C}\sqrt{1-m}]$ such that $\Sigma_{t^*_m}(u) \neq \emptyset$ and*

$$\begin{aligned}\mathcal{F}_\varepsilon(u) \quad &\geq \quad \big(\sigma_W + C(1-m)\log(1-m)\big)P(\Sigma_{t^*_m}(u);\Omega) \\ &\quad + \frac{|\Omega|}{\varepsilon} W\left(\left(m + 2\frac{|\Sigma_{t^*_m}(u)|}{|\Omega|}\right)\wedge 1\right) - \frac{C}{\varepsilon}(1-m)^2|\Sigma_{t^*_m}(u)|.\end{aligned} \tag{3.8}$$

Proof. Let $m \in (\tilde{m}, 1)$ with $\tilde{m}$ a fixed constant greater than $\bar{m}$ in Lemma 3.1 to be chosen later. Fix also $u \in H^1_m(\Omega)$ satisfying $\mathcal{F}_\varepsilon(u) < \mathcal{F}_\varepsilon(m)$.

In order to get the estimate we introduce the following convex modification W_m of the potential W (see Figure 1)

$$W_m(t) := \begin{cases} W(t) & \text{for } t \geq m \\ W(m) + W'(m)(t-m) & \text{for } t \in (-1, m) \\ \max\{W(t), W(m) + W'(m)(t-m)\} & \text{for } t \leq -1. \end{cases}$$

Let us define

$$t_m := -\sup\{t \geq -1 : W_m(t) > W(t)\}. \tag{3.9}$$

It can be checked that $t_m \to 1$ as $m \to 1$; more precisely, from the equality $W(-t_m) = W_m(-t_m)$ we get that $|t_m - 1| \leq \tilde{C}\sqrt{1-m}$ for some $\tilde{C} > 0$ depending only on W. Set

$$\Omega_m := \Big\{x \in \Omega : \ W(u(x)) < W_m(u(x))\Big\}.$$

Then, by (3.9),

$$\Omega_m \subseteq \Big\{x \in \Omega : \ |u(x) + 1| \leq C'\sqrt{1-m}\Big\}, \tag{3.10}$$

for some $C' > 0$. Thus $\mathcal{F}_\varepsilon(u)$ can be rewritten in terms of W_m and Ω_m as

$$\begin{aligned} \mathcal{F}_\varepsilon(u) &= \int_\Omega \left(\varepsilon\frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon}\big(W(u) - W_m(u)\big)^+\right) dx \qquad (3.11)\\ &\quad + \frac{1}{\varepsilon}\int_{\Omega_m} W(u)\, dx + \frac{1}{\varepsilon}\int_{\Omega\setminus\Omega_m} W_m(u)\, dx =: I + II + III. \end{aligned}$$

We will estimate separately the three terms on the right hand side of (3.11). The set $\Omega \setminus \Omega_m$ contains the leading contribution to the energy. Indeed, applying the Coarea formula to the first term we obtain

$$\begin{aligned} I &\geq \int_{-t_m}^{t_m} \left(\int_{\{u=t\}} \left(\varepsilon\frac{|\nabla u|}{2} + \frac{W(t) - W_m(t)}{\varepsilon|\nabla u|}\right) d\mathcal{H}^{n-1}\right) dt \\ &\geq \int_{-t_m}^{t_m} \sqrt{2(W(t) - W_m(t))}P(\Sigma_t(u);\Omega)\, dt \qquad (3.12)\\ &\geq \left(\sigma_W - C\left(1 - m + \int_{-t_m}^{t_m} \frac{W_m}{\sqrt{W}}\, dt\right)\right) \inf_{t\in[-t_m,t_m]} P(\Sigma_t(u);\Omega). \end{aligned}$$

Since

$$\int_{-t_m}^{t_m} \frac{W_m}{\sqrt{W}}\, dt \leq -C(1-m)\log(1-m),$$

we get

$$I \geq \big(\sigma_W + C(1-m)\log(1-m)\big) \inf_{t\in[-t_m,t_m]} P(\Sigma_t(u);\Omega). \tag{3.13}$$

By Jensen's inequality and (3.1) we also have

$$\varepsilon II \geq |\Omega_m| W\left(\frac{1}{|\Omega_m|}\int_{\Omega_m} u\, dx\right) \geq C|\Omega_m|\left(\frac{1}{|\Omega_m|}\int_{\Omega_m} (u+1)\, dx\right)^2,$$

and

$$\varepsilon III \geq |\Omega \setminus \Omega_m| W_m\left(\frac{1}{|\Omega \setminus \Omega_m|}\int_{\Omega\setminus\Omega_m} u\, dx\right). \tag{3.14}$$

Set

$$T_u := \frac{1}{|\Omega \setminus \Omega_m|} \int_{\Omega\setminus\Omega_m} u \; dx.$$

