Approximation to driven motion by crystalline curvature in two dimensions

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Abstract

We study the approximation of driven motion by crystalline curvature in two dimensions with a reaction-diffusion type differential inclusion. A quasi-optimal $\mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$ and an optimal $\mathcal{O}(\varepsilon^2)$ error bound between the original flow and the zero level set of the approximate solution are proved, for the regular and the double obstacle potential respectively. This result is valid before the onset of singularities, and applies when the driving force g does not depend on the space variable x. A comparison principle between crystalline flows and a notion of weak solution for crystalline evolutions, for suitable $g(x,t)$, are also obtained.

1 Introduction

The interest in anisotropic fronts evolutions is motivated by many physical examples where an interface propagation with preferred directions is evident [13], [14], [12], [10]. Anisotropic motion by mean curvature is also strictly related to the geometry of convex bodies [45], [41] and with the theory of Minkowskian and Finsler spaces [3], [33], [31], [8], [47].
When the anisotropy is described by a *smooth* norm ϕ on $\mathbb{R}^N$, a possible way to describe the geometric evolution problem is to consider the asymptotic limit, as $\varepsilon \to 0^+$, of the zero-level set of the solutions to the reaction-diffusion type equation

$$u_t - \operatorname{div}\Big(\phi^o(\nabla u)\phi^o_\xi(\nabla u)\Big) + \frac{1}{2\varepsilon^2}\Psi'(u_\varepsilon) - \frac{c_0}{2\varepsilon}g = 0 \quad \text{in } \mathbb{R}^N \times\,]0,T[. \tag{1}$$

Here ϕ^o is the dual norm of ϕ, $\phi^o_\xi := \nabla\phi^o$, the function Ψ is a double well potential, $c_0 > 0$ is a suitable constant and g is a given forcing term. Unless ϕ is Riemannian, the operator $\operatorname{div}\Big(\phi^o(\cdot)\phi^o_\xi(\cdot)\Big)$ is nonlinear. It is known [8], [20], [19] that the zero level set of the solution to (1) provides an approximation for the motion of an interface evolving with the law "velocity $= -(\kappa_\phi + g)$" along the anisotropic normal direction n_ϕ (the Cahn-Hoffmann vector), where κ_ϕ is the ϕ-mean curvature in the sense of [9]. In [5] is analyzed the case of an unequal double well potential Ψ. We refer to [1], [16], [18], [15], [21], for results concerning the asymptotic behaviour of the solutions to the classical Allen-Cahn equation (i.e. equation (1) for $\phi(\xi) = |\xi|$).

It is important to remark that equation (1) has a variational meaning, since it is the gradient flow of a functional which approximates the perimeter of the front, computed with respect to the distance induced by ϕ (see [9]).

As pointed out by Taylor [43], [44], [45], [46], the convex *non smooth* case plays a central rôle among all anisotropies. In terms of unit balls, this means that the Wulff shape $\mathcal{W}_\phi := \{\phi \leq 1\}$ is a convex body whose boundary may have nondifferentiability points and flat portions. The crystalline case corresponds to the situation in which $\mathcal{W}_\phi$ is completely faceted, i.e. is a convex polytope. This particular case presents interesting mathematical questions, especially in $N \geq 3$ dimensions. Recently, results on crystalline motion by curvature in two dimensions have been obtained, among others, in [39], [40], [29], [28], [42], [30], [38], [2], [23], [26], [24], [22], [25]; concerning the three dimensional case we refer to [27].

The aim of this paper is to define a reaction-diffusion approximation to the crystalline motion of fronts, and to rigorously prove an optimal interface error estimate, valid before the onset of singularities, for driven crystalline evolutions in two dimensions. Our optimal result is valid provided no new facets develop or disappear during the evolution, and if the forcing term g does not depend on space. We also discuss some aspects of the (considerably more difficult) space-dependent case $g = g(x,t)$. In particular, we prove a comparison result for two dimensional driven crystalline flows, which slightly extends a result of Giga and Gurtin [26]. Using this result, we define a weak solution for the driven crystalline motion by curvature by using the barriers method of De Giorgi [17], [6]. Uniqueness and comparison principle for this weak evolution are a direct consequence of its definition.

We remark that we consider, in the class of smooth crystalline flows, not only polygonal domains, but also domains with piecewise $\mathcal{C}^2$ curvilinear boundaries. This turns out to be useful especially when g depends on space, when a flat portion of the front does not remain flat during the evolution.

In the non smooth case, the map $T^o(\xi) := \frac{1}{2}\nabla(\phi^o(\xi)^2)$ is a maximal monotone possibly multivalued operator, and we propose to modify (1) into the differential inclusion

$$u_t - \text{div}\,\zeta + \frac{1}{2\varepsilon^2}\Psi'(u_\varepsilon) - \frac{c_0}{2\varepsilon}g = 0, \qquad \zeta(x,t) \in T^o(\nabla u(x,t)). \tag{2}$$

We prove that, if $N = 2$ and $g = g(t)$, the zero level set of the solutions to (2) approximates, with a quasi-optimal error estimate of order $\varepsilon^2|\log\varepsilon|^2$, the original evolution. The parallel result in the case of the double obstacle potential is given in Section 6.

The basic steps of the error estimate proof are the validity of a comparison lemma for (2) (holding in any space dimension and for $g = g(x,t)$) and the construction of suitable sub- and supersolutions to (2). This construction is not a straightforward extension of the one used in the smooth anisotropic case, and is based on a careful extension of the Cahn-Hoffmann vector field n_ϕ out of the front.

The outline of the paper is as follows. In Section 2 we give some notation and the definitions of ϕ-regular set (Definition 2.1) and ϕ-curvature (Definition 2.2) in the crystalline case. We make several remarks to motivate and clarify these two concepts. In Section 3 we define what is a ϕ-regular flow, and we prove short time existence of such a flow (see Theorem 3.1):

uniqueness of the flow (particularly for $g = g(x,t)$) is not a direct consequence of the proof of Theorem 3.1, but follows a posteriori by Theorems 5.1 and 5.2. In Section 4 we introduce the reaction-diffusion inclusion (2) and we prove (Lemma 4.1) the comparison result for sub- and supersolutions to (2). Theorem 4.1 concerns existence and uniqueness of solutions to (2). In Section 4 no restriction on the dimension is required. Also, g is allowed to depend on space and the assumption that ϕ is crystalline is irrelevant. In Section 5 we prove the quasi-optimal error estimate for the smooth potential and for $g = g(t)$ (see Theorem 5.1). As a consequence, a comparison result between crystalline flows is proven (see Theorem 5.2). Since Theorem 5.1 does not cover the space dependent case, the proof of the comparison for suitable $g(x,t)$ is sketched in the Appendix, under the no-fracture condition (15). Based on Theorem 5.2, in Definition 5.2 we introduce the weak driven crystalline evolutions. The double obstacle formulation is discussed in Section 6.

The extension of the error estimate to the more difficult cases in which g depends on x and $N \geq 3$ will be object of further investigation.

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2 Setting

In what follows $\Omega \subset \mathbb{R}^2$ is a bounded convex open set with smooth boundary. We denote by $\cdot$ the euclidean scalar product in $\mathbb{R}^2$, by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure, and by d_H the euclidean Hausdorff distance between subsets of $\mathbb{R}^2$.

We indicate by $\phi : \mathbb{R}^2 \to [0, +\infty[$ a convex function satisfying the properties

$$\phi(a\xi) = a\phi(\xi), \qquad \xi \in \mathbb{R}^2, \ a \geq 0, \tag{3}$$

$$\lambda|\xi| \leq \phi(\xi) \leq \Lambda|\xi|, \qquad \xi \in \mathbb{R}^2, \tag{4}$$

for two suitable positive constants $0 < \lambda \leq \Lambda < +\infty$. We do not assume that ϕ is even.

The dual function $\phi^o : \mathbb{R}^2 \to [0, +\infty[$ of ϕ is defined [37] by

$$\phi^o(\xi^*) := \sup\{\xi^* \cdot \xi \ : \ \phi(\xi) \leq 1\}, \qquad \xi^* \in \mathbb{R}^2.$$

Notice that ϕ^o is convex, satisfies properties (3) and (4), and $\phi^{oo} = \phi$.

We set

$$\mathcal{F}_\phi := \{\xi^* \in \mathbb{R}^2 : \phi^o(\xi^*) \leq 1\}, \qquad \mathcal{W}_\phi := \{\xi \in \mathbb{R}^2 : \phi(\xi) \leq 1\}.$$

$\mathcal{F}_\phi$ is usually called the Frank diagram and $\mathcal{W}_\phi$ the Wulff shape; they are two convex sets whose interior parts contain the origin.

We say that ϕ is crystalline if $\mathcal{W}_\phi$ is a (convex) polygon. If ϕ is crystalline then also $\mathcal{F}_\phi$ is a (convex) polygon. In this case the edges of $\mathcal{W}_\phi$ and $\mathcal{F}_\phi$ will be called facets; $\mathcal{F}_\phi$ is the convex hull of the normal vectors to the facets of $\mathcal{W}_\phi$ (normalized to have $\phi^o = 1$), see [41, Lemma 2.4.5].

Unless otherwise specified, from now on we assume that ϕ is crystalline. Given $E \subset \mathbb{R}^2$, we set

$$\mathrm{dist}_\phi(x, E) := \inf_{y \in E} \phi(x - y), \qquad \mathrm{dist}_\phi(E, x) := \inf_{y \in E} \phi(y - x), \qquad x \in \mathbb{R}^2,$$

and we let d_ϕ^E be the oriented ϕ-distance function from ∂E negative inside E, i.e.

$$d_\phi^E(x) := \mathrm{dist}_\phi(x, E) - \mathrm{dist}_\phi(\mathbb{R}^2 \setminus E, x).$$

Reasoning as in [9] one can prove that

$$\phi^o(\nabla d_\phi^E) = 1 \tag{5}$$

at each point where d_ϕ^E is differentiable. We let also $\nu_E : \partial E \to \mathbb{R}^2$ be the usual outward euclidean unit normal to ∂E (when it exists); we have

$$\nabla d_\phi^E(x) = \frac{\nu_E(x)}{\phi^o(\nu_E(x))}, \qquad x \in \partial E.$$

If $t \in [a,b] \to E(t) \subset \mathbb{R}^2$ is a parametrized family of subsets of $\mathbb{R}^2$, we let

$$d_\phi^{E(t)}(x) := \mathrm{dist}_\phi(x, E(t)) - \mathrm{dist}_\phi(\mathbb{R}^2 \setminus E(t), x).$$

From now on, the symbols E and $E(t)$ will denote always closed sets with compact boundary, such that $\partial E, \partial E(t) \subset \Omega$.
Let $T^o : \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2)$ be the map defined by

$$T^o(\xi^*) := \frac{1}{2}\partial(\phi^o(\xi^*))^2, \qquad \xi^* \in \mathbb{R}^2,$$

where $\mathcal{P}(\mathbb{R}^2)$ is the class of all subsets of $\mathbb{R}^2$ and ∂ denotes the subdifferential. T^o is usually called the duality mapping (see [4]); it is a maximal monotone multivalued operator, hence

$$(\xi_1^* - \xi_2^*) \cdot (\eta_1 - \eta_2) \geq 0, \qquad \xi_1^*, \xi_2^* \in \mathbb{R}^2,\ \eta_1 \in T^o(\xi_1^*),\ \eta_2 \in T^o(\xi_2^*); \tag{6}$$

moreover

$$T^o(a\xi^*) = aT^o(\xi^*), \qquad \xi^* \in \mathbb{R}^2,\ a \geq 0. \tag{7}$$

Notice that

$$T^o(\mathcal{F}_\phi) = \mathcal{W}_\phi.$$

One can show that

$$\xi^* \cdot \xi = \phi^o(\xi^*)^2 = \phi(\xi)^2, \qquad \xi^* \in \mathbb{R}^2,\ \xi \in T^o(\xi^*), \tag{8}$$

and

$$\xi^* \cdot \xi \geq \lambda^2 |\xi|^2, \qquad \xi^* \in \mathbb{R}^2,\ \xi \in T^o(\xi^*).$$

When $\mathcal{F}_\phi$ and $\mathcal{W}_\phi$ are symmetric and smooth, we can define the so-called Cahn-Hoffmann vector n_ϕ as follows. Let E be of class $\mathcal{C}^2$; then $n_\phi := T^o(\nabla d_\phi^E)$ on ∂E (see [8]). In the crystalline case the pointwise definition of n_ϕ is more delicate. It turns out to be convenient also to redefine what we mean by a smooth boundary with respect to ϕ. In Definition 2.1 we give the notion of ϕ-regular set: a ϕ regular set can have a polygonal boundary (with edges suitably oriented) but also a boundary with a finite number of arcs of class $\mathcal{C}^2$: this enlargement of the class of "admissible" domains is necessary if the driving force g depends on space, when a flat portion of the boundary does not remain flat during the evolution.

