Γ-type estimates for the one-dimensional Allen-Cahn's action

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Abstract

In this paper we prove an asymptotic estimate, up to the second order included, on the behaviour of the one-dimensional Allen-Cahn's action functionals, around a periodic function with bounded variation and taking values in $\{\pm 1\}$. The leading term of this estimate justifies and confirms, from a variational point of view, the results of Fusco-Hale [11] and Carr-Pego [8] on the exponentially slow motion of metastable patterns coexisting at the transition temperature.

1 Introduction

In this paper we are interested in the asymptotic behaviour as $\varepsilon \to 0^+$ of the one-dimensional Allen-Cahn's action functionals

$$F_\varepsilon(u) := \int_{\mathbb{T}} \left(\frac{\varepsilon}{2}(u')^2 + \frac{W(u)}{\varepsilon} \right) dx,$$

where $\mathbb{T}$ is the one-dimensional unit torus, W is a smooth double well potential with zeroes at ± 1, and $u : \mathbb{T} \to \mathbb{R}$. These functionals arise in several models of phase transitions in materials science, see for instance [4, 12, 13, 11, 8] and references therein. In particular, two phases $u = \pm 1$, coexisting at the transition temperature, exhibit metastable patterns which slowly evolve according to the L^2-gradient flow of F_ε,

$$u_t = \varepsilon^2 u_{xx} - W'(u), \tag{1.1}$$

where a time rescaling has been performed. Equation (1.1) is perhaps the simplest partial differential equation modelling nonlinear relaxation to equilibrium in the presence of competing stable states. In [11, 8] the authors showed that, as $\varepsilon \to 0^+$, a solution u of (1.1) is locally equal to ± 1 and the transition points evolve, exponentially slowly, in accordance to a specific system of ODEs (see [11, Eq. (3.11)] and [8, Eq. (1.2)]). The exponential speed is dictated by the qualitative properties of W, in particular by its nondegeneracy at ± 1.

In this paper we aim to provide a variational couterpart of the dynamical results of [11, 8], recovering an analogous ODEs system obtained as a by-product of the behaviour, at the

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leading order, of the action functionals F_ε for $\varepsilon << 1$, around piecewise constant functions u with values in $\{\pm 1\}$, which correspond to the metastable patterns in the two-phase model described above. It is well-known [10, 14] that the sequence (F_ε) is equicoercive in $L^1(\mathbb{T})$ and Γ-$L^1(\mathbb{T})$-converges, as $\varepsilon \to 0^+$, to the functional $F_0 : L^1(\mathbb{T}) \to [0, +\infty]$ defined as

$$F_0(u) := \begin{cases} N(u)\sigma & \text{if} \quad u \in BV(\mathbb{T}; \{\pm 1\}), \\ +\infty & \text{otherwise}, \end{cases} \tag{1.2}$$

where $N(u)$ is the number of jump points of u, and where

$$\sigma := \inf \left\{ \int_{\mathbb{R}} \left(\frac{1}{2}(v')^2 + W(v) \right) \, dy : v \in H^1_{\mathrm{loc}}(\mathbb{R}), v(0) = 0, \lim_{y \to \pm\infty} v(y) = \pm 1 \right\} \tag{1.3}$$

is sometimes called surface tension.

The main results of this paper are the following asymptotic estimates. Firstly (Theorem 6.1) we prove that

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \geq N(v)\sigma - \alpha_+ \kappa_+^2 \sum_{k=1}^{N(v)} e^{-\alpha_+ \frac{d_k^\varepsilon}{\varepsilon}} - \alpha_- \kappa_-^2 \sum_{k=1}^{N(v)} e^{-\alpha_- \frac{d_k^\varepsilon}{\varepsilon}} \\ + o\left(\sum_{k=1}^{N(v)} e^{-\alpha_+ \frac{d_k^\varepsilon}{\varepsilon}} \right) + o\left(\sum_{k=1}^{N(v)} e^{-\alpha_- \frac{d_k^\varepsilon}{\varepsilon}} \right) \end{aligned} \tag{1.4}$$

as $\varepsilon \to 0^+$, where (v_ε) is any sequence converging to $v \in BV(\mathbb{T}; \{\pm 1\})$ in $L^1(\mathbb{T})$, $\alpha_\pm$, $\kappa_\pm$ are constants[1] depending on W, in particular $\alpha_\pm := \sqrt{W''(\pm 1)}$, and d_k^ε is the distance between the k-th and the $(k+1)$-th transition of v^ε (see (5.9) and (6.2)). Notice that the terms appearing on the right-hand side of (1.4) scale differently in ε, as soon as the limits $d_k(v) := \lim_\varepsilon d_k^\varepsilon$ are different for different k's, and in particular we cannot substitute the approximate distance d_k^ε with the distance $d_k(v) = x_{k+1}(v) - x_k(v)$ between the consecutive k-th and $(k+1)$-th jump point of the limit function v. Estimate (1.4) is sharp, in the sense that for any $v \in BV(\mathbb{T}; \{\pm 1\})$ there exists a sequence (v_ε) such that the equality holds in (1.4) (Theorem 6.5).

Secondly (Theorem 8.1) we show that if W is a parabola near ± 1 then we can improve (1.4), obtaining the (sharp) estimate

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \geq N(v)\sigma - \alpha_+ \kappa_+^2 \sum_{k=1}^{N(v)} e^{-\alpha_+ \frac{d_k^\varepsilon}{\varepsilon}} - \alpha_- \kappa_-^2 \sum_{k=1}^{N(v)} e^{-\alpha_- \frac{d_k^\varepsilon}{\varepsilon}} \\ + o\left(\sum_{k=1}^{N(v)} e^{-\frac{3\alpha_+}{2} \frac{d_k^\varepsilon}{\varepsilon}} \right) + o\left(\sum_{k=1}^{N(v)} e^{-\frac{3\alpha_-}{2} \frac{d_k^\varepsilon}{\varepsilon}} \right). \end{aligned} \tag{1.5}$$

Observe that (1.5) provides a sort of second order asymptotic expansion (with vanishing second order term) of F_ε around functions $v \in BV(\mathbb{T}; \{\pm 1\})$, which is reminiscent of a Γ-expansion of F_ε in the sense of [1, 2, 5, 6]. However, our results cannot be straightforwardly

[1] $\kappa_\pm$ are defined in (3.7), (3.8).

framed in that setting: for instance, we do not restrict ourselves to expand around a global minimizer of F_0 (which would be the constant state $u \equiv 1$ or $u \equiv -1$), but we need to work around an $L^1(\mathbb{T})$-local minimizer, more specifically around a periodic bounded variation function taking values in ± 1. In Section 7 we associate to our first order estimate (1.4) a Γ-limit, which turns out to be unbounded from below.
Now, let us introduce the functional $\mathcal{G}_\varepsilon : BV(\mathbb{T}; \{\pm 1\}) \to \mathbb{R}$ as

$$\mathcal{G}_\varepsilon(v) := N(v)\sigma - \sum_{k=1}^{N(v)} \Big(\alpha_+ \kappa_+^2 \ e^{-\alpha_+ \frac{d_k(v)}{\varepsilon}} + \alpha_- \kappa_-^2 \ e^{-\alpha_- \frac{d_k(v)}{\varepsilon}}\Big). \tag{1.6}$$

Notice that the right-hand side of (1.4) coincides, at the leading order, with $\mathcal{G}_\varepsilon(v)$ where however, the approximate distance d_k^ε is replaced by the distance $d_k(v)$.
The functional $\mathcal{G}_\varepsilon$ may be considered as a function of $N(v)$ variables, that is

$$\mathcal{G}_\varepsilon(v) = \mathcal{G}_\varepsilon(x_1(v), x_2(v), \cdots, x_{N(v)}(v)), \tag{1.7}$$

and the gradient flow of $\mathcal{G}_\varepsilon$ is given by the system of ODEs

$$\begin{aligned} \dot{x}_j(v) = \frac{\partial \mathcal{G}_\varepsilon(v)}{\partial x_j(v)} =& \frac{\alpha_+^2 \kappa_+^2}{\varepsilon} \Big(e^{-\alpha_+ \frac{(x_j(v) - x_{j-1}(v))}{\varepsilon}} - e^{-\alpha_+ \frac{(x_{j+1}(v) - x_j(v))}{\varepsilon}}\Big) \\ &+ \frac{\alpha_-^2 \kappa_-^2}{\varepsilon} \Big(e^{-\alpha_- \frac{(x_j(v) - x_{j-1}(v))}{\varepsilon}} - e^{-\alpha_- \frac{(x_{j+1}(v) - x_j(v))}{\varepsilon}}\Big) \end{aligned} \tag{1.8}$$

for all $j = 1, \cdots, N(v)$, where $\dot{x}_j$ stands for the derivative of x_j with respect to t. Notice from (1.8) that the jump point $x_j(v)$ interacts only with $x_{j-1}(v)$ and $x_{j+1}(v)$.
An interesting observation is that system (1.8) coincides, up to a multiplicative constant[2], with the evolution equations obtained in [11, Eq. (3.11)], [8, Eq. (1.2)], thus showing the consistency of (1.4) with the behaviour of (1.1) as $\varepsilon \to 0^+$. This is in accordance with the general principle outlined in [15, 16], where the authors relate the gradient flow of the Γ-limit of a sequence of functionals with the limit of the gradient flows.
We observe that, not surprisingly, only the terms of order $e^{-\alpha_\pm \frac{d_k^\varepsilon}{\varepsilon}}$ are relevant for the evolution law of the jump points $x_j(v)$.
We conclude this introduction by mentioning that the results of [11] and [8] have been generalized to a vector setting (in the target space) in the paper [3]; generalizing estimate (1.4) to this more general situation seems, however, not easy.
The content of the paper is the following. In Section 2 we set the notation. In Section 3 we introduce the contants $c_\pm$ and hence the constants $\kappa_\pm$ appearing in (1.4), (1.5) and (1.8). In Section 4 we introduce various functions, which are useful to prove the main results. The expansions of those functions are computed in Section 4.1. Two lemmas, based on variational arguments, and necessary to the main results are next proven in Section 5. We prove estimate (1.4) and its sharpness in Theorems 6.1 and 6.5. In Section 7 we find a related first order Γ-limit, under the additional assumption that W is even. This is the only place of the paper where we make such an assumption. Eventually, in Section 8 we prove (1.5): as mentioned

[2] The presence of a multiplicative constant is not surprising, for instance, a similar phenomenon happens in mean curvature flow when approximated with the parabolic Allen-Cahn's equation. Such constants can be normalized to one by a time scaling.

above, we are able to show this estimate supposing that W is a parabola near its minimum point, and this makes easy to treat the various singular integrals involved (in particular, the derivative of the function D_+, defined in (4.5), evaluated at the point $s = 1$.