Possibly choosing $\tilde{m}$ sufficiently close to one, we have that $T_u \geq m$. Moreover

$$T_u \leq 1 + C(1-m). \tag{3.15}$$

Indeed, using Jensen's inequality and the hypothesis $\mathcal{F}_\varepsilon(u) \leq \mathcal{F}_\varepsilon(m)$, we have

$$\begin{aligned} W(T_u) = W_m(T_u) &\leq \frac{1}{|\Omega \setminus \Omega_m|} \int_{\Omega\setminus\Omega_m} W_m(u) \; dx \\ &\leq \frac{1}{|\Omega \setminus \Omega_m|} \int_{\Omega\setminus\Omega_m} W(u) \; dx \leq C(1-m)^2. \end{aligned} \tag{3.16}$$

Since, by definition of Ω_m, $T_u \leq 1 + c\sqrt{1-m}$, we can use (3.1) to get from (3.16) that $(1-T_u)^2 \leq C(1-m)^2$, which implies (3.15).

From (3.14) and (3.15) we obtain

$$\varepsilon III \geq |\Omega \setminus \Omega_m| W(T_u) \geq |\Omega| W(T_u) - C|\Omega_m|(1-m)^2 \tag{3.17}$$

Let us now show that

$$\frac{|\Omega_m|}{|\Omega|} \leq C(1-m). \tag{3.18}$$

From (3.15) we deduce

$$\int_{\Omega\setminus\Omega_m} u \; dx \leq |\Omega| - |\Omega_m| + C|\Omega|(1-m).$$

Hence

$$\begin{aligned} m|\Omega| = \int_{\Omega_m} u \; dx + \int_{\Omega\setminus\Omega_m} u \; dx &\leq \int_{\Omega_m} u \; dx + |\Omega| - |\Omega_m| + C|\Omega|(1-m) \\ &\leq -|\Omega_m| + C\sqrt{1-m}|\Omega_m| + |\Omega| - |\Omega_m| + C|\Omega|(1-m). \end{aligned}$$

We can rewrite the above inequality as

$$(2 - C\sqrt{1-m})|\Omega_m| \leq (C+1)|\Omega|(1-m),$$

which implies (3.18).

Let us now define

$$M_u := m + 2\frac{|\Omega_m|}{|\Omega|} - \frac{1}{|\Omega|} \int_{\Omega_m} (u+1) \; dx\,.$$

Recalling (3.10) and (3.18), and choosing $1-m$ small enough, we get

$$m \leq M_u \leq m + C\frac{|\Omega_m|}{|\Omega|} \leq 1 + C(1-m)$$

and

$$M_u - T_u = \frac{|\Omega_m|}{|\Omega|}(1-m) + o\left(|\Omega_m|(1-m)\right) .$$

Thus, we obtain

$$W\left(T_u\right) \geq W\left(M_u\right) - C|\Omega_m|(1-m)^2. \tag{3.19}$$

Hence, using (3.17), (3.14) yields

$$\begin{aligned}\varepsilon III &\geq |\Omega| W\left(M_u\right) - C|\Omega_m|(1-m)^2 \\ &\geq |\Omega| W\left(m + 2\frac{|\Omega_m|}{|\Omega|}\right) - W'(m)\int_{\Omega_m}(u+1)\ dx - C|\Omega_m|(1-m)^2 .\end{aligned}$$

Finally, it follows

$$\begin{aligned}\varepsilon III + \varepsilon II \geq\ & |\Omega| W\left(m + 2\frac{|\Omega_m|}{|\Omega|}\right) \\ & + |\Omega_m|\left(C\left(\frac{1}{|\Omega_m|}\int_{\Omega_m}(u+1)\ dx\right)^2 - C(1-m)^2 \right. \\ & \left. - W'(m)\frac{1}{|\Omega_m|}\int_{\Omega_m}(u+1)\ dx\right) .\end{aligned} \tag{3.20}$$

Note that, if $\left|\frac{1}{|\Omega_m|}\int_{\Omega_m}(u+1)\ dx\right| \leq C(1-m)$, from (3.20) we get

$$\varepsilon III + \varepsilon II \geq |\Omega| W\left(m + 2\frac{|\Omega_m|}{|\Omega|}\right) - C|\Omega_m|(1-m)^2. \tag{3.21}$$

On the other hand, there exists a constant $C > 0$, depending only on W, such that if $\left|\frac{1}{|\Omega_m|}\int_{\Omega_m}(u+1)\ dx\right| \geq C(1-m)$ then the last term in the right hand side of (3.20) is positive. It follows that (3.21) always holds.
The level t_m^* can be chosen as a quasi-minizing level for $P(\Sigma_t(u);\Omega)$, possibly adjusting the constant C. The inequality (3.8) now follows by plugging (3.21) and (3.12) in (3.11), recalling that $|\Omega_m| \leq |\Sigma_{t_m^*}(u)|$ and using also the monotonicity of W.
The fact that $\Sigma_{t_m^*}(u) \neq \emptyset$ follows at once, since otherwise (3.8) would imply $\mathcal{F}_\varepsilon(u) \geq \mathcal{F}_\varepsilon(m)$, leading to a contradiction. □