Definition 2.1. *A ϕ-regular set is a pair (E, n_ϕ) which satisfies the following properties:*

(i) $\partial E \subset \Omega$ *is a simple continuous closed curve, which is union of a finite number* $F_1, \ldots, F_m$ *of* $\mathcal{C}^2$ *closed arcs;*

(ii) the vector field $n_\phi : \partial E \to \mathbb{R}^2$ *is continuous,* $n_{\phi|F_i}$ *is of class* $\mathcal{C}^1$ *for any* $i = 1, \ldots, m$, *and*

$$\phi(n_\phi(x)) = 1, \quad x \in \partial E; \tag{9}$$

(iii) there holds

$$n_\phi(x) \in T^o\left(\nabla d_\phi^E(x)\right), \qquad x \in \bigcup_{i=1}^m \mathrm{int}(F_i), \tag{10}$$

where $\mathrm{int}(F_i)$ *stands for the relative interior of* F_i*;*

(iv) there exists $\delta > 0$ *such that* $|\nu_E(x) - \nu_E(y)| > \delta$ *for* $x \in \mathrm{int}(F_i)$, $y \in \mathrm{int}(F_j)$, $j = i \pm 1$, $i = 1, \ldots, m$ *(we set* $F_{m+1} := F_1$ *and* $F_0 := F_m$*);*

(v) let $i \in \{1, \ldots, m\}$*; if there exists* $x \in \mathrm{int}(F_i)$ *such that* $\nabla d_\phi^E(x)$ *is a vertex of* $\mathcal{F}_\phi$*, then* $\nu_E(\cdot)$ *is constant on* F_i *(i.e.* F_i *is flat).*

If a set E *satisfies (i), (iv), (v) and there exists a vector field* n_ϕ *such that* (E, n_ϕ) *verifies (ii), (iii), we shortly say that* E *is* ϕ*-regular. The arcs of* ∂E *which are straight segments will be called edges, and the endpoints of* $F_1, \ldots, F_m$ *will be called vertices of* ∂E*.*

Conditions (ii)-(iii) are crucial requirements on the vector field n_ϕ. Conditions (iv)-(v) are technical requirements on ∂E, which probably could be weakened: they are however needed to prove the desired $O(\varepsilon^2 |\log \varepsilon|^2)$ interface error estimate of Theorem 5.1.
In general the same set E may admit more than one vector field n_ϕ so that it becomes ϕ-regular. However, as we shall see at the end of this section, there will be a natural choice of a special vector field.
The first example of ϕ-regular set is given by $\mathcal{W}_\phi$, coupled with the vector field $x/\phi(x)$.
Notice that, if (E, n_ϕ) is ϕ-regular, then by (10), (9) and (8)

$$\nabla d_\phi^E(x) \cdot n_\phi(x) = 1, \qquad x \in \bigcup_{i=1}^m \mathrm{int}(F_i).$$

Definition 2.2. *Let* (E, n_ϕ) *be a* ϕ*-regular set and let* $F_1, \ldots, F_m$ *be the arcs of* ∂E*. We define the* ϕ*-curvature* κ_ϕ *of* ∂E *at each point* $x \in \bigcup_{i=1}^m \mathrm{int}(F_i)$ *as*

$$\kappa_\phi(x) := \mathrm{div}_\tau \, n_\phi(x),$$

where div_τ *denotes the tangential divergence.*

In the case $(\mathcal{W}_\phi, x/\phi(x))$, it turns out that $\kappa_\phi(x) \equiv 1$. Note that the ϕ-curvature of ∂E depens on the choice of n_ϕ. Observe also that if F is an edge of ∂E parallel to some facets W of $\mathcal{W}_\phi$ and with the same exterior euclidean normal vector, and if $x \in \mathrm{int}(F)$, then $T^o(\nabla d_\phi^E(x)) = W$ and is independent of $x \in \mathrm{int}(F)$.
The following remark shows that the class of ϕ-regular sets contains suitable polygonal domains.

Remark 2.1. *Let ∂E be a closed simple polygonal curve and let $F_1, \dots, F_m$ be the edges of ∂E. Assume that $\nabla d^E_{\phi\,|\mathrm{int}(F_i)}$ is a vertex of $\mathcal{F}_\phi$ (this condition implies that F_i is parallel to some facet of $\mathcal{W}_\phi$) and that $\nabla d^E_{\phi\,|\mathrm{int}(F_i)}$ and $\nabla d^E_{\phi\,|\mathrm{int}(F_{i+1})}$ are consecutive vertices of $\mathcal{F}_\phi$, for any $i = 1, \dots, m$. Then E is ϕ-regular.*

The following remark shows that, at any vertex v of a ϕ-regular set E, there is a uniquely determined vector, which is the natural choice of the Cahn-Hoffmann vector at v.

Remark 2.2. *Let (E, n_ϕ) be a ϕ-regular set and let v be a vertex of ∂E. Let us consider the set*

$$n_v := \bigcap T^o\big(\nabla d_\phi^E(x_i)\big), \tag{11}$$

where the intersection is taken over all arcs of ∂E meeting at v, and where x_i belongs to interior of such arcs. Then n_v is a singleton and $\phi(n_v) = 1$.

Let E be a ϕ-regular set. The arcs of ∂E can be divided into two groups $\mathcal{G}_1, \mathcal{G}_2$: the group $\mathcal{G}_1$ consists of the edges which are parallel to some facet of $\mathcal{W}_\phi$ and having the same exterior euclidean normal vector (in this case we say that the edge corresponds to some facet of $\mathcal{W}_\phi$); $\mathcal{G}_2$ consists of the remaining arcs (some of which can be flat). From Definition 2.1 it follows that for any facet W of $\mathcal{W}_\phi$, there exists an edge of ∂E corresponding to W. Moreover, for an arc F of ∂E in $\mathcal{G}_2$, from Definition 2.1 it follows that there is only one possible choice of the Cahn-Hoffman vector n_ϕ in F, given by $T^o(\nu_E)/\phi^o(\nu_E)$ (which is constant on F), hence F has zero ϕ-curvature. For an edge F_i of $\mathcal{G}_1$, there exists a special choice of n_ϕ on F_i such that F_i has constant ϕ-curvature. If P_i and Q_i are the endpoints of F_i, this choice corresponds to take the convex combination of $n_\phi(P_i)$ and $n_\phi(Q_i)$ (which are uniquely defined, recall (11)), i.e.

$$n_\phi(x) = n_\phi(P_i) + \delta_i \frac{l_i}{L_i}(x - P_i), \qquad x \in F_i, \tag{12}$$

where L_i is the length of F_i, l_i is the length of the facet of $\mathcal{W}_\phi$ corresponding to F_i, and the quantity $\delta_i \in \{0, \pm 1\}$ is defined as follows: $\delta_i = 1$ (resp. $\delta_i = -1$) if the tern of consecutive arcs (F_{i-1}, F_i, F_{i+1}) is a convex (resp. concave) tern, $\delta_i = 0$ otherwise. With this choice of n_ϕ the curvature κ_ϕ is constant on any edge of ∂E, more precisely $\kappa_\phi = (\delta_i l_i)/L_i$ on F_i. Notice that if F_i is an arc of $\mathcal{G}_2$, we can consistently define $\delta_i = 0$.
In the following we shall always consider ϕ-regular sets E equipped with the uniquely determined choice of the Cahn-Hoffman vector n_ϕ described above. The notation δ_i, l_i, L_i will be systematically used in the paper.

3 Geometric evolution law

Throughout the paper, we will specify time by time if the function g depends on space or not. In the general case, we assume that g is continuous on $\Omega \times [0, +\infty[$. A severe restriction on g, assumed in Sections 5, 6 and in the Appendix, is condition (15) below.
We now define the notion of ϕ-regular flow as a ϕ-regular evolution of a boundary moving with velocity, in the n_ϕ-direction, equal to $-(\kappa_\phi + g)$.

Definition 3.1. *Let $T > 0$; a ϕ-regular flow on $[0,T]$ is a family of pairs $(E(t), n_\phi(\cdot,t))$, $t \in [0,T]$, which satisfies the following properties:*

(1) $(E(t), n_\phi(\cdot,t))$ is a ϕ-regular set for any $t \in [0,T]$;

(2) neither a new arc of $\partial E(t)$ can develop nor an arc can disappear during the evolution, hence $\partial E(t)$ consists of the same number m of arcs for any $t \in [0,T]$;

(3) for any $x \in \bigcup_{t\in[0,T]} \partial E(t)$ the function $d_\phi^{E(t)}(x)$ is differentiable in t for any $t \in [0,T]$ and the following equation holds:

$$\frac{\partial d_\phi^{E(t)}}{\partial t}(x) = div_\tau \ n_\phi(x,t) + g(x,t), \qquad t \in [0,T], \ x \in \bigcup_{i=1}^{m} \mathrm{int}(F_i(t)), \tag{13}$$

where $F_1(t), \dots, F_m(t)$ are the arcs of $\partial E(t)$.

In the following, we sometimes denote by $E(t)$ a ϕ-regular flow.
Condition (2) could be weakened (or even dropped); however, using (2), we are able to prove the $O(\varepsilon^2 |\log \varepsilon|^2)$ interface error estimate of Theorem 5.1. In Theorem 7.2 we shall see that, under condition (15) below, the flow $E(t)$ does not depend on the choice of $n_\phi(\cdot,t)$, but only on $E(0)$.
Now, we want to derive some necessary conditions on g in order to guarantee the existence of a ϕ-regular flow.
Let $(E(t), n_\phi(\cdot,t))$ be a ϕ-regular flow on $[0,T]$. From equation (13) and condition (2) of Definition 3.1, it follows that $\mathrm{div}_\tau n_\phi(x,t) + g(x,t)$ must be constant on any edge of $\partial E(t)$ corresponding to some facet of $\mathcal{W}_\phi$. Let $F_i(t)$ be such an edge and let $P_i(t)$, $Q_i(t)$ be its endpoints. Set $\overline{g}_i(t) := \int_{F_i(t)} g(z,t)\, d\mathcal{H}^1(z)/L_i(t)$. For any $x \in \mathrm{int}(F_i(t))$, we have

$$\begin{aligned} \kappa_\phi(x,t) &= \delta_i \frac{l_i}{L_i(t)} + \overline{g}_i(t) - g(x,t), \\ n_\phi(x,t) &= n_\phi(P_i(t)) + \tau_i(t) \int_{P_i(t)}^{x} \kappa_\phi(z,t)\, d\mathcal{H}^1(z), \end{aligned} \tag{14}$$

where $\tau_i(t) := \frac{Q_i(t)-P_i(t)}{L_i(t)}$, $\int_{P_i(t)}^{x}$ stands for the integration on the segment with endpoints $P_i(t)$ and x, and δ_i, l_i, $L_i(t)$ are introduced after formula (12). Using (14), the condition that n_ϕ belongs to the convex hull of $n_\phi(P_i(t))$ and $n_\phi(Q_i(t))$ reads as follows:

$$\begin{aligned} &-\frac{l_i|x-P_i(t)|}{L_i(t)} \le \int_{P_i(t)}^{x} \delta_i(\overline{g}_i(t) - g(z,t))\, d\mathcal{H}^1(z) \le l_i - \frac{l_i|x-P_i(t)|}{L_i(t)} && \text{if } \delta_i \in \{\pm 1\}, \\ &g(z,t) = \overline{g}_i(t) \quad \forall z \in F_i(t) && \text{if } \delta_i = 0. \end{aligned} \tag{15}$$

Recall that conditions (15) are assumed on the edges belonging to $\mathcal{G}_1$. Notice that, if $F_i(t) \in \mathcal{G}_1$, at one of its endpoints, say $P_i(t)$, we deduce from (15) that

$$-l_i/L_i(t) \leq \delta_i \overline{g}_i(t) - \delta_i g(P_i(t)). \tag{16}$$

The following result shows that, under suitable conditions on $g(x,t)$, there exists a ϕ-regular flow.