2 Notation

The assumptions on the double well potential W are the following:

(W1) $W : \mathbb{R} \to [0, +\infty)$ and $W \in \mathcal{C}^\infty(\mathbb{R})$;

(W2) $W^{-1}(0) = \{\pm 1\}$;

(W3) $W''(\pm 1) > 0$. We set

$$\alpha_+ := \sqrt{W''(1)}, \qquad \alpha_- := \sqrt{W''(-1)}. \tag{2.1}$$

Notice that we do not suppose that W is even. We define

$$\beta_\pm := W'''(\pm 1).$$

A Taylor expansion around $s = 1$ gives

$$2W(s) = \alpha_+^2 (1-s)^2 \left[1 + \frac{\beta_+}{3\alpha_+^2}(s-1)\right] + o\left((1-s)^3\right) \qquad \text{as } s \to 1^-, \tag{2.2}$$

and similarly in a right neighborhood of -1 with α_- replacing α_+ and β_- replacing β_+. From (2.2) it follows $\frac{1}{\sqrt{2W(s))}} = \frac{1}{\alpha_+(1-s)}\Big(1 + \frac{\beta_+}{6\alpha_+^2}(1-s) + o(1-s)\Big)$ as $s \to 1^-$, so that

$$\frac{1}{\sqrt{2W(s))}} - \frac{1}{\alpha_+(1-s)} = \frac{\beta_+}{6\alpha_+^3} + \rho(s), \tag{2.3}$$

where the reminder

$$\rho : [0,1] \to \mathbb{R} \ \text{ is continuous}, \qquad \lim_{s \to 1^-} \rho(s) = 0.$$

We define

$$\phi(\eta) := \int_{-1}^{\eta} \sqrt{2W(s)}\, ds, \qquad \eta \in [-1, 1], \tag{2.4}$$

It is known that σ in (1.3) satisfies

$$\sigma = \phi(1) = \sigma_- + \sigma_+,$$

where we have set

$$\sigma_- := \int_{-1}^{0} \sqrt{2W(s)}\, ds, \qquad \sigma_+ := \int_{0}^{1} \sqrt{2W(s)}\, ds.$$

Remark 2.1. Our assumptions on W ensure that for $\eta \in (0,1)$ (resp. $\eta \in (-1,0)$) sufficiently close to 1 (resp. to -1) we have

$$W(\eta) < W(s), \quad s \in (0, \eta) \qquad (\text{resp. } s \in (\eta, 0)),$$

and $W'(\eta) < 0$ (resp. $W'(\eta) > 0$).

2.1 Periodic BV functions

Let $\mathbb{T}$ be the one-dimensional unit torus. We denote by $BV(\mathbb{T};\{\pm 1\})$ the space of functions of bounded variation in $\mathbb{T}$ taking values ± 1. For a function $u \in BV(\mathbb{T};\{\pm 1\})$ with nonempty jump set $S(u) \subset \mathbb{T}$, we write $S(u) = \{x_1(u), \cdots, x_{N(u)}(u)\}$, where $N(u) \in (0,+\infty)$ is the number of the jump points of u, and

$$x_1(u) < x_2(u) < \cdots < x_{N(u)}(u) < x_{N(u)+1}(u) := x_1(u). \tag{2.5}$$

We let $N_+(u)$ be the number of increasing jumps from -1 to 1 (resp. $N_-(u)$ be the number of decreasing jumps from 1 to -1). Due to the periodicity of functions in $BV(\mathbb{T};\{\pm 1\})$, $N(u)$ is even (or zero) and $N_+(u) = N_-(u)$. If $S(u) = \emptyset$ we set $N(u) = N_+(u) = N_-(u) = 0$.

Definition 2.2. *Let $u \in BV(\mathbb{T};\{\pm 1\})$ be nonconstant. We define*

$$\begin{cases} d_k(u) := x_{k+1}(u) - x_k(u), & k = 1, \cdots, N(u)-1, \\ d_{N(u)} := 1 - (x_{N(u)} - x_1(u)) = d_0(u), & \end{cases} \tag{2.6}$$

and

$$\begin{aligned} \mathcal{I}_+(u) :=& \Big\{ k \in \{1, \cdots, N(u)\} : u \text{ jumps from } -1 \text{ to } 1 \text{ at } x_k(u) \Big\}, \\ \mathcal{I}_-(u) :=& \{1, \cdots, N(u)\} \setminus \mathcal{I}_+(u). \end{aligned}$$

2.2 The functionals F_ε and the minimizer γ

For any $\varepsilon \in (0,1)$ let $F_\varepsilon : L^1(\mathbb{T}) \to [0,+\infty]$ be defined by

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_{\mathbb{T}} \left(\frac{\varepsilon}{2}(u')^2 + \frac{W(u)}{\varepsilon} \right) dx & \text{if } u \in H^1(\mathbb{T}) \text{ and } W(u) \in L^1(\mathbb{T}), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

When I is a measurable subset of $\mathbb{T}$, we denote by $F_\varepsilon(\cdot, I)$ the localization of $F_\varepsilon(\cdot)$ on I (obtained by replacing $\mathbb{T}$ with I in (2.7)) and we set $F_\varepsilon(\cdot, \mathbb{T}) = F_\varepsilon(\cdot)$.
If $J \subset \mathbb{R}$ is a bounded interval and $v \in H^1(J)$, we set

$$\mathcal{F}(v, J) := \int_J \left(\frac{1}{2}(v')^2 + W(v) \right) \, dy, \tag{2.8}$$

and for $v \in H^1_{\mathrm{loc}}(\mathbb{R})$, we let $\mathcal{F}(v) := \int_{\mathbb{R}} \left(\frac{1}{2}(v')^2 + W(v) \right) \, dy$.
It is well-known that the infimum in (1.3) is a minimum and is attained by the function $\gamma \in \mathcal{C}^\infty(\mathbb{R})$ solving

$$\begin{cases} \gamma' = \sqrt{2W(\gamma)} & \text{in } \mathbb{R}, \\ \gamma(0) = 0. \end{cases} \tag{2.9}$$

3 The functions $\mathrm{B}_\pm$, the constants $c_\pm$ and $\kappa_\pm$

We let

$$\mathrm{B}_+(\eta) := \int_0^\eta \frac{1}{\sqrt{2W(s)}}\, ds, \qquad \eta \in (0,1). \tag{3.1}$$

Note that

$$\mathrm{B}_+(\eta) = \frac{1}{\alpha_+}\log\left(\frac{1}{1-\eta}\right) + o(1)\log(1-\eta) \qquad \text{as } \eta \to 1^-. \tag{3.2}$$

Indeed, from (2.3) and de l'Hôpital theorem, we deduce

$$\lim_{\eta\to 1^-} \frac{\mathrm{B}_+(\eta)}{-\log(1-\eta)} = \lim_{\eta\to 1^-} \frac{1-\eta}{\sqrt{2W(\eta)}} = \frac{1}{\alpha_+},$$

and (3.2) follows.

Lemma 3.1 (**Expansion of** B_+)**.** *Let* B_+ *be the function defined in* (3.1)*. Then*

$$\exists \lim_{\eta\to 1^-} \left(\mathrm{B}_+(\eta) - \frac{1}{\alpha_+}\log\left(\frac{1}{1-\eta}\right)\right) =: c_+ \in \mathbb{R}. \tag{3.3}$$

Hence

$$\mathrm{B}_+(\eta) = \frac{1}{\alpha_+}\log\left(\frac{1}{1-\eta}\right) + c_+ + o(1) \qquad \text{as } \eta \to 1^-. \tag{3.4}$$

Proof. We have

$$\mathrm{B}_+(\eta) - \frac{1}{\alpha_+}\log\left(\frac{1}{1-\eta}\right) = \int_0^\eta \left(\frac{1}{\sqrt{2W(s)}} - \frac{1}{\alpha_+(1-s)}\right) ds, \qquad \eta \in (0,1). \tag{3.5}$$

Coupling (2.3) with (3.5) we get

$$\mathrm{B}_+(\eta) - \frac{1}{\alpha_+}\log\left(\frac{1}{1-\eta}\right) = \frac{\beta_+}{6\alpha_+^3}\eta + \int_0^\eta \rho(s)\, ds, \qquad \eta \in (0,1). \tag{3.6}$$

Then formula (3.3) follows, and

$$\begin{aligned} c_+ = \lim_{\eta\to 1^-} \int_0^\eta \left(\frac{1}{\sqrt{2W(s)}} - \frac{1}{\alpha_+(1-s)}\right) ds = \int_0^1 \left(\frac{1}{\sqrt{2W(s)}} - \frac{1}{\alpha_+(1-s)}\right) ds \\ = \frac{\beta_+}{6\alpha_+^3} + \int_0^1 \rho(s)\, ds. \end{aligned} \tag{3.7}$$

□

The minimizer γ tends to its asymptotic values with an exponential rate given by $\alpha_\pm$. For convenience of the reader and for future reference (see the proof of Theorem 6.1) we give the proof of the following result (see for instance [8]).

Corollary 3.2 (**Asymptotic behaviour of** γ). *There exist the limits*

$$\lim_{y\to+\infty} \frac{1-\gamma(y)}{e^{-\alpha_+ y}} =: \kappa_+ \in (0,+\infty), \qquad \lim_{y\to-\infty} \frac{1+\gamma(y)}{e^{\alpha_- y}} =: \kappa_- \in (0,+\infty). \tag{3.8}$$

Proof. We consider the case $y > 0$, the case $y < 0$ being similar. From (2.9) it follows

$$\int_0^y \frac{\gamma'}{\sqrt{2W(\gamma)}}\,dz = \mathrm{B}_+(\gamma(y)) = y, \qquad y > 0. \tag{3.9}$$

Therefore, using (3.4) we find $y = -\frac{1}{\alpha_+}\log(1-\gamma(y)) + c_+ + o(1)$ as $y \to +\infty$. This implies the assertion in (3.8) with

$$\kappa_+ = e^{\alpha_+ c_+}. \tag{3.10}$$

□

We set $\mathrm{B}_-(\eta) := \int_\eta^0 \frac{1}{\sqrt{2W(s)}}\,ds$ for $\eta \in (-1,0)$, and

$$c_- := \lim_{\eta\to 1^-} \int_0^\eta \left(\frac{1}{\sqrt{2W(-s)}} - \frac{1}{\alpha_-(1+s)} \right) ds, \qquad \kappa_- := e^{\alpha_- c_-}.$$