Remark 3.3. Notice that, from (3.8) and the isoperimetric inequality, we get

$$|\Sigma_{t_m^*}|^{\frac{n-1}{n}} \leq CP(\Sigma_{t_m^*};\Omega) \leq C|\Omega|\frac{(1-m)^2}{\varepsilon} . \tag{3.22}$$

Corollary 3.4. *Under the same assumptions of Theorem 3.2 we have*

$$
\begin{aligned}
|\Sigma_{-t_m}(u)|^{\frac{1}{n}} \quad &\geq \quad |\Omega_m|^{\frac{1}{n}} \geq \frac{\sigma_W C_{\rm iso}(\Omega) + C(1-m)\log(1-m)}{-2W'(m) + C(1-m)^2}\, \varepsilon \\
&\geq \quad \left(\frac{\sigma_W C_{\rm iso}(\Omega) + C(1-m)\log(1-m)}{2W''(1) + C(1-m)} \right) \frac{\varepsilon}{1-m}. \quad (3.23)
\end{aligned}
$$

Proof. Notice that, if we apply the isoperimetric inequality after the second inequality in (3.12), recalling Lemma 3.1, using the fact that $|\Omega_m| \leq |\Sigma_{-t_m}(u)| \leq |\Sigma_t(u)|$ for any $t \in [-t_m, t_m]$, and taking into account (3.21), we get

$$
\begin{aligned}
\mathcal{F}_\varepsilon(m) \quad \geq \quad & \mathcal{F}_\varepsilon(u) \geq \sigma_W C_{\rm iso}(\Omega)|\Omega_m|^{\frac{n-1}{n}} + \frac{|\Omega|}{\varepsilon} W\left(m + 2\frac{|\Omega_m|}{|\Omega|}\right) \\
& -C|\Omega_m|\frac{(1-m)^2}{\varepsilon} + C|\Omega_m|^{\frac{n-1}{n}}(1-m)\log(1-m),
\end{aligned}
$$

which gives (3.23). □

4 Proof of the main results

This section is devoted to the proof of Theorems 2.1, 2.3 and 2.4.

4.1 Proof of Theorem 2.1

Let us prove assertion (i). We can suppose $m < 1$, since the thesis is trivial for $m = 1$. Assume by contradiction that for any C_1 and any ε_0 there exist $\varepsilon \in (0, \varepsilon_0)$, $m \in (1 - C_1\varepsilon^{\frac{n}{n+1}}, 1)$ and $u \in H^1_m(\Omega)$ such that $\mathcal{F}_\varepsilon(u) < \mathcal{F}_\varepsilon(m)$. From (3.22) and (3.23) (recall (3.9)) it follows

$$
\begin{aligned}
&\left(\left(\frac{\sigma_W C_{\rm iso}(\Omega)}{2W''(1) + C(1-m)} + C\log(1-m)^{1-m} \right) \frac{\varepsilon}{1-m} \right)^{n-1} \\
&\leq |\Sigma_{-t_m}(u)|^{\frac{n-1}{n}} \leq C\frac{(1-m)^2}{\varepsilon}.
\end{aligned} \tag{4.1}
$$

It follows

$$
CC_1^{-(n-1)}\varepsilon^{\frac{n-1}{n+1}} \leq CC_1^2\varepsilon^{\frac{n-1}{n+1}}. \tag{4.2}
$$

Then we are lead to a contradiction, since if $C_1 > 0$ is small enough, then there cannot be any $m \in [1 - C_1\varepsilon^{\frac{n}{n+1}}, 1]$ for which (4.2) holds. This proves assertion (i).

To prove assertion (ii), let us denote by $\gamma : \mathbb{R} \to (-1, 1)$ the (smooth strictly increasing) absolute minimizer of the problem

$$
\inf\left\{ \int_{\mathbb{R}} \left(\frac{1}{2}|\zeta'|^2 + W(\zeta) \right) dy : \ \zeta \in H^1_{\rm loc}(\mathbb{R}), \zeta(0) = 0, \lim_{y\to\pm\infty} \zeta(y) = \pm 1 \right\}. \tag{4.3}
$$

Recall that $-\gamma'' + W'(\gamma) = 0$, which gives $\gamma' = \sqrt{2W(\gamma)}$.

Lemma 4.1. *Let* $m \in [1-\beta(\Omega), 1-C_2\varepsilon^{\frac{n}{n+1}}]$ *for some constant* $C_2 > 0$. *Then there exists* $\bar{\varepsilon} > 0$ *such that*

$$\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,m}) \le \sigma_W n \omega_n^{\frac{1}{n}} \left(\frac{|\Omega|(1-m)}{2}\right)^{\frac{n-1}{n}} (1 + O(\varepsilon|\log\varepsilon|)) \qquad \forall \varepsilon \in (0, \bar{\varepsilon}). \tag{4.4}$$

