COMPARISON RESULTS BETWEEN MINIMAL BARRIERS AND VISCOSITY SOLUTIONS FOR GEOMETRIC EVOLUTIONS

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1. Introduction

In [12] De Giorgi introduced a notion of weak solution, called minimal barrier, for a wide class of evolution problems. In the particular case of geometric flows of subsets of $\mathbf{R}^n$, the concept of minimal barrier can be described as follows (see Section 2.1 for precise definitions). First we choose a nonempty family $\mathcal{F}$ of maps which take some time interval into the set $\mathcal{P}(\mathbf{R}^n)$ of all subsets of $\mathbf{R}^n$: for instance $\mathcal{F}$ can be the family of all smooth local evolutions with respect to a given geometric law. Then we define the class $\mathcal{B}(\mathcal{F})$ of all maps $\phi : [0,+\infty[\to \mathcal{P}(\mathbf{R}^n)$ which are barriers for $\mathcal{F}$ in $[0,+\infty[$ with respect to the inclusion of sets, that is, if $f : [a,b] \subseteq [0,+\infty[\to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$, then it must hold $f(b) \subseteq \phi(b)$. Finally, we define the minimal barrier $\mathcal{M}(E,\mathcal{F})(t)$ with origin in $E \subseteq \mathbf{R}^n$, with respect to $\mathcal{F}$, at time $t \in [0,+\infty[$ as

$$\mathcal{M}(E,\mathcal{F})(t) := \bigcap\{\phi(t) : \phi : [0,+\infty[\to \mathcal{P}(\mathbf{R}^n), \phi \in \mathcal{B}(\mathcal{F}), \phi(0) \supseteq E\}. \tag{1.1}$$

We stress the dependence on $\mathcal{F}$ of the minimal barrier (see Example 2.1) and also the fact that the minimal barrier is unique and globally defined, for an arbitrary initial set E. Therefore, given any initial function $u_0 : \mathbf{R}^n \to \mathbf{R}$, (1.1) yields a unique global evolution function $\mathcal{M}_{u_0,\mathcal{F}}(t,x)$ (assuming u_0 as initial datum), defined as the function which, for any $\lambda \in \mathbf{R}$, has $\mathcal{M}(\{u_0 < \lambda\},\mathcal{F})(t)$ as λ-sublevel set at time $t \in [0,+\infty[$.

The aim of this paper is to compare the minimal barrier with the viscosity solution of geometric fully nonlinear parabolic problems of the form

$$\frac{\partial u}{\partial t} + F(t,x,\nabla u,\nabla^2 u) = 0. \tag{1.2}$$

The definition of viscosity solution has been introduced by Crandall and P.-L. Lions [11] (we refer to [10] for a bibliography on this argument). It has been exploited by Evans-Spruck [13] in the case of motion by mean curvature and by Chen-Giga-Goto [9], Giga-Goto-Ishii-Sato [16] in the case of geometric evolutions of the form (1.2). We recall that, in order to define the viscosity evolution $V(E)(t)$ of a bounded open set $E \subseteq \mathbf{R}^n$ for problem (1.2), first we find the unique continuous viscosity solution of (1.2) (with a suitable initial datum negative inside E) and then we recover $V(E)(t)$ by setting $V(E)(t) := \{x \in \mathbf{R}^n : v(t,x) < 0\}$.

A comparison result for sets E with compact boundary in case of driven motion by mean curvature (whose corresponding function F is given by $F(t,x,p,X) =$

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$-\mathrm{tr}((\mathrm{Id} - p \otimes p/|p|^2)X) + g(t,x)|p|$, g being the driving force) has been proved in [7], and shows that the two weak definitions are essentially equivalent. The proofs of [7] rely on a paper by Ilmanen [20], where viscosity solutions are compared, in the case of motion by mean curvature, with the so called set theoretic subsolutions.

The results of [20,7] are based on Ilmanen's interposition lemma and on Huisken's estimates [18] of the existence time for the evolution of a smooth compact hypersurface in dependence on the L^∞ norm of its second fundamental form, without requiring bounds on further derivatives of the curvatures. The above results of Ilmanen and Huisken apply basically to the case of motion by mean curvature; it seems difficult to recover the time estimates of [18] for a general evolution law of the form (1.2) (some generalizations of Huisken's results can be found in [2,3]). This is the main reason for which we follow, in this paper, a completely different approach to the problem, which allows us to compare minimal barriers with viscosity solutions for a general F.

