MINIMIZERS OF ANISOTROPIC PERIMETERS WITH CYLINDRICAL NORMS

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ABSTRACT. We study various regularity properties of minimizers of the Φ–perimeter, where Φ is a norm. Under suitable assumptions on Φ and on the dimension of the ambient space, we prove that the boundary of a cartesian minimizer is locally a Lipschitz graph out of a closed singular set of small Hausdorff dimension. Moreover, we show the following anisotropic Bernstein-type result: any entire cartesian minimizer is the subgraph of a monotone function depending only on one variable.

CONTENTS

1. INTRODUCTION

In this paper we are interested in regularity properties of minimizers of the anisotropic perimeter

$$P_\Phi(E,\Omega) = \int_{\Omega\cap\partial^* E} \Phi^o(\nu_E)\, d\mathcal{H}^n,$$

of E in Ω, and of the related area-type functional

$$\mathcal{G}_{\Phi^o}(v,\widehat{\Omega}) = \int_{\widehat{\Omega}} \Phi^o(-Dv,1).$$

Here $\Omega \subseteq \mathbb{R}^{n+1}$ is an open set, $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is a norm (called anisotropy), Φ^o is its dual, $E \subset \mathbb{R}^{n+1}$ is a set of locally finite perimeter, $\partial^* E$ is its reduced boundary, ν_E is the outward (generalized) unit normal to $\partial^* E$, and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. On the other hand, $\widehat{\Omega} \subseteq \mathbb{R}^n$, v belongs to the space $BV_{\rm loc}(\widehat{\Omega})$ of functions with locally bounded total variation in $\widehat{\Omega}$, and Dv is the distributional derivative of v. When $\Omega = \widehat{\Omega}\times\mathbb{R}$ the two functionals coincide provided E is cartesian, *i.e.* E is the subgraph ${\rm sg}(v) \subset \widehat{\Omega}\times\mathbb{R}$ of the function $v \in BV_{\rm loc}(\widehat{\Omega})$ (see (4.1)).

Anisotropic perimeters appear in many models in material science and phase transitions [21, 37], in crystal growth [7, 8, 12, 13, 39, 3], and in boundary detection and tracking [15]. Functionals like $\mathcal{G}_{\Phi^o}$, having linear growth in the gradient, appear quite frequently in calculus of variations [20, 9, 6].

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The one-homogeneous case is particularly relevant, since it is related to the *anisotropic total variation functional*

$$TV_\varphi(v,\widehat{\Omega}) = \int_{\widehat{\Omega}} \varphi^o(Dv), \tag{1.1}$$

a useful functional appearing, for example, in image reconstruction and denoising [34, 16, 17, 5, 28]. Here $\varphi : \mathbb{R}^n \to [0,+\infty)$ is a norm, and its dual φ^o is typically the restriction of Φ^o on the "horizontal" $\mathbb{R}^n$.

Minimizers of P_Φ have been widely studied [37, 4]; in particular, it is known [10, 2] that if Φ^2 is smooth and uniformly convex, (boundaries of) minimizers are smooth out of a "small" closed singular set. In contrast to the classical case, where perimeter minimizers are smooth out of a closed set of Hausdorff dimension at most $n-7$, the behaviour of minimizers of anisotropic perimeters is more irregular: for instance, there exist singular minimizing cones even for smooth and uniformly convex anisotropies in $\mathbb{R}^4$ [29]. Referring to functionals of the form (1.1), we recall that, if $n \le 7$, Hölder continuity of minimizers for the image denoising functional [34], consisting of the Euclidean total variation TV plus the usual quadratic fidelity term, has been studied in [14]. In [26] such result is extended to the anisotropic total variation TV_φ.

One of the remarkable results in the classical theory of minimal surfaces is the classification of entire minimizers of the Euclidean perimeter P: if $n \le 6$ the only entire minimizers are hyperplanes, while for $n = 7$ there are nonlinear entire minimizers (see for instance [20, Chapter 17] and references therein); in the cartesian case (sometimes called the non parametric case), this is the well-known Bernstein problem. In the anisotropic setting, to our best knowledge, only a few results are available: entire minimizers in $\mathbb{R}^2$ are classified in [32], and minimizing cones in $\mathbb{R}^3$ for crystalline anisotropies are classified in [38]. In [23, 35] the authors show that if $n \le 2$ and Φ^2 is smooth, the only entire cartesian minimizers are the subgraphs of linear functions (anisotropic Bernstein problem), and the same result holds up to dimension $n \le 6$ if Φ is close enough to the Euclidean norm [35]. However, the anisotropic Bernstein problem seems to be still open in dimensions $4 \le n \le 6$, even for smooth and uniformly convex norms (see [33] for recent results in this direction).

The above discussion shows the difficulty of describing perimeter minimizers in the presence of an anisotropy; it seems therefore rather natural to look for reasonable assumptions on Φ that allow to simplify the classification problem. A possible requirement, which will be often (but not always) assumed in the sequel of the paper, is that Φ is cylindrical over φ, *i.e.*

$$\Phi(\widehat{\xi},\xi_{n+1}) = \max\{\varphi(\widehat{\xi}), |\xi_{n+1}|\}, \quad (\widehat{\xi},\xi_{n+1}) \in \mathbb{R}^{n+1}. \tag{1.2}$$

Despite its splitted expression, a cylindrical anisotropy is neither smooth nor strictly convex, and this still makes the above mentioned classification rather complicated. For instance, in Examples 2.7 and 2.9 we show that there exist singular cones minimizing P_Φ in any dimension $n \ge 1$. Moreover, while it can be proved that if horizontal and vertical sections of E are minimizers of P_φ and P respectively then E is a minimizer of P_Φ (Remark 2.6), in general sections of a minimizer of P_Φ need not satisfy this minimality property (Examples 2.8 and 2.9).

These phenomena lead us to investigate the classification problem under some simplifying assumptions on the structure of minimizers. We shall consider two cases: cylindrical minimizers (Definition 3.1), and cartesian minimizers (Definition 4.1), the latter being our main interest. Cylindrical minimizers of P_Φ are studied in Section 3: in particular, in Example 3.6 we classify all cylindrical minimizers of P_Φ when $n = 2$ and the unit ball B_Φ of Φ (sometimes called Wulff shape) is a cube. Cartesian minimizers are studied in Sections 4, 5 and 6. In Section 4 we investigate the relationships between cartesian minimizers of P_Φ and minimizers of $\mathcal{G}_{\Phi^o}$, provided Φ is partially monotone (Definition 4.4). In Theorem 4.6 we show that the subgraph of a minimizer of $\mathcal{G}_{\Phi^o}$ is also a minimizer of P_Φ among all perturbations not preserving the cartesian structure. In particular, for Φ satisfying (1.2) the subgraph E of some function $u : \widehat{\Omega} \to \mathbb{R}$ is a cartesian minimizer of P_Φ in $\widehat{\Omega} \times \mathbb{R}$ if and only if u is a minimizer of TV_φ.

Sections 5 and 6 contain our main results, valid under the assumptions that

$$\Phi \text{ is cylindrical over } \varphi \text{ and } E \text{ is cartesian.}$$

In Theorem 5.8 (see also Corollary 5.12) we prove the following Bernstein-type classification result: *if eiter $n \leq 7$ and φ is Euclidean, or if $n = 2$ and φ^o is strictly convex, then any entire cartesian minimizer of P_Φ in $\mathbb{R}^{n+1}$ (i.e. the subgraph of a minimizer of TV_φ) is the subgraph of the composition of a monotone function on $\mathbb{R}$ with a linear function on $\mathbb{R}^n$.* We notice that this result is sharp: if $n = 8$, there are entire cartesian minimizers of P in $\mathbb{R}^9$ which cannot be represented as the subgraph of the composition of a monotone and a linear function (see Remark 5.11).

In view of our assumptions, also the regularity results of Section 6 are concerned with the anisotropic total variation functional. For our purposes, it is useful to remark that, even if the anisotropy φ is smooth and uniformly convex, in general minimizers of TV_φ are not necessarily continuous. In contrast, we remark that minimizers of TV_φ with continuous boundary data on bounded domains are continuous, see [22, 26, 25]. Nevertheless, in Theorems 6.2 and 6.4 we show that, *if $\varphi^2 \in C^3$ is uniformly convex, then the boundary of the subgraph of a minimizer of TV_φ is locally Lipschitz (that is, locally a Lipschitz graph) out of a closed singular set with a suitable Hausdorff dimension depending on φ*. As observed in Remark 6.3, for φ Euclidean these statements are optimal, while the statement is false already in dimension $n = 2$ for φ the square norm.

2. Notation and preliminaries

In what follows $n \geq 1$, $\Omega \subseteq \mathbb{R}^{n+1}$ and $\widehat{\Omega} \subseteq \mathbb{R}^n$ are open sets. $BV(\Omega)$ (resp. $BV_{\rm loc}(\Omega)$) stands for the space of functions with bounded (resp. locally bounded) variation in Ω [6]. The characteristic function of a (measurable) set $E \subset \Omega$ is denoted by χ_E; we write $E \in BV(\Omega)$ (resp. $E \in BV_{\rm loc}(\Omega)$) when $\chi_E \in BV(\Omega)$ (resp. $\chi_E \in BV_{\rm loc}(\Omega)$). Similar notation holds in $\widehat{\Omega}$. $P(E, A)$ denotes the Euclidean perimeter of the set E in the open set A. Recall that the perimeter of $E \in BV_{\rm loc}(\Omega)$ does not change if we change E into another set in the same Lebesgue equivalence class; henceforth we shall always assume that any set E coincides with its points of density one [6, 19]. The outward generalized unit normal to the reduced boundary $\partial^* E$ of $E \in BV_{\rm loc}(\Omega)$ is denoted by ν_E. We often use the splitting $\mathbb{R}^{n+1} = \{(x,t) : \ x \in \mathbb{R}^n, \ t \in \mathbb{R}\}$ and write $\nu_E = (\widehat{\nu}_E, (\nu_E)_t)$ and $e_{n+1} = (0, \dots, 0, 1)$. If $F \subseteq \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ we let

$$F_t := \{y \in \mathbb{R}^n : \quad (y,t) \in F\}, \qquad F_x := \{s \in \mathbb{R} : \quad (x,s) \in F\}. \tag{2.1}$$

Unless otherwise specified, in the sequel we take $m \in \{n, n+1\}$.

$\mathcal{A}_c(\Omega)$ is the collection of all open relatively compact subsets of Ω. The sequence $\{E_h\}$ of subsets of $\mathbb{R}^m$ converges to set $E \subset \mathbb{R}^m$ in $L^1_{\rm loc}(\Omega)$ if $\chi_{E_h} \to \chi_E$ as $h \to +\infty$ in $L^1(A)$ for any $A \in \mathcal{A}_c(\Omega)$.

Finally, for a function $u : \widehat{\Omega} \to \mathbb{R}$ we let

$$\mathrm{sg}(u) := \{(x,t) \in \mathbb{R}^{n+1} : \ x \in \widehat{\Omega}, \ u(x) > t\}$$

be the subgraph of u. We recall [20, 27] that

$$P_\Phi(\mathrm{sg}(u), \widehat{A} \times \mathbb{R}) < +\infty \qquad \forall \widehat{A} \in \mathcal{A}_c(\widehat{\Omega}). \tag{2.2}$$

if and only if $u \in BV_{\rm loc}(\widehat{\Omega})$.

2.1. **Norms.** A norm on $\mathbb{R}^m$ is a convex function $\Psi : \mathbb{R}^m \to [0, +\infty)$ satisfying $\Psi(\lambda\xi) = |\lambda|\Psi(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^m$, and for which there exists a constant $c > 0$ such that

$$c|\xi| \leq \Psi(\xi), \qquad \xi \in \mathbb{R}^m. \tag{2.3}$$

We let $B_\Psi := \{\xi \in \mathbb{R}^m : \Psi(\xi) \leq 1\}$, which is sometimes called Wulff shape, and $\Psi^o : (\mathbb{R}^m)^* \to [0, +\infty)$ the dual norm of Ψ,

$$\Psi^o(\xi^*) = \sup\{\xi^* \cdot \xi : \ \xi \in B_\Psi\}, \qquad \xi^* \in (\mathbb{R}^m)^*,$$

where $(\mathbb{R}^m)^*$ is the dual of $\mathbb{R}^m$, and $\cdot$ is the Euclidean scalar product. We have

$$\xi^* \cdot \xi \leq \Psi^o(\xi^*)\Psi(\xi), \quad \xi^* \in (\mathbb{R}^m)^*, \ \xi \in \mathbb{R}^m, \tag{2.4}$$

and $\Psi^{oo} = \Psi$. When $m = n+1$ we often split $\xi \in \mathbb{R}^{n+1}$ as $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$, and employ the symbol Φ (resp. φ) to denote a norm in $\mathbb{R}^{n+1}$ (resp. in $\mathbb{R}^n$). In $\mathbb{R}^{n+1}$ we frequently exploit the restriction $\Phi_{|\{\xi_{n+1}=0\}}$ of Φ to the horizontal hyperplane $\{\xi_{n+1}=0\}$, which is a norm on $\mathbb{R}^n$. Note that

$$\left(\Phi_{|\{\xi_{n+1}=0\}}\right)^o \leq \Phi^o_{|\{\xi^*_{n+1}=0\}}. \tag{2.5}$$

Indeed, let

$$\varphi := \Phi_{|\{\xi_{n+1}=0\}} \quad \text{and} \quad \phi := \left(\Phi^o_{|\{\xi^*_{n+1}=0\}}\right)^o.$$

Fix $\widehat{\xi}^* \in \mathbb{R}^n$ and choose $\widehat{\xi} \in \mathbb{R}^n$ such that $\varphi(\widehat{\xi}) = \Phi(\widehat{\xi}, 0) = 1$ and $\varphi^o(\widehat{\xi}^*) = \widehat{\xi} \cdot \widehat{\xi}^*$. Thus,

$$\varphi^o(\widehat{\xi}^*) = (\widehat{\xi}, 0) \cdot (\widehat{\xi}^*, 0) \leq \Phi^o(\widehat{\xi}^*, 0) = \phi^o(\widehat{\xi}^*).$$

Remark 2.1. Inequality (2.5) may be strict. For $\alpha \in (0, \pi/2)$ consider the symmetric parallelogram with vertices at $(1 \pm \cot\alpha, \pm 1)$, $(-1 \pm \cot\alpha, \pm 1)$, and let Φ_α be the Minkowski functional of P_α. Notice that

$$(\Phi_\alpha)_{|\{\xi_2=0\}}(\xi_1) = |\xi_1|$$

and

$$(\Phi^o_\alpha)_{|\{\xi^*_2=0\}}(1) = \Phi^o_\alpha(1,0) = \sup\{\xi_1 : \ (\xi_1, \xi_2) \in P_\alpha\} = 1 + \cot\alpha,$$

thus

$$\left((\Phi_\alpha)_{|\{\xi_2=0\}}\right)^o(1) = 1 < 1 + \cot\alpha = (\Phi^o_\alpha)_{|\xi^*_2=0}(1).$$

In Lemma A.4 we give necessary and sufficient conditions on Φ ensuring that equality in (2.5) holds.