4 The functions $\mathcal{Q}_\pm$, $\mathcal{A}_\pm$, $D_\pm$, $\mathcal{L}_\pm$

From Remark 2.1 we have that the function

$$\mathcal{Q}_+(\eta) := \int_0^\eta \frac{1}{\sqrt{2W(s) - 2W(\eta)}}\,ds, \qquad \eta \in (0,1) \text{ close enough to } 1 \tag{4.1}$$

is well defined (one checks that $\frac{1}{\sqrt{2W(\cdot)-2W(\eta)}} \in L^1(0,\eta)$).
We let

$$\mathcal{A}_+(\eta) := \frac{1}{\alpha_+}\int_0^\eta \frac{1}{\sqrt{(1-s)^2 - (1-\eta)^2}}\,ds, \qquad \eta \in (0,1). \tag{4.2}$$

Setting $1-\eta = \xi$, changing variable with $1-s = t$ and then $t/\xi = x$ we get $\mathcal{A}_+(\eta) = \frac{1}{\alpha_+}\int_1^{\frac{1}{1-\eta}} \frac{1}{\sqrt{x^2-1}}\,dx$. With a direct integration we have

$$\mathcal{A}_+(\eta) = \frac{1}{\alpha_+}\log\left(\frac{1-\eta}{1-\sqrt{1-(1-\eta)^2}}\right), \qquad \eta \in (0,1). \tag{4.3}$$

Hence[3]

$$\mathcal{A}_+(\eta) = \frac{1}{\alpha_+}\log\left(\frac{2}{1-\eta}\right) + o(1) \qquad \text{as } \eta \to 1^-. \tag{4.4}$$

For $\eta \in (0,1)$ sufficiently close to 1, we also consider the difference

$$D_+(\eta) := \mathcal{Q}_+(\eta) - \mathcal{A}_+(\eta). \tag{4.5}$$

[3]Notice that if we put $\eta = 1$ inside the integrand of (4.2) we get $\frac{1}{\alpha_+}\int_0^\eta \frac{1}{1-s}\,ds$, which is not equal to the leading term on the right hand side of (4.4).

Finally

$$\mathcal{L}_+(\eta) := \int_0^\eta \sqrt{2W(s) - 2W(\eta)}\, ds, \qquad \eta \in (0,1] \text{ close enough to } 1, \tag{4.6}$$

Notice that

$$\lim_{\eta \to 1^-} \mathcal{L}_+(\eta) = \mathcal{L}_+(1) = \sigma_+,$$

and

$$\mathcal{L}'_+(\eta) = -W'(\eta)\mathcal{Q}_+(\eta), \qquad \eta \in (0,1) \text{ close enough to } 1. \tag{4.7}$$

We set $\mathcal{Q}_-(\eta) := \int_\eta^0 \frac{1}{\sqrt{2W(s)-2W(\eta)}}\, ds$, $\mathcal{L}_-(\eta) := \int_\eta^0 \sqrt{2W(s) - 2W(\eta)}\, ds$, and $D_-(\eta) := \mathcal{Q}_-(\eta) - \mathcal{A}_-(\eta)$ for $\eta \in (-1, 0)$ sufficiently close to -1, where $\mathcal{A}_-(\eta) := \frac{1}{\alpha_-} \int_\eta^0 \frac{1}{\sqrt{(1-s)^2-(1-\eta)^2}}\, ds$ for $\eta \in (-1, 0)$.

4.1 Expansions of D_+, $\mathcal{Q}_+$ and $\mathcal{L}_+$

We shall need the following result.

Lemma 4.1 (**Expansion of D_+**)**.** *Let c_+ be as in* (3.3)*. Then*

$$\lim_{\eta \to 1^-} D_+(\eta) = c_+ = D_+(1). \tag{4.8}$$

Proof. Recalling (2.2), write

$$f(x) := 2W(x) - \alpha_+^2(1-x)^2 - \frac{\beta_+}{3}(x-1)^3, \qquad x \in \mathbb{R}.$$

For $s < \eta < 1$ we write $f(s) - f(\eta) = f'(\xi)(s - \eta)$ for a suitable $\xi \in (s, \eta)$: for s sufficiently close to 1 we deduce

$$\begin{aligned} &2W(s) - 2W(\eta) \\ =&\alpha_+^2\Big((1-s)^2 - (1-\eta)^2\Big) + \frac{\beta_+}{3}\Big((s-1)^3 - (\eta-1)^3\Big) \\ &+ \big(2W'(\xi) + 2\alpha_+^2(1-\xi) - \beta_+(1-\xi)^2\big)(s-\eta) \\ =&\alpha_+^2\Big((1-s)^2 - (1-\eta)^2\Big)\Big(1 + \frac{\beta_+}{3\alpha_+^2}\psi(s,\eta) + R(s,\eta)\Big), \end{aligned} \tag{4.9}$$

where

$$\begin{cases} \psi(s,\eta) = \dfrac{(s-1)^3 - (\eta-1)^3}{(1-s)^2 - (1-\eta)^2} = -\dfrac{(1-s)^2 + (1-s)(1-\eta) + (1-\eta)^2}{(1-s) + (1-\eta)}, \\ \\ R(s,\eta) = \dfrac{\big(2W'(\xi) + 2\alpha_+^2(1-\xi) - \beta_+(1-\xi)^2\big)}{\alpha_+^2\,(s+\eta-2)}. \end{cases} \tag{4.10}$$

Since $1 - \eta \le 1 - s$ we have

$$\psi(s,\eta) \le 3(1-s). \tag{4.11}$$

Taylor expanding W' around $s=1$, we have

$$2W'(\xi)+2\alpha_+^2(1-\xi)-\beta_+(1-\xi)^2=\frac{W''''(\zeta)}{6},$$

for a suitable $\zeta\in(\xi,1)$. Hence

$$R(s,\eta)=\frac{O\left((1-\xi)^3\right)}{2-(s+\eta)}=O\left((1-s)^2\right).$$

It then certainly follows from (4.9)

$$\frac{1}{\sqrt{2W(s)-2W(\eta)}}=\frac{1+O(1-s)}{\alpha_+\sqrt{(1-s)^2-(1-\eta)^2}}.$$

Therefore, for $s<\eta$ and as $(s,\eta)\to(1^-,1^-)$,

$$\frac{1}{\sqrt{2W(s)-2W(\eta)}}-\frac{1}{\alpha_+\sqrt{(1-s)^2-(1-\eta)^2}}=\frac{O(1-s)}{\sqrt{(1-s)^2-(1-\eta)^2}}. \tag{4.12}$$

For $\eta\in(0,1)$ sufficiently close to 1 we have

$$\begin{aligned}D_+(\eta)&=\int_0^\eta\left(\frac{1}{\sqrt{2W(s)-2W(\eta)}}-\frac{1}{\alpha_+\sqrt{(1-s)^2-(1-\eta)^2}}\right)\,ds\\&=\eta\int_0^1\left(\frac{1}{\sqrt{2W(s\eta)-2W(\eta)}}-\frac{1}{\alpha_+\sqrt{(1-s\eta)^2-(1-\eta)^2}}\right)\,ds.\end{aligned} \tag{4.13}$$

From (4.12) applied with $s\eta$ in place of s it follows

$$\begin{aligned}\left|\frac{1}{\sqrt{2W(s\eta)-2W(\eta)}}-\frac{1}{\alpha_+\sqrt{(1-s\eta)^2-(1-\eta)^2}}\right|&\le\frac{C(1-s\eta)}{\sqrt{(1-s\eta)^2-(1-\eta)^2}}\\&=\frac{C}{\sqrt{1-\left(\frac{1-\eta}{1-s\eta}\right)^2}},\end{aligned}$$

for a suitable absolute positive constant C. Since the function $\eta\in(0,1)\to\frac{1}{\sqrt{1-\left(\frac{1-\eta}{1-s\eta}\right)^2}}$ is decreasing, the integrands on the right member of (4.13) are equiintegrable. Thus, by Lebesgue's dominated convergence theorem, we can pass to the limit in (4.13) as $\eta\to1^-$ and obtain

$$\lim_{\eta\to1^-}D_+(t)=\int_0^1\left(\frac{1}{\sqrt{2W(s)}}-\frac{1}{\alpha_+\sqrt{(1-s)^2}}\right)\,ds=c_+. \tag{4.14}$$

$\square$

Lemma 4.2 (Expansion of $\mathcal{Q}_+$ at first order). *Let $\mathcal{Q}_+$ be the function defined in (4.1). Then*

$$\mathcal{Q}_+(\eta)=\frac{1}{\alpha_+}\log\left(\frac{1}{1-\eta}\right)+\frac{\log 2}{\alpha_+}+c_++o(1)\quad\text{as }\eta\to1^-. \tag{4.15}$$

Proof. Using (4.4) and (4.5) we have

$$\mathcal{Q}_+(\eta) = \frac{1}{\alpha_+} \log\left(\frac{2}{1-\eta}\right) + D_+(1) + o(1) \quad \text{as } \eta \to 1^-, \tag{4.16}$$

and the assertion follows from (4.8). □

Lemma 4.3 (**Expansion of $\mathcal{L}_+$ at first order**). *Let $\mathcal{L}_+$ be the function defined in* (4.6)*. Then*

$$\begin{aligned}\mathcal{L}_+(\eta) =& \sigma_+ - \frac{\alpha_+}{2}(1-\eta)^2 \log\left(\frac{1}{1-\eta}\right) \\ &- \frac{\alpha_+}{2}\left(\log(2\kappa_+) + \frac{1}{2}\right)(1-\eta)^2 + o\left((1-\eta)^2\right) \quad \text{as } \eta \to 1^-.\end{aligned} \tag{4.17}$$

Proof. Using de l'Hôpital theorem, (4.7), (2.2) and (4.15), we compute

$$\begin{aligned}&\lim_{\eta\to 1^-} \frac{\mathcal{L}_+(\eta) - \sigma_+ + \frac{\alpha_+}{2}(1-\eta)^2 \log\left(\frac{1}{1-\eta}\right)}{(1-\eta)^2} \\ =& \frac{1}{2}\lim_{\eta\to 1^-} \frac{W'(\eta)\mathcal{Q}_+(\eta) + \alpha_+(1-\eta)\log\left(\frac{1}{1-\eta}\right) - \frac{\alpha_+}{2}(1-\eta)}{1-\eta} \\ =& \frac{1}{2}\lim_{\eta\to 1^-} \frac{-\alpha_+^2(1-\eta)\left[\frac{1}{\alpha_+}\log\left(\frac{1}{1-\eta}\right) + \frac{\log 2}{\alpha_+} + c_+\right] + \alpha_+(1-\eta)\log\left(\frac{1}{1-\eta}\right) - \frac{\alpha_+}{2}(1-\eta)}{1-\eta} \\ =& -\frac{\alpha_+}{2}\left[\log 2 + \alpha_+ c_+ + \frac{1}{2}\right].\end{aligned}$$

Then formula (4.17) follows, recalling also (3.10). □

We shall use expansions (4.15) and (4.17) in formulas (6.13) and (6.16) below.