Proof. Let $\bar{x} \in \Omega$ be such that $B_{\bar{r}}(\bar{x}) \subset \Omega$. Let $r(m) \in (0, \bar{r}/2)$ and take $\varepsilon > 0$ small enough such that $\varepsilon|\log\varepsilon| < r(m)$ (which implies $B_{r(m)+\varepsilon|\log\varepsilon|}(\bar{x}) \subset B_{\bar{r}}(\bar{x}) \subset \Omega$). Define

$$u_{\varepsilon,m}(x) := \begin{cases} \gamma_\varepsilon\left(\frac{|x-\bar{x}|-r(m)}{\varepsilon}\right) & \text{if } r(m)-\varepsilon|\log\varepsilon| \le |x-\bar{x}| \le r(m)+\varepsilon|\log\varepsilon| \\ -1 & \text{if } 0 \le |x-\bar{x}| \le r(m)-\varepsilon|\log\varepsilon| \\ 1 & \text{otherwise,} \end{cases} \tag{4.5}$$

where γ_ε joins smoothly γ with its asymptotic values ± 1, so that

$$\int_{-|\log\varepsilon|}^{|\log\varepsilon|} (\frac{1}{2}(\gamma_\varepsilon')^2 + W(\gamma_\varepsilon))\, ds = \sigma_W + O(\varepsilon^2)$$

and $\gamma_\varepsilon(\pm|\log\varepsilon|) = \pm 1$. We now fix $r(m)$ in such a way that $\int_\Omega u_{\varepsilon,m}\, dx = |\Omega| m$. Note that

$$1 - \frac{2\omega_n r(m)^n}{|\Omega|} \in (m - C\varepsilon|\log\varepsilon|, m + C\varepsilon|\log\varepsilon|) \tag{4.6}$$

for some contant $C > 0$. A direct computation gives

$$\begin{aligned}
\mathcal{F}_\varepsilon(u_{\varepsilon,m}) &= \mathcal{F}_\varepsilon(u_{\varepsilon,m}; \{r(m)-\varepsilon|\log\varepsilon| \le |x-\bar{x}| \le r(m)+\varepsilon|\log\varepsilon|\}) \\
&= \int_{-\varepsilon|\log\varepsilon|}^{\varepsilon|\log\varepsilon|} dt \int_{\{|x-\bar{x}|=r(m)+t\}} \left(\frac{1}{2\varepsilon}|\gamma'\left(\frac{t}{\varepsilon}\right)|^2 + \frac{1}{\varepsilon} W\left(\gamma\left(\frac{t}{\varepsilon}\right)\right)\right) d\mathcal{H}^{n-1} \\
&= \int_{\partial B_{r(m)}(0)} d\mathcal{H}^{n-1} \int_{-|\log\varepsilon|}^{|\log\varepsilon|} \left(\frac{1}{2}(\gamma_\varepsilon')^2 + W(\gamma_\varepsilon)\right) ds \\
&= \sigma_W n\omega_n r(m)^{n-1} + O(\varepsilon|\log\varepsilon|) \\
&= \sigma_W n \omega_n \left(\frac{(1-m)|\Omega|}{2\omega_n}\right)^{\frac{n-1}{n}} (1 + O(\varepsilon|\log\varepsilon|)),
\end{aligned}$$

where the last equality is a consequence of (4.6). Then (4.4) follows from $\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,m}) \le \mathcal{F}_\varepsilon(u_{\varepsilon,n})$. □

Let us now prove assertion (ii). Assume

$$m \in \left[1-\beta(\Omega), 1-C_2\varepsilon^{\frac{n}{n+1}}\right], \tag{4.7}$$

where C_2 is a positive constant large enough that will be chosen later on independently of ε. It is sufficient to prove that the function $\overline{u}_{\varepsilon,m}$ changes sign for any $\varepsilon > 0$ small enough.

Assume by contradiction that $\min \overline{u}_{\varepsilon,m} \geq 0$. By (3.1), we have

$$\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,m}) \geq \frac{1}{\varepsilon}\int_\Omega W(\overline{u}_{\varepsilon,m})\, dx \geq \frac{C}{\varepsilon}\int_\Omega (\overline{u}_{\varepsilon,m}-1)^2\, dx \geq \frac{C|\Omega|}{\varepsilon}(1-m)^2, \quad (4.8)$$

where the last inequality follows from Jensen's inequality. On the other hand, by Lemma 4.1

$$(1-m)^{-\frac{n-1}{n}}\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,m}) \leq C|\Omega|^{\frac{n-1}{n}} \qquad \forall \varepsilon \in (0,\bar{\varepsilon}). \quad (4.9)$$

Combining (4.7), (4.8) and (4.9) we obtain

$$\begin{aligned} C_2^{\frac{n+1}{n}}\varepsilon &\leq (1-m)^{\frac{n+1}{n}} = (1-m)^{2-\frac{n-1}{n}} \\ &\leq \frac{C\,\varepsilon}{|\Omega|}\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,m})(1-m)^{-\frac{n-1}{n}} \leq \frac{C}{|\Omega|^{\frac{1}{n}}}\,\varepsilon, \end{aligned}$$

which gives a contradiction if $C_2 > \frac{C}{|\Omega|^{\frac{1}{n+1}}}$. □

4.2 Proof of Theorem 2.3

Let us first show that

$$\min_{u\in H^1_m(\Omega)} \mathcal{F}_\varepsilon(u) \geq \min_{E\subseteq\Omega} P_{\varepsilon,m}(E) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right). \quad (4.10)$$

Indeed, if $\mathcal{F}_\varepsilon(m) = \min_{u\in H^1_m(\Omega)}\mathcal{F}_\varepsilon(u)$, then (4.10) is satisfied by choosing $E=\emptyset$, since $\mathcal{F}_\varepsilon(m) = \frac{|\Omega|}{\varepsilon}W(m) = P_{\varepsilon,m}(\emptyset) \geq \min_{E\subseteq\Omega} P_{\varepsilon,m}(E)$.