Definition 2.2 (Cylindrical and conical norms). We say that the norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is *cylindrical* over φ if

$$\Phi(\widehat{\xi}, \xi_{n+1}) = \max\{\varphi(\widehat{\xi}), |\xi_{n+1}|\}, \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}, \tag{2.6}$$

where $\varphi : \mathbb{R}^n \to [0, +\infty)$ is a norm. We say that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is *conical* over φ, if

$$\Phi(\xi) = \varphi(\widehat{\xi}) + |\xi_{n+1}|, \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}.$$

Notice that if Φ is cylindrical over φ then Φ^o is conical over φ^o, and vice-versa.

2.2. **Perimeters.** Let $\Psi : \mathbb{R}^m \to [0, +\infty)$ be a norm and $O \subseteq \mathbb{R}^m$ be an open set. For any $E \in BV_{\mathrm{loc}}(O)$ and for any $A \in \mathcal{A}_c(O)$ we define [5] the Ψ-perimeter of E in A as

$$P_\Psi(E, A) := \int_A \Psi^o(D\chi_E) = \sup\left\{-\int_E \operatorname{div}\eta \, dx : \ \eta \in C^1_c(A, B_\Psi)\right\}.$$

It is known [5] that

$$P_\Psi(E, A) = \int_{A \cap \partial^* E} \Psi^o(\nu_E) \, d\mathcal{H}^{m-1}. \tag{2.7}$$

Definition 2.3 (Minimizer of anisotropic perimeter). We say that $E \in BV_{\mathrm{loc}}(O)$ is a *minimizer* of P_Ψ *by compact perturbations in* O (briefly, a *minimizer of* P_Ψ *in* O) if

$$P_\Psi(E, A) \leq P_\Psi(F, A) \tag{2.8}$$

for any $A \in \mathcal{A}_c(O)$ and $F \in BV_{\mathrm{loc}}(O)$ such that $E \Delta F \subset\subset A$.

From (2.7) it follows that if E is minimizer of P_Ψ in O, then so is $\mathbb{R}^m \setminus E$. If $m = 1$, then $\Phi(\xi) = \Phi(1)|\xi|$, thus $E \subset \mathbb{R}$ is a minimizer of P_Ψ in an open interval I if and only if it is a minimizer of the Euclidean perimeter, so E is of the form

$$\emptyset, \quad I, \quad (-\infty, \lambda) \cap I, \quad (\lambda, +\infty) \cap I, \qquad \lambda \in I. \tag{2.9}$$

The following example is based on a standard calibration argument[1].

[1]See for instance [1] for some definitions, results and references concerning calibrations.

Example 2.4 (Half-spaces). Let $H \subset \mathbb{R}^m$ be a half-space and $O \subseteq \mathbb{R}^m$ be open. Then $E = H \cap O$ is a minimizer of P_Ψ in O. Indeed, let $\zeta \in \mathbb{R}^m$ be such that $\Psi(\zeta) = 1$ and $\nu_H \cdot \zeta = \Psi^o(\nu_H)$. Consider $F \in BV_{\rm loc}(O)$ with $E\Delta F \subset\subset A \subset\subset O$. Observe that $\partial^*(E\backslash F)$ can be written as a pairwise disjoint[2] union of $(\mathbb{R}^m \setminus F) \cap \partial E$, $E \cap \partial^* F$ and $J := \{z \in \partial E \cap \partial^* F : \nu_H(z) = -\nu_F(z)\}$ (see for example [24, Theorem 16.3]). For the vector field $N : \mathbb{R}^m \to \mathbb{R}^m$ constantly equal to ζ, we have

$$
\begin{aligned}
0 &= \int_{E\backslash F} \operatorname{div} N \, dz \\
&= \int_{(\mathbb{R}^m\backslash F)\cap A\cap\partial E} \nu_H \cdot N \, d\mathcal{H}^{m-1} - \int_{E\cap A\cap\partial^* F} \nu_F \cdot N \, d\mathcal{H}^{m-1} + \int_{J\cap A} \nu_H \cdot N \, d\mathcal{H}^{m-1} \\
&=: \mathrm{I} - \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{2.10}
$$

Similarly,

$$
\begin{aligned}
0 &= \int_{F\backslash E} \operatorname{div}(-N) \, dz \\
&= -\int_{(\mathbb{R}^m\backslash E)\cap A\cap\partial^* F} \nu_F \cdot N \, d\mathcal{H}^{m-1} + \int_{F\cap A\cap\partial E} \nu_H \cdot N \, d\mathcal{H}^{m-1} - \int_{J\cap A} \nu_F \cdot N \, d\mathcal{H}^{m-1} \\
&=: -\mathrm{IV} + \mathrm{V} - \mathrm{VI}.
\end{aligned} \tag{2.11}
$$

Adding (2.10)-(2.11) and using $\nu_F \cdot N \le \Psi^o(\nu_F)$ we obtain

$$
\int_{A\cap\partial E} \Psi^o(\nu_H) \, d\mathcal{H}^{m-1} = \int_{A\cap\partial E} \nu_H \cdot N \, d\mathcal{H}^{m-1} = \mathrm{I} + \mathrm{III} + \mathrm{V}
$$

$$
= \mathrm{II} + \mathrm{IV} + \mathrm{VI} = \int_{A\cap\partial^* F} \nu_F \cdot N \, d\mathcal{H}^{m-1} \le \int_{A\cap\partial^* F} \Psi^o(\nu_F) \, d\mathcal{H}^{m-1}.
$$

The previous argument does not apply to a strip between two parallel planes.

Example 2.5 (Parallel planes). Let $n = 2$, and let $\Phi : \mathbb{R}^3 \to [0, +\infty)$ be cylindrical over the Euclidean norm. Given $a < b$ consider $E = \{(x,t) \in \mathbb{R}^3 : a < t < b\}$. Then E is not a minimizer of P_Φ in $\mathbb{R}^3$. Indeed, it is sufficient to compare E with the set $E \setminus C$, obtained from E by removing a sufficiently large cylinder $C = B_R \times [a,b]$ homothetic to B_Φ, where $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$. Then $P_\Phi(E)$ is reduced by $2\pi R^2$ (the sum of the areas of the top and bottom facets of C), while it is increased by the lateral area $2\pi(b-a)R$ of C. Hence, for $R > 0$ sufficiently large, (2.8) is not satisfied. Notice that the horizontal sections of E are either empty or a plane, which both are minimizers of the Euclidean perimeter in $\mathbb{R}^2$.

Remark 2.6. Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ. Assume that $E \in BV_{\rm loc}(\Omega)$ has the following property: for almost every $t \in \mathbb{R}$ the set E_t (horizontal section) is a minimizer of P_φ in Ω_t and for almost every $x \in \mathbb{R}^n$ the set E_x (vertical section) is a minimizer of Euclidean perimeter in Ω_x. Then by Remark A.2 we get that E is a minimizer of P_Φ in Ω.

Example 2.7. For any $l, \gamma \in \mathbb{R}$ we define the cones[3] in $\mathbb{R}^{n+1}$

$$
C_1^{(n)}(l,\gamma) := (-\infty, l) \times \mathbb{R}^{n-1} \times (\gamma, +\infty), \quad C_2^{(n)}(l,\gamma) := (l, +\infty) \times \mathbb{R}^{n-1} \times (-\infty, \gamma). \tag{2.12}
$$

From Example 2.4 and Remark 2.6 it follows that the following sets are minimizers of P_Φ in $\mathbb{R}^{n+1}$ provided that Φ satisfies (2.6):

a) $C_1^{(n)}(l_1,\gamma_1) \cup C_2^{(n)}(l_2,\gamma_2) \subset \mathbb{R}^{n+1}$, where $l_1 \le l_2$, $\gamma_1 \ge \gamma_2$ (see Figure 1).

b) The union of $C_1^{(n)}(l_1,\gamma)$ and the rotation of $C_2^{(n)}(l_2,\gamma)$ around the vertical axis x_{n+1} of α radiants (see Figure 2).

[2] Up to sets of zero $\mathcal{H}^{m-1}$-measure.

[3] A set $E \subseteq \mathbb{R}^m$ is a *cone* if there exists $x_0 \in \partial E$ such that for any $x \in E$ and $\lambda > 0$ it holds $x_0 + \lambda(x - x_0) \in E$.

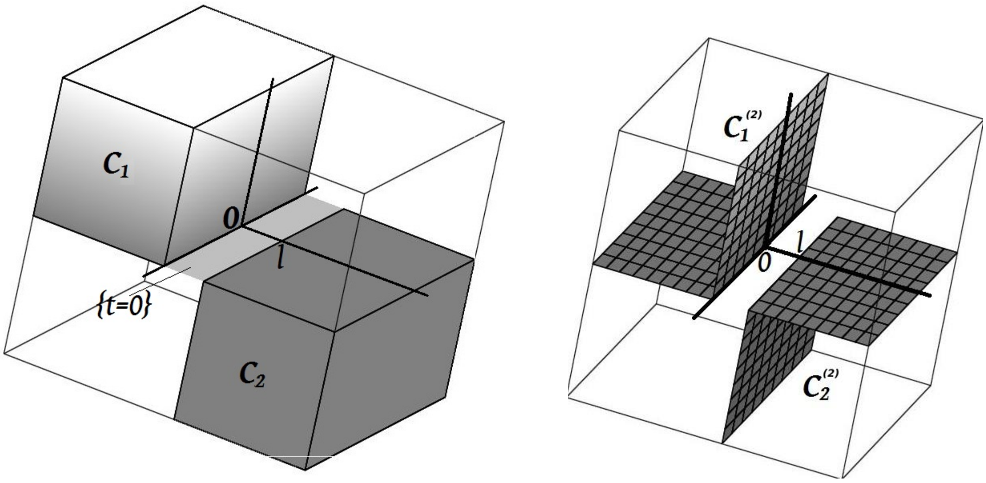

FIGURE 1. $C_1^{(2)}(0,0)\cup C_2^{(2)}(l,0)$ with $l>0$ in Example 2.7(a) and its boundary. The right picture is a slight rotation of left picture.

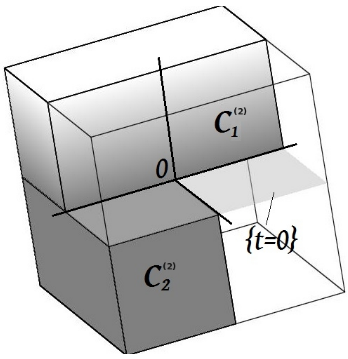

FIGURE 2. Union C of $C_1^{(2)}(0,0)$ and the $(-\pi/2)$ -rotation of $C_2^{(2)}(0,0)$ in Example 2.7(b). Notice that C_t for $t=0$ is not a minimizer of the Euclidean perimeter in $\mathbb{R}^2$; however, this does not affect the minimality of C.

In general, a minimizer of P_Φ in Ω for a cylindrical Φ, need *not* satisfy the minimality property of horizontal sections in Remark 2.6.

Example 2.8 (Strips). Let $n=2$, and let $\widehat\Omega=\mathbb{R}\times(0,\gamma)\subset\mathbb{R}^2$ with $\gamma>0$. Take $\varphi^o(\xi_1^*,\xi_2^*)=|\xi_1^*|+|\xi_2^*|$, so that

$$P_\varphi(\widehat F,\widehat A)=\int_{\widehat A}|D_{x_1}\chi_{\widehat F}|+\int_{\widehat A}|D_{x_2}\chi_{\widehat F}|,\qquad \widehat F\in BV_{\rm loc}(\widehat\Omega),\ \widehat A\in\mathcal{A}_c(\widehat\Omega).$$

We prove that if $l>\gamma>0$ then the rectangle $\widehat E=(0,l)\times(0,\gamma)$ is a minimizer of P_φ in the strip $\widehat\Omega$. Let $\widehat F\in BV_{\rm loc}(\widehat\Omega)$ be such that $\widehat E\Delta\widehat F\subset\subset\widehat A\subset\subset\widehat\Omega$. Let L_{x_1} stand for the vertical line passing through $(x_1,0)$. If $\mathcal{H}^1(\widehat F\cap L_{x_1})=0$ or $\mathcal{H}^1(\widehat F\cap L_{x_1})=\gamma$ for some $0<x_1<l$, then

$$P_\varphi(\widehat F,\widehat A)=P_\varphi(\widehat F,\widehat A\cap[(-\infty,x_1)\times(0,\gamma)])+P_\varphi(\widehat F,\widehat A\cap[(x_1,+\infty)\times(0,\gamma)]).$$

Hence

$$P_\varphi(\widehat F,\widehat A\cap[(-\infty,x_1)\times(0,\gamma)])\ge P_\varphi(\widehat E,\widehat A\cap[(-\infty,x_1)\times(0,\gamma)]),$$
$$P_\varphi(\widehat F,\widehat A\cap[(x_1,+\infty)\times(0,\gamma)])\ge P_\varphi(\widehat E,\widehat A\cap[(x_1,+\infty)\times(0,\gamma)]),$$

thus $P_\varphi(\widehat F,\widehat A)\ge P_\varphi(\widehat E,\widehat A)$. Now assume that $0<\mathcal{H}^1(\widehat F\cap L_{x_1})<\gamma$ for all $x_1\in(0,l)$. In this case $\int_{\widehat A}|D_{x_2}\chi_{\widehat F}|\ge 2l$. Indeed, since $\widehat E\Delta\widehat F\subset\subset\widehat A$, each vertical line L_{x_1}, $x_1\in(0,l)$ should cross $\partial^*\widehat F$ at least twice. For a similar reason, taking into account the term $\int_{\widehat A}|D_{x_1}\chi_{\widehat F}|$ we may assume that

$\widehat{E} \cap \partial^* \widehat{F}$ lies on two horizontal parallel lines at distance $\varepsilon \in (0,\gamma)$. Then by definition of φ-perimeter

$$P_\varphi(\widehat{F},\widehat{A}) - P_\varphi(\widehat{E},\widehat{A}) \ge 2l - 2\varepsilon \ge 2(l-\gamma) > 0.$$

This implies that $\widehat{E}$ is a minimizer of P_φ in $\widehat{\Omega}$. Notice that every horizontal section of $\widehat{E}$ is $(0,l)$, which is not a minimizer of the perimeter in $\mathbb{R}$.