5 Two useful lemmas

In this section we prove two useful lemma, which are preliminary for the results of Section 6.

Lemma 5.1 (The functions z_ε). *Let $v \in BV(\mathbb{T};\{\pm 1\})$ be a function with $N(v) > 0$. For any $k = 1, \cdots, N(v)$ with k even, suppose that $v = -1$ in $(x_{k-1}(v), x_k(v))$ and $v = 1$ in $(x_k(v), x_{k+1}(v))$. For any $k = 1, \cdots, N(v)$, let $(x_k^\varepsilon) \subset \mathbb{T}$ be a sequence of points converging to $x_k(v)$ as $\varepsilon \to 0^+$, where we set $k^\varepsilon_{N(v)+1} := x_1^\varepsilon$. Let*

$$s_0 \in (-1, 0), \tag{5.1}$$

and define

$$A_k^\varepsilon(s_0) := \Big\{ z \in H^1(x_k^\varepsilon, x_{k+1}^\varepsilon) : z(x_k^\varepsilon) = 0, z(x_{k+1}^\varepsilon) = 0,\ (-1)^k z(x) \geq s_0 \text{ for any } x \in (x_k^\varepsilon, x_{k+1}^\varepsilon) \Big\}.$$

Then there exists a function $z_\varepsilon \in H^1(\mathbb{T}; (-1,1))$ with the following properties:

(i) for any $k = 1, \cdots, N(v)$

$$F\left(z_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)\right) = \min_{z \in A_k^\varepsilon(s_0)} F_\varepsilon\left(z, (x_k^\varepsilon, x_{k+1}^\varepsilon)\right); \tag{5.2}$$

(ii) there is a positive constant C depending only on W and v such that

$$\sup_{\varepsilon \in (0,1)} F_\varepsilon(z_\varepsilon) \leq C; \tag{5.3}$$

(iii) for any $k = 1, \cdots, N(v)$ we have $z_\varepsilon \in \mathcal{C}^{1,1}([x_k^\varepsilon, x_{k+1}^\varepsilon])$;

(iv) for any $k = 1, \cdots, N(v)$ we have that, for $\varepsilon \in (0,1)$ sufficiently small, $z_\varepsilon \in \mathcal{C}^\infty(x_k^\varepsilon, x_{k+1}^\varepsilon)$ is a classical solution to

$$\begin{cases} -\varepsilon z_\varepsilon'' + \varepsilon^{-1} W'(z_\varepsilon) = 0 & \text{in } (x_k^\varepsilon, x_{k+1}^\varepsilon), \\ z_\varepsilon(x_k^\varepsilon) = z_\varepsilon(x_{k+1}^\varepsilon) = 0, & \\ z_\varepsilon > 0 & \text{in } (x_k^\varepsilon, x_{k+1}^\varepsilon) \text{ if } k \text{ is even}, \\ z_\varepsilon < 0 & \text{in } (x_k^\varepsilon, x_{k+1}^\varepsilon) \text{ if } k \text{ is odd}. \end{cases} \tag{5.4}$$

Morover, z_k^ε is even with respect to the mid point of $(x_k^\varepsilon, x_{k+1}^\varepsilon)$;

(v) for any $k = 1, \cdots, N(v)$

$$\lim_{\varepsilon \to 0^+} \max_{x \in [x_k^\varepsilon, x_{k+1}^\varepsilon]} |z_\varepsilon(x)| = 1. \tag{5.5}$$

Proof. Given $k = 1, \cdots, N(v)$, the minimum problem on the right hand side of (5.2) has a solution z_k^ε by direct methods. Hence[4], setting

$$z_\varepsilon := z_k^\varepsilon \qquad \text{on } [x_k^\varepsilon, x_{k+1}^\varepsilon], \qquad k = 1, \cdots, N(v),$$

we have that z_ε satisfies (i); note that, by truncating with the constants -1 and 1, we can suppose that $z_\varepsilon(x) \in [-1, 1]$ for any $x \in \mathbb{T}$.

[4] If $a < b < c < d$, $u \in H^1(a,c)$, $v \in H^1(c,d)$, and $u(c) = v(c)$, then the function w defined in (a,d) as $w := u$ in (a,b) and $w := v$ in (c,d) belongs to $H^1(a,d)$.

Assertion (ii) follows by comparing $F_\varepsilon(z_\varepsilon)$ with the value of F_ε, on each interval $(x_k^\varepsilon, x_{k+1}^\varepsilon)$, of a competitor which, for k even (resp. k odd) takes values in $[0,1]$ (resp. in $[-1,0]$) and grows (resp. decreases) linearly from 0 to 1 (resp. from 0 to -1) in $(x_k^\varepsilon, x_k^\varepsilon+\varepsilon)$, it is 1 (resp. -1) in $(x_k^\varepsilon+\varepsilon, x_{k+1}^\varepsilon-\varepsilon)$, and then decreases (resp. grows) to 0 (resp. to -1) in $(x_{k+1}^\varepsilon-\varepsilon, x_{k+1}^\varepsilon)$.
Let us show (iii) and (iv). Without loss of generality, we fix k even. The minimality of z_ε in $(x_k^\varepsilon, x_{k+1}^\varepsilon)$ entails $-\varepsilon z_\varepsilon'' + \varepsilon^{-1}W'(z_\varepsilon) \geq 0$ in the distributional sense in $(x_k^\varepsilon, x_{k+1}^\varepsilon)$. It follows that $z_\varepsilon'' \leq \varepsilon^{-2}\min_{s\in[-1,1]} W'(s)$ in the distributional sense in $(x_k^\varepsilon, x_{k+1}^\varepsilon)$. Therefore z_ε is semiconcave [7] in $(x_k^\varepsilon, x_{k+1}^\varepsilon)$ and, even more, it is semiconcave in $[x_k^\varepsilon, x_{k+1}^\varepsilon]$. As a consequence, the inequality $-\varepsilon z_\varepsilon'' + \varepsilon^{-1}W'(z_\varepsilon) \geq 0$ holds in $[x_k^\varepsilon, x_{k+1}^\varepsilon]$ in the viscosity sense.
We also have that z_ε is classical solution to $-\varepsilon z_\varepsilon'' + \varepsilon^{-1}W'(z_\varepsilon) = 0$ in the set $\{z_\varepsilon > s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon)$ and, in particular, $-\varepsilon z_\varepsilon'' + \varepsilon^{-1}W'(z_\varepsilon) \leq 0$ in the viscosity sense in $\{z_\varepsilon > s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon)$, so that

$$z_\varepsilon \in \mathcal{C}^{1,1}\left(\{z_\varepsilon > s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon)\right) \cap \mathcal{C}^\infty\left(\{z_\varepsilon > s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon)\right).$$

On $\{z_\varepsilon = s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon)$, the function z_ε has a minimum, and therefore $-z_\varepsilon'' \leq 0$ in the viscosity sense. Coupled with the previous observation, we deduce

$$z_\varepsilon \in \mathcal{C}^{1,1}([x_k^\varepsilon, x_{k+1}^\varepsilon]).$$

The energy conservation implies that $\varepsilon\frac{(z_\varepsilon')^2}{2} - \varepsilon^{-1}W(z_\varepsilon)$ is constant in any interval contained in $\{z_\varepsilon > s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon)$, therefore

$$\varepsilon\frac{(z_\varepsilon')^2}{2} - \varepsilon^{-1}W(z_\varepsilon) \text{ is a constant } e(z_k^\varepsilon) \text{ in } [x_k^\varepsilon, x_{k+1}^\varepsilon]. \tag{5.6}$$

In particular

$$\varepsilon^{-1}W(z_\varepsilon) \geq -e(z_k^\varepsilon) \qquad \text{in } [x_k^\varepsilon, x_{k+1}^\varepsilon]. \tag{5.7}$$

We claim that

$$z_\varepsilon > s_0 \qquad \text{in } [x_k^\varepsilon, x_{k+1}^\varepsilon]. \tag{5.8}$$

Suppose by contradiction that $\{z_\varepsilon = s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon) \neq \emptyset$. From (5.6) it follows that $-e(z_k^\varepsilon) = \varepsilon^{-1}W(s_0)$ because on the set $\{z_\varepsilon = s_0\}$ there holds $z_\varepsilon' = 0$. We deduce from (5.7)

$$F_\varepsilon(z_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)) \geq \varepsilon^{-1}W(s_0)(x_{k+1}^\varepsilon - x_k^\varepsilon) > C,$$

for $\varepsilon \in (0,1)$ sufficiently small depending only on v and s_0, in contradiction with (5.3). We conclude that $\{z_\varepsilon = s_0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon) = \emptyset$, and this proves our claim (5.8). Notice that the same argument shows that z_ε cannot have critical points in $\{z_\varepsilon < 0\} \cap (x_k^\varepsilon, x_{k+1}^\varepsilon)$, hence in particular

$$z_\varepsilon > 0 \qquad \text{in } (x_k^\varepsilon, x_{k+1}^\varepsilon).$$

The proof of the validity of the ordinary differential equation in (5.4) then follows, and hence by uniqueness $z_\varepsilon(x) \in (-1,1)$ for any $x \in \mathbb{T}$.
Let us show that z_k^ε is even with respect to the mid point of $(x_k^\varepsilon, x_{k+1}^\varepsilon)$. Let $\overline{x} \in (x_k^\varepsilon, x_{k+1}^\varepsilon)$ be a point where z_k^ε takes the maximum value in $[x_k^\varepsilon, x_{k+1}^\varepsilon]$. Observe that $\widetilde{z}_k^\varepsilon(x) := z_k^\varepsilon(2\overline{x} - x)$ solves the ordinary differential equation in (5.4), with $\widetilde{z}_k^\varepsilon(\overline{x}) = z_k^\varepsilon(\overline{x})$ and $\widetilde{z}_k^{\varepsilon\prime}(\overline{x}) = z_k^{\varepsilon\prime}(\overline{x}) = 0$. Hence by uniqueness $\widetilde{z}_k^\varepsilon = z_k^\varepsilon$. If by contradiction $\overline{x}$ is not the mid point of $(x_k^\varepsilon, x_{k+1}^\varepsilon)$, we have that z_k^ε vanishes somewhere in $(x_k^\varepsilon, x_{k+1}^\varepsilon)$, which is impossible, because $z_k^\varepsilon > 0$ by (5.4).
Assertion (v) follows, because contradicting (5.5) would contradict estimate (5.3). $\square$

Note that assertions (ii)-(v) are valid independently of s_0; we shall make use of s_0 in the second case of the proof of the next lemma. We need the following preliminary observation. Let $v \in BV(\mathbb{T};\{\pm 1\})$ and $(v_\varepsilon) \subset H^1(\mathbb{T})$ be a sequence converging to v in $L^1(\mathbb{T})$ as $\varepsilon \to 0^+$. The continuity of v_ε and the convergence of (v_ε) to v imply that, for any $k = 1, \cdots, N(v)$, there exists a sequence $(x_k^\varepsilon) \subset \mathbb{T}$ of points converging to $x_k(v)$, such that

$$v_\varepsilon(x_k^\varepsilon) = 0, \tag{5.9}$$

where $x_{N(v)+1}^\varepsilon := x_1^\varepsilon$.