Theorem 3.1. *Let (E, n_ϕ) be a ϕ-regular set. Assume that the function $g(x,t)$ satisfies conditions (15) at $t = 0$ (on each edge of ∂E corresponding to some facet of $\mathcal{W}_\phi$) with $\leq$ replaced by $<$. Then there exist $T > 0$ and a ϕ-regular flow $(E(t), n_\phi(\cdot, t))$ on $[0,T]$, such that $(E(0), n_\phi(\cdot, 0)) = (E, n_\phi(\cdot))$.*

Proof. Let $F_1, \dots, F_m$ be the arcs of ∂E. We first consider arcs of $\mathcal{G}_2$; let F_i be such an arc. We define γ_{F_i} as the $\mathcal{C}^{1,1}$ curve whose trace is composed by F_i and the two half straight lines tangent to F_i at its endpoints. Let us now flow γ_{F_i} under equation (13). It admits a unique local in time solution $\gamma_{F_i}(t)$ defined on $[0, T_1]$ (see [32]), for some $T_1 > 0$. We can assume that T_1 is independent of F_i varying among the arcs of ∂E in $\mathcal{G}_1$. Let $\gamma(t)$ be the vector composed of such $\gamma_{F_i}(t)$.
Let now $F_1, \dots, F_k$ be the edges of ∂E corresponding to facets of $\mathcal{W}_\phi$ (having lengths $l_1, \dots, l_k$) and let $\sigma(t) := (\sigma_1(t), \dots, \sigma_k(t))$, $t \in [0,T]$, be the solution to the following system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma_i(t) = H(\sigma(t), \gamma(t)) := -\big(\delta_i \dfrac{l_i}{L_i(t)} + \overline{g}_i(t)\big), \\ \\ \sigma(0) = 0, \end{cases} \tag{17}$$

where $F_i(t)$ are segments such that

$$\begin{cases} \inf\ \big\{\phi(x-y):\ x \in \gamma_{F_i}(t),\ y \in \gamma_{F_i}\big\} = \sigma_i(t), & \text{if } \sigma_i(t) \geq 0, \\ \\ \inf\ \big\{\phi(y-x):\ x \in \gamma_{F_i}(t),\ y \in \gamma_{F_i}\big\} = -\sigma_i(t), & \text{if } \sigma_i(t) < 0 \end{cases}$$

and $L_i(t)$ is the length of $F_i(t)$.
Observe that $E(t)$, the set enclosed by all $\gamma_{F_i}(t)$ and $F_l(t)$, is uniquely determined by $\sigma(t)$ and $\gamma(t)$.
Notice also that, since the function H is Lipschitz continuous in σ in a neighbourhood of $\sigma(0) = 0$, equation (17) admits a solution for a suitable $T \in]0, T_1]$.
We have to show that $E(t)$ is ϕ-regular. Possibly reducing T we can assume that conditions (15) and conditions (iv) and (v) of Definition 2.1 hold on $\partial E(t)$ for any $t \in [0,T]$. Then, if we define $n_\phi(x,t)$ as in (14), it is easy to check that $(E(t), n_\phi(\cdot, t))$ is a ϕ-regular flow starting from $(E, n_\phi(\cdot))$. $\square$

Summing up, if $E(t)$ is a ϕ-regular flow, the velocity of an edge $F_i(t)$ of $\partial E(t)$, with $F_i(t) \in \mathcal{G}_1$, is $-(\kappa^i_\phi(t) + \overline{g}_i(t)) n_\phi(x,t)$, where $\kappa^i_\phi(t) := \delta_i l_i / L_i(t)$ and $\overline{g}_i(t) := \int_{F_i(t)} g(z,t)\ d\mathcal{H}^1(z)/L_i(t)$, while the velocity of a point x belonging to an arc $F(t) \in \mathcal{G}_2$, is simply $-g(x,t)n_\phi(x,t)$.

Remark 3.1. *In principle, the evolution constructed in the proof of Theorem 3.1 may depend on the extension γ_{F_i}. Actually, the evolution is unique, see Theorem 7.2.*

Remark 3.2. *It is not difficult to check that conditions (15) are equivalent to requiring that no new arcs create during the evolution (no-fracture condition).*

4 Relaxed evolution law

The results of this section are valid in any space dimension $N \geq 2$. The extension of the required definitions of Section 2 to the N-dimensional case are obvious. The function g is here assumed to belong to $H^1(0,T;L^2(\Omega))$; also the restriction on ϕ to be crystalline can be dropped, thus assuming on ϕ only convexity and conditions (3) and (4) in $\mathbb{R}^N$.
The double well potential $\Psi : \mathbb{R} \to [0, +\infty[$ is, as usual, an even function of class $\mathcal{C}^2$ having only two zeroes at $\{-1,1\}$, say $\Psi(s) = (1-s^2)^2$. We set $\psi := \Psi'/2$, and

$$\alpha := \psi'(\pm 1), \qquad \beta := \psi''(1) = -\psi''(-1).$$

We also define the functional $\mathcal{E}_\phi$ as

$$\mathcal{E}_\phi(v) := \int_\Omega (\phi^o(\nabla v))^2 + \Psi(v)\ dx, \qquad v \in H^1(\Omega).$$

We denote by γ the unique minimizer of the functional $v \to \int_{\mathbb{R}} |v'|^2 + \Psi(v)dx$, over all $v \in H^1_{\rm loc}(\mathbb{R})$ such that $v(\pm\infty) = \pm 1$ and $v(0) = 0$; γ is a smooth strictly increasing function exponentially asymptotic, at $\pm\infty$, to the two stable zeroes ± 1 of ψ, and

$$-\gamma'' + \psi(\gamma) = 0. \tag{18}$$

We denote by $\mathcal{L} : H^2_{\rm loc}(\mathbb{R}) \to L^2_{\rm loc}(\mathbb{R})$ the selfadjoint operator obtained by linearization of (18) around γ, i.e.

$$\mathcal{L}\zeta := -\zeta'' + \psi'(\gamma)\zeta.$$

We set

$$c_0 := \int_{\mathbb{R}} (\gamma')^2\ dy.$$

We recall the definitions of the shape functions $\eta, \chi \in H^2_{\rm loc}(\mathbb{R})$ and $\omega \in H^2(\mathbb{R})$, which have been introduced and studied in [7]. They solve the problems

$$\mathcal{L}\eta = -\gamma' + \frac{c_0}{2}, \qquad \mathcal{L}\chi = -\eta' - \frac{1}{2}\psi''(\gamma)\eta^2, \qquad \mathcal{L}\omega = -y\gamma'.$$

Since the right hand side (say f) satisfy the orthogonality condition $\int_{\mathbb{R}} f\gamma' dy = 0$, any such solution (say v) exists and is unique if we further require $v(0) = 0$ and a polynomial growth at infinity [7]. Moreover η is even, χ, ω are odd, $\lim_{y\to\pm\infty} \eta(y) = c_0/(2\alpha) =: \eta_\infty$, $\lim_{y\to\pm\infty} \chi(y) = \mp c_0^2\beta/(8\alpha^3) =: \pm\chi_\infty$,

$$|\eta - \eta_\infty|, |\eta'| \leq C(1+|y|)\gamma', \quad |\chi - \chi_\infty|,\ |\chi'|, |\omega|, |\omega'| \leq C(1+|y|^2)\gamma', \quad y \in \mathbb{R}, \tag{19}$$

where C is an absolute positive constant.

Let us now introduce the relaxed evolution problems. Let be given $\varepsilon > 0$, $T > 0$, and u_0 such that $\mathcal{E}_\phi(u_0) < +\infty$. Let us consider the problem

$$\begin{cases} u_t - \operatorname{div}(T^o(\nabla u)) + \dfrac{1}{\varepsilon^2}\psi(u) \ni \dfrac{c_0}{2\varepsilon} g & \text{in } Q, \\ \\ u(\cdot,0) = u_0(\cdot) & \text{in } \Omega, \\ \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0 & \text{in } \partial\Omega\times\,]0,T[, \end{cases} \tag{20}$$

where ν_Ω is the outward unit normal to $\partial\Omega$ and $Q := \Omega\times\,]0,T[$. The notion of variational sub- and supersolution of (20) reads as follows.

Definition 4.1. *A couple (u,ζ) is a subsolution of (20) if, for any $T > 0$, the following properties hold:*

(i) $u \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and $\zeta \in (L^2(Q))^N$;

(ii) for any $\varphi \in H^1(\Omega;[0,+\infty[)$ and a.e. $t \in\,]0,T[$ there holds

$$\int_\Omega \left(u_t\varphi + \zeta\cdot\nabla\varphi + \frac{1}{\varepsilon^2}\psi(u)\varphi - \frac{c_0}{2\varepsilon} g\varphi \right) dx \le 0; \tag{21}$$

(iii) for a.e. $x \in \Omega$ there holds $u(x,0) \le u_0(x)$;

(iv) for a.e. $(x,t) \in Q$ there holds

$$\zeta(x,t) \in T^o(\nabla u(x,t)). \tag{22}$$

The couple (u,ζ) is a supersolution of (20) if (i) and (iv) hold, and conditions (ii) and (iii) hold with $\ge$ in place of $\le$. The couple (u,ζ) is a solution of (20) if it is both a subsolution and a supersolution.

Notice that by (i), (22) and (4), we have $\zeta \in L^\infty\big(0,T;(L^2(\Omega))^N\big)$.
The following elementary comparison lemma is crucial for proving the main results.

Lemma 4.1. *Let (u_1,ζ_1) and (u_2,ζ_2) be respectively a subsolution and a supersolution of (20). Then*

$$u_1 \le u_2 \qquad \text{a.e. in } Q.$$

Proof. Define $e := u_1 - u_2$ and $e^+ := \max(e,0) \in L^\infty(0,T;H^1(\Omega;[0,+\infty[))$. Denote for simplicity by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$. Fix $\tau \in\,]0,T[$ such that (21) holds for u_1 and with $\ge$ for u_2, and $e^+(\cdot,\tau) \in H^1(\Omega)$. Choosing $\varphi(\cdot) = e^+(\cdot,\tau)$, subtracting the two resulting inequalities and recalling that $\psi = \psi_m + \psi_l$ where ψ_m is monotone and ψ_l is Lipschitz continuous with Lipschitz constant L, we have

$$
\begin{aligned}
(e_t, e^+) + &(\zeta_1 - \zeta_2, \nabla e^+) + \frac{1}{\varepsilon^2}(\psi_i(u_1) - \psi_i(u_2), e^+) + \frac{1}{\varepsilon^2}(\psi_l(u_1) - \psi_l(u_2), e^+) \\
&=: I + II + III + IV \leq 0.
\end{aligned}
$$

For a.e. $t \in]0,T[$, let us integrate with respect to $\tau \in]0,t[$, and let us analyze separately each resulting term. We have

$$
\begin{aligned}
\int_0^t I \, d\tau =& \frac{1}{2}\int_0^t \frac{d}{d\tau}\int_\Omega (e^+)^2 \, dx \, d\tau \\
&= \frac{1}{2}\Big(\|e^+(\cdot,t)\|^2_{L^2(\Omega)} - \|e^+(\cdot,0)\|^2_{L^2(\Omega)}\Big) = \frac{1}{2}\|e^+(\cdot,t)\|^2_{L^2(\Omega)},
\end{aligned}
$$

as $e^+(\cdot,0) = 0$. Concerning II, as T^o is a monotone operator, by (22) and (6) we have

$$
\int_0^t II \, d\tau = \int_0^t \int_{\{e^+>0\}} (\zeta_1 - \zeta_2) \cdot (\nabla u_1 - \nabla u_2) \, dx \, d\tau \geq 0,
$$

and, as ψ_i is monotone,

$$
\int_0^t III \, d\tau = \frac{1}{\varepsilon^2}\int_0^t\int_{\{e^+>0\}} (\psi_m(u_1) - \psi_m(u_2))(u_1 - u_2) \, d\tau \geq 0.
$$

Therefore

$$
\frac{1}{2}\|e^+(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t IV \, d\tau \leq 0,
$$

which, combined with the fact that ψ_l is L-Lipschitz, gives

$$
\begin{aligned}
\|e^+(\cdot,t)\|^2_{L^2(\Omega)} \leq 2\int_0^t |IV| \, d\tau &\leq \frac{2}{\varepsilon^2}\int_0^t\int_\Omega |\psi_l(u_1) - \psi_l(u_2)| e^+ \, dx \, d\tau \\
&\leq \frac{2L}{\varepsilon^2}\int_0^t\int_\Omega |e^+|^2 \, dx d\tau = \frac{2L}{\varepsilon^2}\int_0^t \|e^+(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau.
\end{aligned}
$$

Applying Gronwall's Lemma, we have $\|e^+(\cdot,t)\|_{L^2(\Omega)} = 0$, and thus $e^+ = 0$ a.e. in Q. □

Notice that to prove Lemma 4.1 no relations between ζ_1 and ζ_2 are required.
The existence of a solution to (20) can be proved by using the methods of [11], [48]. Uniqueness of the solution is a consequence of Lemma 4.1. We thus have the following result.

Theorem 4.1. *Problem (20) admits a solution (u,ζ). Moreover, if (u_1,ζ_1) and (u_2,ζ_2) are two solutions of (20), then $u_1 = u_2$ a.e. in Q.*

One can prove [48] that, if there exists $\zeta_0 \in T^o(\nabla u_0)$ such that $\mathrm{div}\zeta_0 \in L^2(\Omega)$, then the solution (u,ζ) to (20) is such that $u \in W^{1,\infty}(0,T;L^2(\Omega))$ and $\mathcal{E}_\phi(u) \in W^{1,\infty}(0,T)$. If u is bounded on Q then one gets also $\mathrm{div}\zeta \in L^\infty(0,T;L^2(\Omega))$.

Remark 4.1. *Assume that $N = 2$ and that ϕ is crystalline. We do not know whether the solution u belongs to $C^0(Q)$. This regularity result is however irrelevant in the proof of Theorem 5.1.*

5 Main results

All results of this section are valid in $N=2$ dimensions and for a crystalline ϕ. The proof of the following theorem is not a straightforward modification of the proof for motion by curvature with a smooth homogeneous anisotropy (see [5]).