Now, let $\Phi^o(\widehat{\xi^*},\xi_3^*) = \varphi^o(\widehat{\xi^*}) + |\xi_3^*|$. By Proposition 3.4 (b) below, $\widehat{E}\times\mathbb{R}$ is a minimizer of P_Φ in $\widehat{\Omega}\times\mathbb{R}$. Since Φ is symmetric with respect to relabelling the coordinate axis, the set $E = (0,l)\times\mathbb{R}\times(0,\gamma)$ is also a minimizer of P_Φ in $\mathbb{R}\times\mathbb{R}\times(0,\gamma)$. Notice that every horizontal section of E is a translation of the strip $(0,l)\times\mathbb{R}$, which is not a minimizer of P_φ in $\mathbb{R}^2$ according to Example 2.5.

Example 2.9. Let $\Phi^o(\xi_1^*,\xi_2^*) = |\xi_1^*| + |\xi_2^*|$. Given $l,\gamma\in\mathbb{R}$ suppose one of the following:

a) $l = 0$;
b) $l \ge 0 \ge \gamma$;
c) $l \ge \gamma > 0$.

Then the set $E = C_1^{(1)}(0,0) \cup C_2^{(1)}(l,\gamma)$ is a minimizer of P_Φ in $\mathbb{R}^2$ even though in case (c) for any $t\in(0,\gamma)$, the horizontal section E_t is not a minimizer of the perimeter in $\mathbb{R}$ (see (2.9)).

Indeed, if $l \ge 0 \ge \gamma$ then E satisfies the property in Remark 2.6. If $l = 0$ and $\gamma > 0$, then $\mathbb{R}^2\setminus E$ is union of two disjoint cones satisfying property stated in Remark 2.6. Thus, in both cases E is a minimizer of P_Φ in $\mathbb{R}^2$.

Assume (c). By Remark 2.6 both $C_1 = (-\infty,0)\times(0,+\infty)$ and $C_2 = (-\infty,\gamma)\times(l,+\infty)$ are minimizers of P_Φ in $\mathbb{R}^2$ (for brevity we do not write the dependence on l and γ). Consider arbitrary $F\in BV_{\mathrm{loc}}(\mathbb{R}^2)$ with $E\Delta F \subset\subset (-M,M)^2$, for some $M>0$.

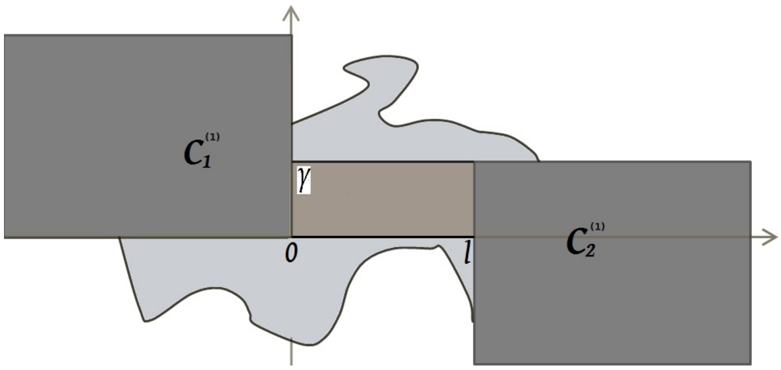

FIGURE 3. In case $0<\gamma\le l$, among all sets connecting two components of E the strip parallel to ξ_1-axis has the "smallest" Φ-perimeter.

If F perturbs the components C_1, C_2 of E separately, *i.e.* $F = F_1\cup F_2$ and there exist disjoint open sets $A_1,A_2\subset\mathbb{R}^2$ such that $C_i\Delta F_i \subset A_i$, $i=1,2$, then by minimality of C_1, C_2 we have

$$\begin{aligned} P_\Phi(F,A_1\cup A_2) =& P_\Phi(F_1,A_1) + P_\Phi(F_2,A_2) \ge P_\Phi(C_1,A_1) + P_\Phi(C_2,A_2) \\ =& P_\Phi(E,A_1\cup A_2). \end{aligned}$$

On the other hand, it is not difficult to see that among all perturbations of E involving both components, the best one is obtained by inserting an horizontal strip as in Figure 3. However, because of the assumption $0<\gamma\le l$, this perturbation has larger Φ-perimeter than E. Consequently, E is a minimizer of P_Φ in $\mathbb{R}^2$.

3. Cylindrical minimizers

Let Φ be a norm on $\mathbb{R}^{n+1}$ and $\Omega = \widehat{\Omega}\times\mathbb{R}$.

Definition 3.1 (Cylindrical minimizers). We say that a minimizer $E\subseteq\Omega$ of P_Φ in Ω is *cylindrical* over $\widehat{E}$ if $E = \widehat{E}\times\mathbb{R}$, where $\widehat{E}\subseteq\widehat{\Omega}$.

The aim of this section is to characterize cylindrical minimizers of P_Φ. The idea here is that the (Euclidean) normal to the boundary of a cylindrical minimizer is horizontal, and therefore what matters, in the computation of the anisotropic perimeter, is only the horizontal section of the anisotropy. For this reason it is natural to introduce the following property, which informally requires the upper (and the lower) part of the boundary of the Wulff shape to be a generalized graph (hence possibly with vertical parts) over its projection on the horizontal hyperplane $\mathbb{R}^n \times \{0\}$.

Definition 3.2 (**Unit ball as a generalized graph in the vertical direction**). We say that the boundary of the unit ball B_Φ of the norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is a *generalized graph in the vertical direction* if

$$\Phi(\widehat{\xi}, \xi_{n+1}) \geq \Phi(\widehat{\xi}, 0), \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}. \tag{3.1}$$

In Lemma A.3 we show that ∂B_Φ is a generalized graph in the vertical direction if and only if so is ∂B_{Φ^o}.

Example 3.3. (a) If $\Phi(\widehat{\xi}, -\xi_{n+1}) = \Phi(\widehat{\xi}, \xi_{n+1})$ for all $(\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$, then ∂B_Φ is a generalized graph in the vertical direction. Indeed, from convexity

$$\Phi(\widehat{\xi}, 0) \leq \Phi\left(\frac{\widehat{\xi}}{2}, \frac{\xi_{n+1}}{2}\right) + \Phi\left(\frac{\widehat{\xi}}{2}, -\frac{\xi_{n+1}}{2}\right) = \Phi(\widehat{\xi}, \xi_{n+1}), \qquad (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}.$$

(b) There exists ∂B_Φ which is a generalized graph in the vertical direction, but Φ does not satisfy $\Phi(\widehat{\xi}, -\xi_{n+1}) = \Phi(\widehat{\xi}, \xi_{n+1})$. Fix some $\varepsilon \in (1/\sqrt{2}, \sqrt{2}]$ and consider the (symmetric convex) plane hexagon K_ε with vertices at $(1, 0)$, $(\varepsilon, -\varepsilon)$, $(0, -1)$, $(-1, 0)$, $(-\varepsilon, \varepsilon)$, $(0, 1)$. Let $\Phi_\varepsilon : \mathbb{R}^2 \to [0, +\infty)$ be the Minkowski functional of K_ε. Then Φ_ε does not satisfy $\Phi_\varepsilon(\xi_1, -\xi_2) = \Phi_\varepsilon(\xi_1, \xi_2)$. But $\partial B_{\Phi_\varepsilon}$ is a generalized graph in the vertical direction. Indeed, consider the straight line passing through $(1, 0)$ and parallel to ξ_2 - axis. This line does not cross the interior of K_ε. Thus $\Phi_\varepsilon(1, \xi_2) \geq 1 = \Phi_\varepsilon(1, 0)$. If $\xi_1 \neq 0$, then

$$\Phi_\varepsilon(\xi_1, \xi_2) = |\xi_1| \Phi_\varepsilon(1, \xi_2/\xi_1) \geq |\xi_1| \Phi_\varepsilon(1, 0) = \Phi_\varepsilon(\xi_1, 0).$$

If $\xi_1 = 0$, the inequality $\Phi_\varepsilon(\xi_1, \xi_2) \geq \Phi_\varepsilon(\xi_1, 0)$ is obvious.

c) The norm $\Phi : \mathbb{R}^2 \to [0, +\infty)$, $\Phi(\xi_1, \xi_2) = \sqrt{\xi_1^2 + \xi_1 \xi_2 + \xi_2^2}$ has a unit ball the boundary of which is not a generalized graph in the vertical direction, since $\Phi(2, 0) = 2 > \sqrt{3} = \Phi(2, -1)$.

Proposition 3.4 (**Cylindrical minimizers**). *Let $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ be a norm. Let $\widehat{E} \in BV_{\mathrm{loc}}(\widehat{\Omega})$. The following assertions hold:*

(a) *if $\widehat{E} \times \mathbb{R}$ is a minimizer of P_Φ in $\widehat{\Omega} \times \mathbb{R}$, then $\widehat{E}$ is a minimizer of P_ϕ in $\widehat{\Omega}$, where*

$$\phi := \left(\Phi^o_{|_{\{\xi^*_{n+1}=0\}}}\right)^o.$$

(b) *if ∂B_Φ is a generalized graph in the vertical direction and $\widehat{E}$ is a minimizer of P_φ in $\widehat{\Omega}$, where $\varphi := \Phi_{|_{\{\xi_{n+1}=0\}}}$, then $\widehat{E} \times \mathbb{R}$ is a minimizer of P_Φ in $\widehat{\Omega} \times \mathbb{R}$.*

Remark 3.5. In general $\phi \neq \varphi$ (see Remark 2.1 and Lemma A.4).

Proof. (a) Take $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$, $\widehat{F} \in BV_{\mathrm{loc}}(\widehat{\Omega})$ with $\widehat{E} \Delta \widehat{F} \subset\subset \widehat{A}$. For any $m > 0$ set $I_m := (-m, m)$, and define

$$F_m := [E \setminus (\mathbb{R}^n \times I_m)] \cup [\widehat{F} \times I_m].$$

Then $E \Delta F_m \subset\subset \widehat{A} \times I_{m+1} \subset\subset \widehat{\Omega} \times \mathbb{R}$ and, by minimality,

$$P_\Phi(E, \widehat{A} \times I_{m+1}) \leq P_\Phi(F_m, \widehat{A} \times I_{m+1}). \tag{3.2}$$

Writing $\nu_E = (\widehat{\nu_E}, (\nu_E)_t)$, we have $\nu_E = (\nu_{\widehat{E}}, 0)$ $\mathcal{H}^n$-almost everywhere on $\partial^* E$. Hence

$$
\begin{aligned}
P_\Phi\left(E, \widehat{A} \times I_{m+1}\right) &= \int_{[\widehat{A} \times I_{m+1}] \cap \partial^* E} \Phi^o\left(\widehat{\nu_E}, (\nu_E)_t\right) d\mathcal{H}^n \\
&= \int_{[\widehat{A} \times I_{m+1}] \cap \partial^* E} \phi^o(\nu_{\widehat{E}})\, d\mathcal{H}^n = 2(m+1) \int_{\widehat{A} \cap \partial^* \widehat{E}} \phi^o(\nu_{\widehat{E}})\, d\mathcal{H}^{n-1} \\
&= 2(m+1) P_\phi(\widehat{E}, \widehat{A}).
\end{aligned} \tag{3.3}
$$

Similarly, $\nu_{F_m} = (\nu_{\widehat{F}}, 0)$ on $(\partial^* \widehat{F}) \times I_m$, $\nu_{F_m} = (0, \pm 1)$ on $(\widehat{E} \Delta \widehat{F}) \times \{\pm m\}$ and $\nu_{F_m} = (\nu_{\widehat{E}}, 0)$ on $(\partial^* \widehat{F}) \times (I_{m+1} \setminus \overline{I_m})$. As a consequence,

$$
\begin{aligned}
P_\Phi(F_m, \widehat{A} \times I_{m+1}) &= \int_{[\widehat{A} \times I_m] \cap \partial^* F_m} \Phi^o(\nu_{F_m})\, d\mathcal{H}^n + \int_{[\widehat{A} \times \{\pm m\}] \cap \partial^* F_m} \Phi^o(\nu_{F_m})\, d\mathcal{H}^n \\
&\quad + \int_{[\widehat{A} \times (I_{m+1} \setminus \overline{I_m})] \cap \partial^* F_m} \Phi^o(\nu_{F_m})\, d\mathcal{H}^n \\
&= 2m P_\phi(\widehat{F}, \widehat{A}) + 2\Phi^o(0,1) \mathcal{H}^n(\widehat{E} \Delta \widehat{F}) + 2 P_\phi(\widehat{E}, \widehat{A}).
\end{aligned} \tag{3.4}
$$

From (3.3), (3.4) and (3.2), it follows

$$
P_\phi(\widehat{E}, \widehat{A}) \le P_\phi(\widehat{F}, \widehat{A}) + \frac{\Phi^o(0,1)}{m} \mathcal{H}^n(\widehat{F} \Delta \widehat{E}).
$$

Letting $m \to +\infty$ we get $P_\phi(\widehat{E}, \widehat{A}) \le P_\phi(\widehat{F}, \widehat{A})$, and assertion (a) follows.