Assume now that there exists a function $u \in H^1_m(\Omega)$ such that $\mathcal{F}_\varepsilon(u) < \mathcal{F}_\varepsilon(m)$. Then $m<1$, since for $m=1$ the unique minimizer of $\mathcal{F}_\varepsilon$ is $u\equiv m$. Let $\widetilde{E}_{\varepsilon,m}$ be a minimizer of

$$\min_{E\subseteq\Omega}\left\{\sigma_W P(E;\Omega) + \frac{|\Omega|}{\varepsilon}W\left(\left(m+2\frac{|E|}{|\Omega|}\right)\wedge 1\right)\right\}.$$

From (3.8) and (3.22) it follows

$$\begin{aligned} \mathcal{F}_\varepsilon(u) &\geq \sigma_W P(\Sigma_{t^*_m}(u);\Omega) + \frac{|\Omega|}{\varepsilon}W\left(\left(m+2\frac{|\Sigma_{t^*_m}(u)|}{|\Omega|}\right)\wedge 1\right) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right) \\ &\geq \sigma_W P(\widetilde{E}_{\varepsilon,m};\Omega) + \frac{|\Omega|}{\varepsilon}W\left(\left(m+2\frac{|\widetilde{E}_{\varepsilon,m}|}{|\Omega|}\right)\wedge 1\right) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right) \end{aligned} \quad (4.11)$$

Recall that, as in Remark (3.3), we have

$$|\widetilde{E}_{\varepsilon,m}| \leq C|\Omega|\varepsilon^{\frac{n}{n+1}}. \quad (4.12)$$

To conclude it remains to prove that

$$\frac{|\widetilde{E}_{\varepsilon,m}|}{|\Omega|} \leq \frac{1-m}{2}|\Omega| + o(1-m). \tag{4.13}$$

Indeed, assume by contradiction that $|\widetilde{E}_{\varepsilon,m}| \geq C(1-m)|\Omega|$, with $C > 1/2$. Then, by definition of $\widetilde{E}_{\varepsilon,m}$, we have

$$P(\widetilde{E}_{\varepsilon,m};\Omega) \leq P(E;\Omega) \qquad \forall E \text{ such that } \frac{|E|}{|\Omega|} = \frac{1-m}{2}. \tag{4.14}$$

The estimate (4.13) follows from (4.12) and (4.14) recalling that, by assumption (H), it holds

$$P(E;\Omega) = n(c_\Omega \omega_n)^{\frac{1}{n}} \left(\frac{|E|}{|\Omega|}\right)^{\frac{n-1}{n}} + o\left(\varepsilon^{\frac{n-1}{n+1}}\right),$$

for all $E \subseteq \Omega$ such that $|E| \leq C|\Omega|\varepsilon^{\frac{n}{n+1}}$. The inequality (4.10) now follows from (4.13) and (4.11).

Notice that, by assumption (H), there exists $r_{\varepsilon,m} \geq 0$ such that

$$\min_{u\in H^1_m(\Omega)} \mathcal{F}_\varepsilon(u) \geq P_{\varepsilon,m}(B_{r_{\varepsilon,m}}(x_0)) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right)$$

and

$$c_1\ \varepsilon^{\frac{1}{n+1}} \leq r_{\varepsilon,m} \ \leq\ c_2\ \varepsilon^{\frac{1}{n+1}}. \tag{4.15}$$

To prove the opposite inequality in (4.10), we first notice that, fixed any $E_{\varepsilon,m}$ minimizer of $P_{\varepsilon,m}$ among all $E \subseteq \Omega$, we have

$$P(E_{\varepsilon,m};\Omega) \leq \frac{1}{\sigma_W} P_{\varepsilon,m}(E_{\varepsilon,m}) \leq \frac{1}{\sigma_W} P_{\varepsilon,m}(\emptyset) \leq C|\Omega|\frac{(1-m)^2}{\varepsilon} \leq C|\Omega|\varepsilon^{\frac{n-1}{n+1}}. \tag{4.16}$$

In addition, from the inequality $P_{\varepsilon,m}(E_{\varepsilon.m}) \leq P_{\varepsilon,m}(\emptyset)$, it follows

$$|E_{\varepsilon,m}| \leq C|\Omega|(1-m) \leq C|\Omega|\varepsilon^{\frac{n}{n+1}}.$$

