Approximation and comparison for non–smooth anisotropic motion by mean curvature in $\mathbb{R}^N$

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Abstract

Assuming the existence of a regular flow, we prove that a reaction-diffusion inclusion provide a sub-optimal approximation for anisotropic motion by mean curvature in the non-smooth case. The result is valid in any space dimension and with a time-dependent driving force. The crystalline case is included. As a by-product of our analysis, a comparison theorem between regular flows is obtained. This result implies uniqueness of the original flow.

1 Introduction

It is well known [1], [9], [10], [8], [12] that motion by mean curvature, and more generally anisotropic motion by mean curvature [6], [11] can be viewed as the limit of suitably scaled reaction-diffusion equations. These results are valid in the smooth case, that is when the anisotropy is described by a *smooth* Finsler metric $\phi : \mathbb{R}^N \to [0, +\infty[$. The non-smooth case corresponds to the situation in which the boundary of the convex set $\mathcal{W}_\phi := \{\xi : \phi(\xi) \le 1\}$ has nondifferentiability points and flat portions; in particular, the crystalline case corresponds to a completely faceted $\mathcal{W}_\phi$. As pointed out by Taylor (see for instance [18], [20], [21]), the non-smooth case has relevant applications, and presents interesting mathematical questions [19]. Here the situation becomes quite delicate and has been analyzed mostly for crystalline anisotropies in two dimensions. We refer, among others, to the papers [17], [16], [2], [13], [14] for results in this direction. In [3] it is proved that crystalline motion by curvature in $N = 2$ dimensions can be approximated by a scaled reaction-diffusion inclusion, with a quasi-optimal error estimate of order $O(\varepsilon^2 |\log \varepsilon|^2)$. The aim of this note is to extend the analysis of [3] to arbitrary space dimensions and for general non-smooth anisotropies: in this framework we prove an approximation theorem for anisotropic motion by mean curvature with a sub-optimal error estimate of order $O(\varepsilon |\log \varepsilon|^2)$ (see Theorem 3.1). As a consequence, we obtain a comparison principle between limit evolutions (Theorem 3.2) which, in turn, provides uniqueness of the original flow. We thus extend to arbitrary dimensions and anisotropies a theorem proved in [14] in the two-dimensional crystalline case, and in [15] in the three-dimensional crystalline case. In order to get the approximation result, we assume the existence of a regular flow (see Definition 2.2). To the best knowledge of the authors, the characterization of those sets which admit a local in time evolution in presence of a non-smooth anisotropy is an open problem, which deserves further investigation. In the two-dimensional crystalline case this problem is completely solved (except for the driven motion under a non-uniform force), see the papers [14], [2] and references therein. In the crystalline three-dimensional case the problem seems to be difficult, see the paper [15] for related results.

2 Setting

In what follows $\Omega \subset \mathbb{R}^N$ is a bounded convex open set with smooth boundary, $N \geq 2$. We denote by $\cdot$ the euclidean scalar product in $\mathbb{R}^N$ and by d_H the euclidean Hausdorff distance between subsets of $\mathbb{R}^N$. By ∇ and div we always mean the gradient and the divergence with respect to the space variables.
For any set $C \subset \mathbb{R}^N$, we define $C^* := \{x \in \mathbb{R}^N : \exists \rho > 0 : |B_\rho(x) \setminus C| = 0\}$, where $|\cdot|$ is the Lebesgue measure and $B_\rho(x)$ denotes the euclidean open ball of radius ρ centered at x.
We indicate by $\phi : \mathbb{R}^N \to [0, +\infty[$ a convex function satisfying the properties

$$\Lambda^{-1}|\xi| \leq \phi(\xi) \leq \Lambda|\xi|, \qquad \phi(a\xi) = a\phi(\xi), \qquad \xi \in \mathbb{R}^N,\ a \geq 0, \tag{1}$$

for a suitable constant $\Lambda \in]0, +\infty[$, and by $\phi^o : \mathbb{R}^N \to [0, +\infty[$, $\phi^o(\xi^*) := \sup\{\xi^* \cdot \xi \ : \ \phi(\xi) \leq 1\}$ the dual of ϕ. We set

$$\mathcal{F}_\phi := \{\xi^* \in \mathbb{R}^N : \phi^o(\xi^*) \leq 1\}, \qquad \mathcal{W}_\phi := \{\xi \in \mathbb{R}^N : \phi(\xi) \leq 1\}.$$