(b) By Lemma A.4, $\varphi^o = \Phi^o_{|\{\xi^*_{n+1} = 0\}}$. Take $F \in BV_{\mathrm{loc}}(\widehat{\Omega} \times \mathbb{R})$, and let $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$ and $M > 0$ be such that $E \Delta F \subset\subset \widehat{A} \times I_M$, where $I_M := (-M, M)$. Then $\widehat{E} \Delta F_t \subset\subset \widehat{A}$ for all $t \in (-M, M)$ and since $\widehat{E}$ is a minimizer of P_φ in $\widehat{\Omega}$, using (3.1) and (A.1) we get

$$
\begin{aligned}
P_\Phi(F, \widehat{A} \times I_M) &= \int_{(\widehat{A} \times I_M) \cap \partial^* F} \Phi^o(\widehat{\nu_F}, (\nu_F)_t)\, d\mathcal{H}^n \ge \int_{(\widehat{A} \times I_M) \cap \partial^* F} \Phi^o(\widehat{\nu_F}, 0)\, d\mathcal{H}^n \\
&= \int_{-M}^{M} P_\varphi(F_t, \widehat{A})\, dt \ge \int_{-M}^{M} P_\varphi(\widehat{E}, \widehat{A})\, dt = P_\Phi(E, \widehat{A} \times I_M),
\end{aligned}
$$

and assertion (b) follows. □

Example 3.6 (Characterization of cylindrical minimizers for a cubic anisotropy). Proposition 3.4 allows us to classify the cylindrical minimizers of P_Φ for suitable choices of the dimension and of the anisotropy. Take $n = 2$, $\widehat{\Omega} = \mathbb{R}^2$, and let

$$
B_\Phi = [-1, 1]^3;
$$

in particular, ∂B_Φ is a generalized graph in the vertical direction and B_φ is the square $[-1, 1]^2$ in the (horizontal) plane. The minimizers of P_φ are classified as follows [32, Theorems 3.8 (ii) and 3.11 (2)]: the infinite cross $\widehat{C} = \{|x_1| > |x_2|\}$ and its complement, the subgraphs and epigraphs $\widehat{S}$ of monotone functions of one variable, and suitable unions $\widehat{U}$ of two connected components, each of which is the subgraph of a monotone function of one variable. Then Proposition 3.4 (b) implies that

$$
\widehat{C} \times \mathbb{R}, \quad (\mathbb{R}^2 \setminus \widehat{C}) \times \mathbb{R}, \quad \widehat{S} \times \mathbb{R}, \quad \widehat{U} \times \mathbb{R}
$$

are the *only* cylindrical minimizers of P_Φ in $\mathbb{R}^3$. The same result holds if B_φ is a parallelogram centered at the origin, and ∂B_Φ is any generalized graph in the vertical direction such that $B_\varphi = B_\Phi \cap \{\xi_3 = 0\}$.

Let $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ be a norm.

Definition 4.1 (Cartesian minimizers). We call a minimizer $E \subseteq \Omega = \widehat{\Omega} \times \mathbb{R}$ a *cartesian minimizer* of P_Φ in $\Omega = \widehat{\Omega} \times \mathbb{R}$ if $E = \mathrm{sg}(u)$ for some function $u : \widehat{\Omega} \to \mathbb{R}$.

Let $v \in BV_{\mathrm{loc}}(\widehat{\Omega})$; in what follows the symbol $\int_{\widehat{A}} \Phi^o(-Dv, 1)$ means

$$\int_{\widehat{A}} \Phi^o(-Dv, 1) = \sup \Big\{ \int_{\widehat{A}} \Big(v \sum_{j=1}^{n} \frac{\partial \eta_j}{\partial x_j} + \eta_{n+1} \Big)\, dx : \qquad \eta = (\eta_1, \dots, \eta_{n+1}) \in C^1_c(\widehat{A}, B_\Phi) \Big\}.$$

If $v \in W^{1,1}_{\mathrm{loc}}(\widehat{\Omega})$ we have [6, Theorem 2.91]

$$\begin{aligned} P_\Phi(\mathrm{sg}(v), \widehat{A} \times \mathbb{R}) &= \int_{(\widehat{A}\times\mathbb{R})\cap\partial \mathrm{sg}(v)} \Phi^o(\nu_{\mathrm{sg}(v)})\, d\mathcal{H}^n \\ &= \int_{(\widehat{A}\times\mathbb{R})\cap\partial \mathrm{sg}(v)} \Phi^o(-\nabla v, 1) \frac{d\mathcal{H}^n}{\sqrt{1+|\nabla v|^2}} = \int_{\widehat{A}} \Phi^o(-\nabla v, 1)\, dx. \end{aligned}$$

Using the techniques in [18], the previous equality extends to any $v \in BV_{\mathrm{loc}}(\widehat{\Omega})$,

$$P_\Phi(\mathrm{sg}(v), \widehat{A} \times \mathbb{R}) = \int_{(\widehat{A}\times\mathbb{R})\cap\partial^* \mathrm{sg}(v)} \Phi^o(\nu_{\mathrm{sg}(v)})\, d\mathcal{H}^n = \int_{\widehat{A}} \Phi^o(-Dv, 1). \tag{4.1}$$

Accordingly, we define the functional $\mathcal{G}_{\Phi^o} : BV_{\mathrm{loc}}(\widehat{\Omega}) \times \mathcal{A}_c(\widehat{\Omega}) \to [0, +\infty)$ as follows:

$$\mathcal{G}_{\Phi^o}(v, \widehat{A}) := \int_{\widehat{A}} \Phi^o(-Dv, 1), \qquad v \in BV_{\mathrm{loc}}(\widehat{\Omega}),\ \widehat{A} \in \mathcal{A}_c(\widehat{\Omega}).$$

Definition 4.2. We say that $u \in BV_{\mathrm{loc}}(\widehat{\Omega})$ is a *minimizer* of $\mathcal{G}_{\Phi^o}$ by *compact perturbations* in $\widehat{\Omega}$ (briefly, a *minimizer* of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$), and we write

$$u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}),$$

if for any $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$ and $v \in BV_{\mathrm{loc}}(\widehat{\Omega})$ with $\mathrm{supp}\,(u - v) \subset\subset \widehat{A}$ one has

$$\mathcal{G}_{\Phi^o}(u, \widehat{A}) \le \mathcal{G}_{\Phi^o}(v, \widehat{A}).$$

Note that $\mathcal{M}_{\Phi^o}(\widehat{\Omega}) \neq \emptyset$ since linear functions on $\widehat{\Omega}$ belong[4] to $\mathcal{M}_{\Phi^o}(\widehat{\Omega})$. Observe also that if $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ then $u + c \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ for any $c \in \mathbb{R}$.

We shall need the following standard result.

Theorem 4.3 (Compactness). *Let $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ be a norm. If $u_k \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$, $u \in L^1_{\mathrm{loc}}(\widehat{\Omega})$ and $u_k \to u$ in $L^1_{\mathrm{loc}}(\widehat{\Omega})$ as $k \to +\infty$, then $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$.*

Proof. The proof is the same as in [31, Theorem 3.4] making use of lower semicontinuity of P_Φ, (2.3) and the inequality $\Phi^o(-Dw, 1) \le \Phi^o(Dw, 0) + \Phi^o(0, 1)$. □

The aim of this section is to show the relations between minimizers and cartesian minimizers, under a special assumption on the norm.

Definition 4.4 (Partially monotone norm). The norm $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is called *partially monotone* if given $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ and $\eta = (\widehat{\eta}, \eta_{n+1}) \in \mathbb{R}^{n+1}$ we have

$$\Phi(\widehat{\xi}, 0) \le \Phi(\widehat{\eta}, 0), \quad \Phi(0, \xi_{n+1}) \le \Phi(0, \eta_{n+1}) \quad \Longrightarrow \quad \Phi(\xi) \le \Phi(\eta). \tag{4.2}$$

Partially monotone norms are characterized in Section A.3.

[4] If u is linear, then $\mathrm{sg}(u)$ is the intersection of a half-space with $\widehat{\Omega} \times \mathbb{R}$, hence $\mathrm{sg}(u)$ is a minimizer of P_Φ in $\widehat{\Omega} \times \mathbb{R}$ (Example 2.4) and $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ (see Theorem 4.6(a) below).

Example 4.5. The following norms on $\mathbb{R}^{n+1}$ are partially monotone: $\Phi(\widehat{\xi},\xi_{n+1}) = \max\{\varphi(\widehat{\xi}), |\xi_{n+1}|\}$; $\Phi(\widehat{\xi},\xi_{n+1}) = ([\varphi(\widehat{\xi})]^p + |\xi_{n+1}|^p)^{1/p}$, where $\varphi : \mathbb{R}^n \to [0,+\infty)$ is a norm and $p \in [1,+\infty)$.

Theorem 4.6 (Minimizers and cartesian minimizers). *Let $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ be a norm, and $u \in BV_{\rm loc}(\widehat{\Omega})$. The following assertions hold:*

(a) *if $\mathrm{sg}(u)$ is a minimizer of P_Φ in $\widehat{\Omega}\times\mathbb{R}$, then u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$;*

(b) *if Φ is partially monotone and u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$, then $\mathrm{sg}(u)$ is a minimizer of P_Φ in $\Omega = \widehat{\Omega}\times\mathbb{R}$.*

Proof. (a) Let $\psi \in C^1_c(\widehat{\Omega})$ be such that $\mathrm{supp}\,(\psi) \subset\subset \widehat{A}$ for some $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$. Then there exists $H > 0$ such that $\mathrm{sg}(u)\Delta\mathrm{sg}(u+\psi) \subset\subset \widehat{A}\times(-H,H)$. If $\mathrm{sg}(u)$ is a minimizer of P_Φ, then $P_\Phi(\mathrm{sg}(u), \widehat{A}\times\mathbb{R}) \le P_\Phi(\mathrm{sg}(u+\psi), \widehat{A}\times\mathbb{R})$ and so, by virtue of (4.1),

$$\mathcal{G}_{\Phi^o}(u,\widehat{A}) \le \mathcal{G}_{\Phi^o}(u+\psi,\widehat{A}). \tag{4.3}$$

For general $\psi \in BV_{\rm loc}(\widehat{\Omega})$ inequality (4.3) can be proven by approximation.

(b) Let u be a minimizer of $\mathcal{G}_{\Phi^o}$ and $F \in BV_{\rm loc}(\Omega)$ be such that $\mathrm{sg}(u)\Delta F \subset\subset A = \widehat{A}\times(-M,M)$ with $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$ and $M > 0$. Then (2.2) yields that

$$P_\Phi(F, \widehat{B}\times\mathbb{R}) < +\infty \qquad \forall \widehat{B} \in \mathcal{A}_c(\widehat{\Omega}).$$

We shall closely follow [20, 27], where the argument is done in the Euclidean setting. For simplicity let $\varphi_1^o(\cdot) = \Phi^o(\cdot,0)$ and $\varphi_2^o(\cdot) = \Phi^o(0,\cdot)$. We claim that there exists $v \in BV_{\rm loc}(\widehat{\Omega})$ with $\mathrm{supp}\,(u-v) \subset\subset \widehat{A}$ such that for any $\widehat{B} \in \mathcal{A}_c(\widehat{\Omega})$

$$\begin{aligned} &\int_{\widehat{B}\times\mathbb{R}} \varphi_1^o(D_x\chi_{\mathrm{sg}(v)}) \le \int_{\widehat{B}\times\mathbb{R}} \varphi_1^o(D_x\chi_F), \\ &\int_{\widehat{B}\times\mathbb{R}} \varphi_2^o(D_t\chi_{\mathrm{sg}(v)})) \le \int_{\widehat{B}\times\mathbb{R}} \varphi_2^o(D_t\chi_F). \end{aligned} \tag{4.4}$$

Supposing that the claim is true, from (4.4) and from Lemma A.7 we deduce

$$P_\Phi(\mathrm{sg}(v), \widehat{A}\times\mathbb{R}) = \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_{\mathrm{sg}(v)}) \le \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_F) = P_\Phi(F, \widehat{A}\times\mathbb{R}).$$

Then by the minimality of u and (4.1) we get

$$\begin{aligned} P_\Phi(\mathrm{sg}(u), \widehat{A}\times\mathbb{R}) = &\int_{\widehat{A}} \Phi^o(-Du,1) \le \int_{\widehat{A}} \Phi^o(-Dv,1) \\ &= P_\Phi(\mathrm{sg}(v), \widehat{A}\times\mathbb{R}) \le P_\Phi(F, \widehat{A}\times\mathbb{R}). \end{aligned}$$

Let us prove our claim. Since $\mathrm{sg}(u)\Delta F \subset\subset A$, we have

$$\lim_{t\to+\infty} \chi_F(x,t) = 0, \quad \lim_{t\to-\infty} \chi_F(x,t) = 1 \quad \text{for a.e. } x \in \widehat{\Omega}. \tag{4.5}$$

Then, by [20, Lemma 14.7 and Theorem 14.8] (see also [27, Theorem 2.3]) the function

$$v_h(x) := \int_{-h}^{h} \chi_F(x,t)dt - h, \qquad x \in \widehat{\Omega},$$

belongs to $L^1_{\rm loc}(\widehat{\Omega})$ and the sequence $\{v_h\}$ converges pointwise to $v \in BV_{\rm loc}(\widehat{\Omega})$ as $h \to +\infty$. To show that $u - v$ is compactly supported in $\widehat{A}$ it is enough to take $\widehat{A}' \in \mathcal{A}_c(\widehat{\Omega})$ such that $\widehat{A}' \subset\subset \widehat{A}$ and $\mathrm{sg}(u)\Delta F \subset\subset \widehat{A}'\times(-M,M)$, and to observe that since $\mathrm{sg}(u)\cap\big((\widehat{\Omega}\setminus\widehat{A}')\times\mathbb{R}\big) = F\cap\big((\widehat{\Omega}\setminus\widehat{A}')\times\mathbb{R}\big)$, if $x \in \widehat{\Omega}\setminus\widehat{A}'$, for h sufficiently large we have

$$v_h(x) = \int_{-h}^{u(x)} \chi_F(x,t)dt - h = u(x).$$

Now, define $\eta_h : \mathbb{R} \to [0,+\infty)$ as $\eta_h := 1$ on $[-h,h]$, $\eta_h := 0$ on $(-\infty,-h-1] \cup [h+1,+\infty)$, and

$$\eta_h(t) := \begin{cases} h+1-t & \text{if } h \le t \le h+1, \\ h+1+t & \text{if } -h-1 \le t \le -h. \end{cases}$$