Theorem 5.1. *Assume that $(E(t), n_\phi(\cdot,t))$ is a $\phi-$regular flow on $[0,T]$, with $g=g(t)$. For any $\varepsilon>0$ let u_ε be the solution to problem (20) with initial datum as in (32) below. Let $\Sigma_\varepsilon(t) := \{x\in\Omega : u_\varepsilon(x,t)=0\}$. Then there exist $\varepsilon_0 \in]0,1]$ and a constant C depending on $\partial E(0)$, g, T, and independent of $\varepsilon \in]0,\varepsilon_0]$, such that for all $\varepsilon\in]0,\varepsilon_0]$ the following quasi-optimal interface error estimate holds:*

$$\sup_{t\in[0,T]} d_H(\Sigma_\varepsilon(t), \partial E(t)) \le C\varepsilon^2 |\log\varepsilon|^2. \tag{23}$$

A comment is in order about the above definition of $\Sigma_\varepsilon(t)$. Since we do not know if u_ε is continuous, $\Sigma_\varepsilon(t)$ must be intended as follows. The set $\{x\in\Omega : u_\varepsilon(x,t)=0\}$ is defined as the complement in Ω of $\{x : u_\varepsilon(x,t)>0\}^* \cup \{x : u_\varepsilon(x,t)<0\}^*$, where, for any set $C\subseteq\Omega$, we define $C^* := \{x\in\Omega : \exists\rho>0 : |B_\rho(x)\setminus C| = 0\}$, and $B_\rho(x)$ denotes the euclidean ball of radius ρ centered at x.

Proof. For any $t\in[0,T]$ denote by $v_1(t),\dots,v_m(t)$ the vertices of $\partial E(t)$, set $v_0(t) := v_m(t)$, $v_{m+1}(t) := v_1(t)$, and let $F_i(t)$ be the arc of $\partial E(t)$ having endpoints $v_{i-1}(t), v_i(t)$ for $i = 1,\dots,m$. Let $\gamma_{F_i}(t)$ be the $\mathcal{C}^{1,1}$ curve whose trace contains $F_i(t)$ and the two half straight lines tangent to $F_i(t)$ at $v_{i-1}(t)$ and $v_i(t)$. Denote by d_i the oriented ϕ-distance function from $\gamma_{F_i}(t)$ negative in the half plane which, in a neighbourhood of $F_i(t)$, contains $E(t)$; we set $d_\phi(x,t) := d_\phi^{E(t)}(x)$, and

$$\begin{aligned} y = y(x,t) &:= \frac{d_\phi(x,t)}{\varepsilon}, \quad y_\epsilon = y_\epsilon(x,t) := y(x,t) - \theta(t)\varepsilon|\log\varepsilon|^2, \\ y^i = y^i(x,t) &:= \frac{d_i(x,t)}{\varepsilon}, \quad y^i_\epsilon = y^i_\epsilon(x,t) := y^i(x,t) - \theta(t)\varepsilon|\log\varepsilon|^2, \end{aligned} \tag{24}$$

where

$$\theta(t) := c\exp\big(1 + t\max_{\tau\in[0,T]} \|\kappa_\phi^2\|_{L^\infty(\partial E(\tau))}\big), \qquad t\in[0,T],$$

and $c>0$ is a constant large enough to be defined later on independently of ε (see *Step 6*). Let $\delta\ge 3$ be a fixed natural number such that, if for any $\varepsilon\in]0,1]$ we let $z_\varepsilon := \delta|\log\varepsilon|$, then $\gamma(\pm z_\varepsilon) = \pm 1 + O(\varepsilon^{2\delta})$, $\gamma'(\pm z_\varepsilon) = O(\varepsilon^{2\delta})$, and

$$\begin{aligned} &|\eta(\pm z_\varepsilon) - \eta_\infty|, |\eta'(\pm z_\varepsilon)| = |\log\varepsilon| O(\varepsilon^{2\delta}), \\ &|\omega(\pm z_\varepsilon)|, |\omega'(\pm z_\varepsilon)|, |\chi(\pm z_\varepsilon) - \chi_\infty|, |\chi'(\pm z_\varepsilon)| = |\log\varepsilon|^2 O(\varepsilon^{2\delta}). \end{aligned}$$

Arguing as in [7], we can construct four functions

$$\gamma_\epsilon, \eta_\epsilon, \omega_\varepsilon, \chi_\varepsilon \in \mathcal{C}^{1,1}(\mathrm{I\!R}) \cap \mathcal{C}^\infty(\mathrm{I\!R}\setminus\{\pm z_\varepsilon, \pm 2z_\varepsilon\}), \tag{25}$$

which coincide, respectively, with $\gamma, \eta, \omega, \chi$ on $[-z_\varepsilon, z_\varepsilon]$ and are constant (and assume the corresponding asymptotic values $\pm 1, \eta_\infty, 0, \pm\chi_\infty$) outside the interval $]-2z_\varepsilon, 2z_\varepsilon[$. We can also assume that these functions satisfy (19), and that γ_ϵ is strictly increasing on $]-2z_\varepsilon, 2z_\varepsilon[$ (provided ε is small enough).

For any $t \in [0,T]$, we set

$$\mathcal{T}_\varepsilon(t) := \{x \in \Omega : |y_\epsilon(x,t)| < 2z_\varepsilon\}, \qquad \mathcal{T}_\varepsilon := \bigcup_{t\in[0,T]} \mathcal{T}_\varepsilon(t) \times \{t\},$$

$$\mathcal{T}_\varepsilon^-(t) := \{x \in \Omega : y_\epsilon(x,t) \le -2z_\varepsilon\}, \quad \mathcal{T}_\varepsilon^+(t) := \{x \in \Omega : y_\epsilon(x,t) \ge 2z_\varepsilon\},$$

and, for any $i \in \{1, \ldots, m\}$,

$$\mathcal{S}_\varepsilon^i(t) := \{x \in \Omega : |y_\epsilon^i(x,t)| < 2z_\varepsilon\}, \qquad \mathcal{Q}_\varepsilon^i(t) := \mathcal{S}_\varepsilon^i(t) \cap \mathcal{S}_\varepsilon^{i+1}(t),$$
$$T_\varepsilon^i(t) := \mathcal{S}_\varepsilon^i(t) \cap \mathcal{T}_\varepsilon(t) \cap \Big(\Omega \setminus \bigcup_{j=1}^m \mathcal{Q}_\varepsilon^j(t)\Big).$$

Notice that, since $\mathcal{H}^1(F_i(t)) > 0$ for any i and for any $t \in [0,T]$, for $i \neq j$ one can find $\overline{\varepsilon} > 0$ small enough such that $\mathcal{Q}_\varepsilon^i(t) \cap \mathcal{Q}_\varepsilon^j(t) = \emptyset$, $t \in [0,T]$, $\varepsilon \in]0, \overline{\varepsilon}]$. Observe also that the neighbourhood $\mathcal{T}_\varepsilon(t)$ of $\partial E(t)$ is the disjoint union of the m boxes $\mathcal{Q}_\varepsilon^i(t)$ around the vertices, and of the m portions of strips $T_\varepsilon^i(t)$.

In the sequel we shall assume that $\varepsilon \in]0, \overline{\varepsilon}]$ is small enough such that the closure of $\cup_{t\in[0,T]}\mathcal{T}_\varepsilon(t)$ is contained in Ω.

Following the suggestion of a formal inner asymptotic expansion, we want to define on $\Omega \times [0,T]$ a subsolution $(v_\varepsilon^-, \zeta_\varepsilon^-)$ and a supersolution $(v_\varepsilon^+, \zeta_\varepsilon^+)$ to (20), and we want $v_\varepsilon^\pm$ to be *continuous*. To get the optimal error estimate, we need to carefully match the level lines of $v_\varepsilon^\pm$ around the vertices of $\partial E(t)$.

For any $i \in \{1, \ldots, m\}$ we denote by $\kappa_\phi^i(t)$ the ϕ-curvature of $F_i(t)$. We divide the proof into six steps.

Step 1. Definition of v_ε^-.

For any $i \in \{1, \ldots, m\}$ and $x \in \mathcal{S}_\varepsilon^i(t)$ set

$$\Gamma_\epsilon^i(x,t) := \gamma_\epsilon(y_\epsilon^i) + \varepsilon\eta_\epsilon(y_\epsilon^i)g(t) + \varepsilon^2\big[(\kappa_\phi^i(t))^2\omega_\varepsilon(y_\epsilon^i) + g^2(t)\chi_\varepsilon(y_\epsilon^i)\big] - \Theta\varepsilon^3|\log\varepsilon|^2 \tag{26}$$

where $\Theta > 0$ is a constant which will be determined later on independently of ε (see *Steps 5,6*). Let $I \subseteq \{1, \ldots, m\}$ be such that $i \in I$ if and only if $E(t)$ is convex in a neighbourhood of $v_i(t)$ (observe that I does not depend on t). We define $v_\varepsilon^- : \Omega \times [0,T] \to \mathbb{R}$ as follows:

$$\begin{cases} i \in \{1, \ldots, m\}, \ x \in T_\varepsilon^i(t) \Longrightarrow v_\varepsilon^-(x,t) := \Gamma_\epsilon^i(x,t), \\[2mm] i \in I, \ x \in \mathcal{Q}_\varepsilon^i(t) \Longrightarrow v_\varepsilon^-(x,t) := \max\big(\Gamma_\epsilon^i(x,t), \Gamma_\epsilon^{i+1}(x,t)\big), \\[2mm] i \in \{1, \ldots, m\} \setminus I, \ x \in \mathcal{Q}_\varepsilon^i(t) \Longrightarrow v_\varepsilon^-(x,t) := \min\big(\Gamma_\epsilon^i(x,t), \Gamma_\epsilon^{i+1}(x,t)\big), \\[2mm] x \in \mathcal{T}_\varepsilon^-(t) \Longrightarrow v_\varepsilon^-(x,t) := -1 + \varepsilon\eta_\infty g(t) + \varepsilon^2\chi_\infty g^2(t) - \Theta\varepsilon^3|\log\varepsilon|^2, \\[2mm] x \in \mathcal{T}_\varepsilon^+(t) \Longrightarrow v_\varepsilon^-(x,t) := 1 + \varepsilon\eta_\infty g(t) + \varepsilon^2\chi_\infty g^2(t) - \Theta\varepsilon^3|\log\varepsilon|^2. \end{cases} \tag{27}$$

Notice that $v_\varepsilon^- \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ for any $\varepsilon > 0$. Moreover, taking ε small enough, one can check that v_ε^- is continuous on $\Omega \times [0,T]$; this follows from the fact that the function Γ_ε^i, if considered as a function of y_ε^i, is strictly increasing on $]-2z_\varepsilon, 2z_\varepsilon[$. Recall that, in general, $\kappa_\phi^i(t) \neq \kappa_\phi^{i+1}(t)$ (hence $\kappa_\phi(t)$ is discontinuous along $\partial E(t)$): this is the main reason of the above definition of v_ε^- on $\mathcal{Q}_\varepsilon^i(t)$. The effect is to perturb the level lines of v_ε^- in $\mathcal{Q}_\varepsilon^i(t)$: the perturbation can however be controlled in terms of ε, see *Step 3* below. In a similar fashion we can define v_ε^+ by changing the sign in front of $\theta(t)$ in (24) and in front of Θ in (26) and (27).