We are mainly interested in the case $N \geq 3$ and when $\partial\mathcal{F}_\phi$, $\partial\mathcal{W}_\phi$ contain nondifferentiability points and/or flat portions.
Let $T^o : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ be the duality mapping defined by

$$T^o(\xi^*) := \frac{1}{2}\partial(\phi^o(\xi^*))^2, \qquad \xi^* \in \mathbb{R}^N,$$

where $\mathcal{P}(\mathbb{R}^N)$ is the class of all subsets of $\mathbb{R}^N$ and ∂ denotes the subdifferential in the sense of convex analysis. T^o is a possibly multivalued maximal monotone operator, and

$$T^o(a\xi^*) = aT^o(\xi^*), \qquad a \geq 0. \tag{2}$$

One can show that

$$\xi^* \cdot \xi = \phi^o(\xi^*)^2 = \phi(\xi)^2, \qquad \xi^* \in \mathbb{R}^N,\ \xi \in T^o(\xi^*). \tag{3}$$

Given $E \subset \mathbb{R}^N$ and $x \in \Omega$, we set

$$\mathrm{dist}_\phi(x,E) := \inf_{y\in E} \phi(x-y), \qquad \mathrm{dist}_\phi(E,x) := \inf_{y\in E} \phi(y-x),$$

$$d_\phi^E(x) := \mathrm{dist}_\phi(x,E) - \mathrm{dist}_\phi(\mathbb{R}^N \setminus E, x).$$

At each point where d_ϕ^E is differentiable one can prove [7] that

$$\phi^o(\nabla d_\phi^E) = 1. \tag{4}$$

If $t \in [0,T] \to E(t) \subset \mathbb{R}^N$ is a parametrized family of subsets of $\mathbb{R}^N$, we let

$$d_\phi^{E(t)}(x) := \mathrm{dist}_\phi(x, E(t)) - \mathrm{dist}_\phi(\mathbb{R}^N \setminus E(t), x).$$

Whenever no confusion is possible, we set $d_\phi(x,t) := d_\phi^{E(t)}(x)$. From now on, the symbols E, $E(t)$ will denote subsets of $\mathbb{R}^N$ whose boundary is contained in Ω.
The following definition closely follows an idea of M. Paolini.

Definition 2.1. *We say that the pair (E, n_ϕ) is ϕ-regular if*

(i) ∂E is a Lipschitz hypersurface;

(ii) there exists an open set $A \supset \partial E$ such that $n_\phi : A \to \mathbb{R}^N$ and

$$\begin{array}{lll} n_\phi \in L^\infty(A;\mathbb{R}^N), & \mathrm{div}\ n_\phi \in L^\infty(A), & \\ \phi(n_\phi(x)) = 1, & n_\phi(x) \in T^o\left(\nabla d_\phi^E(x)\right) & a.e.\ x \in A. \end{array}$$

The above definition imposes a sort of regularity of ∂E in a very weak sense. For technical reasons, we find more convenient to require the existence of a "normal" vector field n_ϕ in a tubular neighbourhood of ∂E (the set A), rather than on ∂E. In the smooth situation, n_ϕ is the Cahn-Hoffmann vector field and div n_ϕ is the correct notion of mean curvature depending on ϕ, see [7].
In the two-dimensional crystalline case one can easily construct ϕ-regular pairs, see [21], [14], [3]. In the three-dimensional crystalline case the situation is much more complicated, see [15].
Notice that, if (E, n_ϕ) is ϕ-regular, then

$$\nabla d_\phi^E \cdot n_\phi = 1 \qquad \text{a.e. on } A. \tag{5}$$

We now define a ϕ-regular flow as an evolution of ϕ-regular pairs moving with velocity, in the n_ϕ-direction, equals $-(\mathrm{div}\ n_\phi + g)$, where $g \in W^{1,\infty}([0,+\infty[)$ is given, and stands for the driving force of the flow.