Being $1/2 = \int_{-h-1}^{-h} [h+1+t]dt$, we have

$$\begin{aligned}\Big| \int_{\mathbb{R}} \eta_h(t)\chi_F(x,t)dt - h - \frac{1}{2} - v(x) \Big| &= \Big| - \int_{-h-1}^{-h} [h+1+t](1-\chi_F(x,t))dt \\ &\quad + \int_h^{h+1} [h+1-t]\chi_F(x,t)dt + v_h(x) - v(x) \Big| \\ &\le \Big| v_h(x) - v(x) \Big| + \int_{-h-1}^{-h} (1-\chi_F(x,t))dt + \int_h^{h+1} \chi_F(x,t)dt.\end{aligned}$$

Hence, from (4.5), for almost every $x \in \widehat{\Omega}$ we get

$$\lim_{h\to+\infty} \Big| \int_{\mathbb{R}} \eta_h(t)\chi_F(x,t)dt - h - \frac{1}{2} - v(x) \Big| = 0. \tag{4.6}$$

Let us fix $\psi \in C_c^1(\widehat{\Omega})$ and $1 \le j \le n$. Then, using $\int_{\widehat{\Omega}} D_{x_j}\psi(x)dx = 0$, the dominated convergence theorem (see [20, 27] for more details) and (4.6) we find

$$\begin{aligned}\int_{\widehat{\Omega}\times\mathbb{R}} \psi(x) D_{x_j}\chi_F(x,t) &= \lim_{h\to+\infty} \int_{\widehat{\Omega}\times\mathbb{R}} \eta_h(t)\psi(x)D_{x_j}\chi_F(x,t) \\ &= - \lim_{h\to+\infty} \int_{\widehat{\Omega}} D_{x_j}\psi(x)dx \int_{\mathbb{R}} \eta_h(t)\chi_F(x,t)dt \\ &= - \lim_{h\to+\infty} \int_{\widehat{\Omega}} D_{x_j}\psi(x) \left[\int_{\mathbb{R}} \eta_h(t)\chi_F(x,t)dt - h - \frac{1}{2} \right] dx \\ &= - \int_{\widehat{\Omega}} v(x) D_{x_j}\psi(x)dx.\end{aligned}$$

Hence for any $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$ and $\eta \in C_c^1(\widehat{A}; B_{\varphi_1})$ one has

$$- \int_{\widehat{A}} v(x) \sum_{j=1}^n D_{x_j}\eta(x)dx = \int_{\widehat{A}\times\mathbb{R}} \eta(x) \cdot D_x\chi_F(x,t) \le \int_{\widehat{A}\times\mathbb{R}} \varphi_1^o(D_x\chi_F(x,t)).$$

Since η is arbitrary, the definition of $\int_{\widehat{A}} \varphi_1^o(D_x v)$ implies

$$\int_{\widehat{A}} \varphi_1^o(D_x v) \le \int_{\widehat{A}\times\mathbb{R}} \varphi_1^o(D_x\chi_F). \tag{4.7}$$

Being $|D_t\chi_F|$ a counting measure, we have [27]

$$\int_{\widehat{A}\times\mathbb{R}} \varphi_2^o(D_t\chi_F) = \varphi_2^o(1) \int_{\widehat{A}\times\mathbb{R}} |D_t\chi_F| \ge \varphi_2^o(1)|\widehat{A}|. \tag{4.8}$$

Moreover, one checks that

$$\int_{\widehat{A}\times\mathbb{R}} \varphi_1^o(D_x\chi_{\mathrm{sg}(v)}) = \int_{\widehat{A}} \varphi_1^o(D_x v), \qquad \int_{\widehat{A}\times\mathbb{R}} \varphi_2^o(D_t\chi_{\mathrm{sg}(v)}) = \varphi_2^o(1)\,|\widehat{A}|. \tag{4.9}$$

Now our claim (4.4) follows from (4.7)-(4.9). □

Corollary 4.7. *Let $\Phi^o : (\mathbb{R}^{n+1})^* \to [0,+\infty)$ be a partially monotone norm and $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$. Then $u \in L^\infty_{\mathrm{loc}}(\widehat{\Omega})$.*

Proof. It follows repeating essentially the same arguments in the proof of [20, Theorem 14.10], using Theorem 4.6(b) and the density estimates (see for instance [26, Proposition 1.10] for the anisotropic setting). □

5. Classification of cartesian minimizers for cylindrical norms

The aim of this section is to give a rather complete classification of entire cartesian minimizers, supposing the norm Φ cylindrical. As explained in the introduction, this case covers, in particular, the study of minimizers of the total variation functional. We start with a couple of observations.

Remark 5.1. Suppose that $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is cylindrical over φ. Then

$$u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \Longrightarrow \lambda u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \qquad \forall \lambda \in \mathbb{R}, \tag{5.1}$$

since

$$\mathcal{G}_{\Phi^o}(v,\widehat{A}) = \int_{\widehat{A}} \varphi^o(Dv) + |\widehat{A}|, \qquad (v,\widehat{A}) \in BV_{\rm loc}(\widehat{\Omega}) \times \mathcal{A}_c(\widehat{\Omega}).$$

On the other hand, (5.1) is expected to hold not for all non cylindrical norms Φ. For example, let Φ be Euclidean, $n \geq 8$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth nonlinear solution [11] of the minimal surface equation ${\rm div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$. Then[5] $u \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$, but if $\Delta u |\nabla u|^2$ is not identically zero, then $\lambda u \notin \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$ for any $\lambda \in \mathbb{R} \setminus \{0,\pm 1\}$. Indeed, otherwise λu solves the minimal surface equation, hence

$$\begin{aligned}
0 =& \lambda {\rm div}\left(\frac{\nabla u}{\sqrt{1+\lambda^2|\nabla u|^2}}\right) \\
=& \frac{\lambda}{\sqrt{1+\lambda^2|\nabla u|^2}}\left(\Delta u - \frac{\lambda^2}{1+\lambda^2|\nabla u|^2}\sum_{i,j=1}^n \nabla_i u \cdot \nabla_j u \nabla_{ij} u\right) \\
=& \frac{\lambda}{\sqrt{1+\lambda^2|\nabla u|^2}}\left(\Delta u - \lambda^2 \Delta u \frac{1+|\nabla u|^2}{1+\lambda^2|\nabla u|^2}\right) = \frac{\lambda(1-\lambda^2)\Delta u\, |\nabla u|^2}{(1+\lambda^2|\nabla u|^2)^{3/2}}.
\end{aligned}$$

If $\Delta u |\nabla u|^2$ is not identically zero, we get $\lambda(1-\lambda^2) = 0$, a contradiction.

Remark 5.2. Suppose that $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is cylindrical over φ. Then

$$u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \implies \max\{u,\lambda\}, \min\{u,\lambda\} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega}) \quad \forall \lambda \in \mathbb{R}. \tag{5.2}$$

Indeed, suppose first $\lambda = 0$. If $\{u \geq 0\} \in BV_{\rm loc}(\widehat{\Omega})$, then (5.2) can be proven as in [31, Lemma 3.5], using [6, Theorem 3.84]. In the general case, by the coarea formula there exists a sequence $\lambda_j \uparrow 0$ such that $\{u \geq \lambda_j\} \in BV_{\rm loc}(\widehat{\Omega})$. Clearly $u_j := u - \lambda_j \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$, hence $u_j^+ \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$. Since $u_j^+ \to u^+$ in $L^1_{\rm loc}(\widehat{\Omega})$, Theorem 4.3 implies $u^+ \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$. The case $\lambda \neq 0$ is implied by the previous proof and the identity $\max\{u,\lambda\} = (u-\lambda)^+ + \lambda$. The relation $\min\{u,\lambda\} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ then follows from the identity $\min\{u,\lambda\} = -\max\{-u,-\lambda\}$ and from Remark 5.1.

Further properties of cartesian minimizers are listed in the following proposition, which in particular (when φ is Euclidean) asserts some properties of minimizers of the total variation functional [17].

Proposition 5.3 (**Cartesian minimizers for cylindrical norms**). *Suppose that $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is cylindrical over φ. The following assertions hold:*

(a) *if $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ and $\lambda \in \mathbb{R}$ then $\chi_{\{u>\lambda\}}, \chi_{\{u\geq\lambda\}} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$;*
(b) *if $\widehat{E} \subset \widehat{\Omega}$ and $\chi_{\widehat{E}} \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ then $\widehat{E}$ is a minimizer of P_φ in $\widehat{\Omega}$;*
(c) *if $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ and $\lambda \in \mathbb{R}$ then $\{u > \lambda\}$ and $\{u \geq \lambda\}$ are minimizers of P_φ in $\widehat{\Omega}$;*
(d) *if $u \in BV_{\rm loc}(\widehat{\Omega})$ and for almost every $\lambda \in \mathbb{R}$ the sets $\{u > \lambda\}$ (resp. $\{u \geq \lambda\}$) are minimizers of P_φ in $\widehat{\Omega}$, then $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$;*
(e) *if $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$ and $f : \mathbb{R} \to \mathbb{R}$ is monotone then $f \circ u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$;*
(f) *let $\zeta \in \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ be a monotone function, and define $u(x) := f(x \cdot \zeta)$ for any $x \in \widehat{\Omega}$. Then $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$.*

[5] u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^n$, since the Euclidean unit normal (pointing upwards) to ${\rm graph}(u)$, constantly extended in the e_{n+1} direction, provides a calibration for ${\rm graph}(u)$ in the whole of $\mathbb{R}^{n+1}$.

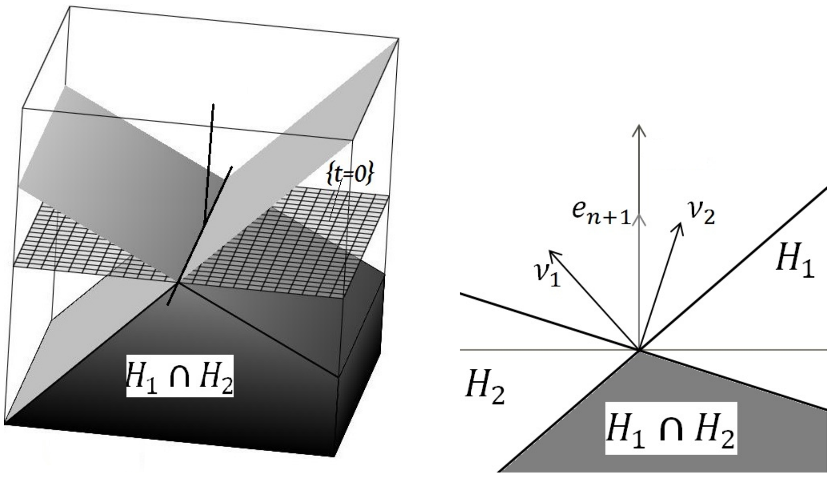

FIGURE 4. "Roof" like cone (left) and its section (right) along $(\partial H_1 \cap \partial H_2)^\perp$.

Clearly, assertion (e) generalizes (5.1) and (5.2). We also anticipate here that the converse of statement (f) is considered in Theorem 5.8 below.

Proof. The proof of (a) is the same as in [11, Theorem 1] and (b) is immediate. (c) follows from (a) and (b), while (d) follows from the coarea formula

$$\int_{\widehat{A}} \varphi^o(Dv) = \int_{\mathbb{R}} P_\varphi(\{v > \lambda\}, \widehat{A})\, d\lambda, \qquad v \in BV(\widehat{A}).$$

Let us prove (e). Without loss of generality assume that f is nondecreasing. Suppose first that f is Lipschitz and strictly increasing. Set $v = f \circ u$, and let $\lambda \in \mathbb{R}$. Since $u \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$, by (c) it follows that $\{u \geq f^{-1}(\lambda)\}$ and $\{u > f^{-1}(\lambda)\}$ are minimizers of P_φ in $\widehat{\Omega}$, hence $\{v > \lambda\}$ are minimizers of P_φ in $\widehat{\Omega}$. Then (d) implies $v \in \mathcal{M}_{\Phi^o}(\widehat{\Omega})$. In the general case, it is sufficient to approximate f with a sequence of strictly increasing Lipschitz functions, and use Theorem 4.3. (f) follows from (e), since the linear function $u_0(x) = x \cdot \zeta$, $x \in \widehat{\Omega}$, is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$. □

Now, we show that Proposition 5.3(f) implies the minimality of certain cones; the same conclusion could be obtained by applying Remark 2.6.

Proposition 5.4 (Cones minimizing the anisotropic perimeter). *Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ. Let $H_1, H_2 \subset \mathbb{R}^{n+1}$ be two half-spaces, with outer unit normals $\nu_1, \nu_2 \in \mathbb{S}^n$ respectively. Suppose that*

$$\{0\} \in \partial H_1 \cap \partial H_2 \subset \{t = 0\}, \tag{5.3}$$

and that

(a) $\nu_1 \cdot \nu_2 \geq 0$, $\nu_2 \cdot e_{n+1} \geq \nu_1 \cdot e_{n+1} \geq 0$*;*
(b) $\arccos(\nu_1 \cdot \nu_2) + \arccos(\nu_2 \cdot e_{n+1}) = \arccos(\nu_1 \cdot e_{n+1})$.

Then the cones $E := H_1 \cap H_2$ and $F := H_1 \cup H_2$ are minimizers of P_Φ in $\mathbb{R}^{n+1}$.