Step 2. Definition of n_ϕ^ε. We now define a suitable extension n_ϕ^ε, on the whole of $\mathcal{T}_\varepsilon$, of the Cahn-Hoffmann vector field $n_\phi(\cdot,t) : \partial E(t) \to \mathbb{R}^2$. Let $t \in [0,T]$, $x \in \mathcal{T}_\varepsilon(t)$ and let $i(x) \in \{1,\dots,m\}$ be such that $v_\varepsilon^-(x,t) = \Gamma_\varepsilon^{i(x)}(x,t)$; if $i(x)$ is not uniquely determined we just choose one of the two possible indices. Let $F_\varepsilon(x,t)$ be the closure of the set $\{z \in \mathcal{T}_\varepsilon(t) : v_\varepsilon^-(z,t) = v_\varepsilon^-(x,t), i(z) = i(x)\}$, and let $P_\varepsilon(x,t)$, $Q_\varepsilon(x,t)$ be the endpoints of $F_\varepsilon(x,t)$ corresponding to the endpoints $P_{i(x)}(t)$ and $Q_{i(x)}(t)$ of $F_{i(x)}(t)$. It is enough to consider the case when $F_{i(x)}(t)$ corresponds to some facet of $\mathcal{W}_\phi$; in the other cases, as the Cahn-Hoffmann vector field is constant, the extension is trivial. Recalling that $n_\phi(P_{i(x)}(t))$ and $n_\phi(Q_{i(x)}(t))$ are uniquely determined, we set

$$n_\phi^\varepsilon(P_\varepsilon(x,t)) := n_\phi(P_{i(x)}(t)), \qquad n_\phi^\varepsilon(Q_\varepsilon(x,t)) := n_\phi(Q_{i(x)}(t)), \tag{28}$$

and finally we define n_ϕ^ε on $F_\varepsilon(x,t)$ as the linear combination of $n_\phi^\varepsilon(P_\varepsilon(x,t))$ and $n_\phi^\varepsilon(Q_\varepsilon(x,t))$. In this way $n_\phi^\varepsilon : \mathcal{T}_\varepsilon \to \mathbb{R}^2$, and $n_\phi^\varepsilon(\cdot,t) = n_\phi(\cdot,t)$ on $\partial E(t)$, $t \in [0,T]$.
It is immediate to check that

$$n_\phi^\varepsilon(x,t) \in T^o(\nabla d_{i(x)}(x,t)), \qquad \text{a.e. } (x,t) \in \mathcal{T}_\varepsilon. \tag{29}$$

Notice that, with this choice of n_ϕ^ε, the level line $\{z \in \mathcal{T}_\varepsilon(t) : v_\varepsilon^-(z,t) = v_\varepsilon^-(x,t)\}$ is the boundary of a ϕ-regular set for ε small enough.
Set for simplicity

$$h_\phi(x,t) := \left(\kappa_\phi^{i(x)}(t)\right)^2, \quad \mathbf{b}(x,t) := (h_\phi(x,t), g^2(t)), \quad \mathbf{p}_\varepsilon := (\omega_\varepsilon, \chi_\varepsilon).$$

The next step is crucial in order to get the desired $O(\varepsilon^2|\log\varepsilon|^2)$ error estimate.

Step 3. The extension n_ϕ^ε defined in *Step 2* satisfies

$$\mathrm{div}_\tau n_\phi^\varepsilon(x,t) = \kappa_\phi^{i(x)}(t) - d_{i(x)}(x,t)h_\phi(x,t) + O(\varepsilon^2|\log\varepsilon|^2), \qquad \text{a.e. } (x,t) \in \mathcal{T}_\varepsilon. \tag{30}$$

Let $t \in [0,T]$, $x \in \mathcal{T}_\varepsilon(t)$, and $i(x)$ be as in the proof of *Step 2*. Let $L_{i(x)}(t) := |P_{i(x)}(t) - Q_{i(x)}(t)|$ denote the length of $F_{i(x)}(t)$, $\tau_{i(x)}(t) := (Q_{i(x)}(t) - P_{i(x)}(t))/L_{i(x)}(t)$, let $l_{i(x)}$ be the length of the facet of $\mathcal{W}_\phi$ corresponding to $F_{i(x)}(t)$ (if there are no facets of $\mathcal{W}_\phi$ corresponding to $F_{i(x)}(t)$ we set $l_{i(x)} = 0$). By (28) and the definition of ${n_\phi}^\varepsilon$ we have

$$\begin{aligned} n_\phi^\varepsilon(x,t) &= n_\phi(P_{i(x)}(t)) + \frac{|P_\varepsilon(x,t) - x|}{|P_\varepsilon(x,t) - Q_\varepsilon(x,t)|}[n_\phi(Q_{i(x)}(t)) - n_\phi(P_{i(x)}(t))] \\ &= n_\phi(P_{i(x)}(t)) + \frac{|P_\varepsilon(x,t) - x|}{|P_\varepsilon(x,t) - Q_\varepsilon(x,t)|}\, \delta_{i(x)} l_{i(x)} \tau_{i(x)}(t). \end{aligned}$$

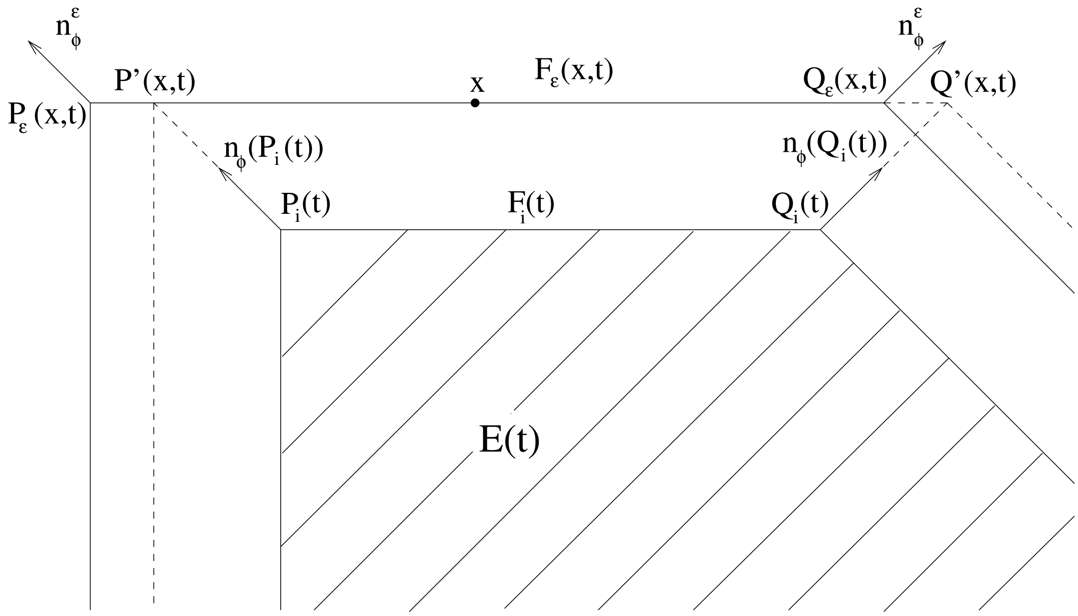

Figure 1: in this example we have $i := i(x)$, $\delta_i = 1$ and $\Delta L = \big(Q_\varepsilon(x,t) - Q'(x,t) - P_\varepsilon(x,t) + P'(x,t)\big) \cdot \tau_i(t)$

Let us write

$$|P_\varepsilon(x,t) - Q_\varepsilon(x,t)| = L_{i(x)}(t) + d_{i(x)}(x,t)\ \delta_{i(x)} l_{i(x)} + \Delta L,$$

see Figure 1. Recalling that $\kappa_\phi^{i(x)}(t) = \delta_{i(x)} l_{i(x)}/L_{i(x)}(t)$, we get

$$\begin{aligned} n_\phi^\varepsilon(x,t) &= n_\phi(P_{i(x)}(t)) + \frac{\delta_{i(x)} l_{i(x)}}{L_{i(x)}(t) + d_{i(x)}(x,t)\delta_{i(x)} l_{i(x)} + \Delta L}\ \big(x - P(x,t)\big) \\ &= n_\phi(P_{i(x)}(t)) + [\kappa_\phi^{i(x)}(t) - d_{i(x)}(x,t) h_\phi(x,t)]\ \big(x - P(x,t)\big) + O((d_{i(x)}(x,t))^2) + O(\Delta L). \end{aligned}$$

Since $\mathrm{div}_\tau n_\phi^\varepsilon$ is the derivative of $n_\phi^\varepsilon \cdot \tau_{i(x)}$ with respect to the arc-length parameter on $F_\varepsilon(x,t)$, we get

$$\mathrm{div}_\tau n_\phi^\varepsilon = \kappa_\phi^{i(x)}(t) - d_{i(x)}(x,t) h_\phi(x,t) + O((d_{i(x)}(x,t))^2) + O(\Delta L).$$

As $x \in \mathcal{T}_\varepsilon(t)$ we have $(d_{i(x)}(x,t))^2 = O(\varepsilon^2 |\log\varepsilon|^2)$, therefore to prove (30) it is enough to show that $\Delta L = O(\varepsilon^2|\log\varepsilon|^2)$; actually we shall prove that $\Delta L = O(\varepsilon^3|\log\varepsilon|^2)$.
For $i \in \{1, \ldots, m\}$, we let

$$\gamma_\varepsilon^i(x,t) := \gamma_\epsilon(x) + \varepsilon\eta_\epsilon(x) g(t) + \varepsilon^2\big[(\kappa_\phi^i(t))^2 \omega_\varepsilon(x) + g^2(t)\chi_\varepsilon(x)\big] - \Theta\varepsilon^3|\log\varepsilon|^2, \quad x \in \mathbb{R}.$$

Notice that $\gamma_\varepsilon^i(\cdot,t)$ is strictly increasing on $]-2z_\varepsilon, 2z_\varepsilon[$ for ε small enough. Fix now $c \in v_\varepsilon^-(\mathcal{T}_\varepsilon)$ and let $z := \gamma_\epsilon^{-1}(c)$. From the relation

$$\begin{aligned} (p+\varepsilon q)^{-1}(c) = p^{-1}(c) - \varepsilon\frac{q(p^{-1}(c)}{p'(p^{-1}(c))} \\ + \frac{\varepsilon^2}{2}\frac{2q(p^{-1}(c))q'(p^{-1}(c))p'(p^{-1}(c)) - p''(p^{-1}(c))q^2(p^{-1}(c))}{\big(p'(p^{-1}(c))\big)^3} + O(\varepsilon^3), \end{aligned}$$

we get

$$(\gamma_\varepsilon^i)^{-1}(c) + O(\varepsilon^3) = z - \varepsilon\frac{\eta_\varepsilon(z)g(t)}{\gamma_\varepsilon'(z)} - \varepsilon^2\frac{\big(\kappa_\phi^i(t)\big)^2\omega_\varepsilon(z) + g^2(t)\chi_\varepsilon(z)}{\gamma_\varepsilon'(z)}$$

$$+\frac{\varepsilon^2}{2}\frac{2\eta_\varepsilon(z)\eta_\varepsilon'(z)\gamma_\varepsilon'(z)g^2(t) - 2\big(\gamma_\varepsilon'(z)\big)^2\eta_\varepsilon(z)g(t) - \gamma_\varepsilon''(z)\big(\eta_\varepsilon(z)\big)^2g^2(t)}{\big(\gamma_\varepsilon'(z)\big)^3}.$$

As $\varepsilon^{-1}\Delta L = O(\sup_{i,j}|(\gamma_\varepsilon^i)^{-1}(c) - (\gamma_\varepsilon^j)^{-1}(c)|)$, from the above equality and (19) we obtain

$$\varepsilon^{-1}\Delta L = \varepsilon^2 O\Big(\sup_{i,j}\big[(\kappa_\phi^i(t))^2 - (\kappa_\phi^j(t))^2\big]\ \frac{\omega_\varepsilon(z)}{\gamma_\varepsilon'(z)}\Big) = \varepsilon^2(1+z^2)O(1) = O(\varepsilon^2|\log\varepsilon|^2).$$

Therefore $\Delta L = O(\varepsilon^3|\log\varepsilon|^2)$. This concludes the proof of (30).
Notice that

$$\mathrm{div}_\tau n_\phi^\varepsilon(x,t) = \mathrm{div} n_\phi^\varepsilon(x,t) \qquad t\in[0,T],\ \ \text{a.e. } x\in\mathcal{T}_\varepsilon(t). \tag{31}$$

Step 4. Definition of ζ_ε^-.
Let

$$\zeta_\varepsilon^-(x,t) := \begin{cases} [\varepsilon^{-1}\gamma_\epsilon'(y_\epsilon^{i(x)}) + \eta_\epsilon'(y_\epsilon^{i(x)})g(t) + \varepsilon\mathbf{b}(x,t)\cdot\mathbf{p}_\varepsilon'(y_\epsilon^{i(x)})]n_\phi^\varepsilon(x,t) & \text{on } \mathcal{T}_\varepsilon, \\ 0 & \text{elsewhere in } \Omega\times[0,T]. \end{cases}$$

Notice that $\zeta_\varepsilon^- \in (L^2(\Omega\times[0,T]))^2$. Let us check that $\zeta_\varepsilon^-(x,t) \in T^o(\nabla v_\varepsilon^-(x,t))$ for any $t\in[0,T]$ and for a.e. $x\in\Omega$. A direct computation yields

$$\begin{aligned} &T^o(\nabla v_\varepsilon^-) \\ &= T^o\Big([\varepsilon^{-1}\gamma_\epsilon'(y_\epsilon^{i(x)}) + \eta_\epsilon'(y_\epsilon^{i(x)})g(t) + \varepsilon\mathbf{b}(x,t)\cdot\mathbf{p}_\varepsilon'(y_\epsilon^{i(x)})]\nabla d_{i(x)}(x,t) + \varepsilon^2\omega_\varepsilon(y_\epsilon^{i(x)})\nabla h_\phi(x,t)\Big). \end{aligned}$$

Since κ_ϕ is constant on each arc of $\partial E(t)$, recalling the definition of h_ϕ we obtain that for almost every $x\in\mathcal{T}_\varepsilon(t)$ there holds $\nabla h_\phi(x,t)=0$. Therefore, using (7), we get

$$T^o(\nabla v_\varepsilon^-) = [\varepsilon^{-1}\gamma_\epsilon'(y_\epsilon^{i(x)}) + \eta_\epsilon'(y_\epsilon^{i(x)})g(t) + \varepsilon\mathbf{b}(x,t)\cdot\mathbf{p}_\varepsilon'(y_\epsilon^{i(x)})]T^o(\nabla d_{i(x)}(x,t)) \ni \zeta_\varepsilon^-(x,t).$$

In a similar fashion we can define ζ_ε^+ by changing the sign in front of $\theta(t)$ in (24) and in front of Θ in (26) and (27).
The crucial result is to prove that $v_\varepsilon^- \le u_\varepsilon \le v_\varepsilon^+$, u_ε being the solution to (20) with the initial datum u_ε^0 defined in (32) below. To this aim we shall focus our attention to v_ε^-.
The initial datum for problem (20) is fixed as follows. For $\varepsilon>0$ we set

$$u_\varepsilon(x,0) = u_\varepsilon^0(x) := \gamma_\varepsilon(y(x,0)) + \varepsilon\eta_\varepsilon(y(x,0))g(0) + \varepsilon^2\chi_\varepsilon(y(x,0))g^2(0). \tag{32}$$

Step 5. There exist a real number $\Theta > 0$, independent of ε, and $\varepsilon_0 > 0$ such that $v_\varepsilon^-(\cdot, 0) \leq u_\varepsilon^0(\cdot)$ in Ω, for any $0 < \varepsilon < \varepsilon_0$.
Indeed, it is enough to argue as in [5, Section 4.1].