Lemma 5.2 (**Action comparison between** v_ε **and** z_ε**).** *Let v be as in Lemma 5.1. Let $(v_\varepsilon) \subset H^1(\mathbb{T})$ be a sequence converging to v in $L^1(\mathbb{T})$ as $\varepsilon \to 0^+$. For any $k = 1, \cdots, N(v)$, select a sequence $(x_k^\varepsilon) \subset \mathbb{T}$ of points converging to $x_k(v)$ such that $v_\varepsilon(x_k^\varepsilon) = 0$, where we have set $x_{N(v)+1}^\varepsilon := x_1^\varepsilon$. With s_0 as in* (5.1)*, let (z_ε) be the sequence of functions given by Lemma 5.1. Then*

$$F_\varepsilon(v_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)) \geq F_\varepsilon(z_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)), \qquad k = 1, \cdots, N(v) \tag{5.10}$$

for $\varepsilon \in (0,1)$ small enough.

Proof. Without loss of generality, let us fix k even. We divide the proof into two cases.
Case 1. $v_\varepsilon \geq s_0$ in $(x_k^\varepsilon, x_{k+1}^\varepsilon)$.
In this case we have that $v_\varepsilon \in A_k^\varepsilon(s_0)$, and (5.10) follows by the minimality of z_ε (see (5.2)).
Case 2. Suppose that $v_\varepsilon(\overline{x}) < s_0$ for some $\overline{x} \in (x_k^\varepsilon, x_{k+1}^\varepsilon)$.
We have

$$\begin{aligned} F_\varepsilon\left(v_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)\right) \geq & \int_{(x_k^\varepsilon, x_{k+1}^\varepsilon)} \sqrt{2W(v_\varepsilon)}|v_\varepsilon'|\; dx = \int_{(x_k^\varepsilon, x_{k+1}^\varepsilon)} |\phi(v_\varepsilon)'|\; dx \\ \geq & (\phi(0) - \phi(m_k^\varepsilon)) + (\phi(M_k^\varepsilon) - \phi(m_k^\varepsilon)) + (\phi(M_k^\varepsilon) - \phi(0)) \\ = & 2\left(\phi(M_k^\varepsilon) - \phi(m_k^\varepsilon)\right), \end{aligned}$$

where ϕ is defined in (2.4), and

$$M_k^\varepsilon := \max\{v_\varepsilon(x) : x \in [x_k^\varepsilon, x_{k+1}^\varepsilon]\} > m_k^\varepsilon := \min\{v_\varepsilon(x) : x \in [x_k^\varepsilon, x_{k+1}^\varepsilon]\}.$$

Since ϕ is strictly increasing and $m_k^\varepsilon < s_0$, we deduce

$$F_\varepsilon\left(v_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)\right) \geq 2\left(\phi(M_k^\varepsilon) - \phi(s_0)\right) \geq 2\left(\phi(1) - \phi(s_0)\right) + o(1) \tag{5.11}$$

as $\varepsilon \to 0^+$, where in the last inequality we have used that $\lim_{\varepsilon\to 0^+} \|v_\varepsilon - 1\|_{L^1(x_k^\varepsilon, x_{k+1}^\varepsilon)} = 0$.
For any $k = 1, \cdots, N(v) - 1$ let now $d_k^\varepsilon := x_{k+1}^\varepsilon - x_k^\varepsilon$ and $d_{N(v)}^\varepsilon := 1 - (x_{N(v)}^\varepsilon - x_1^\varepsilon)$, so that $\lim_{\varepsilon\to 0^+} d_k^\varepsilon = d_k(v)$.
Define

$$\widetilde{z}_k^\varepsilon(x) := \begin{cases} \gamma\left(\frac{x - x_k^\varepsilon}{\varepsilon}\right) & \text{if } x \in (x_k^\varepsilon, x_k^\varepsilon + \frac{d_k^\varepsilon}{2}), \\ \\ \gamma\left(\frac{x_{k+1}^\varepsilon - x}{\varepsilon}\right) & \text{if } x \in (x_k^\varepsilon + \frac{d_k^\varepsilon}{2}, x_{k+1}^\varepsilon), \end{cases}$$

where γ solves (2.9). We have $\widetilde{z}_k^\varepsilon \in A_k^\varepsilon(s_0)$ and, as $\varepsilon \to 0^+$,

$$F_\varepsilon\left(\widetilde{z}_k^\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)\right) = 2\left(\phi\left(\max_{x\in[x_k^\varepsilon, x_{k+1}^\varepsilon]} \widetilde{z}_k^\varepsilon(x)\right) - \phi(0)\right) = 2\left(\phi(1) - \phi(0)\right) + o(1). \tag{5.12}$$

In addition, by minimality,

$$F_\varepsilon\left(\widetilde{z}_k^\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)\right) \geq F_\varepsilon\left(z_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)\right). \tag{5.13}$$

Then (5.10) follows from (5.11), (5.12) and (5.13) provided $\varepsilon > 0$ is sufficiently small since, being ϕ strictly increasing and $s_0 \in (-1, 0)$,

$$\begin{aligned} F_\varepsilon\left(v_\varepsilon, (x_k^\varepsilon, x_k^\varepsilon)\right) \geq & 2(\phi(1) - \phi(s_0)) + o(1) > 2(\phi(1) - \phi(0)) \\ = & F_\varepsilon(\widetilde{z}_k^\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)) + o(1) \geq F_\varepsilon(z_k^\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)) + o(1). \end{aligned}$$

□

6 First order estimate for F_ε

In this section we prove the first order expansion for F_ε, in the sense specified by Theorems 6.1 and Theorem 6.5.

Theorem 6.1 (First order estimate from below). *Suppose that assumptions* (W1) – (W3) *hold. Let* $(v_\varepsilon) \subset H^1(\mathbb{T})$ *be a sequence converging in* $L^1(\mathbb{T})$ *to a non constant function* $v \in BV(\mathbb{T}; \{\pm 1\})$*. Then, for any* $k = 1, \cdots, N(v)$*, there exists a sequence* (d_k^ε) *satisfying* $\lim\limits_{\varepsilon\to 0^+} d_k^\varepsilon = d_k(v)$ *such that*

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \geq & N(v)\sigma - \alpha_+\kappa_+^2 \sum_{k=1}^{N(v)} e^{-\alpha_+ \frac{d_k^\varepsilon}{\varepsilon}} - \alpha_-\kappa_-^2 \sum_{k=1}^{N(v)} e^{-\alpha_- \frac{d_k^\varepsilon}{\varepsilon}} \\ & + o\left(\sum_{k=1}^{N(v)} e^{-\alpha_+ \frac{d_k^\varepsilon}{\varepsilon}}\right) + o\left(\sum_{k=1}^{N(v)} e^{-\alpha_- \frac{d_k^\varepsilon}{\varepsilon}}\right) \qquad \text{as } \varepsilon \to 0^+. \end{aligned} \tag{6.1}$$

Proof. Without loss of generality, we can assume $N(v) \geq 2$, and that $v = -1$ in $(x_{k-1}(v), x_k(v))$ and $v = 1$ in $(x_k(v), x_{k+1}(v))$ for any $k = 1, \cdots, N(v)$, k even. For any $k = 1, \cdots, N(v)$ select a sequence (x_k^ε) of points of $\mathbb{T}$ satisfying $\lim_{\varepsilon\to 0^+} x_k^\varepsilon = x_k(v)$ and equality (5.9), where $x_{N(v)+1}^\varepsilon := x_1^\varepsilon$.
Now, let d_k^ε be defined as

$$\begin{cases} d_k^\varepsilon := x_{k+1}^\varepsilon - x_k^\varepsilon, & k = 1, \cdots, N(v) - 1, \\ d_{N(v)}^\varepsilon := 1 - (x_{N(v)}^\varepsilon - x_1^\varepsilon) =: d_0^\varepsilon, & \end{cases} \tag{6.2}$$

and set

$$I_k^\varepsilon(x_k^\varepsilon) := \left(x_k^\varepsilon - \frac{d_{k-1}^\varepsilon}{2}, x_k^\varepsilon + \frac{d_k^\varepsilon}{2}\right).$$

From inequality (5.10) of Lemma 5.2 it follows

$$
\begin{aligned}
F_\varepsilon(v_\varepsilon) &= \sum_{k=1}^{N(v)} F_\varepsilon\left(v_\varepsilon,(x_k^\varepsilon,x_{k+1}^\varepsilon)\right) \geq \sum_{k=1}^{N(v)} F_\varepsilon\left(z_\varepsilon,(x_k^\varepsilon,x_{k+1}^\varepsilon)\right) \\
&= F_\varepsilon(z_\varepsilon) = \sum_{k=1}^{N(v)} F_\varepsilon\left(z_\varepsilon, I_k^\varepsilon(x_k^\varepsilon)\right),
\end{aligned}
\tag{6.3}
$$

for $\varepsilon \in (0,1)$ small enough. With the change of variable $x = \varepsilon y + x_k^\varepsilon$ we get

$$
F_\varepsilon\left(z_\varepsilon, I_k^\varepsilon(x_k^\varepsilon)\right) = \int_{\frac{1}{\varepsilon}I_k^\varepsilon(0)} \left(\frac{\varepsilon^2}{2}\left(z_\varepsilon'(\varepsilon y + x_k^\varepsilon)\right)^2 + W\left(z_\varepsilon(\varepsilon y + x_k^\varepsilon)\right)\right)\, dy, \tag{6.4}
$$

where

$$
\frac{1}{\varepsilon}I_k^\varepsilon(0) = \left(\frac{-d_{k-1}^\varepsilon}{2\varepsilon}, \frac{d_k^\varepsilon}{2\varepsilon}\right). \tag{6.5}
$$