Let us first assume

$$|E_{\varepsilon,m}| = o\left(\varepsilon^{\frac{n}{n+1}}\right), \tag{4.17}$$

then

$$W\left(m + 2\frac{|E_{\varepsilon,m}|}{|\Omega|}\right) = W(m) + o\left(\varepsilon^{\frac{2n}{n+1}}\right),$$

which implies, using the isoperimetric inequality and the equality $P_{\varepsilon,m}(\emptyset) = \mathcal{F}_\varepsilon(m)$,

$$P_{\varepsilon,m}(E_{\varepsilon,m}) = P_{\varepsilon,m}(\emptyset) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right) = \mathcal{F}_\varepsilon(m) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right),$$

which gives

$$\min_{E\subseteq\Omega} P_{\varepsilon,m}(E) \geq \min_{u\in H^1_m(\Omega} \mathcal{F}_\varepsilon(u) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right).$$

If (4.17) is not satisfied, we can assume that $|E_{\varepsilon_k,m}| \geq c\varepsilon_k^{\frac{n}{n+1}}$, for some $c > 0$ and some sequence $\varepsilon_k \to 0$ (for simplicity we shall drop the explicit dependence on k). By assumption (H) and (4.16) there exists $x_0 \in \partial\Omega$ and $r_\varepsilon > 0$ such that

$$|E_{\varepsilon,m}| = |B_{r_\varepsilon}(x_0)\cap\Omega|, \quad P_{\varepsilon,m}(E_{\varepsilon,m}) = P_{\varepsilon,m}(B_{r_\varepsilon}(x_0)) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right). \quad (4.18)$$

Let us now consider the function $w_\varepsilon \in H^1(\Omega)$ defined as

$$w_\varepsilon(x) := \min\left\{\gamma_\varepsilon\left(\frac{d_{B_{r_\varepsilon}(x_0)}(x)}{\varepsilon}\right), m + 2\frac{|E_{\varepsilon,m}|}{|\Omega|} + c_\varepsilon\right\}, \qquad \forall\, x \in \Omega, \quad (4.19)$$

where $d_{B_{r_\varepsilon}(x_0)}$ is the signed distance from $\partial B_{r_\varepsilon}(x_0)$, γ_ε is as in Lemma 4.1, and $c_\varepsilon = o\left(\varepsilon^{\frac{n}{n+1}}\right)$ is chosen in such a way that $\frac{1}{|\Omega|}\int_\Omega w_\varepsilon\, dx = m$. Reasoning as in the proof of Theorem 2.1 (ii) and using (4.13), (4.18) , we then get

$$\mathcal{F}_\varepsilon(w_\varepsilon) = P_{\varepsilon,m}(B_{r_\varepsilon}(x_0)) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right) \leq P_{\varepsilon,m}(E_{\varepsilon,m}) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right),$$

which gives (2.8). □

We conclude this section with the following result.

Proposition 4.2. *Let $\varepsilon \in (0,\varepsilon_0)$ and let $\overline{m}_\varepsilon$ be defined as in* (2.3)*. Then there exists a solution $\overline{u}_{\varepsilon,\overline{m}_\varepsilon}$ of* (2.2) *such that $\overline{u}_{\varepsilon,\overline{m}_\varepsilon} \not\equiv \overline{m}_\varepsilon$. Moreover, if Ω has the regularity property (H) we also have*

$$\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,\overline{m}_\varepsilon}) = \mathcal{F}_\varepsilon(\overline{m}_\varepsilon) = P_{\varepsilon,\overline{m}_\varepsilon}(B_{r_{\varepsilon,\overline{m}_\varepsilon}}(x_0)\cap\Omega) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right), \quad (4.20)$$

where $r_{\varepsilon,\overline{m}_\varepsilon}$ is as in (4.15)*.*

Proof. Let us prove the first equality in (4.20), namely

$$\mathcal{F}_\varepsilon(\overline{m}_\varepsilon) = \min_{v\in H^1_{\overline{m}_\varepsilon}(\Omega)} \mathcal{F}_\varepsilon(v).$$

Assume by contradiction there exists $\delta > 0$ such that

$$\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,\overline{m}_\varepsilon}) = \min_{v\in H^1_{\overline{m}_\varepsilon}(\Omega)} \mathcal{F}_\varepsilon(v) = \mathcal{F}_\varepsilon(\overline{m}_\varepsilon) - \delta.$$

Recall that $\overline{u}_{\varepsilon,\overline{m}_\varepsilon} \in \mathcal{C}^\infty(\Omega)$ and

$$-\varepsilon^2\Delta\overline{u}_{\varepsilon,\overline{m}} + W'(\overline{u}_{\varepsilon,\overline{m}}) = \frac{1}{\Omega}\int_\Omega W'(\overline{u}_{\varepsilon,\overline{m}})\, dx \qquad \text{in } \Omega. \quad (4.21)$$

Therefore $W'(\max_\Omega \overline{u}_{\varepsilon,\overline{m}}) \leq \int_\Omega W'(\overline{u}_{\varepsilon,\overline{m}})\ dx \leq W'(\overline{u}_{\varepsilon,\overline{m}})$, which in particular implies that $|\overline{u}_{\varepsilon,\overline{m}_\varepsilon}| \leq C$ where C is independent of ε. Therefore $\mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,\overline{m}_\varepsilon} + \lambda) \leq \mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,\overline{m}_\varepsilon}) + C\lambda$.