Step 6. There exist $\varepsilon_0 > 0$, and real numbers $c, \Theta > 0$, both independent of ε, such that for any $\varepsilon \in]0, \varepsilon_0[$ and $\varphi \in H^1(\Omega; [0, +\infty[)$ there holds

$$\int_\Omega \left(\partial_t v_\varepsilon^- \varphi + \zeta_\varepsilon^- \cdot \nabla\varphi + \frac{1}{\varepsilon^2}\psi(v_\varepsilon^-)\varphi - \frac{c_0}{2\varepsilon} g\varphi \right) dx \leq 0. \tag{33}$$

For simplicity we use the notation $(v_\varepsilon, \zeta_\varepsilon)$ in place of $(v_\varepsilon^-, \zeta_\varepsilon^-)$. Inequality (33) can be equivalently written as

$$\int_{\mathcal{T}_\varepsilon(t)} \left(\partial_t v_\varepsilon - \mathrm{div}\zeta_\varepsilon + \frac{1}{\varepsilon^2}\psi(v_\varepsilon) - \frac{c_0}{2\varepsilon} g \right) \varphi \; dx + \int_{\Omega \setminus \mathcal{T}_\varepsilon(t)} (\partial_t v_\varepsilon + \frac{1}{\varepsilon^2}\psi(v_\varepsilon) - \frac{c_0}{2\varepsilon} g)\varphi \; dx$$
$$+ \int_{\partial\mathcal{T}_\varepsilon(t)} \varphi \; \zeta_\varepsilon \cdot \nu_\varepsilon \; d\mathcal{H}^1 =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where ν_ε denotes the a.e. defined euclidean outward unit normal to $\partial\mathcal{T}_\varepsilon(t)$.
Evaluation of the integral $\mathcal{I}_1$.
If $\mathbf{b} = (b_1, b_2)$ we set $\mathbf{b}_t = (\partial_t b_1, \partial_t b_2)$. Direct computations yield, for a.e. $(x, t) \in \mathcal{T}_\varepsilon$,

$$\partial_t v_\varepsilon = (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g + \varepsilon\mathbf{b} \cdot \mathbf{p}_\varepsilon')(\partial_t d_{i(x)} - \theta'\varepsilon^2|\log\varepsilon|^2) + \varepsilon\eta g_t + \varepsilon^2\mathbf{b}_t \cdot \mathbf{p}_\varepsilon,$$

where $\gamma_\epsilon, \eta_\epsilon, \omega_\varepsilon$ and χ_ε are evaluated at $y_\varepsilon^{i(x)}$. Hence by (13)

$$\partial_t v_\varepsilon = (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g)\kappa_\phi + (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g)g - \gamma_\epsilon'\varepsilon|\log\varepsilon|^2\theta' + O(\varepsilon),$$

where for simplicity of notation we set $\kappa_\phi := \kappa_\phi^{i(x)}$. Furthermore

$$\mathrm{div}\zeta_\varepsilon = (\varepsilon^{-2}\gamma_\epsilon'' + \varepsilon^{-1}\eta_\epsilon'' g + \mathbf{b} \cdot \mathbf{p}_\varepsilon'')\nabla d_{i(x)} \cdot n_\phi^\varepsilon + (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g + \varepsilon\mathbf{b} \cdot \mathbf{p}_\varepsilon')\mathrm{div}\; n_\phi^\varepsilon.$$

Using (5), (8), (30), (31), recalling the definition of h_ϕ, and setting $\tilde{y}_\varepsilon := y_\varepsilon^{i(x)}$, we get

$$\begin{aligned}
\mathrm{div}\zeta_\varepsilon &= \varepsilon^{-2}\gamma_\epsilon'' + \varepsilon^{-1}\eta_\epsilon'' g + \mathbf{b} \cdot \mathbf{p}_\varepsilon'' + (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g + \varepsilon\mathbf{b} \cdot \mathbf{p}_\varepsilon')(\kappa_\phi - d_{i(x)}h_\phi) + O(\varepsilon|\log\varepsilon|^2) \\
&= \varepsilon^{-2}\gamma_\epsilon'' + \varepsilon^{-1}\eta_\epsilon'' g + \mathbf{b} \cdot \mathbf{p}_\varepsilon'' + (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g)\kappa_\phi \\
&\quad + (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g)(-\varepsilon\tilde{y}_\varepsilon - \theta\varepsilon^2|\log\varepsilon|^2)h_\phi + O(\varepsilon|\log\varepsilon|^2) \\
&= \varepsilon^{-2}\gamma_\epsilon'' + \varepsilon^{-1}\eta_\epsilon'' g + \mathbf{b} \cdot \mathbf{p}_\varepsilon'' + (\varepsilon^{-1}\gamma_\epsilon' + \eta_\epsilon' g)\kappa_\phi \\
&\quad - \gamma_\epsilon'\tilde{y}_\varepsilon h_\phi - \gamma_\epsilon'\theta\varepsilon|\log\varepsilon|^2 h_\phi + O(\varepsilon|\log\varepsilon|^2).
\end{aligned}$$

Moreover we can expand

$$\begin{aligned}
\varepsilon^{-2}\psi(v_\varepsilon) =& \varepsilon^{-2}\psi(\gamma_\epsilon) + \varepsilon^{-1}g\eta_\epsilon\psi'(\gamma_\epsilon) + \mathbf{b} \cdot \mathbf{p}_\varepsilon\psi'(\gamma_\epsilon) \\
&+ \frac{1}{2}g^2\eta_\epsilon^2\psi''(\gamma_\epsilon) - \Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon) + O(\varepsilon).
\end{aligned}$$

Hence, using (24) we get

$$\partial_t v_\varepsilon - \mathrm{div}\zeta_\varepsilon + \frac{1}{\varepsilon^2}\psi(v_\varepsilon) - \frac{c_0}{2\varepsilon}g = \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon + O(\varepsilon|\log\varepsilon|^2), \tag{34}$$

where

$$\begin{aligned}
\mathrm{I}_\varepsilon &= \varepsilon^{-2}(-\gamma_\epsilon'' + \psi(\gamma_\epsilon)),\\
\mathrm{II}_\varepsilon &= \varepsilon^{-1} g(\mathcal{L}_\varepsilon \eta_\epsilon - \frac{c_0}{2} + \gamma_\epsilon'),\\
\mathrm{III}_\varepsilon &= -\mathbf{b}\cdot\mathbf{p}_\varepsilon'' + \mathbf{b}\cdot\mathbf{p}_\varepsilon\psi'(\gamma_\epsilon) + \frac{1}{2}g^2\eta_\epsilon^2\psi''(\gamma_\epsilon) + \tilde{y}_\epsilon\gamma_\epsilon' h_\phi + \eta_\epsilon' g^2\\
&= h_\phi[\mathcal{L}_\varepsilon\omega_\varepsilon + \tilde{y}_\epsilon\gamma_\epsilon'] + g^2[\mathcal{L}_\varepsilon\chi_\varepsilon + \eta_\epsilon' + \frac{1}{2}\eta_\epsilon^2\psi''(\gamma_\epsilon)],\\
\mathrm{IV}_\varepsilon &= -\Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon) - \gamma_\epsilon'\varepsilon|\log\varepsilon|^2\theta' + \gamma_\epsilon'\theta\varepsilon|\log\varepsilon|^2 h_\phi,
\end{aligned}$$

and we have denoted by $\mathcal{L}_\varepsilon : H^2_{\mathrm{loc}}(\mathrm{I\!R}) \to L^2_{\mathrm{loc}}(\mathrm{I\!R})$ the operator defined by $\mathcal{L}_\varepsilon\zeta := -\zeta'' + \psi'(\gamma_\varepsilon)\zeta$. Notice that the $O(\varepsilon|\log\varepsilon|^2)$ at the right hand side of (34) does not depend on c and Θ.
Reasoning exactly as in [8] we have $\mathrm{I}_\varepsilon = \mathrm{II}_\varepsilon = \mathrm{III}_\varepsilon = o(\varepsilon^{2\delta-3})$. For the sake of completeness, we now repeat the argument of [7] concerning the term IV_ε. We have

$$\begin{aligned}
\mathrm{IV}_\varepsilon &= (\theta h_\phi - \theta')\varepsilon|\log\varepsilon|^2\gamma_\epsilon' - \Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon)\\
&\leq -\theta\varepsilon|\log\varepsilon|^2\gamma_\epsilon' - \Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon) = -\varepsilon|\log\varepsilon|^2\big(\theta\gamma_\epsilon' + \Theta\psi'(\gamma_\epsilon)\big).
\end{aligned}$$

As $c_1\gamma_\epsilon' + \psi'(\gamma_\epsilon)$ is uniformly positive for a proper choice of the positive constant c_1, we have that, if c and Θ are large enough (independently of ε), the expression in (34) is non positive, hence $\mathcal{I}_1 \leq 0$.
Similarly, following [7, Section 6.3] we have $\mathcal{I}_2 \leq 0$. Moreover, from the definition of ζ_ε it follows that $\zeta_\varepsilon(x,t)_{|\partial\mathcal{T}_\varepsilon(t)} = 0$ and then $\mathcal{I}_3 = 0$.
The proof of *Step 6* is concluded.

Summing up, we have proved the following result: there exist $\varepsilon_0 > 0$, an exponentially increasing continuous function $\theta : [0,T] \to]0,+\infty[$ and a real number $\Theta > 0$, both independent of ε, such that, if u_ε denotes the solution to (20) with initial datum (32), then $v_\varepsilon^-(x,t) \leq u_\varepsilon(x,t)$ for a.e. $(x,t) \in Q$ and for $\varepsilon \in]0,\varepsilon_0]$.
Theorem 5.1 follows now arguing as in [8, Theorem 6.1]. □

The extension of Theorem 5.1 to the case $g = g(x,t)$ seems to be not easy, even if one looks for the suboptimal $O(\varepsilon|\log\varepsilon|)$ interface error estimate.
We now prove a comparison result between ϕ-regular flows: when $g = g(t)$ we easily obtain the comparison using Theorem 5.1; if $g = g(x,t)$ the proof follows from the results of the Appendix.

Theorem 5.2. *Let $g(x,t)$ satisfy conditions (15) with $\leq$ replaced by $<$. Let $(E_1(t), n_\phi^{(1)}(\cdot,t))$, $(E_2(t), n_\phi^{(2)}(\cdot,t))$ be two ϕ-regular flows on $[0,T]$. Then*

$$E_1(0) \subseteq E_2(0) \Rightarrow E_1(t) \subseteq E_2(t), \qquad t \in [0,T]. \tag{35}$$

Proof. Assume first that $g = g(t)$. Let $i \in \{1,2\}$ and let $u_\varepsilon^{(i)}$ be the function given by Theorem 5.1, where the initial datum $u_{\varepsilon i}^0$ is fixed as in (32), correspondingly to E_i. From (32), for $\varepsilon > 0$ small enough we have $u_{01}^\varepsilon \geq u_{02}^\varepsilon$ in Ω. Hence, by Lemma 4.1, it follows that

$$u_\varepsilon^{(1)} \geq u_\varepsilon^{(2)} \qquad \text{a.e. in } Q. \tag{36}$$

Applying (23) of Theorem 5.1, from (36) we get (35).
If $g = g(x,t)$, the result is proved in the Appendix. □

Starting from Theorem 5.2, we can give a weak definition of evolution by crystalline curvature valid for any initial set $E \subseteq \mathbb{R}^2$ and defined for any $t \in [0, +\infty[$, by means of the barriers method of De Giorgi [17] (see [6]).
Let $\mathcal{F}$ be the family of all ϕ-regular flows, i.e. $f \in \mathcal{F}$ if and only if there exist $a, b \in [0, +\infty[$, $a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbb{R}^2)$, and there exists a vector field $n_\phi : \bigcup_{t\in[a,b]} \partial f(t) \times \{t\} \to \mathbb{R}^2$ such that $(f(t), n_\phi(\cdot, t))$ is a ϕ-regular flow on $[a,b]$.