Let $w_k^\varepsilon \in H^1\left(\frac{1}{\varepsilon}I_k^\varepsilon(0)\right)$ be the function defined as

$$
w_k^\varepsilon(y) := z_\varepsilon(\varepsilon y + x_k^\varepsilon), \qquad y \in \frac{1}{\varepsilon}I_k^\varepsilon(0), \tag{6.6}
$$

where we set $w_0^\varepsilon := w_{N(v)}^\varepsilon$. We deduce

$$
F_\varepsilon\left(z_\varepsilon, I_k^\varepsilon(x_k^\varepsilon)\right) = \int_{\frac{1}{\varepsilon}I_k^\varepsilon(0)} \left(\frac{1}{2}(w_k^{\varepsilon\prime}(y))^2 + W(w_k^\varepsilon(y))\right)\, dy = \mathcal{F}\left(w_k^\varepsilon, \frac{1}{\varepsilon}I_k^\varepsilon(0)\right), \tag{6.7}
$$

where $\mathcal{F}$ is defined in (2.8). Hence, from (6.3),

$$
\begin{aligned}
F_\varepsilon(v_\varepsilon) &\geq \sum_{k=1}^{N(v)} \mathcal{F}\Big(w_k^\varepsilon, \frac{1}{\varepsilon}I_k^\varepsilon(0)\Big) \\
&= \sum_{k\in\mathcal{I}_+(v)} \mathcal{F}\Big(w_k^\varepsilon, \frac{1}{\varepsilon}I_k^\varepsilon(0)\Big) + \sum_{k\in\mathcal{I}_-(v)} \mathcal{F}\Big(w_k^\varepsilon, \frac{1}{\varepsilon}I_k^\varepsilon(0)\Big),
\end{aligned}
\tag{6.8}
$$

for $\varepsilon \in (0,1)$ small enough. Observe from (5.4) that w_k^ε solves

$$
\begin{cases}
-w_k^{\varepsilon\prime\prime} + W'(w_k^\varepsilon) = 0 & \text{in } \left(\frac{-d_{k-1}^\varepsilon}{2\varepsilon}, \frac{d_k^\varepsilon}{2\varepsilon}\right) \setminus \{0\}, \\
w_k^\varepsilon(0) = 0.
\end{cases}
\tag{6.9}
$$

Moreover, from (5.5) we get

$$
\lim_{\varepsilon\to 0^+} \left| w_k^\varepsilon\left(\frac{-d_{k-1}^\varepsilon}{2\varepsilon}\right)\right| = \lim_{\varepsilon\to 0^+} \left| w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right| = 1. \tag{6.10}
$$

Define $e(w_{k-1}^\varepsilon)$ and $e(w_k^\varepsilon)$ as the (conserved) energy densities of w_k^ε in $\left(-\frac{d_{k-1}^\varepsilon}{2\varepsilon}, 0\right)$ and $\left(0, \frac{d_k^\varepsilon}{2\varepsilon}\right)$ respectively, namely

$$
\begin{aligned}
e(w_{k-1}^\varepsilon) &:= \frac{(w_k^{\varepsilon\prime-}(0))^2}{2} - W(0) = -W\left(w_k^\varepsilon\Big(-\frac{d_{k-1}^\varepsilon}{2\varepsilon}\Big)\right) < 0, \\
e(w_k^\varepsilon) &:= \frac{(w_k^{\varepsilon\prime+}(0))^2}{2} - W(0) = -W\left(w_k^\varepsilon\Big(\frac{d_k^\varepsilon}{2\varepsilon}\Big)\right) < 0,
\end{aligned}
\tag{6.11}
$$

where ${w_k^\varepsilon}'^-(0)$ (resp. ${w_k^\varepsilon}'^+(0)$) stands for the left (resp. right) derivative of w_k^ε at 0. Set also $e(w_0^\varepsilon) := e(w_{N(v)}^\varepsilon)$. We have, for $k \in \mathcal{I}_+(v)$,

$$
{w_k^\varepsilon}' = \begin{cases} \sqrt{2W(w_k^\varepsilon) + 2e(w_{k-1}^\varepsilon)} & \text{in } \left(\frac{-d_{k-1}^\varepsilon}{2\varepsilon}, 0\right), \\ \sqrt{2W(w_k^\varepsilon) + 2e(w_k^\varepsilon)} & \text{in } \left(0, \frac{d_k^\varepsilon}{2\varepsilon}\right), \\ w_k^\varepsilon(0) = 0. \end{cases} \tag{6.12}
$$

Hence

$$
\frac{d_k^\varepsilon}{2\varepsilon} = \int_0^{w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)} \frac{1}{\sqrt{2W(s) + 2e(w_k^\varepsilon)}}\, ds.
$$

From (6.11) and the expression of $\mathcal{Q}_+$ in (4.1), we get

$$
\frac{d_k^\varepsilon}{2\varepsilon} = \mathcal{Q}_+\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right). \tag{6.13}
$$

From (6.13), (3.10) and (4.15) we deduce, as $\varepsilon \to 0^+$,

$$
\log\left(\frac{1}{1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)}\right) = \alpha_+ \frac{d_k^\varepsilon}{2\varepsilon} - \log(2\kappa_+) + o(1), \tag{6.14}
$$

and therefore[5]

$$
1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right) = 2\kappa_+ e^{-\alpha_+ \frac{d_k^\varepsilon}{2\varepsilon} + o(1)} \qquad \text{as } \varepsilon \to 0^+. \tag{6.15}
$$

Adding and subtracting the term $W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)$ inside the integral, and taking advantage of (6.12), for $k \in \mathcal{I}_+(v)$ we write

$$
\begin{aligned}
&\mathcal{F}\left(w_k^\varepsilon, \left(0, \frac{d_k^\varepsilon}{2\varepsilon}\right)\right) \\
&= \int_0^{\frac{d_k^\varepsilon}{2\varepsilon}} \left(\frac{({w_k^\varepsilon}')^2}{2} + W(w_k^\varepsilon) - W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)\right) dy + \frac{d_k^\varepsilon}{2\varepsilon} W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right) \\
&= \int_0^{\frac{d_k^\varepsilon}{2\varepsilon}} \sqrt{2W(w_k^\varepsilon(y)) - 2W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)}\, {w_k^\varepsilon}'(y)\, dy + \frac{d_k^\varepsilon}{2\varepsilon} W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right) \\
&= \int_0^{w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)} \sqrt{2W(s) - 2W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)}\, ds + \frac{d_k^\varepsilon}{2\varepsilon} W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right) \\
&= \mathcal{L}_+\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right) + \frac{d_k^\varepsilon}{2\varepsilon} W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right),
\end{aligned} \tag{6.16}
$$

where we recall that $\mathcal{L}_+$ is defined in (4.6).

[5]From (6.15) it follows $\lim_{\varepsilon \to 0^+} \frac{\left(1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)^2}{e^{-\alpha_+ \frac{d_k^\varepsilon}{\varepsilon}}} = 4\kappa_+^2$, a formula also proven in [8, Prop. 3.4].

Substituting (4.17) and (2.2) into (6.16) we deduce, using also (6.14) and (6.15),

$$
\begin{aligned}
&\mathcal{F}\left(w_k^\varepsilon, \left(0, \frac{d_k^\varepsilon}{2\varepsilon}\right)\right) \\
=&\sigma_+ - \frac{\alpha_+}{2}\left(1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)^2 \log\left(\frac{1}{1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)}\right) \\
&- \frac{\alpha_+}{2}\left(\log(2\kappa_+) + \frac{1}{2}\right)\left(1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)^2 \\
&+ \frac{d_k^\varepsilon}{2\varepsilon}\frac{\alpha_+^2}{2}\left(1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)^2 + o\left(\left(1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)^2\right) \\
=&\sigma_+ - \frac{\alpha_+}{2}\left(\log(2\kappa_+) + \frac{1}{2} - \log(2\kappa_+)\right)\left(1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)^2 \\
&+ o\left(\left(1 - w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)^2\right) \\
=&\sigma_+ - \alpha_+\kappa_+^2 e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}} + o\left(e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}}\right)
\end{aligned}
$$

as $\varepsilon \to 0^+$.
With similar arguments one can prove that

$$
\mathcal{F}\left(w_k^\varepsilon, \left(-\frac{d_{k-1}^\varepsilon}{2\varepsilon}, 0\right)\right) = \sigma_- - \alpha_-\kappa_-^2 e^{-\alpha_-\frac{d_{k-1}^\varepsilon}{\varepsilon}} + o\left(e^{-\alpha_-\frac{d_{k-1}^\varepsilon}{\varepsilon}}\right).
$$

Hence, for $k \in \mathcal{I}_+(v)$ we get

$$
\mathcal{F}\left(w_k^\varepsilon, \frac{1}{\varepsilon}I_k^\varepsilon(0)\right) = \sigma - \alpha_+\kappa_+^2 e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}} - \alpha_-\kappa_-^2 e^{-\alpha_-\frac{d_{k-1}^\varepsilon}{\varepsilon}} + o\left(e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}}\right) + o\left(e^{-\alpha_-\frac{d_{k-1}^\varepsilon}{\varepsilon}}\right). \tag{6.17}
$$

Similarly, for $k \in \mathcal{I}_-(v)$, we have

$$
\mathcal{F}\left(w_k^\varepsilon, \frac{1}{\varepsilon}I_k^\varepsilon(0)\right) = \sigma - \alpha_-\kappa_-^2 e^{-\alpha_-\frac{d_k^\varepsilon}{\varepsilon}} - \alpha_+\kappa_+^2 e^{-\alpha_+\frac{d_{k-1}^\varepsilon}{\varepsilon}} + o\left(e^{-\alpha_-\frac{d_k^\varepsilon}{\varepsilon}}\right) + o\left(e^{-\alpha_+\frac{d_{k-1}^\varepsilon}{\varepsilon}}\right). \tag{6.18}
$$

From (6.8), (6.17) and (6.18) the assertion of the theorem follows. □

Remark 6.2 (W **even**)**.** When W is even, in order to prove Theorem 6.1 there is no need to introduce s_0 as in (5.1), and there is no need to use Lemmas 5.1 and 5.2. Indeed, if W is even, we can define z_k^ε a a solution to (5.2) where s_0 is replaced by 0, and we can directly prove inequality (5.10), since

$$
F_\varepsilon(v_\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)) = F_\varepsilon(|v_\varepsilon|, (x_k^\varepsilon, x_{k+1}^\varepsilon)) \geq F_\varepsilon(z_k^\varepsilon, (x_k^\varepsilon, x_{k+1}^\varepsilon)).
$$

Remark 6.3. As soon as $d_k(v) \neq d_h(v)$, the corresponding infinitesimals $e^{-\alpha_\pm \frac{d_k^\varepsilon}{\varepsilon}}$, $e^{-\alpha_\pm \frac{d_h^\varepsilon}{\varepsilon}}$ on the right hand side of (6.1) are not comparable. It may happen that the error on a term of the sum, say $e^{-\alpha_\pm \frac{d_k^\varepsilon}{\varepsilon}}$, is larger then another term of the sum, say $e^{-\alpha_\pm \frac{d_h^\varepsilon}{\varepsilon}}$. An estimate more rough than (6.1) is obtained by replacing the terms $o\left(\sum_{k=1}^{N(v)} e^{-\alpha_\pm \frac{d_k^\varepsilon}{\varepsilon}}\right)$ with

$$o\left(e^{-\alpha_\pm \min_{k=1,\cdots,N(v)} \frac{d_k^\varepsilon}{\varepsilon}}\right).$$

Corollary 6.4. *Let $(v_\varepsilon) \subset H^1(\mathbb{T})$ be a sequence converging in $L^1(\mathbb{T})$ to a non constant function $v \in BV(\mathbb{T};\{\pm 1\})$. Then, for any $d \in \Big(0, \min\{d_k(v) : k = 1, \cdots, N(v)\}\Big)$ and any $C_+ > 0, C_- > 0$ we have*

$$F_\varepsilon(v_\varepsilon) \geq N(v)\sigma - C_+ e^{-\alpha_+ \frac{d}{\varepsilon}} - C_- e^{-\alpha_- \frac{d}{\varepsilon}} + o\left(e^{-\alpha_+ \frac{d}{\varepsilon}}\right) + o\left(e^{-\alpha_- \frac{d}{\varepsilon}}\right) \tag{6.19}$$

as $\varepsilon \to 0^+$.