Take $0 < \lambda < C\varepsilon\delta/8$ and let $u_\lambda := \overline{u}_{\varepsilon,\overline{m}_\varepsilon} + \frac{\lambda}{|\Omega|}$, so that $\frac{1}{|\Omega|}\int_\Omega u_\lambda = \overline{m}_\varepsilon + \lambda$ for $\lambda > 0$. Then

$$\mathcal{F}_\varepsilon(u_\lambda) \leq \mathcal{F}_\varepsilon(\overline{u}_{\varepsilon,\overline{m}_\varepsilon}) + \frac{C\lambda}{\varepsilon} < \mathcal{F}_\varepsilon(\overline{m}_\varepsilon) - \frac{C\lambda}{\varepsilon} \leq \mathcal{F}_\varepsilon(\overline{m}_\varepsilon + \lambda),$$

which contradicts the fact (that follows from the definition of $\overline{m}_\varepsilon$) that

$$\mathcal{F}_\varepsilon(\overline{m}_\varepsilon + \lambda) = \min_{v \in H^1_{\overline{m}_\varepsilon+\lambda}(\Omega)} \mathcal{F}_\varepsilon(v).$$

Let us now show that there exists a nonconstant solution of (2.2).

Let us take a sequence $m_j \uparrow \overline{m}_\varepsilon$ for which there exist corresponding nonconstant minimizers u_j of (2.2) with $\mathcal{F}_\varepsilon(u_j) < \mathcal{F}_\varepsilon(m_j)$. Note that there exists a constant $C(\varepsilon) > 0$ independent of j such that

$$\|u_j\|_{H^1(\Omega)} \leq \frac{1}{\varepsilon}\mathcal{F}_\varepsilon(u_j) \leq \frac{W(m_j)}{\varepsilon^2} \leq C(\varepsilon).$$

It follows that the sequence (u_j) is precompact in $L^2(\Omega)$ hence, up to a (not relabelled) subsequence, $u_j \to u$ in $L^2(\Omega)$ for some $u \in H^1(\Omega)$ as $j \to +\infty$. By semicontinuity of $\mathcal{F}_\varepsilon$, we have $\mathcal{F}_\varepsilon(u) \leq \liminf_j \mathcal{F}_\varepsilon(u_j) \leq \liminf_j \mathcal{F}_\varepsilon(m_j) = \mathcal{F}_\varepsilon(\overline{m}_\varepsilon)$, and therefore u is a solution of (2.2). Moreover, applying Corollary 3.4 to each u_j we can find t_{m_j} such that $|\Sigma_{-t_{m_j}}(u_j)| \geq c\varepsilon^{\frac{n}{n+1}}$ for some constant $c > 0$ independent of j. Passing to a subsequence, we can find $t_m \geq 0$ such that $|\Sigma_{-t_m}(u)| \geq c\varepsilon^{\frac{n}{n+1}}$. We then deduce that u is not constant.

The last equality in (4.20) follows from Theorem 2.3, applied with $m = m_j$, passing to the limit as $j \to +\infty$. $\square$

4.3 Proof of Theorem 2.4

From Theorem 2.1 it follows

$$\overline{m}_\varepsilon \in \left[1 - C_2\varepsilon^{\frac{n}{n+1}}, 1 - C_1\varepsilon^{\frac{n}{n+1}}\right]. \tag{4.22}$$

By Theorem 2.3 we get

$$\mathcal{F}_\varepsilon(\overline{m}_\varepsilon) + o\left(\varepsilon^{\frac{n-1}{n+1}}\right) = \min_{r \geq 0} P_{\varepsilon,\overline{m}_\varepsilon}(B_r(x_0) \cap \Omega). \tag{4.23}$$

Therefore, recalling that

$$\mathcal{F}_\varepsilon(\overline{m}_\varepsilon) = \frac{|\Omega|}{\varepsilon} W(\overline{m}_\varepsilon) = \frac{|\Omega|}{2\varepsilon} W''(1)(1 - \overline{m}_\varepsilon)^2 + o\left(\varepsilon^{\frac{n-1}{n+1}}\right),$$

we obtain

$$\left(\frac{1-\overline{m}_\varepsilon}{\varepsilon^{\frac{n}{n+1}}}\right)^2 = \frac{2\mathcal{F}_\varepsilon(\overline{m}_\varepsilon)}{|\Omega|W''(1)\varepsilon^{\frac{n-1}{n+1}}} + o(1) = \frac{2\min_{r\geq 0} P_{\varepsilon,\overline{m}_\varepsilon}(B_r(x_0))}{|\Omega|W''(1)\varepsilon^{\frac{n-1}{n+1}}} + o(1). \tag{4.24}$$

Recalling the expression of $P_{\varepsilon,m}$, we define an approximation of it (thanks to assumption (H) and using an expansion of W) as

$$\phi(r) := c_\Omega \sigma_W n\omega_n r^{n-1} + \frac{|\Omega|W''(1)}{2\varepsilon}\left(1-\overline{m}_\varepsilon - \frac{2c_\Omega\omega_n}{|\Omega|}r^n\right)^2, \qquad r\geq 0,$$

where c_Ω is as in (2.7).