A further remark on the definition of minimal barrier is the following: denoting by $\mathcal{F}_F$ the family of all local smooth geometric supersolutions of (1.2) (see Definition 2.5), to ensure that $\mathcal{M}(E, \mathcal{F}_F)$ is well defined we do not need to assume that F, if considered as a function on symmetric matrices, is decreasing (degenerate ellipticity condition); it turns out [6] that when E is open we have

$$\mathcal{M}(E, \mathcal{F}_F) = \mathcal{M}(E, \mathcal{F}_{F^+}), \tag{1.3}$$

where F^+ is defined as the smallest function which is degenerate elliptic and greater than or equal to F, i.e.,

$$F^+(t,x,p,X) := \sup\{F(t,x,p,Y) : Y \geq X\}. \tag{1.4}$$

Such a result is obtained in the present paper by passing through the viscosity theory (Corollary 6.2) and allows to remove the degenerate ellipticity assumption from the hypotheses of all results of Sections 3 and 5, provided that also F^+ satisfies the assumptions listed in [16].

Finally we observe that $\mathcal{M}(E, \mathcal{F}_F)$ and $\mathcal{M}_{u_0, \mathcal{F}_F}$ verify by definition the comparison principle and it is immediate to check that, if ∂E is smooth, $\mathcal{M}(E, \mathcal{F}_F)$ coincides with the classical evolution of E, as long as the latter exists, provided that the classical evolutions are barriers on their time interval of definition (which is the case, for instance, for uniformly elliptic smooth functions F).

Let us briefly summarize the content and the main results of the present paper. In Section 2 we introduce some notation and the notion of minimal barrier and regularized minimal barriers with respect to a family $\mathcal{F}$ (Definitions 2.2, 2.3, 2.4). In Proposition 2.2 we show that the minimal barriers agree with the smooth evolutions whenever the latter exist. We conclude Section 2 with two examples of minimal barriers obtained with particular choices of $\mathcal{F}$: Example 2.1 concerns motion by mean curvature whenever $\mathcal{F}$ consists of smooth convex evolutions; in Example 2.2 we consider the case of inverse mean curvature flow. Sections 3-5 are concerned with geometric evolutions of the form (1.2) where F satisfies some of the assumptions made by Giga-Goto-Ishii-Sato in [16]. In Section 4 we prove some auxiliary results on barriers used throughout the paper. The comparison result between barriers and viscosity solutions is divided into two parts. In Section 3 we prove that the sublevel sets of a viscosity subsolution of (1.2) are barriers (Theorem 3.2) and in

Section 5 we prove that a function whose sublevel sets are barriers is a viscosity subsolution of (1.2) (Theorem 5.1). In Theorems 3.2 and 5.1, in order to simplify the proofs, we distinguish the case in which F does not depend explicitly on x with the general case; if F is not degenerate elliptic we extend the results to the function F^+. In Corollary 6.1 we summarize the comparison results whenever there exists a unique uniformly continuous viscosity solution v of (1.2) having a given initial datum. More precisely, if $E \subseteq \mathbf{R}^n$ is a bounded set, for any $t \in [0, +\infty[$ we have

$$\begin{aligned} \mathcal{M}_*(E, \mathcal{F}_F)(t) &= \{x \in \mathbf{R}^n : v(t,x) < 0\}, \\ \mathcal{M}^*(E, \mathcal{F}_F)(t) &= \{x \in \mathbf{R}^n : v(t,x) \leq 0\}, \end{aligned} \tag{1.5}$$

where $\mathcal{M}_*(E, \mathcal{F}_F)$ and $\mathcal{M}^*(E, \mathcal{F}_F)$ are the lower and upper regularized minimal barriers (see Definition 2.3). In particular

$$\mathcal{M}^*(E, \mathcal{F}_F)(t) \setminus \mathcal{M}_*(E, \mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) = 0\}. \tag{1.6}$$

Equality (1.6) is connected with the so called fattening phenomenon (see (2.5) and Remark 6.1).

In case of nonuniqueness of viscosity solutions, we show in Corollary 6.3 that $\mathcal{M}_{u_0, \mathcal{F}_F}$ coincides with the maximal viscosity subsolution, see also Example 6.1. If F is not degenerate elliptic and if F^+ verifies the assumptions of Corollary 6.1, then (1.5) holds when v is the viscosity solution of (1.2) with F^+ in place of F (Corollary 6.2). In Remark 6.6 we extend our results to the case in which F has superlinear growth and E is unbounded, where the notion of viscosity evolution is the one introduced by Ishii-Souganidis in [22].