Definition 5.1. *A function ψ is a barrier if and only if $\phi : [0, +\infty[\to \mathcal{P}(\mathbb{R}^2)$ and the following property holds: if $f \in \mathcal{F}$, $f : [a,b] \subseteq [0, +\infty[\to \mathcal{P}(\mathbb{R}^2)$, and $f(a) \subseteq \psi(a)$ then $f(b) \subseteq \psi(b)$. We denote by $\mathcal{B}(\mathcal{F})$ the family of all barriers.*

Definition 5.2. *Let $E \subseteq \mathbb{R}^2$ be an arbitrary given set. The minimal barrier $\mathcal{M}(E, \mathcal{F})$: $[0, +\infty[\to \mathcal{P}(\mathbb{R}^2)$ (with origin in E at time 0) at any time $t \geq 0$ is defined by*

$$\mathcal{M}(E)(t) := \bigcap \{\psi(t) : \ \psi \in \mathcal{B}(\mathcal{F}), \ \psi(0) \supseteq E\}.$$

As discussed in [6], to implement the barriers method we need only a comparison result at the level of the elements of $\mathcal{F}$, which is given by Theorem 5.2. Then, the following properties follow: $\mathcal{M}(E) \in \mathcal{B}(\mathcal{F})$ (uniqueness of the minimal barrier), and $E_1 \subseteq E_2$ implies $\mathcal{M}(E_1) \subseteq \mathcal{M}(E_2)$ (comparison property).
One can also check that $\mathcal{M}(E)(0) = E$, that the minimal barrier agrees with the elements of $\mathcal{F}$ and that satisfies the semigroup property in time.

6 The double obstacle potential

In this section we want to discuss a different formulation of the relaxed evolution problem (2), useful for computational purposes, showing an optimal interface error estimate valid in $N = 2$ dimensions and for $g = g(t)$.
More precisely, we define $\Psi(s) := 1 - s^2$ if $s \in [-1,1]$ and $\Psi(s) := +\infty$ if $s \notin [-1,1]$ (see [34], [35], [36]). Then $\psi := \frac{1}{2}\Psi'$ is defined as $\psi(-1) = (-\infty, 1]$, $\psi(s) = -s$ if $s \in (-1,1)$, $\psi(1) = [-1, \infty)$. The analogous to (18) is $\gamma'' + \gamma \ni 0$, whose solution in $\mathbb{R}$ is the nondecreasing function γ, defined as $\gamma(y) = -1$ if $y < \frac{-\pi}{2}$, $\gamma(y) = \sin y$ if $y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, $\gamma(y) = 1$ if $y > \frac{\pi}{2}$.
Therefore, the first relation in (20) becomes

$$u_t - \operatorname{div}(T^o(\nabla u)) - \frac{1}{\varepsilon^2} u \ni \frac{\pi}{4\varepsilon} g \qquad \text{in } Q. \tag{37}$$

We address (20) with the first relation replaced by (37) as the double obstacle problem DOP.

Definition 6.1. *A couple (u,ζ) is a subsolution of DOP if, for any $T>0$, the following properties hold:*

(i) $u \in L^\infty(0,T;H^1(\Omega;(-\infty,1])) \cap H^1(0,T;L^2(\Omega))$ *and* $\zeta \in (L^2(Q))^N$*;*

(ii) for any $\varphi \in H^1(\Omega;[0,+\infty[)$ *with spt*$(\varphi) \subseteq \{x \in \Omega : u(x,t) > -1\}$ *and a.e.* $t \in]0,T[$ *there holds*

$$\int_\Omega (u_t\varphi + \zeta \cdot \nabla\varphi - \frac{1}{\varepsilon^2}u\varphi - \frac{\pi}{4\varepsilon}g\varphi)\,dx \le 0; \tag{38}$$

(iii) conditions (iii) and (iv) of Definition (4.1) hold.

The couple (u,ζ) is a supersolution of DOP if (i) holds with $H^1(\Omega;(-\infty,1])$ replaced by $H^1(\Omega;[-1,+\infty))$; condition (ii) holds with spt$(\varphi) \subseteq \{x \in \Omega : u(x,t) > -1\}$ replaced by spt$(\varphi) \subseteq \{x \in \Omega : u(x,t) < 1\}$ and with $\le$ replaced by $\ge$ in (38); condition (iii) of Definition 4.1 holds with $\ge$ in place of $\le$, and condition (iv) of Definition 4.1 holds. The couple (u,ζ) is a solution of DOP if it is both a subsolution and a supersolution.

The analogous of Lemma 4.1 (and hence uniqueness of a solution to DOP) holds true also (in arbitrary N dimension) for sub- and supersolutions of the double obstacle problem, with a similar proof (observe that $\mathrm{spt}(e^+) \subseteq \{u_1 > -1\} \cap \{u_2 < 1\}$). Also, performing the minimization of $\mathcal{E}_\phi$ on the convex subset of all $v \in H^1(\Omega)$ with $|v| \le 1$, we can prove the existence of a solution to DOP.
The main result of this section is the analogous of Theorem 5.1:

Theorem 6.1. *Assume that $(E(t), n_\phi(\cdot,t))$ is a $\phi-$regular flow on $[0,T]$, with $g=g(t)$. For any $\varepsilon>0$ let u_ε be the solution to the double obstacle problem with initial datum as in (44) below. Let $\Sigma_\varepsilon(t) := \{x \in \Omega : u_\varepsilon(x,t) = 0\}$. Then there exist $\varepsilon_0 \in]0,1]$ and a constant C depending on $\partial E(0)$, g, T, and independent of $\varepsilon \in]0,\varepsilon_0]$, such that for all $\varepsilon \in]0,\varepsilon_0]$ the following optimal interface error estimate holds:*

$$\sup_{t\in[0,T]} d_H(\Sigma_\varepsilon(t), \partial E(t)) \le C\varepsilon^2. \tag{39}$$

As in the regular case, the main steps for the validity of (39) are a comparison result and a proper construction of the lower and upper barriers. Since they can be obtained by a suitable modification of the proof of Theorem 5.1, in the following we will recall just the sketch of the proof, pointing out the steps which differ from the regular case.
The modified distance functions in (24) have to be changed as follows:

$$y_\epsilon = y_\epsilon(x,t) := y(x,t) - \varepsilon\theta(t)\big(\varepsilon^2 C_1 y(x,t)^2 + C_2\big),$$

$$y^i_\epsilon = y^i_\epsilon(x,t) := y^i(x,t) - \varepsilon\theta(t)\big(\varepsilon^2 C_1 y^i(x,t)^2 + C_2\big),$$

where $y(x,t)$ and $y^i(x,t)$ are defined in (24), $C_1, C_2 \ge 1$ are suitable constants independent of ε and $\theta(t) := e^{2Kt}$, with $K := \max_{0\le t\le T}\|\kappa_\phi^2\|_{L^\infty(\partial E(t))}$.
For any $t \in [0,T]$, the sets $\mathcal{T}_\varepsilon(t)$, $\mathcal{T}_\varepsilon$, $\mathcal{T}_\varepsilon^\pm(t)$, $\mathcal{S}^i_\varepsilon(t)$, $\mathcal{Q}^i_\varepsilon(t)$ and $T^i_\varepsilon(t)$ are defined as in Section 5, with z_ε replaced by $\pi/4$.

We recall the definition of the shape functions, introduced and studied in [35]. Let $I_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ denote the characteristic function of $[-\frac{\pi}{2},\frac{\pi}{2}]$ and $\tilde{\mathcal{L}}\zeta := \zeta''+\zeta$, for $\zeta \in H^1(-\frac{\pi}{2},\frac{\pi}{2})$, $\zeta(\pm\frac{\pi}{2}) = 0$. The shape functions $\eta \in C^{1,1}(\mathbb{R})$ and $\chi \in C^{0,1}(\mathbb{R})$, solve the problems

$$\tilde{\mathcal{L}}\eta = \gamma' - \frac{\pi}{4}, \qquad \tilde{\mathcal{L}}\chi = y\gamma' \qquad \text{in }]-\frac{\pi}{2},\frac{\pi}{2}[, \tag{40}$$

The function $\eta = \frac{1}{2}(y\gamma + \gamma' - \frac{\pi}{2})I_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ is even and satisfies $\eta(\pm\frac{\pi}{2}) = \eta'(\pm\frac{\pi}{2}) = 0$, and the function $\chi = \frac{1}{16}(4y\gamma' + (4y^2 - \pi^2)\gamma)I_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ is odd and satisfies $\chi(\pm\frac{\pi}{2}) = 0$. Since χ' exhibits a jump discontinuity at $\pm\frac{\pi}{2}$, precisely $[\![\chi']\!](\pm\frac{\pi}{2}) = \mp\frac{\pi}{8}$, we introduce the auxiliary function $\chi_- \in C^{0,1}(\mathbb{R})$ which can be interpreted as a suitable shift of the original χ, defined by $\chi_- = \chi + \frac{\pi}{8}\gamma'$, which still solves the second equality in (40). Furthermore, χ_-' is discontinuous at $-\frac{\pi}{2}$ because $[\![\chi_-']\!](-\frac{\pi}{2}) = \frac{\pi}{4}$, but $\chi_-'(\frac{\pi}{2}) = 0$.
For any $i \in \{1,\dots,m\}$ and $x \in \mathcal{S}_\varepsilon^i(t)$ set

$$\Gamma_\epsilon^i(x,t) := \gamma(y_\epsilon^i) + \varepsilon\eta(y_\epsilon^i)g(t) + \varepsilon^2\Big((\kappa_\phi^i(t))^2 + \frac{1}{2}g^2(t)\Big)\chi_-(y_\epsilon^i). \tag{41}$$

We define $v_\varepsilon^- : \Omega\times[0,T] \to [-1,1]$ as in (27) where the last two relations are replaced by

$$x \in \mathcal{T}_\varepsilon^-(t) \Longrightarrow v_\varepsilon^-(x,t) := -1, \qquad x \in \mathcal{T}_\varepsilon^+(t) \Longrightarrow v_\varepsilon^-(x,t) := 1. \tag{42}$$

The extension n_ϕ^ε of the Cahn-Hoffman vector n_ϕ is done as in the regular case (*Step 2*, Section 5). In particular, arguing exactly as in *Step 3*, we can show that

$$\mathrm{div}_\tau n_\phi^\varepsilon(x,t) = \kappa_\phi^{i(x)}(t) - d_{i(x)}(x,t)h_\phi(x,t) + O(\varepsilon^2), \qquad \text{a.e. } (x,t) \in \mathcal{T}_\varepsilon. \tag{43}$$

Let

$$\zeta_\varepsilon^-(x,t) := \Big(\varepsilon^{-1}\gamma'(y_\epsilon^{i(x)}) + g\eta'(y_\epsilon^{i(x)}) + \varepsilon\Big(h_\phi(x,t) + \frac{1}{2}g^2\Big)\chi_-'\Big)\Big(1 - 2\varepsilon^3\theta C_1 y_\epsilon^{i(x)}\Big)n_\phi^\varepsilon(x,t)$$

on $\mathcal{T}_\varepsilon(t)$, and $\zeta_\varepsilon^-(x,t) = 0$ on $\Omega\setminus\mathcal{T}_\varepsilon(t)$. Then $\zeta_\varepsilon^-(x,t) \in T^o(\nabla v_\varepsilon^-(x,t))$ for any $t \in [0,T]$ and for a.e. $x \in \Omega$.
The initial datum is fixed as

$$u_\varepsilon(x,0) = u_\varepsilon^0(x) := \gamma(y(x,0)). \tag{44}$$

We have that $(v_\varepsilon^-, \zeta_\varepsilon^-) := (v_\varepsilon, \zeta_\varepsilon)$ is a subsolution of DOP: precisely there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0,\varepsilon_0[$ and $\varphi \in H^1(\Omega;[0,+\infty[)$ with $\mathrm{spt}(\varphi) \subseteq \{x \in \Omega : v_\varepsilon^-(x,t) > -1\}$ there holds

$$\int_\Omega \Big(\partial_t v_\varepsilon^-\varphi + \zeta_\varepsilon^-\cdot\nabla\varphi - \frac{1}{\varepsilon^2}v_\varepsilon^-\varphi - \frac{\pi}{4\varepsilon}g\varphi\Big)\,dx \le 0. \tag{45}$$

The left hand side of (45) can be written as $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$, where

$$\mathcal{I}_1 = \int_{\mathcal{T}_\varepsilon(t)} \varphi\Big(\partial_t v_\varepsilon - \mathrm{div}\zeta_\varepsilon - \frac{1}{\varepsilon^2}v_\varepsilon - \frac{\pi}{4\varepsilon}g\Big)\,dx, \qquad \mathcal{I}_2 = -\int_{\{v_\varepsilon^-(x,t)=1\}} (\frac{\pi}{4\varepsilon}g + \frac{1}{\varepsilon^2})\varphi\,dx,$$

and $\mathcal{I}_3 = \int_{\partial\mathcal{T}_\varepsilon(t)} \zeta_\varepsilon \cdot \nu_\varepsilon \varphi \, d\sigma$. For ε sufficiently small [35] we have $\mathcal{I}_2 < 0$ and $\mathcal{I}_3 \le 0$. Moreover, by (40) we can write

$$\mathcal{I}_1 = \int_{\mathcal{T}_\varepsilon(t)} \varphi\Big(\varepsilon^{-2}\mathcal{I}_1^a + \varepsilon^{-1}\mathcal{I}_1^b + \mathcal{I}_1^c + \varepsilon\mathcal{I}_1^d + O(\varepsilon)\Big),$$

where $\mathcal{I}_1^a = -\tilde{\mathcal{L}}\gamma = 0$, $\mathcal{I}_1^b = -g\left(\tilde{\mathcal{L}}\eta - \gamma' + \frac{\pi}{4}\right) = 0$, $\mathcal{I}_1^c = -\left(h_\phi + \frac{1}{2}g^2\right)\left(\mathcal{L}\chi_- - y_\epsilon\gamma'\right) = 0$, and

$$\mathcal{I}_1^d = -\theta(t)\Big(- 4C_1 y_\epsilon \gamma'' + C_2\gamma'(2K - h_\phi) - 2C_1\gamma'\Big).$$

Arguing exactly as in [35], we can deduce that $\mathcal{I}_1^d \le 0$. Therefore the lower barrier verifies the comparison lemma conditions. As for the regular case, the proof of the Theorem 6.1 follows arguing as in [7].