Before proving the proposition, some comments are in order. Our assumptions on H_1 and H_2 exclude, in particular, that E is a "roof-like" cone (as the one depicted in Figure 4). More specifically, in case $\nu_1 \neq \nu_2$, the inclusion $\partial H_1 \cap \partial H_2 \subset \{t = 0\}$ in (5.3) implies that the orthogonal complement to $\{t = 0\}$ is contained in the span of the orthogonal complements of ∂H_i, *i.e.*

$$e_{n+1} \in \operatorname{span}(\nu_1, \nu_2).$$

Next, assumption (a) implies that ν_1 and ν_2 lie "on the same side" with respect to e_{n+1}, while assumption (b) implies that ν_2 lies between ν_1 and e_{n+1} (a condition not satisfied in Figure 4, and satisfied in Figure 5). We shall see in Example 5.6 that, if condition (b) is not satisfied, then E and F need not be minimizers.

Proof. For $i = 1, 2,$ define

$$\lambda_i := \begin{cases} \dfrac{\sqrt{1-(\nu_i \cdot e_{n+1})^2}}{\nu_i \cdot e_{n+1}} = \tan \alpha_i & \text{if } \nu_i \cdot e_{n+1} \neq 0, \\ +\infty & \text{if } \nu_i \cdot e_{n+1} = 0, \end{cases}$$

see Figure 5, left. By (a) we have $\lambda_2 \le \lambda_1$. Let $\widehat{\nu} \in \mathbb{S}^{n-1} \subset \{t = 0\}$ be a unit normal to $\partial H_1 \cap \partial H_2$ which, according to (b), can be chosen so that $(\widehat{\nu}, 0) \cdot \nu_i \le 0, \quad i = 1, 2.$

If $\lambda_2 = +\infty$, then by conditions (a) and (b) we have $H_1 = H_2 = H$, where H is the half-space whose outer unit normal is $-(\widehat{\nu}, 0)$. By Example 2.4 it follows that $H = E = F$ is a minimizer of P_Φ.

Assume that $\lambda_2 \le \lambda_1 < +\infty$. Define

$$f(\sigma) := \begin{cases} \lambda_2 \sigma, & \sigma \ge 0 \\ \lambda_1 \sigma, & \sigma < 0 \end{cases}, \qquad g(\sigma) := \begin{cases} \lambda_1 \sigma, & \sigma \ge 0 \\ \lambda_2 \sigma, & \sigma < 0 \end{cases}.$$

Then $E = \mathrm{sg}(u)$, $F = \mathrm{sg}(v)$, where $u(x) := f(x \cdot \widehat{\nu})$, $v(x) := g(x \cdot \widehat{\nu})$, $x \in \mathbb{R}^n$. Since f, g are *monotone*, by Proposition 5.3(f) we have $u, v \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$. Since Φ^o is partially monotone (recall Example 4.5), Theorem 4.6(b) yields that E and F are minimizers of P_Φ in $\mathbb{R}^{n+1}$. Now, assume that

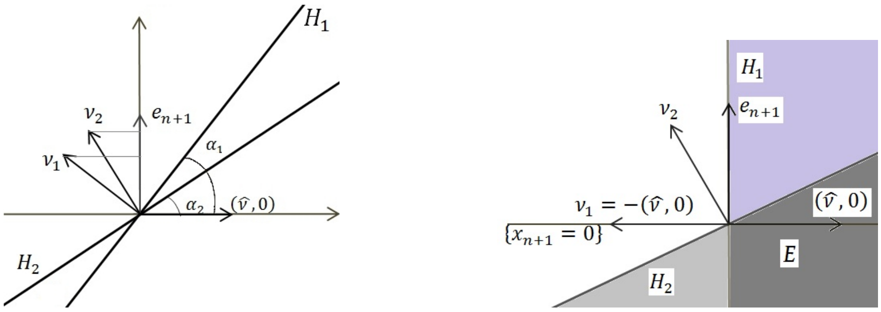

FIGURE 5. Sections of cones when $\lambda_1 < +\infty$ and $\lambda_1 = +\infty$.

$0 \le \lambda_2 < \lambda_1 = +\infty$. Then $\nu_1 = -(\widehat{\nu}, 0)$. We prove that F is a minimizer of P_Φ in $\mathbb{R}^{n+1}$ (the proof for E being similar). It is enough to show minimality of F inside every strip $S_m = \mathbb{R}^n \times (-m, m)$, $m > 0$. Define $h : \mathbb{R} \to \mathbb{R}$ as $h(\sigma) := m\chi_{(0,+\infty)}(\sigma)$ if $\lambda_2 = 0$ and

$$h(\sigma) := \begin{cases} -m & \text{if } \sigma < -\frac{m}{\lambda_2}, \\ \lambda_2 \sigma & \text{if } -\frac{m}{\lambda_2} \le \sigma < 0, \\ m & \text{if } \sigma \ge 0 \end{cases}$$

if $\lambda_2 > 0$. Let $w(x) := h(x \cdot \widehat{\nu})$, $x \in \mathbb{R}^n$. As before, the subgraph $\mathrm{sg}(w)$ of w is a minimizer of P_Φ in $\mathbb{R}^{n+1}$. Since $\mathrm{sg}(w) \cap S_m = F \cap S_m$, it follows that F is a minimizer of P_Φ in S_m. $\square$

Remark 5.5. It is not difficult to see that in Proposition 5.4 the assumption $\partial H_1 \cap \partial H_2 \subset \{t = 0\}$ is in general not necessary. Indeed, assume $n = 2$, $\Phi^o(\xi_1^*, \xi_2^*, \xi_3^*) = |\xi_1^*| + |\xi_2^*| + |\xi_3^*|$ and H_i, $i = 1, 2$ are half-spaces with outer unit normals $\nu_1 = (\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$, $\nu_2 = (\frac{1}{\sqrt{10}}, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{10}})$ respectively. Then both $H_1 \cap H_2$ and $H_1 \cup H_2$ are minimizers of P_Φ in $\mathbb{R}^3$. Indeed, for the Euclidean isometry $U(x, t) := (x_1, t, x_2)$, one sees that UH_1 and UH_2 satisfy the assumptions of Proposition 5.4, hence $UH_1 \cap UH_2$ and $UH_1 \cup UH_2$ are minimizers of P_Φ. Since $\Phi \circ U = \Phi$, the thesis follows.

Example 5.6 (Non minimal cones). Let $n = 1$, $\Phi^o(\xi_1^*, \xi_2^*) = |\xi_1^*| + |\xi_2^*|$ and H_1 and H_2 be half-planes of $\mathbb{R}^2$ with outer unit normals $\nu_1, \nu_2 \in \mathbb{S}^1$ such that

(a) $\partial H_1 \cap \partial H_2 = \{0\}$;
(b) $\nu_1 \cdot \nu_2 \ge 0$, $\nu_2 \cdot e_2 \ge \nu_1 \cdot e_2 \ge 0$, and if $\nu_2 \cdot e_2 = 1$ then $0 < \nu_1 \cdot e_2 < 1$;
(d) $\arccos(\nu_1 \cdot \nu_2) = \arccos(\nu_1 \cdot e_2) + \arccos(e_2 \cdot \nu_2)$.

Then the cones $E := H_1 \cap H_2$ and $F := H_1 \cup H_2$ are not minimizers of P_Φ. Let us prove the assertion for E, the statement for F being similar. The lines ∂H_1, ∂H_2 and $\{t = -1\}$ compose a nondegenerate triangle $T \subset E$ with sides $a_1, a_2, b > 0$, b the horizontal side. For any $A \in \mathcal{A}_c(\mathbb{R}^2)$ with $T \subset\subset A$ we have

$$P_\Phi(E, A) - P_\Phi(E \setminus T, A) = a_1\Phi^o(\nu_1) + a_2\Phi^o(\nu_2) - b\Phi^o(e_2) \geq a_1 + a_2 - b > 0,$$

since $\Phi^o(\nu) \geq 1$ for all $\nu \in \mathbb{S}^1$. Hence, E is not a minimizer of P_Φ.

We shall need the following relevant result (see for instance [20, Theorem 17.3] and references therein).

Theorem 5.7. *Let $\widehat{E}$ be a minimizer of the Euclidean perimeter in $\mathbb{R}^n$. Then either $n \geq 8$ or $\partial\widehat{E}$ is a hyperplane.*

Our classification result of minimizers of $\mathcal{G}_{\Phi^o}$ reads as follows:

Theorem 5.8 (Entire cartesian minimizers). *Suppose that $\Phi : \mathbb{R}^{n+1} \to [0, +\infty)$ is cylindrical over φ. Assume one of the two following alternatives:*

(a) $1 \leq n \leq 7$ *and φ is Euclidean;*
(b) $n = 2$ *and φ^o is strictly convex.*

If u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^n$ then there exists $\zeta \in \mathbb{S}^{n-1}$ and a monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = f(x \cdot \zeta), \qquad x \in \mathbb{R}^n. \tag{5.4}$$

Remark 5.9. If φ is a noneuclidean smooth and uniformly convex norm, the conclusion of Theorem 5.8 under assumption (a) does not necessarily hold. For example, if $n = 4$ and K is the cone over the Clifford torus [29] – a minimizer of P_φ in $\mathbb{R}^4$ for some uniformly convex smooth norm φ – then by Proposition 5.3(d), $u = \chi_K$ is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^4$ which cannot be represented as in (5.4). We don't know if there are counterexamples also for $n = 3$.

Proof. Let $u \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$. By Corollary 4.7, $u \in L^\infty_{\rm loc}(\mathbb{R}^n)$. Let

$$c_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} u(x) \in [-\infty, +\infty), \qquad c_1 := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} u(x) \in (-\infty, +\infty].$$

If $c_0 = c_1$, then $u \equiv c_0$ a.e. on $\mathbb{R}^n$. In this case $\zeta \in \mathbb{S}^{n-1}$ can be chosen arbitrarily and $f \equiv c_0$.

Assume that $-\infty \leq c_0 < c_1 \leq +\infty$. Given $\lambda \in \mathbb{R}$, Proposition 5.3(c) implies that $\{u > \lambda\}$ is a minimizer of P_φ in $\mathbb{R}^n$. We claim that either $\partial^*\{u > \lambda\}$ is a hyperplane or $\partial^*\{u > \lambda\} = \emptyset$. Indeed, if $n = 1$ the claim is trivial. If $n = 2$ and φ strictly convex, the claim follows from [32, Theorem 3.11]. When $3 \leq n \leq 7$ and φ is Euclidean, the claim is implied by Theorem 5.7. Hence for any $\lambda \in (c_0, c_1)$ there exist $\zeta_\lambda \in \mathbb{S}^{n-1}$ and $a_\lambda \in \mathbb{R}$ such that

$$\{u > \lambda\} = \{x \in \mathbb{R}^n : \ x \cdot \zeta_\lambda < a_\lambda\}. \tag{5.5}$$

In addition, these hyperplanes cannot intersect transversely, hence there exists $\zeta \in \mathbb{S}^{n-1}$ such that $\zeta_\lambda = \zeta$ for all $\lambda \in (c_0, c_1)$. Since the function $\lambda \in (c_0, c_1) \mapsto a_\lambda$ is monotone, it remains to construct the function f. We may assume that $\lambda \mapsto a_\lambda$ is nonincreasing, the nondecreasing case being similar. Extend a_λ to $\mathbb{R} \setminus [c_0, c_1]$ setting $a_\lambda := +\infty$ for $\lambda < c_0$ if $c_0 \in \mathbb{R}$, and $a_\lambda := -\infty$ for $\lambda > c_1$ if $c_1 \in \mathbb{R}$. Then, we define

$$f(\sigma) := \sup\{\lambda : \ \sigma < a_\lambda\}, \quad \sigma \in \mathbb{R},$$

which is nonincreasing. Note that f is real valued. Indeed, if $f(\sigma) = -\infty$ for some $\sigma \in \mathbb{R}$, then $\sigma \geq a_\lambda$ for all $\lambda \in \mathbb{R}$ which is impossible since $a_\lambda \to +\infty$ as $\lambda \to -\infty$. Similarly, $f(\sigma) < +\infty$ for any $\sigma \in \mathbb{R}$.

Set $v(x) := f(x \cdot \zeta)$. By construction, we have $\{v > \lambda\} = \{u > \lambda\}$ for a.e. $\lambda \in \mathbb{R}$. It is easy to check that if $w \in L^1_{\rm loc}(\mathbb{R}^n)$ then for a.e. $x \in \mathbb{R}^n$ one has

$$w(x) = \int_0^{+\infty} \chi_{\{w>\lambda\}}(x)d\lambda + \int_{-\infty}^0 (1 - \chi_{\{w>\lambda\}}(x))d\lambda,$$

hence $u = v$ almost everywhere on $\mathbb{R}^n$. $\square$

Remark 5.10. It seems not easy to generalize Theorem 5.8 to noneuclidean φ (for some $n \in \{3, \dots, 7\}$)[6], since our argument was based on Theorem 5.7.

Remark 5.11. Assumption (a) of Theorem 5.8 is optimal in the sense that if $n \geq 8$ there exist minimizers of $\mathcal{G}_{\Phi^o}$ on $\mathbb{R}^n$ which cannot be written as in (5.4). Indeed, let $C \subset \mathbb{R}^8$ be the Simons cone minimizing the Euclidean perimeter [11, Theorem A]. By Proposition 5.3(d) $u = \chi_C \in \mathcal{M}_{\Phi^o}(\mathbb{R}^n)$, however u does not admit the representation (5.4).

From Theorem 5.8 and Proposition 5.3 (f) we deduce the following result.

Corollary 5.12 (**Composition of linear and monotone functions**). *Under the assumptions of Theorem 5.8, u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^n$ if and only if there exists $\zeta \in \mathbb{S}^{n-1}$ and a monotone function $f : \mathbb{R} \to \mathbb{R}$ such that $u(x) = f(x \cdot \zeta)$ for any $x \in \mathbb{R}^n$.*

6. Lipschitz regularity of cartesian minimizers for cylindrical norms

We recall from [31, Theorem 3.12] that if $n = 2$ and if ∂B_φ either does not contain segments, or it is locally a graph in a neighborhood of its segments, then the graph of a minimizer of $\mathcal{G}_{\Phi^o}$ in $\mathbb{R}^2$ is locally Lipschitz. On the other hand, an example in [31, Sect. 4] shows that such a regularity result cannot be expected for a general anisotropy. More precisely, for Φ^o cylindrical as in (2.6) with $\varphi^o(\widehat{\xi}^*) = |\xi_1^*| + |\xi_2^*|$, that example exhibits a function $u \in \mathcal{M}_{\Phi^o}(\mathbb{R}^2)$ such that the set of points where the boundary of $\mathrm{sg}(u)$ is not locally the graph of a Lipschitz function has positive $\mathcal{H}^2$-measure. We look for sufficient conditions on φ which exclude such pathological example.