Definition 2.2. *Let $T > 0$. A ϕ-regular flow on $[0,T]$ is a family of pairs $(E(t), n_\phi(\cdot,t))$, $t \in [0,T]$ such that*

(i) for any $t \in [0,T]$ the pair $(E(t), n_\phi(\cdot,t))$ is ϕ-regular with the set A of Definition 2.1 independent of $t \in [0,T]$;

(ii) $d_\phi \in Lip(A \times [0,T])$ and

$$\frac{\partial d_\phi}{\partial t}(x,t) = \mathrm{div}\ n_\phi(x,t) + g(t) + O\big(d_\phi(x,t)\big) \qquad a.e.\ (x,t) \in A \times [0,T].$$

As in Definition 2.1, we prefer to let evolve a tubular neighbourhood of the front, rather than the front itself. In the smooth case, the term $O(d_\phi(x,t))$ arises from the expansion of the differential of the Cahn-Hoffmann vector field near the front.
As a consequence of Theorem 3.2 below, it follows that a ϕ-regular flow depends only on $E(0)$, i.e., it does not depend on n_ϕ. The problem of characterizing those sets E and anisotropies ϕ such that there exists a ϕ-regular flow starting from E seems to be open.

Let us now introduce the relaxed evolution law. The double well potential $\Psi : \mathbb{R} \to [0,+\infty[$ is an even function of class $\mathcal{C}^2$ having only two zeroes at $\{-1,1\}$, say $\Psi(s) = (1-s^2)^2$. We set $\psi := \Psi'/2$.
We denote by γ the unique smooth strictly increasing function exponentially asymptotic, at $\pm\infty$, to the two stable zeroes ± 1 of ψ, satisfying

$$-\gamma'' + \psi(\gamma) = 0, \qquad \gamma(0) = 0. \tag{6}$$

We set $c_0 := \int_{\mathbb{R}} (\gamma')^2\, dy$. We denote [5] by $\eta \in H^2_{\mathrm{loc}}(\mathbb{R})$ the unique solution of the problem

$$-\eta'' + \psi'(\gamma)\eta = -\gamma' + \frac{c_0}{2}, \qquad \eta(0) = 0,$$

in the class of all functions in $H^2_{\mathrm{loc}}(\mathbb{R})$ with polynomial growth at infinity. η is even, $\lim_{y\to\pm\infty} \eta(y) = c_0/(2\psi'(1)) =: \eta_\infty$, and

$$|\eta - \eta_\infty|, |\eta'| \le C(1+|y|)\gamma', \qquad y \in \mathbb{R}, \tag{7}$$

where C is a positive constant.
Let $\delta \ge 3$ be a fixed natural number such that, if for any $\varepsilon \in]0,1]$ we let $z_\varepsilon := \delta|\log\varepsilon|$, then $\gamma(\pm z_\varepsilon) = \pm 1 + O(\varepsilon^{2\delta})$, $\gamma'(\pm z_\varepsilon) = O(\varepsilon^{2\delta})$, and

$$|\eta(\pm z_\varepsilon) - \eta_\infty|, |\eta'(\pm z_\varepsilon)| = O(\varepsilon^{2\delta-1}).$$

We construct [5] two functions $\gamma_\epsilon, \eta_\epsilon \in \mathcal{C}^{1,1}(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R} \setminus \{\pm z_\varepsilon, \pm 2z_\varepsilon\})$ which coincide, respectively, with γ, η on $[-z_\varepsilon, z_\varepsilon]$ and assume the corresponding asymptotic values $\pm 1, \eta_\infty$ outside the interval $]-2z_\varepsilon, 2z_\varepsilon[$. We can also assume that γ_ϵ and η_ϵ satisfy (7), and that γ_ϵ is strictly increasing on $]-2z_\varepsilon, 2z_\varepsilon[$ (provided ε is small enough).