Theorem 6.5 (First order estimate from above). *Suppose that assumptions* (W1)−(W3) *hold. Let $v \in BV(\mathbb{T};\{\pm 1\})$ be a non constant function. Then there exists a sequence $(v_\varepsilon) \subset H^1(\mathbb{T})$ converging to v in $L^1(\mathbb{T})$ and satisfying the inequality*

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \leq N(v)\sigma - \alpha_+\kappa_+^2 \sum_{k=1}^{N(v)} e^{-\alpha_+ \frac{d_k(v)}{\varepsilon}} - \alpha_-\kappa_-^2 \sum_{k=1}^{N(v)} e^{-\alpha_- \frac{d_k(v)}{\varepsilon}} \\ + o\left(\sum_{k=1}^{N(v)} e^{-\alpha_+ \frac{d_k(v)}{\varepsilon}}\right) + o\left(\sum_{k=1}^{N(v)} e^{-\alpha_- \frac{d_k(v)}{\varepsilon}}\right) \qquad \text{as } \varepsilon \to 0^+. \end{aligned} \tag{6.20}$$

Proof. By standard arguments, it is sufficient to prove the statement for a function v having only two jumps $x_1(v) < x_2(v)$. Let $d_1(v) := x_2(v) - x_1(v)$ and $d_2(v) := 1 - d_1(v) =: d_0(v)$. Without loss of generality, we can assume

$$v = \begin{cases} -1 & \text{in } (x_1(v), x_2(v)), \\ 1 & \text{in } (x_2(v), x_1(v)). \end{cases}$$

Set

$$I_1(x_1(v)) := \left(x_1(v) - \frac{d_2(v)}{2}, x_1(v) + \frac{d_1(v)}{2}\right), \qquad I_2(x_2(v)) := \left(x_2(v) - \frac{d_1(v)}{2}, x_2(v) + \frac{d_2(v)}{2}\right).$$

Let z_1^ε, z_2^ε and z_ε be as in Lemma 5.1 (with $N(v) = 2$) with the choice

$$x_1^\varepsilon := x_1(v), \qquad x_2^\varepsilon := x_2(v), \qquad \varepsilon \in (0,1).$$

Write

$$\frac{1}{\varepsilon} I_1(0) := \left(\frac{-d_0(v)}{2\varepsilon}, \frac{d_1(v)}{2\varepsilon}\right), \qquad \left(\frac{-d_1(v)}{2\varepsilon}, \frac{d_2(v)}{2\varepsilon}\right),$$

and let $w_1^\varepsilon \in H^1(\frac{1}{\varepsilon}I_1(0))$ and $w_2^\varepsilon = w_0^\varepsilon \in H^1(\frac{1}{\varepsilon}I_2(0))$ be defined as in (6.6). Then w_1^ε and w_2^ε satisfy (6.11) and (6.12). We define

$$v_\varepsilon(x) := \begin{cases} w_2^\varepsilon\left(\frac{x_2(v)-x}{\varepsilon}\right) & \text{if } x \in I_2(x_2(v)), \\ \\ w_1^\varepsilon\left(\frac{x-x_1(v)}{\varepsilon}\right) & \text{if } x \in I_1(x_1(v)). \end{cases}$$

Then $v_\varepsilon \in H^1(\mathbb{T})$, (v_ε) converges to v in $L^1(\mathbb{T})$ as $\varepsilon \to 0^+$, and

$$F_\varepsilon(v_\varepsilon) = F_\varepsilon(v_\varepsilon, I_1(x_1(v))) + F_\varepsilon(v_\varepsilon, I_2(x_2(v))). \tag{6.21}$$

With the change of variable $y = \frac{x-x_2(v)}{\varepsilon}$, we have $F_\varepsilon(v_\varepsilon, I_2(x_2(v))) = \mathcal{F}\left(w_2^\varepsilon, \frac{1}{\varepsilon}I_2(0)\right)$, and, as in (6.16),

$$\mathcal{F}\left(w_2^\varepsilon, \left(0, \frac{d_2(v)}{2\varepsilon}\right)\right) = \mathcal{L}_+\left(w_2^\varepsilon\left(\frac{d_2(v)}{2\varepsilon}\right)\right) + \frac{d_2(v)}{2\varepsilon}W\left(w_2^\varepsilon\left(\frac{d_2(v)}{2\varepsilon}\right)\right). \tag{6.22}$$

Then the proof follows along the same lines as the proof of Theorem 6.1. $\square$

Remark 6.6. With slight modifications in the proof of Theorem 6.5, one can show that for any sequence (d_k^ε) converging to $d_k(v)$, there exists a sequence $(v_\varepsilon) \subset H^1(\mathbb{T})$ converging to v in $L^1(\mathbb{T})$ and satisfying the equality in (6.1) with $d_k^\varepsilon = d_k(v)$ for any $k = 1, \cdots, N(v)$.

Remark 6.7. From Theorems 6.1 and 6.5 it follows that, given $\gamma > 0$, for any $v \in BV(\mathbb{T}; \{\pm 1\})$ with $N(v) > 0$ and for any sequence $(v_\varepsilon) \in H^1(\mathbb{T})$ such that $v_\varepsilon \to v$ in $L^1(\mathbb{T})$ as $\varepsilon \to 0^+$, there holds

$$\liminf_{\varepsilon \to 0^+} \frac{F_\varepsilon(v_\varepsilon) - N(v)\sigma}{\varepsilon^\gamma} \geq 0,$$

with the equality along a particular sequence. Hence the Γ-expansion of the functionals F_ε in the sense of [5], whose zeroth-order is given by $N(\cdot)\sigma$, contains no terms of order ε^γ for any $\gamma > 0$.

7 Γ-convergence

Throughout this short section, $N \in \mathbb{N}$ and $m > 0$ are fixed, and we assume for simplicity that W is even. We set $\alpha := \alpha_- = \alpha_+$ (see (2.1)) and $\kappa := \kappa_- = \kappa_+$ (see (3.10)). For any $\varepsilon \in (0, 1]$ we define the functionals $T_\varepsilon^{N,m} : L^1(\mathbb{T}) \to (-\infty, +\infty]$ as

$$T_\varepsilon^{N,m}(v) := e^{\alpha\frac{m}{\varepsilon}}\Big(F_\varepsilon(v) - N\sigma\Big).$$

Observe that $T_\varepsilon^{N,m}$ may take negative values.

Remark 7.1. Let $(v_\varepsilon) \subset H^1(\mathbb{T})$ be such that

$$\sup_{\varepsilon \in (0,1]} T_\varepsilon^{N,m}(v_\varepsilon) < +\infty. \tag{7.1}$$

Then $\sup\limits_{\varepsilon \in (0,1]} F_\varepsilon(v_\varepsilon) < +\infty$. Hence (v_ε) admits a (not relabeled) subsequence converging in $L^1(\mathbb{T})$ to a function $v \in BV(\mathbb{T}; \{\pm 1\})$, and

$$N(v)\sigma \leq \Gamma\text{–}L^1(\mathbb{T}) \liminf_{\varepsilon \to 0^+} F_\varepsilon(v) \leq \limsup_{\varepsilon \to 0^+} F_\varepsilon(v_\varepsilon) \leq N\sigma,$$

where the last iequality follows from (7.1). Hence

$$N \geq N(v).$$

Theorem 7.2 (First order Γ-limit). *Suppose that assumptions* (W1) – (W3) *hold, and that in addition W is even. Then the sequence $(T_\varepsilon^{N,m})$ Γ-$L^1(\mathbb{T})$-converges, as $\varepsilon \to 0^+$, to the functional $T^{N,m} : L^1(\mathbb{T}) \to [-\infty, +\infty]$ given by*

$$T^{N,m}(v) = \begin{cases} 0 & \text{if } v \in BV(\mathbb{T}; \{\pm 1\}),\ N = N(v) \text{ and } m < m(v), \\ -\infty & \text{if } v \in BV(\mathbb{T}; \{\pm 1\}),\ N = N(v) \text{ and } m \geq m(v), \\ +\infty & \text{if } v \in BV(\mathbb{T}; \{\pm 1\}) \text{ and } N < N(v), \\ -\infty & \text{if } v \in BV(\mathbb{T}; \{\pm 1\}) \text{ and } N > N(v), \\ +\infty & \text{if } v \in L^1(\mathbb{T}) \setminus BV(\mathbb{T}; \{\pm 1\}), \end{cases}$$

where, for any $v \in BV(\mathbb{T}; \{\pm 1\})$, we have set $m(v) := \min\{d_k(v) : k = 1, \cdots, N(v)\}$.