Let us start with a regularity property of cartesian minimizers of $\mathcal{G}_{\Phi^o}$ for *cylindrical norms over the Euclidean norm*, namely for

$$\Phi(\widehat{\xi}, \xi_{n+1}) = \max(|\widehat{\xi}|, |\xi_{n+1}|),$$

which is exactly the case of the total variation functional.

We need the following regularity result, a special case of [36, Theorem 1].

Theorem 6.1. *Let $\{\widehat{E}_h\}$ be a sequence of minimizers of the Euclidean perimeter in $\widehat{\Omega}$ locally converging to a set $\widehat{E}$ in $\widehat{\Omega}$, and let $x_h \in \partial \widehat{E}_h$ be such that $\lim\limits_{h \to +\infty} x_h = x \in \partial^* \widehat{E}$. Then there exists $\overline{h} \in \mathbb{N}$ such that $x_h \in \partial^* \widehat{E}_h$ for any $h \in \mathbb{N}$, $h \geq \overline{h}$, and $\lim\limits_{h \to +\infty} \nu_{\widehat{E}_h}(x_h) = \nu_{\widehat{E}}(x)$.*

Theorem 6.2 (**Local Lipschitz regularity**). *Suppose that $u \in BV_{\mathrm{loc}}(\widehat{\Omega})$ is a minimizer of the total variation functional*

$$TV(v, \widehat{\Omega}) := \int_{\widehat{\Omega}} |Dv|, \qquad v \in BV_{\mathrm{loc}}(\widehat{\Omega}).$$

Then there exists a closed set $\Sigma(u) \subseteq \partial \mathrm{sg}(u)$ of Hausdorff dimension at most $n-7$, with $\Sigma(u) = \emptyset$ if $n \leq 7$, such that $\partial \mathrm{sg}(u) \setminus \Sigma(u)$ is locally Lipschitz.

Proof. By Proposition 5.3(c) the sets $\{u > \lambda\}$ and $\{u \geq \lambda\}$ are minimizers of the Euclidean perimeter in $\widehat{\Omega}$ for every $\lambda \in \mathbb{R}$. Let $\lambda \in \mathbb{R}$ be such that $\partial\{u > \lambda\}$ (resp. $\{u \geq \lambda\}$) is nonempty. From classical regularity results (see for instance [20, Theorem 11.8] and references therein) it follows that $\partial\{u > \lambda\}$ (resp. $\partial\{u \geq \lambda\}$) *is of class* C^∞ out of a closed set $\Sigma_\lambda^>(u)$ (resp. $\Sigma_\lambda^\geq(u)$) of Hausdorff dimension at most $n-8$. Define

$$\Sigma(u) := \{(x, \lambda) \in \partial \mathrm{sg}(u) : \ x \in \Sigma_\lambda^>(u) \text{ or } x \in \Sigma_\lambda^\geq(u)\},$$

so that $\Sigma(u)$ has dimension at most $n-7$. From Theorem 6.1 it follows that $\Sigma(u)$ is closed.

Fix

$$(x, \lambda) \in \partial \mathrm{sg}(u) \setminus \Sigma(u). \tag{6.1}$$

One of the following three (not necessarily mutually exclusive) cases holds:

a) $x \in \mathrm{int}(\{u = \lambda\})$;
b) $x \in \partial\{u > \lambda\}$;
c) $x \in \partial\{u \geq \lambda\}$.

[6] If φ is $\mathcal{C}^\infty$-uniformly convex norm and $n = 3$, then $\{u \geq \lambda\}$ is smooth [2, Theorem II.7].

In case a) u is locally constant around x, thus, the assertion is immediate.

Assume b). We prove that there exists $r_x > 0$ such that $\partial\{u > \mu\}$ is a graph in direction $\nu_{\{u>\lambda\}}(x)$ for every $\mu \in \mathbb{R}$ such that $\partial\{u > \mu\} \cap B_{r_x}(x) \neq \emptyset$. Indeed, otherwise there would exist $\varepsilon > 0$ and an infinitesimal sequence $(r_h) \subset (0, +\infty)$, and sequences $(\mu_h) \subset \mathbb{R}$, (x_h) with $x_h \in \partial^*\{u > \mu_h\} \cap B_{r_h}(x)$ and

$$|\nu_{\{u>\lambda\}}(x) - \nu_{\{u>\mu_h\}}(x_h)| \geq \varepsilon \qquad \forall h \in \mathbb{N}. \tag{6.2}$$

By Corollary 4.7 u is locally bounded, thus (μ_h) is bounded and we can extract a (not relabelled) subsequence converging to some $\overline{\lambda} \in \mathbb{R}$. There is no loss of generality in assuming (μ_h) nondecreasing. Then $\{u > \mu_h\} \to \{u \geq \overline{\lambda}\}$ in $L^1_{\mathrm{loc}}(\widehat{\Omega})$ as $h \to +\infty$. By (6.1) we have $x \in \partial^*\{u \geq \overline{\lambda}\}$, hence from Theorem 6.1 it follows

$$\nu_{\{u>\mu_h\}}(x_h) \to \nu_{\{u\geq\overline{\lambda}\}}(x) \quad \text{as } h \to +\infty. \tag{6.3}$$

Clearly, either $\{u \geq \overline{\lambda}\} \subseteq \{u > \lambda\}$ or $\{u \geq \overline{\lambda}\} \supseteq \{u > \lambda\}$. Since

$$x \in \partial^*\{u > \lambda\} \cap \partial^*\{u \geq \overline{\lambda}\}$$

and $\partial\{u \geq \overline{\lambda}\}$ and $\partial\{u > \lambda\}$ are smooth around x, necessarily

$$\nu_{\{u>\lambda\}}(x) = \nu_{\{u\geq\overline{\lambda}\}}(x).$$

But then from (6.2) and (6.3) we get

$$\varepsilon \leq |\nu_{\{u>\mu_h\}}(x_h) - \nu_{\{u>\lambda\}}(x)| \to 0 \quad \text{as } h \to +\infty,$$

a contradiction.

Thus, for every $x \in \partial^*\{u > \lambda\}$ there exist $r_x > 0$ and $\varepsilon \in (0, 1)$ such that for any $\mu \in (\lambda - r_x, \lambda + r_x)$ and $y \in \partial^*\{u > \mu\} \cap B_{r_x}(x)$ one has

$$\nu_{\{u>\lambda\}}(x) \cdot \nu_{\{u>\mu\}}(y) \geq \varepsilon.$$

Notice that for any $(y, \mu) \in \partial^*\mathrm{sg}(u) \setminus \Sigma(u)$ one has that

$$\text{either} \quad \nu_{\mathrm{sg}(u)}(y, \mu) = \frac{(\nu_{\{u>\mu\}}(x), \sigma)}{\sqrt{1+\sigma^2}} \text{ for some } \sigma \geq 0, \quad \text{or} \quad \nu_{\mathrm{sg}(u)}(y, \mu) = e_{n+1}.$$

We want to prove that there exist $\rho > 0$, $\eta \in \mathbb{S}^n$ and $c \in (0, 1)$ such that $\mathcal{H}^n$-every $(y, \mu) \in \partial\,\mathrm{sg}(u) \cap B_\rho(x, \lambda)$ there holds

$$\eta \cdot \nu_{\mathrm{sg}(u)}(y, \mu) \geq c, \tag{6.4}$$

so that [31, Lemma 3.10] implies that $\partial\,\mathrm{sg}(u) \cap B_\rho(x, \lambda)$ is a Lipschitz graph in the direction η with Lipschitz constant $L = \sqrt{1/c^2 - 1}$.

Set

$$\rho = r_x, \quad \eta = \frac{1}{\sqrt{2}}(\nu_{\{u>\lambda\}}(x), 1).$$

Then for any $(y, \mu) \in \partial^*\mathrm{sg}(u) \cap B_\rho(x, \lambda)$ we have

$$\nu_{\mathrm{sg}(u)}(y, \mu) \cdot \eta = \frac{1}{\sqrt{2}}, \tag{6.5}$$

if $\nu_{\mathrm{sg}(u)}(y, \mu) = e_{n+1}$, and

$$\nu_{\mathrm{sg}(u)}(y, \mu) \cdot \eta = \frac{\nu_{\{u>\mu\}}(y) \cdot \nu_{\{u>\lambda\}}(x) + s}{\sqrt{2}\sqrt{1+s^2}} \geq \frac{\varepsilon + s}{\sqrt{2}\sqrt{1+s^2}} \geq \frac{\varepsilon}{\sqrt{2}}, \tag{6.6}$$

if $y \in \partial\{u > s\}$ (here we use $\frac{a+s}{\sqrt{1+s^2}} \geq a$ for any $a \in (0, 1)$ and $s \geq 0$). Formulas (6.5) and (6.6) imply (6.4) with $c = \varepsilon/\sqrt{2}$.

Finally, case c) can be treated as case b).

□

Remark 6.3. The assertion of Theorem 6.2 cannot be improved: if $n \geq 8$, there exists a minimizer u of $\mathcal{G}_{\Phi^o}$ such that the points where $\partial\mathrm{sg}(u)$ is not locally Lipschitz have positive $(n-7)$-dimensional Hausdorff measure. For the Simons cone in $\mathbb{R}^8$ (and with the Euclidean norm), the graph of $u = \chi_C$ cannot be represented as the graph of a Lipschitz function in a neighborhood of the origin.

Theorem 6.2 can be generalized as follows.

Theorem 6.4. *Suppose that* $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ *is cylindrical over* φ *with*

$$\varphi^2 \in C^3(\mathbb{R}^n) \quad \text{is uniformly convex.}$$

If u *is a minimizer of* $\mathcal{G}_{\Phi^o}$ *in* $\widehat{\Omega}$, *then* $\partial \mathrm{sg}(u) \setminus \Sigma(u)$ *is locally Lipschitz, where* $\Sigma(u) \subseteq \partial \mathrm{sg}(u)$ *is a closed set of Hausdorff dimension at most* $n-2$ *if* $n > 3$, *and* $\Sigma(u) = \emptyset$ *if* $n = 2,3$.

Proof. The proof is the same as in Theorem 6.2, using [2, Theorems II.7] in place of [20], and [30, Theorem 4.5] in place of [36]. □

Remark 6.5. In [31] it is proven that if $n = 2$, B_φ is not a quadrilateral, and u is a minimizer of $\mathcal{G}_{\Phi^o}$ in $\widehat{\Omega}$, then the graph of u is locally Lipschitz around any point of $\partial \mathrm{sg}(u)$.

Remark 6.6. Using the regularity result in [2, Theorem II.8], under the assumption that φ is uniformly convex, smooth and sufficiently close to the Euclidean norm, one can improve Theorem 6.4 by showing that $\Sigma(u)$ has Hausdorff dimension at most $n-5$.

Appendix A.

A.1. A Fubini-type theorem.

Proposition A.1. *Let* $E \in BV_{\mathrm{loc}}(\Omega)$. *Then for any* $A \in \mathcal{A}_c(\Omega)$

$$\int_{A\cap\partial^* E} \Phi^o(\widehat{\nu}_E, 0) d\mathcal{H}^n = \int_{\mathbb{R}} dt \int_{A_t\cap\partial^* E_t} \Phi^o(\nu_{E_t}, 0) d\mathcal{H}^{n-1}, \tag{A.1}$$

$$\int_{A\cap\partial^* E} \Phi^o(0, (\nu_E)_t) d\mathcal{H}^n = \int_{\mathbb{R}^n} dx \int_{A_x\cap\partial^* E_x} \Phi^o(0,1) d\mathcal{H}^0. \tag{A.2}$$

where E_t *and* E_x *are defined as* (2.1), ν_{E_t} *is a outer unit normal to* $\partial^* E_t$ *and* ν_{E_x} *is a outer unit normal to* $\partial^* E_x$.