Let us now introduce the relaxed evolution problem. Let $\varepsilon > 0$, $T > 0$, and $u_0 \in H^1(\Omega)$ be such that $\mathcal{E}_\phi(u_0) := \int_\Omega \phi^o(\nabla u_0)^2 + \Psi(u_0)\, dx < +\infty$ and $\frac{\partial u_0}{\partial \nu_\Omega} = 0$, where ν_Ω is the outward unit normal to $\partial\Omega$. Let us consider the problem

$$\varepsilon u_t - \varepsilon \mathrm{div}(T^o(\nabla u)) + \frac{1}{\varepsilon}\psi(u) \ni \frac{c_0}{2} g \qquad \text{in } Q, u(\cdot, 0) = u_0(\cdot) \text{in } \Omega, \frac{\partial u}{\partial \nu_\Omega} = 0 \qquad \text{in } \partial\Omega \times]0, T[, \tag{8}$$

where $Q := \Omega \times\ ]0, T[$. The notion of variational sub- and supersolution of (8) reads as follows [3].

Definition 2.3. *A couple (u, ζ) is a subsolution of (8) if, for any $T > 0$, the following properties hold:*

(i) $u \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ *and* $\zeta \in (L^2(Q))^N$*;*

(ii) for any $\varphi \in H^1(\Omega; [0, +\infty[)$ *and a.e.* $t \in]0, T[$ *there holds*

$$\int_\Omega \Big(\varepsilon u_t \varphi + \varepsilon \zeta \cdot \nabla \varphi + \frac{1}{\varepsilon}\psi(u)\varphi - \frac{c_0}{2} g \varphi \Big)\, dx \leq 0; \tag{9}$$

(iii) for a.e. $x \in \Omega$ *there holds* $u(x, 0) \leq u_0(x)$*;*

(iv) for a.e. $(x, t) \in Q$ *there holds*

$$\zeta(x, t) \in T^o(\nabla u(x, t)). \tag{10}$$

The couple (u, ζ) is a supersolution of (8) if (i) and (iv) hold, and conditions (ii) and (iii) hold with $\geq$ in place of $\leq$. The couple (u, ζ) is a solution of (8) if it is both a subsolution and a supersolution.

Notice that by (i), (iv) and (1), we have $\zeta \in L^\infty\big(0, T; (L^2(\Omega))^N\big)$.

The following elementary comparison lemma, proved in [3], is crucial in order to get the main results.

Lemma 2.4. *Let (u^-, ζ^-) and (u^+, ζ^+) be respectively a subsolution and a supersolution of (8). Then $u^- \leq u^+$ a.e. in Q.*

Following [22] we then have

Theorem 2.5. *Problem (8) admits a solution (u, ζ). Moreover, if (u_1, ζ_1) and (u_2, ζ_2) are two solutions of (8), then $u_1 = u_2$ a.e. in Q.*

One can prove [22] that, if there exists $\zeta_0 \in T^o(\nabla u_0)$ such that $\mathrm{div}\zeta_0 \in L^2(\Omega)$, then the solution (u, ζ) to (8) is such that $u \in W^{1,\infty}(0, T; L^2(\Omega))$ and $\mathcal{E}_\phi(u) \in W^{1,\infty}(0, T)$. Moreover, if u is bounded on Q, then $\mathrm{div}\, \zeta \in L^\infty(0, T; L^2(\Omega))$.

We do not know wether the smoothness of u_0 on Ω (even in the case $N = 2$, $g = 0$, ϕ crystalline) implies the continuity of u on Q.

3 Approximation and comparison principle

The main results of the paper are the following two theorems.