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2. Notation and main definitions

In the following we let $I := [t_0, +\infty[$, for a fixed $t_0 \in \mathbf{R}$; in Sections 3-6 we will take $t_0 = 0$. We denote by $\mathcal{P}(\mathbf{R}^n)$ the family of all subsets of $\mathbf{R}^n$, $n \geq 1$. Given a set $C \subseteq \mathbf{R}^n$, we denote by $\text{int}(C)$, $\overline{C}$ and ∂C the interior part, the closure and the boundary of C, respectively; χ_C is the characteristic function of C, i.e., $\chi_C(x) = 1$ if $x \in C$, $\chi_C(x) = 0$ if $x \notin C$. If $C \neq \mathbf{R}^n$ and $C \neq \emptyset$, we set

$$d_C(x) := \text{dist}(x, C) - \text{dist}(x, \mathbf{R}^n \setminus C),$$

and for any $\varrho > 0$

$$C_\varrho^- := \{x \in \mathbf{R}^n : \text{dist}\,(x, \mathbf{R}^n \setminus C) > \varrho\}, \tag{2.1}$$

$$C_\varrho^+ := \{x \in \mathbf{R}^n : \text{dist}(x, C) < \varrho\}. \tag{2.2}$$

Given a map $\phi : J \to \mathcal{P}(\mathbf{R}^n)$, where $J \subseteq \mathbf{R}$ is a convex set, if $\phi(t) \neq \mathbf{R}^n$ and $\phi(t) \neq \emptyset$ for any $t \in J$ we let $d_\phi : J \times \mathbf{R}^n \to \mathbf{R}$ be the function defined by

$$d_\phi(t,x) := \text{dist}(x, \phi(t)) - \text{dist}(x, \mathbf{R}^n \setminus \phi(t)) = d_{\phi(t)}(x).$$

If $\phi_1, \phi_2 : J \to \mathcal{P}(\mathbf{R}^n)$, by $\phi_1 \subseteq \phi_2$ (resp. $\phi_1 = \phi_2$) we mean $\phi_1(t) \subseteq \phi_2(t)$ (resp. $\phi_1(t) = \phi_2(t)$) for any $t \in J$.
Given a function $v : J \times \mathbf{R}^n \to \mathbf{R}$ we denote by v_* (resp. v^*) the lower (resp. upper) semicontinuous envelope of v.
For $x \in \mathbf{R}^n$ and $R > 0$ we set $B_R(x) := \{y \in \mathbf{R}^n : |y - x| < R\}$ and $\mathbf{S}^{n-1} := \{x \in \mathbf{R}^n : |x| = 1\}$.
If $c_1, c_2 \in \mathbf{R}$, we let $c_1 \wedge c_2 = \min(c_1, c_2)$ and $c_1 \vee c_2 = \max(c_1, c_2)$.
We denote by $\mathrm{Sym}(n)$ the space of all symmetric real $(n \times n)$-matrices, endowed with the norm $|X|^2 = \sum_{i,j=1}^{n} X_{ij}^2$, where $X = (X_{ij})$.
Given $p \in \mathbf{R}^n \setminus \{0\}$, we set $P_p := \mathrm{Id} - p \otimes p/|p|^2$. Finally we define

$$J_0 := I \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n), \qquad J_1 := I \times (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n).$$

Remark 2.1. *All results of this paper still hold when I in J_0 and J_1 is replaced by $[t_0, t_0 + T[$, for some $T > 0$.*

2.1. Definitions of minimal barriers. The following two definitions are a particular case of the definitions proposed in [12].

Definition 2.1. *Let $\mathcal{F}$ be a family of functions with the following property: for any $f \in \mathcal{F}$ there exist $a, b \in \mathbf{R}$, $a < b$, such that $f : [a, b] \to \mathcal{P}(\mathbf{R}^n)$. A function ϕ is a barrier with respect to $\mathcal{F}$ if and only if there exists a convex set $L \subseteq I$ such that $\phi : L \to \mathcal{P}(\mathbf{R}^n)$ and the following property holds: if $f : [a, b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$ then $f(b) \subseteq \phi(b)$. We denote by $\mathcal{B}(\mathcal{F})$ the family of all barriers ϕ such that $L = I$ (that is, barriers on the whole of I).*

Definition 2.2. *Let $E \subseteq \mathbf{R}^n$ be a given set. The minimal barrier $\mathcal{M}(E, \mathcal{F}, t_0) : I \to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time t_0) with respect to the family $\mathcal{F}$ at any time $t \in I$ is defined by*

$$\mathcal{M}(E, \mathcal{F}, t_0)(t) := \bigcap \{\phi(t) : \quad \phi : I \to \mathcal{P}(\mathbf{R}^n),\ \phi \in \mathcal{B}(\mathcal{F}),\ \phi(t_0) \supseteq E\}. \tag{2.3}$$

Let us observe that $\mathcal{M}(E, \mathcal{F}, t_0) \in \mathcal{B}(\mathcal{F})$ (uniqueness of the minimal barrier), $\mathcal{M}(E, \mathcal{F}, t_0)(t_0) = E$, and that $E_1 \subseteq E_2$ implies $\mathcal{M}(E_1, \mathcal{F}, t_0) \subseteq \mathcal{M}(E_2, \mathcal{F}, t_0)$ (comparison property).