Proof. Set $T^+ := \Gamma - L^1(\mathbb{T}) \limsup\limits_{\varepsilon \to 0^+} T_\varepsilon^{N,m}$ and $T^- := \Gamma - L^1(\mathbb{T}) \liminf\limits_{\varepsilon \to 0^+} T_\varepsilon^{N,m}$. Let $v \in L^1(\mathbb{T})$, and let $(v_\varepsilon) \subset H^1(\mathbb{T})$ be a sequence satisfying (7.1) and converging to v in $L^1(\mathbb{T})$. Then $v \in BV(\mathbb{T}; \{\pm 1\})$ and $N \geq N(v)$, so that $T^-(v) = +\infty$ if

$$\text{either } v \in L^1(\mathbb{T}) \setminus BV(\mathbb{T}; \{\pm 1\}) \quad \text{or } v \in BV(\mathbb{T}; \{\pm 1\}) \text{ and } N < N(v), \tag{7.2}$$

and therefore

$$\Gamma - L^1(\mathbb{T}) \lim_{\varepsilon \to 0^+} T_\varepsilon^{N,m}(v) = +\infty \quad \text{if } v \text{ satisfies (7.2)}.$$

We can assume from now on that $v \in BV(\mathbb{T}; \{\pm 1\})$. The continuity of v_ε and the convergence of (v_ε) to v imply that there exist an infinitesimal sequence $(\delta_\varepsilon) \subset (0,1)$ and, for any $k = 1, \cdots, N(v)$, a sequence of points $(x_k^\varepsilon) \subset \mathbb{T}$, such that for any $\varepsilon \in (0,1)$,

$$|x_k(v) - x_k^\varepsilon| \leq \delta_\varepsilon,$$

and (5.9) holds. From Theorem 6.1, (6.1), and $d_k^\varepsilon = x_{k+1}^\varepsilon - x_k^\varepsilon \geq x_{k+1}(v) - x_k(v) - 2\delta_\varepsilon \geq m(v) - 2\delta_\varepsilon$, we have

$$F_\varepsilon(v_\varepsilon) \geq N(v)\sigma - 2\alpha\kappa^2 \# \Big\{ k = 1, \cdots, N(v) : d_k(v) = m(v) \Big\} e^{-\alpha \frac{m(v) - 2\delta_\varepsilon}{\varepsilon}} + o\left(e^{-\alpha \frac{m(v) - 2\delta_\varepsilon}{\varepsilon}} \right)$$

since the contribution due to the remaining jump points is of higher order. Moreover

$$\begin{aligned} T_\varepsilon^{N,m}(v_\varepsilon) =& e^{\alpha\frac{m}{\varepsilon}}\Big(F_\varepsilon(v_\varepsilon) - N\sigma\Big) \\ \geq & e^{\alpha\frac{m}{\varepsilon}}(N(v)-N)\sigma - 2\alpha\kappa^2\#\Big\{k=1,\cdots,N(v) : d_k(v) = m(v)\Big\}e^{\alpha\frac{m-m(v)+2\delta_\varepsilon}{\varepsilon}} \\ & + o\left(e^{\alpha\frac{m-m(v)+2\delta_\varepsilon}{\varepsilon}}\right). \end{aligned}$$

Hence

$$N = N(v), \quad m < m(v) \quad \Rightarrow \quad T^-(v) \geq 0. \tag{7.3}$$

If now (v_ε) denotes the sequence constructed in Theorem 6.5, we have

$$T_\varepsilon^{N,m}(v_\varepsilon) \leq e^{\alpha\frac{m}{\varepsilon}}\Big(N(v)-N\Big)\sigma - 2\alpha\kappa^2\sum_{k=1}^{N(v)} e^{\alpha\frac{m-d_k(v)}{\varepsilon}} + o\left(\sum_{k=1}^{N(v)} e^{\alpha\frac{m-d_k(v)}{\varepsilon}}\right). \tag{7.4}$$

Therefore, for a v satisfying (7.3), we have $\lim_{\varepsilon\to0^+} T_\varepsilon^{N,m}(v_\varepsilon) = 0$, hence $T^+(v) \leq 0$, which coupled with (7.3) gives

$$N < N(v), \quad m < m(v) \quad \Rightarrow \quad \Gamma - L^1(\mathbb{T})\lim_{\varepsilon\to0^+} T_\varepsilon^{N,m}(v) = 0.$$

If either $N > N(v)$ or $N = N(v)$ and $m > m(v)$, from (7.4) it follows $\limsup_{\varepsilon\to0^+} T_\varepsilon^{N,m}(v_\varepsilon) = -\infty$, so that $T^+(v) = -\infty$. Eventually, from the $L^1(\mathbb{T})$-lower semicontinuity of T^+, we deduce

$$N = N(v), \quad m \geq m(v) \quad \Rightarrow T^+(v) = -\infty.$$

□

8 Second order estimate for F_ε

This section is devoted to prove estimate (1.5). In what follows, beside the hypotheses on W listed at the beginning of Section 2, we shall suppose also that there exists $\delta \in (0,1)$ so that

$$\begin{cases} W(s) = \dfrac{\alpha_-^2}{2}(1-s)^2, & s \in (-1-\delta, -1+\delta), \\ \\ W(s) = \dfrac{\alpha_+^2}{2}(1-s)^2, & s \in (1-\delta, 1+\delta). \end{cases} \tag{8.1}$$

Notice that, in this case, we have

$$\beta_\pm = 0. \tag{8.2}$$

Theorem 8.1 (Second order estimate from below). *Suppose that assumptions* (W1) – (W3) *hold, and that in addition* (8.1) *holds. Let* (v_ε), v, k *and* (d_k^ε) *be as in Theorem 6.1. Then*

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \geq & N(v)\sigma - \alpha_+\kappa_+^2 \sum_{k=1}^{N(v)} e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}} - \alpha_-\kappa_-^2 \sum_{k=1}^{N(v)} e^{-\alpha_-\frac{d_k^\varepsilon}{\varepsilon}} \\ & + o\left(\sum_{k=1}^{N(v)} e^{-\frac{3\alpha_+}{2}\frac{d_k^\varepsilon}{\varepsilon}}\right) + o\left(\sum_{k=1}^{N(v)} e^{-\frac{3\alpha_-}{2}\frac{d_k^\varepsilon}{\varepsilon}}\right) \end{aligned} \tag{8.3}$$

as $\varepsilon \to 0^+$.

We start the proof of Theorem 8.1 with the following result.

Lemma 8.2 (Computation of $D'_+(1)$). *We have*

$$D'_+(1) := \lim_{\eta\to 1^-} D'_+(\eta) = 0. \tag{8.4}$$

Proof. Using the additional assumption (8.1) on W, for $\eta \in (1-\delta, 1)$ we have for the function D_+ defined in (4.5),

$$D_+(\eta) = \int_0^{1-\delta} \left\{ \frac{1}{\sqrt{2W(s) - 2W(\eta)}} - \frac{1}{\alpha_+\sqrt{(1-s)^2 - (1-\eta)^2}} \right\} ds,$$

and therefore D_+ is of class $\mathcal{C}^\infty$ in a left neighbourhood of $\eta = 1$. Differentiating under the integral sign we get

$$D'_+(\eta) = \int_0^{1-\delta} \left\{ \frac{W'(\eta)}{(2W(s) - 2W(\eta))^{3/2}} + \frac{1-\eta}{\alpha_+((1-s)^2 - (1-\eta)^2)^{3/2}} \right\} ds,$$

and the assertion follows passing to the limit under the integral sign. □

Corollary 8.3 (Second order expansion of $\mathcal{Q}_+$ and $\mathcal{L}_+$). *We have*

$$\begin{aligned} \mathcal{Q}_+(\eta) =& \frac{1}{\alpha_+}\log\left(\frac{2}{1-\eta}\right) + c_+ + o(1-\eta), \\ \mathcal{L}_+(\eta) =& \sigma_+ - \frac{\alpha_+}{2}(1-\eta)^2\log\left(\frac{1}{1-\eta}\right) \\ & - \frac{\alpha_+}{2}\left(\log(2\kappa_+) + \frac{1}{2}\right)(1-\eta)^2 + o\left((1-\eta)^3\right) \qquad \text{as } \eta \to 1^-. \end{aligned} \tag{8.5}$$

as $\eta \to 1^-$.

Proof. The formula for $\mathcal{Q}_+$ follows from (4.16), (4.8) and Lemma 8.2. The formula for $\mathcal{L}_+$ follows by a direct computation as in the proof of Lemma 4.3, considering

$$\lim_{\eta\to 1^-} \frac{\mathcal{L}_+(\eta) - \sigma_+ + \frac{\alpha_+}{2}(1-\eta)^2\log\left(\frac{1}{1-\eta}\right) + \frac{\alpha_+}{2}\left(\log(2\kappa_+) + \frac{1}{2}\right)(1-\eta)^2}{(1-\eta)^3},$$

applying de l'Hôpital's Theorem and using the expansion of $\mathcal{Q}_+$ in (8.5) instead of (4.15), and (8.2). □

Following the notation of equations (6.15) and (6.14), we have the following expansions.

Lemma 8.4. *Let $k \in \mathcal{S}_+(v)$. Then*

$$1 - w_\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right) = 2\kappa_+ e^{-\alpha_+\frac{d_k^\varepsilon}{2\varepsilon}} + o\left(e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}}\right), \tag{8.6}$$

and

$$\log\left(\frac{1}{1 - w_\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)}\right) = \alpha_+\frac{d_k^\varepsilon}{2\varepsilon} - \log(2\kappa_+) + o(1). \tag{8.7}$$

Proof. We prove only the first expansion, the other being similar. From (3.10), (6.13) and (8.5) it follows

$$1 - w_\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right) = 2\kappa_+ e^{-\alpha_+\frac{d_k^\varepsilon}{2\varepsilon}}\, e^{o\left(1-w_\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right)}. \tag{8.8}$$

Hence, from (8.8) it follows

$$1 - w_\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right) = 2\kappa_+ e^{-\alpha_+\frac{d_k^\varepsilon}{2\varepsilon}} + o\left(e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}}\right).$$

□

Now, let (v_ε), v, k and and d_k^ε be as in Theorem 6.1. Following the notation and the proof of the same theorem (see in particular (6.16)), we have to expand the quantity

$$\mathcal{L}_+\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right) + \frac{d_k^\varepsilon}{2\varepsilon} W\left(w_k^\varepsilon\left(\frac{d_k^\varepsilon}{2\varepsilon}\right)\right). \tag{8.9}$$

In view of the computations in the proof of Theorem 6.1, using (8.2) it is sufficient to isolate the coefficients of the terms of order $e^{-\alpha_+\frac{3d_k^\varepsilon}{2\varepsilon}}$ in (8.9), and (8.3) follows, thus showing Theorem 8.1. □

We conclude the paper with following result, the proof of which follows along the same lines of Theorem 8.1, in a much simpler way.

Theorem 8.5 (**Second order estimate from above**)**.** *Suppose that assumptions* (W1) – (W3) *hold, and that in addition* (8.1) *holds. Let v and (v_ε) be as in Theorem 6.5. Then*

$$\begin{aligned} F_\varepsilon(v_\varepsilon) \le & N(v)\sigma - \alpha_+\kappa_+^2 \sum_{k=1}^{N(v)} e^{-\alpha_+\frac{d_k(v)}{\varepsilon}} - \alpha_-\kappa_-^2 \sum_{k=1}^{N(v)} e^{-\alpha_-\frac{d_k(v)}{\varepsilon}} \\ & + o\left(\sum_{k=1}^{N(v)} e^{-\frac{3\alpha_+}{2}\frac{d_k(v)}{\varepsilon}}\right) + o\left(\sum_{k=1}^{N(v)} e^{-\frac{3\alpha_-}{2}\frac{d_k(v)}{\varepsilon}}\right) \end{aligned} \tag{8.10}$$

as $\varepsilon \to 0^+$.

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