Theorem 3.1. *Let $(E(t), n_\phi(\cdot,t))$ be a $\phi-$regular flow on $[0,T]$. For any $\varepsilon > 0$ let u_ε be the solution of problem (8) with initial datum*

$$u_\varepsilon(x,0) = u_\varepsilon^0(x) := \gamma_\varepsilon\left(\frac{d_\phi(x,0)}{\varepsilon}\right) + \varepsilon\eta_\epsilon\left(\frac{d_\phi(x,0)}{\varepsilon}\right)g(0). \tag{11}$$

Let $\Sigma_\varepsilon(t)$ denote the complement in Ω of the set $\{x : u_\varepsilon(x,t) > 0\}^* \cup \{x : u_\varepsilon(x,t) < 0\}^*$. Then there exist $\varepsilon_0 \in]0,1]$ and a constant C depending on $(E(t))_{t\in[0,T]}$, g, and independent of $\varepsilon \in]0,\varepsilon_0]$, such that for all $\varepsilon \in]0,\varepsilon_0]$ there holds

$$\sup_{t\in[0,T]} d_H(\Sigma_\varepsilon(t), \partial E(t)) \le C\varepsilon|\log\varepsilon|^2. \tag{12}$$

Theorem 3.2. *Let $(E_1(t), n_\phi^{(1)}(\cdot,t))$, $(E_2(t), n_\phi^{(2)}(\cdot,t))$ be two ϕ-regular flows on $[0,T]$. Then*

$$E_1(0) \subseteq E_2(0) \Rightarrow E_1(t) \subseteq E_2(t), \qquad t \in [0,T]. \tag{13}$$

Proof of Theorem 3.1. For any $t \in [0,T]$ set

$$y = y(x,t) := \frac{d_\phi(x,t)}{\varepsilon}, \qquad y_\epsilon = y_\epsilon(x,t) := y(x,t) - \theta(t)|\log\varepsilon|^2, \tag{14}$$

$$\theta(t) := c\exp\big(Kt\big), \qquad t \in [0,T], \tag{15}$$

where c and K are two positive constants to be defined later on independently of ε. Let also

$$\mathcal{T}_\varepsilon(t) := \{x \in \Omega : |y_\epsilon(x,t)| < 2z_\varepsilon\}, \qquad \mathcal{T}_\varepsilon := \bigcup_{t\in[0,T]} \mathcal{T}_\varepsilon(t) \times \{t\},$$

$$\mathcal{T}_\varepsilon^-(t) := \{x \in \Omega : y_\epsilon(x,t) \le -2z_\varepsilon\}, \quad \mathcal{T}_\varepsilon^+(t) := \{x \in \Omega : y_\epsilon(x,t) \ge 2z_\varepsilon\}.$$

We assume that ε is small enough such that the closure of $\displaystyle\bigcup_{t\in[0,T]} \mathcal{T}_\varepsilon(t)$ is contained in Ω, $\mathcal{T}_\varepsilon$ is contained in the set A of Definition 2.2, and $-\gamma_\epsilon'' + \psi(\gamma_\epsilon) = -\eta_\epsilon'' + \psi'(\gamma_\epsilon)\eta_\epsilon - \frac{c_0}{2}g + \gamma_\epsilon' = O(\varepsilon^{2\delta-3})$ in $]-2z_\varepsilon, 2z_\varepsilon[$.
We define $v_\varepsilon^- : \Omega \times [0,T] \to \mathbb{R}$ and $\zeta_\varepsilon^- : \Omega \times [0,T] \to \mathbb{R}^N$ as follows:

$$x \in \mathcal{T}_\varepsilon(t) \Longrightarrow \begin{cases} v_\varepsilon^-(x,t) := \gamma_\varepsilon(y_\varepsilon) + \varepsilon\eta_\varepsilon(y_\varepsilon)g(t) - \Theta\varepsilon^2|\log\varepsilon|^2, \\ \\ \zeta_\varepsilon^- := [\varepsilon^{-1}\gamma_\epsilon'(y_\epsilon) + \eta_\epsilon'(y_\epsilon)g(t)]n_\phi(x,t), \end{cases}$$