Observe that $P_{\varepsilon,\overline{m}_\varepsilon}(\emptyset) = \phi(0)$ and

$$P_{\varepsilon,\overline{m}_\varepsilon}(B_r(x_0)) = \phi(r) + o(\varepsilon^{\frac{n-1}{n+1}}) \qquad \forall\, r\geq 0.$$

Notice also that $r=0$ is always a local minimizer of ϕ; in addition, it turns out that ϕ has two other critical points $0<\hat{r}_\varepsilon<\tilde{r}_\varepsilon\leq C\varepsilon^{\frac{1}{n+1}}$, which are a local maximum and a local minimum respectively.

Recall that from (4.23) and Proposition 4.2 we know that $P_{\varepsilon,\overline{m}_\varepsilon}(B_r(x_0))$ has at least another local minimizer $r_\varepsilon>0$ such that $c\varepsilon^{\frac{1}{n+1}}\leq r_\varepsilon\leq C\varepsilon^{\frac{1}{n+1}}$. We claim that

$$r_\varepsilon > \hat{r}_\varepsilon. \tag{4.25}$$

Indeed, assuming by contradiction $r_\varepsilon\leq\hat{r}_\varepsilon$, from the inequality

$$\phi(r)\geq\phi(0)+cr^{n-1} \qquad \forall\, r\in[0,\hat{r}_\varepsilon]$$

it would follow $\phi(r_\varepsilon)\geq\phi(0)+c\varepsilon^{\frac{n-1}{n+1}}$. On the other hand

$$\phi(r_\varepsilon) = P_{\varepsilon,\overline{m}_\varepsilon}(B_{r_\varepsilon}(x_0)) + o(\varepsilon^{\frac{n-1}{n+1}}) = \phi(0)+o(\varepsilon^{\frac{n-1}{n+1}}), \tag{4.26}$$

leading to a contradiction. From (4.25) and (4.26) it follows

$$\phi(\tilde{r}_\varepsilon)\leq\phi(r_\varepsilon)=\phi(0)+o(\varepsilon^{\frac{n-1}{n+1}}).$$

Moreover

$$\phi(\tilde{r}_\varepsilon) = P_{\varepsilon,\overline{m}_\varepsilon}(B_{\tilde{r}_\varepsilon}(x_0)) + o(\varepsilon^{\frac{n-1}{n+1}}) \geq \phi(0)+o(\varepsilon^{\frac{n-1}{n+1}}),$$

which implies

$$\phi(0)=\phi(\tilde{r}_\varepsilon)+o\left(\varepsilon^{\frac{n-1}{n+1}}\right),$$

which is equivalent to

$$\left(\frac{c_\Omega\omega_n\tilde{r}_\varepsilon^{n+1}}{|\Omega|\varepsilon}+\frac{n\sigma_W}{2W''(1)}\right)\frac{\varepsilon}{\tilde{r}_\varepsilon} = 1-\overline{m}_\varepsilon+o\left(\varepsilon^{\frac{n}{n+1}}\right). \tag{4.27}$$

From the equality $\phi'(\tilde{r}_\varepsilon) = 0$ we also obtain

$$2W''(1)\left(1 - \overline{m}_\varepsilon - \frac{2c_\Omega\omega_n}{|\Omega|}\tilde{r}_\varepsilon^n\right)\tilde{r}_\varepsilon = (n-1)\sigma_W\varepsilon,$$

hence

$$1 - \overline{m}_\varepsilon = \frac{(n-1)\sigma_W\varepsilon}{2W''(1)\tilde{r}_\varepsilon} + \frac{2c_\Omega\omega_n}{|\Omega|}\tilde{r}_\varepsilon^n.$$

Recalling (4.27), we then get

$$\tilde{r}_\varepsilon = \left(\frac{\sigma_W|\Omega|}{2c_\Omega\omega_n W''(1)}\right)^{\frac{1}{n+1}}\varepsilon^{\frac{1}{n+1}} + o\left(\varepsilon^{\frac{1}{n+1}}\right), \tag{4.28}$$

which implies

$$1 - \overline{m}_\varepsilon = K_\Omega\varepsilon^{\frac{n}{n+1}} + o\left(\varepsilon^{\frac{n}{n+1}}\right),$$

where K_Ω is defined in (2.9). □

Remark 4.3. Notice that from the equality $\phi(r_\varepsilon) = \phi(\tilde{r}_\varepsilon) + o(\varepsilon^{\frac{n-1}{n+1}})$ it also follows

$$r_\varepsilon - \tilde{r}_\varepsilon = o(\varepsilon^{\frac{1}{n+1}}),$$

hence (4.28) holds also for r_ε.

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