Proof. Let us prove (A.1). Notice that by [24, Theorem 18.11] for a.e. $t \in \mathbb{R}$

$$\mathcal{H}^{n-1}(\partial^* E_t \Delta (\partial^* E)_t) = 0, \quad \widehat{\nu}_E \neq 0, \quad \nu_{E_t} = \frac{\widehat{\nu}_E}{|\widehat{\nu}_E|}.$$

We can use the coarea formula [6, Theorem 2.93] with the function $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x,t) = t$. Then $\nabla f = e_{n+1}$ and its orthogonal projection $\nabla^E f$ on the approximate tangent space to $\partial^* E$ is $\nabla^E f = e_{n+1} - (e_{n+1}\cdot \nu_E)\nu_E$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} dt \int_{A_t\cap\partial^* E_t} \Phi^o(\nu_{E_t}, 0) d\mathcal{H}^{n-1} = \int_{\mathbb{R}} dt \int_{(A\cap\partial^* E)\cap\{f=t\}} \Phi^o(\nu_{E_t}, 0) d\mathcal{H}^{n-1} \\
&= \int_{A\cap\partial^* E} \Phi^o(\nu_{E_t}, 0) |e_{n+1} - (e_{n+1}\cdot\nu_E)\nu_E|\, d\mathcal{H}^{n-1} = \int_{A\cap\partial^* E} \Phi^o(\nu_{E_t}, 0) \sqrt{1 - |(\nu_E)_t|^2}\, d\mathcal{H}^{n-1} \\
&= \int_{A\cap\partial^* E} \Phi^o(\nu_{E_t}, 0) |\widehat{\nu}_E|\, d\mathcal{H}^{n-1} = \int_{A\cap\partial^* E} \Phi^o(\widehat{\nu}_E, 0)\, d\mathcal{H}^{n-1}.
\end{aligned}$$

Now, (A.2) follows from (2.7) and [27, Theorem 3.3]:

$$\begin{aligned}
\int_{A\cap\partial^* E} \Phi^o(0, (\nu_E)_t) d\mathcal{H}^n &= \Phi^o(0,1) \int_{A\cap\partial^* E} |(\nu_E)_t| d\mathcal{H}^n = \Phi^o(0,1) \int_A |D_t \chi_E| \\
&= \Phi^o(0,1) \int_{\mathbb{R}^n} dx \int_{A_x\cap\partial^* E_x} d\mathcal{H}^0.
\end{aligned}$$

□

Remark A.2. Let $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ be a norm. For notational simplicity set $\varphi_1 := \Phi_{|_{\{\xi_{n+1}=0\}}}$, $\varphi_2 := \Phi_{|_{\{\widehat{\xi}=0\}}}$. For $f \in BV_{\rm loc}(\Omega)$ and $A \in \mathcal{A}_c(\Omega)$ we define

$$\int_A \varphi_1^o(D_x f) = \sup\Big\{ \int_A f(x,t) \sum_{i=1}^n \frac{\partial \eta_i(x,t)}{\partial x_i}\, dxdt : \eta \in C^1_c(A; B_{\varphi_1})\Big\},$$

$$\int_A \varphi_2^o(D_t f) = \sup\Big\{ \int_A f(x,s) D_t\eta(x,s)\, dxds : \eta \in C^1_c(A),\ \varphi_2(\eta) \le 1\Big\}.$$

With this notation (A.1) and (A.2) can be rewritten respectively as[7]

$$\int_A \varphi_1^o(D_x\chi_E) = \int_{\mathbb{R}} dt \int_{A_t} \varphi_1^o(D_x\chi_{E_t}),$$

$$\int_A \varphi_2^o(D_t\chi_E) = \int_{\mathbb{R}^n} dx \int_{A_x} \varphi_2^o(D_t\chi_{E_x}).$$

A.2. **Norms with generalized graph property.**

Lemma A.3. *∂B_Φ is a generalized graph in the vertical direction if and only if ∂B_{Φ^o} is a generalized graph in the vertical direction.*

Proof. Suppose that ∂B_Φ is a generalized graph in the vertical direction. Let $\xi^* = (\widehat{\xi}^*, 0) \in \mathbb{R}^{n+1}$, and take $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ such that $\Phi(\xi) = 1$ and

$$\widehat{\xi} \cdot \widehat{\xi}^* = \Phi^o(\widehat{\xi}^*, 0) = (\widehat{\xi}, \xi_{n+1}) \cdot (\widehat{\xi}^*, 0). \tag{A.3}$$

Since ∂B_Φ is a generalized graph in the vertical direction, we have $\Phi(\widehat{\xi}, 0) \le \Phi(\xi) = 1$. Thus, by (2.4) and (A.3) we get $\Phi^o(\widehat{\xi}^*, 0) \le \Phi^o(\widehat{\xi}^*, \xi^*_{n+1})$, hence ∂B_{Φ^o} is a generalized graph in the vertical direction. The converse conclusion follows then from the equality $\Phi^{oo} = \Phi$. □

Lemma A.4. *Equality holds in (2.5) if and only if ∂B_Φ is a generalized graph in the vertical direction.*

Proof. Set $\varphi := \Phi_{|_{\{\xi_{n+1}\}}}$. Assume that ∂B_Φ is a generalized graph in the vertical direction. Let $(\widehat{\xi}^*, 0) \in \mathbb{R}^{n+1}$ and take $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ such that $\Phi(\xi) = 1$ and (A.3) holds. By our assumption on ∂B_Φ it follows that $\varphi(\widehat{\xi}) = \Phi(\widehat{\xi}, 0) \le \Phi(\widehat{\xi}, \xi_{n+1}) = 1$, hence by (2.4)

$$\Phi^o(\widehat{\xi}^*, 0) \le \varphi^o(\widehat{\xi}^*)\varphi(\widehat{\xi}) \le \varphi^o(\widehat{\xi}^*).$$

This and (2.5) imply $\varphi^o(\widehat{\xi}^*) = \Phi^o(\widehat{\xi}^*, 0)$, *i.e.* $\Big(\Phi_{|_{\{\xi_{n+1}=0\}}}\Big)^o = \Phi^o_{|_{\{\xi^*_{n+1}=0\}}}$.

Now assume that equality in (2.5) holds. Take any $\xi = (\widehat{\xi}, \xi_{n+1}) \in \mathbb{R}^{n+1}$ and select $\widehat{\xi}^* \in \mathbb{R}^n$ such that $\varphi^o(\widehat{\xi}^*) = \Phi^o(\widehat{\xi}^*, 0) = 1$ and $\varphi(\widehat{\xi}) = \widehat{\xi} \cdot \widehat{\xi}^*$. Then by (2.4)

$$\Phi(\widehat{\xi}, 0) = \varphi(\widehat{\xi}) = \widehat{\xi} \cdot \widehat{\xi}^* = (\widehat{\xi}, \xi_{n+1}) \cdot (\widehat{\xi}^*, 0) \le \Phi(\widehat{\xi}, \xi_{n+1}).$$

□

A.3. **Partially monotone norms.**

Proposition A.5 (Characterization of partially monotone norms). *The norm $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is partially monotone if and only if there exists a positively one-homogeneous convex function $\omega : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ satisfying*

$$\omega(1,0), \omega(0,1) > 0,\ \ \omega(s_1,s_2) \le \omega(t_1,t_2), \quad 0 \le s_i \le t_i,\ i = 1,2, \tag{A.4}$$

such that

$$\Phi(\widehat{\xi}, \xi_{n+1}) = \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|), \tag{A.5}$$

[7] Following [27, Theorem 3.3] one can prove a more general statement, namely, if $f \in BV_{\rm loc}(A)$, then

$$\int_A \varphi_1^o(D_x f) = \int_{\mathbb{R}} dt \int_{A_t} \varphi_1^o(D_x(f|_{A_t})), \qquad \int_A \varphi_2^o(D_t f) = \int_{\mathbb{R}^n} dx \int_{A_x} \varphi_2^o(D_t(f|_{A_x})).$$

where $\varphi = \Phi_{|_{\{\xi_{n+1}=0\}}}$.

Proof. Concerning the "if" part, one checks that the function Φ defined as (A.5) is a partially monotone norm on $\mathbb{R}^{n+1}$. Now, let us prove the "only if" part. Choose any $\widehat{\eta} \in \mathbb{R}^n$ with $\varphi(\widehat{\eta}) = 1$ and define the function $\omega : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ as $\omega(s,t) := \Phi(s\widehat{\eta},t)$ for $s,t \geq 0$. Since Φ is convex and positively one-homogeneous, so is ω. Moreover, the relations $\Phi(\widehat{\eta},0) = 1$, $\Phi(0,1) > 0$ and partial monotonicity of Φ imply that ω satisfies (A.4). Now it remains to prove (A.5). Comparing $\xi = (\widehat{\xi},\xi_{n+1}) \in \mathbb{R}^{n+1}$ with $\eta = (\varphi(\widehat{\xi})\widehat{\eta}, |\xi_{n+1}|) \in \mathbb{R}^{n+1}$ in (4.2) and using the relation $\Phi(0,\xi_{n+1}) = \Phi(0,|\xi_{n+1}|)$ and partial monotonicity, we find

$$\Phi(\widehat{\xi},\xi_{n+1}) = \Phi(\varphi(\widehat{\xi})\widehat{\eta}, |\xi_{n+1}|) = \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|).$$

□

Notice that for Φ as in (A.5) we have

$$\Phi^o(\widehat{\xi^*},\xi^*_{n+1}) = \omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|), \tag{A.6}$$

where $\omega^o : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is defined as

$$\omega^o(s_1^*,s_2^*) = \sup\{s_1s_1^* + s_2s_2^* : \ s_1,s_2 \in [0,+\infty),\ \omega(s_1,s_2) \leq 1\},\ \ s_1^*,s_2^* \in [0,+\infty). \tag{A.7}$$

Indeed, take any $\xi^* = (\widehat{\xi^*},\xi^*_{n+1}) \in (\mathbb{R}^{n+1})^*$. Let $\xi = (\widehat{\xi},\xi_{n+1}) \in \mathbb{R}^{n+1}$ be such that $\Phi(\xi) = \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|) \leq 1$ and $\xi \cdot \xi^* = \Phi^o(\xi^*)$. Then using (2.4) twice we get

$$\begin{aligned}\Phi^o(\xi^*) =& \widehat{\xi} \cdot \widehat{\xi^*} + \xi_{n+1} \cdot \xi^*_{n+1} \leq \varphi(\widehat{\xi})\varphi^o(\widehat{\xi^*}) + |\xi_{n+1}| \cdot |\xi^*_{n+1}| \\ \leq& \omega(\varphi(\widehat{\xi}), |\xi_{n+1}|)\omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|) \leq \omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|).\end{aligned} \tag{A.8}$$

On the other hand, for any $\xi^* \in (\mathbb{R}^{n+1})^*$ there exist $\widehat{\xi} \in \mathbb{R}^n$ such that $\varphi(\widehat{\xi}) \leq 1$ and

$$\widehat{\xi} \cdot \widehat{\xi^*} = \varphi^o(\widehat{\xi^*}).$$

Moreover, by definition of ω^o one can find $(s_1,s_2) \in [0,+\infty) \times [0,+\infty)$ such that $\omega(s_1,s_2) \leq 1$ and $\omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|) = s_1\varphi^o(\widehat{\xi^*}) + s_2|\xi^*_{n+1}|$. Using (A.4) for $(s_1\varphi(\widehat{\xi}), s_2\operatorname{sign}(\xi^*_{n+1}))$ and (s_1,s_2) one has

$$\Phi(s_1\widehat{\xi}, s_2\operatorname{sign}(\xi^*_{n+1})) = \omega(s_1\varphi(\widehat{\xi}), s_2|\operatorname{sign}(\xi_{n+1})|) \leq \omega(s_1,s_2) \leq 1.$$

Thus,

$$\begin{aligned}\omega^o(\varphi^o(\widehat{\xi^*}), |\xi^*_{n+1}|) =& s_1\varphi^o(\widehat{\xi^*}) + s_2|\xi^*_{n+1}| = (s_1\widehat{\xi}) \cdot \widehat{\xi^*} + (s_2\operatorname{sign}(\xi^*_{n+1})) \cdot \xi^*_{n+1} \\ \leq& \Phi(s_1\widehat{\xi}, s_2\operatorname{sign}(\xi^*_{n+1}))\Phi^o(\widehat{\xi^*},\xi^*_{n+1}) \leq \Phi^o(\widehat{\xi^*},\xi^*_{n+1}).\end{aligned} \tag{A.9}$$

From (A.8)-(A.9) we get (A.6).

Remark A.6. The norm $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ is partially monotone if and only if Φ^o is partially monotone.

We give the proof of the following lemma which is used in the proof Theorem 4.6.

Lemma A.7. *Suppose that* $\Phi : \mathbb{R}^{n+1} \to [0,+\infty)$ *is a partially monotone norm,* $E,F \in BV_{\rm loc}(\widehat{\Omega} \times \mathbb{R})$ *such that for every* $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$

$$\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E,0) \leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F,0), \qquad \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0,D_t\chi_E) \leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0,D_t\chi_F). \tag{A.10}$$

Then for any $\widehat{A} \in \mathcal{A}_c(\widehat{\Omega})$ *we have*

$$\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E,D_t\chi_E) \leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F,D_t\chi_F). \tag{A.11}$$

Proof. We may assume that $\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, D_t\chi_F) < +\infty$. Then $\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, D_t\chi_E) < +\infty$. Indeed, since all norms in $\mathbb{R}^{n+1}$ are comparable, there exists $c, C > 0$ such that

$$c\Phi^o(\xi) \leq \Phi^o(\widehat{\xi}, 0) + \Phi^o(0, \xi_{n+1}) \leq C\Phi^o(\xi), \quad \xi \in \mathbb{R}^{n+1},$$

thus

$$c\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, D_t\chi_E) \leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_E, 0) + \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0, D_t\chi_E)$$

$$\leq \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, 0) + \int_{\widehat{A}\times\mathbb{R}} \Phi^o(0, D_t\chi_F) \leq C\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D_x\chi_F, D_t\chi_F).$$

By definition of Φ-perimeter and Proposition A.5, for any $\varepsilon > 0$ there exists $\eta \in C_c(\widehat{A} \times \mathbb{R}; B_\Phi)$ such that $\Phi(\eta) = \omega(\varphi(\widehat{\eta}), |\eta_{n+1}|) \leq 1$ and

$$\int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_E) - \varepsilon < -\int_{\widehat{A}\times\mathbb{R}} \operatorname{div} \eta dx = \int_{\widehat{A}\times\mathbb{R}} \eta \cdot D\chi_E. \tag{A.12}$$

Then from (A.10), (A.5), (A.7) and (A.6) we get

$$\int_{\widehat{A}\times\mathbb{R}} \eta \cdot D\chi_E = \int_{\widehat{A}\times\mathbb{R}} \left(\sum_{j=1}^{n} \eta_j \cdot D_{x_j}\chi_E + \eta_{n+1}D_t\chi_E\right)$$

$$\leq \int_{\widehat{A}\times\mathbb{R}} (\varphi(\widehat{\eta})\, d\varphi^o(D_x\chi_E) + |\eta_{n+1}|\, d|D_t\chi_E|) \leq \int_{\widehat{A}\times\mathbb{R}} (\varphi(\widehat{\eta})\, d\varphi^o(D_x\chi_F) + |\eta_{n+1}|\, d|D_t\chi_F|)$$

$$\leq \int_{\widehat{A}\times\mathbb{R}} \omega(\varphi(\widehat{\eta}), |\eta_{n+1}|)\, d\omega^o(\varphi^o(D_x\chi_F), |D_t\chi_F|) \leq \int_{\widehat{A}\times\mathbb{R}} \omega^o(\varphi^o(D_x\chi_F), |D_t\chi_F|))$$

$$= \int_{\widehat{A}\times\mathbb{R}} \Phi^o(D\chi_F).$$

This inequality, (A.12) and arbitrariness of ε yield (A.11).

□

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