$$x \in \mathcal{T}_\varepsilon^\pm(t) \Longrightarrow \begin{cases} v_\varepsilon^-(x,t) := \pm 1 + \varepsilon\eta_\infty g(t) - \Theta\varepsilon^2|\log\varepsilon|^2, \\ \\ \zeta_\varepsilon^-(x,t) := 0, \end{cases} \tag{16}$$

where $\Theta > 0$ is a constant to be defined later on independently of ε. Notice that $v_\varepsilon^- \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap \mathcal{C}^0(\Omega \times [0,T])$ and $\zeta_\varepsilon^- \in (L^2(Q))^N$. In a similar fashion we can define $(v_\varepsilon^+, \zeta_\varepsilon^+)$ by changing the sign in front of $\theta(t)$ in (14) and in front of Θ in (16).
We want to show that $(v_\varepsilon^-, \zeta_\varepsilon^-)$ (resp. $(v_\varepsilon^+, \zeta_\varepsilon^+)$) is a subsolution (resp. supersolution) of (8). We shall focus our attention on $(v_\varepsilon^-, \zeta_\varepsilon^-)$.
A direct computation, using (2) and the inclusion $n_\phi(x,t) \in T^o(\nabla d_\phi(x,t))$, yields

$$\begin{aligned} T^o(\nabla v_\varepsilon^-) &= T^o\Big([\varepsilon^{-1}\gamma'_\epsilon(y_\epsilon) + \eta'_\epsilon(y_\epsilon)g(t)]\nabla d_\phi(x,t)\Big) \\ &= [\varepsilon^{-1}\gamma'_\epsilon(y_\epsilon) + \eta'_\epsilon(y_\epsilon)g(t)]T^o(\nabla d_\phi(x,t)) \ni \zeta_\varepsilon^-(x,t). \end{aligned}$$

Hence $\zeta_\varepsilon^-(x,t) \in T^o(\nabla v_\varepsilon^-(x,t))$ for a.e. $(x,t) \in Q$ (condition (10)).
Moreover one can prove [5] that there exist a real number $\Theta > 0$, independent of ε, and $\varepsilon_0 > 0$ such that $v_\varepsilon^-(\cdot,0) \leq u_\varepsilon^0(\cdot)$ in Ω, for any $0 < \varepsilon < \varepsilon_0$.

Claim. There exist $\varepsilon_0 > 0$ and positive real numbers c, K, Θ, independent of ε, such that for any $\varepsilon \in]0, \varepsilon_0[$ and $\varphi \in H^1(\Omega; [0,+\infty[)$ there holds

$$\int_\Omega \Big(\varepsilon \partial_t v_\varepsilon^- \varphi + \varepsilon \zeta_\varepsilon^- \cdot \nabla\varphi + \frac{1}{\varepsilon}\psi(v_\varepsilon^-)\varphi - \frac{c_0}{2} g\varphi\Big)\, dx \leq 0, \qquad \text{a.e. } t \in [0,T]. \quad (17)$$

Once (17) is proved, by Lemma 2.4 we get the crucial inequality $v_\varepsilon^- \leq u_\varepsilon$ (and similarly $u_\varepsilon \leq v_\varepsilon^+$).
Let us prove the claim. For simplicity we use the notation $(v_\varepsilon, \zeta_\varepsilon)$ in place of $(v_\varepsilon^-, \zeta_\varepsilon^-)$. The left hand side of inequality (17) can be equivalently written as

$$\begin{aligned} &\int_{\mathcal{T}_\varepsilon(t)} \Big(\varepsilon\partial_t v_\varepsilon - \varepsilon \mathrm{div}\zeta_\varepsilon + \frac{1}{\varepsilon}\psi(v_\varepsilon) - \frac{c_0}{2}g\Big)\varphi\, dx + \int_{\Omega\setminus\mathcal{T}_\varepsilon(t)} \Big(\varepsilon\partial_t v_\varepsilon + \frac{1}{\varepsilon}\psi(v_\varepsilon) - \frac{c_0}{2}g\Big)\varphi\, dx \\ &\qquad + \int_{\partial\mathcal{T}_\varepsilon(t)} \varepsilon\varphi\, \zeta_\varepsilon \cdot \nu_\varepsilon \, d\mathcal{H}^{N-1} =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \end{aligned}$$