7 Appendix

In this appendix we discuss the maximum principle and the comparison principle for driven crystalline evolutions. In what follows, the forcing term g is always assumed to fulfill (15). The proofs are only sketched, since we slightly extend a known result proved by Giga and Gurtin in [26]. Our result generalizes the one in [26], since ϕ-regular sets may have arcs not corresponding to facets of the Wulff shape (arcs in group $\mathcal{G}_2$), which are necessary because the forcing term depends on space.
Let us introduce some notation. Let (E, n_ϕ) be a ϕ-regular set, where n_ϕ is the (uniquely defined as in Section 2) Cahn-Hoffmann vector field. If F is an arc of ∂E we denote by L_F its length, and if F corresponds to the facet W of $\mathcal{W}_\phi$ we denote by l_W the length of W. We also set $\kappa_\phi^F := \delta_F l_F / L_F$, with $\delta_F \in \{0, \pm 1\}$, depending on the local convexity or concavity of ∂E at F, and $\overline{g}_F(t) := \frac{1}{L_F}\int_F g(z,t)\, d\mathcal{H}^1(z)$.
For a fixed $t \in [0, +\infty[$, let us introduce the function $V_E : \partial E \to \mathbb{R}^2$ defined as follows:

$$\begin{cases} F \subseteq \partial E, \ F \in \mathcal{G}_1, \ x \in \operatorname{int}(F) \Rightarrow V_E(x) := -n_\phi(x)\big(\kappa_\phi^F + \overline{g}_F(t)\big), \\ \\ F \subseteq \partial E, \ F \in \mathcal{G}_2, \ x \in \operatorname{int}(F) \Rightarrow V_E(x) := -n_\phi(x) g(x,t). \end{cases} \tag{46}$$

We now need to define V_E on the vertices of ∂E. Let x be a vertex of ∂E. By (46) there are two vectors $V_E^\pm(x)$ corresponding respectively to the two arcs $F^\pm$ of ∂E meeting at x. Let $r^\pm$ be the line passing through the point $x + V_E^\pm(x)$ and parallel to the tangent vector to $F^\pm$ at x. $V_E(x)$ is then defined as the vector $(r^- \cap r^+) - x$. This vector is well-defined thanks to property (iv) of Definition 2.1.
The function V_E is intuitively the velocity of each point of ∂E.
Let x be a vertex of ∂E. Let $\tau_\pm$ be the tangent half-lines to ∂E at x, pointing locally outside E (resp. inside E) if E is locally convex (resp. concave) at x. We define the set $\hat{E}$ is the half-cone inside τ_- and τ_+. If x is not a vertex of ∂E, then $\hat{E} = E$
A version of the maximum principle can be written as follows.

Theorem 7.1. *Let $(E_i, n_\phi^{(i)})$ be two ϕ-regular sets, $i = 1, 2$. Assume that $E_1 \subseteq E_2$, and let $x \in \partial E_1 \cap \partial E_2$. The following statements hold.*

(i) Let $x \in F_1 \cap F_2$, where $F_i \subseteq \partial E_i$, $F_i \in \mathcal{G}_1$ for $i = 1, 2$, and assume that F_1 and F_2 correspond to the same facet W of $\mathcal{W}_\phi$. Then

$$\delta_{F_1} \geq \delta_{F_2}, \qquad \kappa_\phi^{F_1} + \overline{g}_{F_1}(t) \geq \kappa_\phi^{F_2} + \overline{g}_{F_2}(t). \tag{47}$$

(ii) More generally, the vector $x + V_{E_2}(x) - V_{E_1}(x)$ points locally out of $E_1 \cup \hat{E}_2$.

The complete proof of statement (ii) is long, because several different cases (due to the presence of arcs both in $\mathcal{G}_1$ and in $\mathcal{G}_2$) must be taken into account. A complete classification has been given in the case $g = 0$ and $\mathcal{G}_2 = \emptyset$ in the paper [26]. Here we restrict ourselves to sketch the proof of statement (i), which reveals the rôle played by the no-fracture condition (15).

Proof of (i). Without loss of generality, we can assume that W has unitary length. The inequality $\delta_{F_1} \geq \delta_{F_2}$ is immediate. We suppose that E_1 and E_2 are locally convex at x, i.e. $\delta_{F_1} = \delta_{F_2} = 1$. Let P_1, Q_1 (resp. P_2, Q_2) be the endpoints of F_1 (resp. of F_2). Assume also $P_1 \neq P_2$ and $Q_1 \neq Q_2$; the case of equalities can be obtained as a limit case. By conditions (15) we have

$$-\frac{|x - P_2|}{L_{F_2}} \leq |x - P_2|\, \overline{g}_{F_2} - \int_{P_2}^{x} g \leq 1 - \frac{|x - P_2|}{L_{F_2}}, \qquad x \in F_2, \tag{48}$$

where $\int_{P_2}^{x} g$ means the integration of $g(z,t)$ for z in the segment $[P_2, x]$, and we set for simplicity $\overline{g}_{F_i} := \overline{g}_{F_i}(t)$.
In particular, for $x = P_1$ and $x = Q_1$ we get

$$-\frac{|P_2 - P_1|}{L_{F_2}} \leq |P_2 - P_1|\, \overline{g}_{F_2} - \int_{P_2}^{P_1} g, \tag{49}$$

$$\overline{g}_{F_2} - \frac{1}{|Q_1 - P_2|} \int_{P_2}^{Q_1} g \leq \frac{1}{|Q_1 - P_2|} - \frac{1}{L_{F_2}}. \tag{50}$$

Inserting the obvious relation $\int_{P_2}^{Q_1} g + \int_{Q_1}^{Q_2} g = L_{F_2} \overline{g}_{F_2}$ into (50), we get

$$-\frac{|Q_1 - Q_2|}{L_{F_2}} \leq |Q_1 - Q_2|\, \overline{g}_{F_2} - \int_{Q_1}^{Q_2} g. \tag{51}$$

Summing (49) and (51) we get

$$-\frac{L_{F_2} - L_{F_1}}{L_{F_2}} \leq |P_1 - P_2| \Big(\overline{g}_{F_2} - \frac{1}{|P_2 - P_1|} \int_{P_2}^{P_1} g \Big) + |Q_1 - Q_2| \Big(\overline{g}_{F_2} - \frac{1}{|Q_2 - Q_1|} \int_{Q_1}^{Q_2} g \Big)$$

$$= (L_{F_2} - L_{F_1})\, \overline{g}_{F_2} + L_{F_1}\, \overline{g}_{F_1} - L_{F_2}\, \overline{g}_{F_2} = L_{F_1}\, \big(\overline{g}_{F_1} - \overline{g}_{F_2} \big).$$

Then the last inequality in (47) follows. $\square$

Observe that, if $x \in \mathrm{int}(F_1)$, where $F_1 \subseteq \partial E_1$, $F_1 \in \mathcal{G}_1$ and F_1 corresponds to $W \subset \partial \mathcal{W}_\phi$, then there exists $F_2 \subset \partial E_2$, $F_2 \in \mathcal{G}_1$, F_2 corresponding to W, such that $x \in F_1 \cap F_2$, and statement (i) applies. The same observation holds replacing F_1 by F_2 and vice-versa.
If in statement (i) we assume that $F_i \in \mathcal{G}_2$ for $i = 1, 2$, then (47) reduces to the trivial equality $g(x,t) = g(x,t)$.
Assume that $x \in F_1 \subseteq \partial E_1$ is a vertex of ∂E_1, with $F_1 \in \mathcal{G}_1$, and $x \in \mathrm{int}(F_2)$, where $F_2 \subseteq \partial E_2$, $F_2 \in \mathcal{G}_2$. Then from (16) we get $\kappa_\phi^{F_1}(x,t) + \overline{g}_{F_1}(t) \geq g(x,t)$.

Remark 7.1. *If in Theorem 7.1 the set E_i evolves with forcing term g_i, with $g_1 > g_2$, then the second inequality in (47) is strict and, in statement (ii), the vector $x + V_{E_2}(x) - V_{E_1}(x)$ points locally strictly out of $E_1 \cup \hat{E}_2$.*

The following approximation result can be proved directly.

Lemma 7.1. *Assume that the function $g(x,t)$ satisfies conditions (15) with $\leq$ replaced by $<$. Let E be a ϕ-regular set, and let $E(t)$ be the ϕ-regular flow on $[0,T]$ starting from E constructed in Theorem 3.1. Then there exist $\sigma_0 > 0$ and ϕ-regular flows $E_\sigma(t)$, with $\sigma \in]-\sigma_0, \sigma_0[$, such that $E_\sigma(t)$ is a solution in $[0,T]$ of the evolution equation* velocity $= -(\kappa_\phi + g + \sigma) n_\phi$*, starting from E. Moreover* $\sup_{t\in[0,T]} \lim_{\sigma\to 0} d_H(\partial E_\sigma(t), \partial E(t)) = 0$.

Theorem 7.2. *Let E be a ϕ-regular set and assume that the function $g(x,t)$ satisfies conditions (15) with $\leq$ replaced by $<$. Then there exist $T > 0$ and a* unique *ϕ-regular flow $E(t)$ on $[0,T]$ such that $E(0) = E$.*

Proof. Assume that there exist two ϕ-regular flows $E_1(t)$, $E_2(t)$ on $[0,T]$, such that $E_1(0) = E_2(0) = E$, and let $E(t)$ be the ϕ-regular flow constructed in Theorem 3.1. We shall prove that $E(t) = E_1(t) = E_2(t)$ for any $t \in [0,T]$. Let σ_0, σ and $E_\sigma(t)$ be as in Lemma 7.1. Assume also that $\sigma > 0$. Let us show that $E_\sigma(t) \subseteq E_i(t)$ for $t \in [0,T]$ and $i = 1, 2$. Assume by contradiction that $E_\sigma(t)$ is not contained in $E_i(t)$ for some $t \in [0,T]$ and for i either 1 or 2. Let t_0 be the infimum of such t. Then $E_\sigma(t_0) \subseteq E_i(t_0)$, and setting $\eta(t) := \mathrm{dist}(\partial E_\sigma(t), \partial E_i(t))$, we have $\eta(t_0) = 0$. Moreover, by Theorem 7.1 and Remark 7.1, we get $\liminf_{\tau\to 0^+} \frac{\eta(t_0+\tau)-\eta(t_0)}{\tau} > 0$, which leads to a contradiction. Hence $E_\sigma(t) \subseteq E_i(t)$, and letting $\sigma \to 0^+$ we deduce $E(t) \subseteq E_i(t)$ for any $t \in [0,T]$. Choosing $\sigma \in]-\sigma_0, 0[$ we similarly obtain $E_\sigma(t) \supseteq E_i(t)$, hence $E(t) = E_1(t) = E_2(t)$ for any $t \in [0,T]$. $\square$

Using Theorem 7.2 we can now conclude the proof of Theorem 5.2. Indeed, reasoning as in the proof of Theorem 7.2 with $\sigma \in]0, \sigma_0[$, we have $E_\sigma(t) \subseteq E_2(t)$ for $t \in [0,T]$, and letting $\sigma \to 0^+$ we also get $E_1(t) \subseteq E_2(t)$, for $t \in [0,T]$.

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