where ν_ε denotes the a.e. defined euclidean outward unit normal to $\partial\mathcal{T}_\varepsilon(t)$, and $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.
Using (14), (ii) of Definition 2.2, and the fact that $d_\phi = \theta\varepsilon|\log\varepsilon|^2 + O(\varepsilon|\log\varepsilon|)$ in $\mathcal{T}_\varepsilon$, direct computations yield, for a.e. $(x,t) \in \mathcal{T}_\varepsilon$,

$$\begin{aligned} \varepsilon\partial_t v_\varepsilon &= \varepsilon(\gamma'_\epsilon + \varepsilon\eta'_\epsilon g)\partial_t y_\varepsilon + \varepsilon^2\eta_\epsilon g_t = (\mathrm{div}\ n_\phi + g)(\gamma'_\epsilon + \varepsilon\eta'_\epsilon g) \\ &\quad + \gamma'_\epsilon\theta O(\varepsilon|\log\varepsilon|^2) + O(\varepsilon|\log\varepsilon|) - \theta'\varepsilon|\log\varepsilon|^2(\gamma'_\epsilon + \varepsilon\eta'_\epsilon g) \\ &\quad = (\gamma'_\epsilon + \varepsilon\eta'_\epsilon g)\mathrm{div}\ n_\phi + \gamma'_\epsilon g - \gamma'_\epsilon\theta'\varepsilon|\log\varepsilon|^2 + \gamma'_\epsilon\theta O(\varepsilon|\log\varepsilon|^2) + O(\varepsilon|\log\varepsilon|). \end{aligned}$$

Furthermore,

$$\varepsilon \mathrm{div}\ \zeta_\varepsilon = \varepsilon(\varepsilon^{-1}\gamma''_\epsilon + \eta''_\epsilon g)\nabla y_\varepsilon \cdot n_\phi + \varepsilon(\varepsilon^{-1}\gamma'_\epsilon + \eta'_\epsilon g)\mathrm{div}\ n_\phi.$$

By (4) and (3) we have $\nabla y_\varepsilon \cdot n_\phi = \varepsilon^{-1}\nabla d_\phi \cdot n_\phi = \varepsilon^{-1}$, hence

$$\varepsilon \mathrm{div}\ \zeta_\varepsilon = \varepsilon^{-1}\gamma_\epsilon'' + \eta_\epsilon'' g + (\gamma_\epsilon' + \varepsilon\eta_\epsilon' g)\mathrm{div}\ n_\phi.$$

Expanding $\varepsilon^{-1}\psi(v_\varepsilon) = \varepsilon^{-1}\psi(\gamma_\epsilon) + g\eta_\epsilon\psi'(\gamma_\epsilon) - \Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon) + O(\varepsilon)$, we get

$$\begin{aligned}
&\varepsilon\partial_t v_\varepsilon - \varepsilon \mathrm{div}\ \zeta_\varepsilon + \frac{1}{\varepsilon}\psi(v_\varepsilon) - \frac{c_0}{2}g \\
&= \varepsilon^{-1}(-\gamma_\epsilon'' + \psi(\gamma_\epsilon)) + g\left(-\eta_\epsilon'' + \psi'(\gamma_\epsilon)\eta_\epsilon - \frac{c_0}{2} + \gamma_\epsilon'\right) \\
&\quad - \Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon) - \gamma_\epsilon'\theta'\varepsilon|\log\varepsilon|^2 + \gamma_\epsilon'\theta O(\varepsilon|\log\varepsilon|^2) + O(\varepsilon|\log\varepsilon|) \\
&= \gamma_\epsilon'\varepsilon|\log\varepsilon|^2(\theta O(1) - \theta') - \Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon) + O(\varepsilon^{2\delta-3}) + O(\varepsilon|\log\varepsilon|) \\
&= \gamma_\epsilon'\varepsilon|\log\varepsilon|^2(\theta O(1) - \theta') - \Theta\varepsilon|\log\varepsilon|^2\psi'(\gamma_\epsilon) + O(\varepsilon|\log\varepsilon|).
\end{aligned}$$

Recalling the definition of θ in (15) we have $\theta O(1) - \theta' \le -\theta \le c$ for $K > 0$ sufficiently large (independently of ε), so that

$$\varepsilon\partial_t v_\varepsilon - \varepsilon\mathrm{div}\zeta_\varepsilon + \frac{1}{\varepsilon}\psi(v_\varepsilon) - \frac{c_0}{2}g \le -\varepsilon|\log\varepsilon|^2(c\gamma_\epsilon' + \Theta\psi'(\gamma_\epsilon)) + O(\varepsilon|\log\varepsilon|).$$

As $\sigma\gamma_\epsilon' + \psi'(\gamma_\epsilon)$ is uniformly positive for a proper choice of the positive constant σ, we realize that, if c and Θ are large enough (independently of ε),

$$\varepsilon\partial_t v_\varepsilon - \varepsilon\mathrm{div}\zeta_\varepsilon + \frac{1}{\varepsilon}\psi(v_\varepsilon) - \frac{c_0}{2}g \le 0 \qquad \text{in } \mathcal{T}_\varepsilon.$$

We then have $\mathcal{I}_1 \le 0$, and reasoning as in [3], also $\mathcal{I}_2 \le 0$. Moreover, from the definition of ζ_ε it follows that $\zeta_\varepsilon(x,t)_{|\partial\mathcal{T}_\varepsilon(t)} = 0$ hence $\mathcal{I}_3 = 0$.
The proof of the claim is concluded.

Summing up, we have proved the following result: there exist $\varepsilon_0 > 0$, an exponentially increasing continuous function $\theta : [0,T] \to]0,+\infty[$ and a real number $\Theta > 0$, both independent of ε, such that, if u_ε denotes the solution of (8) with initial datum (11), then $v_\varepsilon^-(x,t) \le u_\varepsilon(x,t) \le v_\varepsilon^+(x,t)$ for a.e. $(x,t) \in Q$ and for $\varepsilon \in]0,\varepsilon_0[$. Theorem 3.1 follows now arguing as in [6, Theorem 6.1]. □

Remark 3.3. *A similar statement of Theorem 3.1 holds for the double obstacle problem [3], where $\varepsilon|\log\varepsilon|^2$ in (12) is replaced by ε.*

Proof of Theorem 3.2. Let $i \in \{1,2\}$ and let $u_\varepsilon^{(i)}$ be the function given by Theorem 3.1, where the initial datum $u_\varepsilon^{0(i)}$ is fixed as in (11), with $d_\phi(x,0) = d_\phi^{E_i}(x)$. Since for $\varepsilon > 0$ small enough the function $z \to \gamma_\epsilon(z) + \varepsilon g(0)\eta_\epsilon(z)$ is strictly increasing on $]-2z_\varepsilon, 2z_\varepsilon[$, from (11) we have that $u_\varepsilon^{0(1)} \ge u_\varepsilon^{0(2)}$ in Ω. Hence, by Lemma 2.4, it follows that

$$u_\varepsilon^{(1)} \ge u_\varepsilon^{(2)} \qquad \text{a.e. in } Q. \tag{18}$$

Applying (12) of Theorem 3.1, from (18) we get (13). □

Corollary 3.4. *Let* $(E_1(t), n_\phi^{(1)}(\cdot, t))$, $(E_2(t), n_\phi^{(2)}(\cdot, t))$ *be two* ϕ*-regular flows on* $[0, T]$*. Then*

$$E_1(0) = E_2(0) \Rightarrow E_1(t) = E_2(t), \qquad t \in [0, T]. \tag{19}$$

Notice that, in view of Theorem 3.2, one can implement the barrier method of De Giorgi (see [4]) to construct a unique global weak solution of anisotropic motion by mean curvature in the non-smooth case.

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