For simplicity of notation, we drop the dependence on t_0 of the minimal barrier, thus we write $\mathcal{M}(E, \mathcal{F})$ in place of $\mathcal{M}(E, \mathcal{F}, t_0)$.

The following regularizations have been introduced in [7] for driven motion by mean curvature, and will be useful in the sequel.

Definition 2.3. *Let $E \subseteq \mathbf{R}^n$. If $t \in I$ we set*

$$\begin{aligned} \mathcal{M}_*(E, \mathcal{F})(t) &:= \bigcup_{\varrho > 0} \mathcal{M}(E_\varrho^-, \mathcal{F})(t), \\ \mathcal{M}^*(E, \mathcal{F})(t) &:= \bigcap_{\varrho > 0} \mathcal{M}(E_\varrho^+, \mathcal{F})(t). \end{aligned} \tag{2.4}$$

Following [5], we say that the set E develops fattening (with respect to $\mathcal{F}$) at time $t_1 \in I$ if,

$$\mathcal{H}^n\Big(\mathcal{M}^*(E, \mathcal{F})(t) \setminus \mathcal{M}_*(E, \mathcal{F})(t)\Big) = 0 \qquad \text{for } t \in [t_0, t_1],$$

$$\mathcal{H}^n\Big(\mathcal{M}^*(E, \mathcal{F})(t) \setminus \mathcal{M}_*(E, \mathcal{F})(t)\Big) > 0 \qquad \text{for some } t \in]t_1, +\infty[\ ,$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure (note that one could define the m-dimensional fattening by replacing $\mathcal{H}^n$ with $\mathcal{H}^m$, $0 < m \le n$).

Once the evolution of an arbitrary set is uniquely defined, we can define the unique evolution of an arbitrary initial function u_0.

Definition 2.4. *Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a given function. The two functions $\mathcal{M}_{u_0,\mathcal{F}}, \overline{\mathcal{M}}_{u_0,\mathcal{F}} : I \times \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ are defined by*

$$\begin{aligned} \mathcal{M}_{u_0,\mathcal{F}}(t,x) &:= \inf\{\lambda \in \mathbf{R} : \mathcal{M}(\{u_0 < \lambda\}, \mathcal{F})(t) \ni x\}, \\ \overline{\mathcal{M}}_{u_0,\mathcal{F}}(t,x) &:= \inf\{\lambda \in \mathbf{R} : \mathcal{M}_*(\{u_0 < \lambda\}, \mathcal{F})(t) \ni x\}. \end{aligned} \tag{2.6}$$

If $\mathcal{F}$ consists of functions $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$ such that $f(t)$ is compact for any $t \in [a,b]$, if $\mathcal{M}(A, \mathcal{F})(t)$ is open for any open set $A \subseteq \mathbf{R}^n$, and if $u_0 : \mathbf{R}^n \to \mathbf{R}$ is upper semicontinuous, then there holds

$$\{x \in \mathbf{R}^n : \mathcal{M}_{u_0,\mathcal{F}}(t,x) < \lambda\} = \mathcal{M}(\{u_0 < \lambda\}, \mathcal{F})(t), \qquad t \in I. \tag{2.7}$$

Hence under these assumptions $\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot)$ is upper semicontinuous; if we drop the upper semicontinuity assumption on u_0, we have $\overline{\mathcal{M}}_{u_0,\mathcal{F}} = \mathcal{M}_{u_0^*,\mathcal{F}}$.
General properties of minimal barriers will appear in a forthcoming paper [6].
The definitions of the minimal barriers for geometric evolutions described by a function F are a particular case of the previous definitions, by choosing a suitable family $\mathcal{F}_F$, and read as follows.

Let $F : J_0 \to \mathbf{R}$ be an arbitrary function.

Definition 2.5. *Let $a, b \in \mathbf{R}$, $a < b$, $[a,b] \subseteq I$ and let $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$. We write $f \in \mathcal{F}_F$ if and only if the following conditions hold: $f(t)$ is compact for any $t \in [a,b]$, there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a,b] \times A)$, $\partial f(t) \subseteq A$ for any $t \in [a,b]$, and*

$$\frac{\partial d_f}{\partial t}(t,x) + F(t,x,\nabla d_f(t,x), \nabla^2 d_f(t,x)) \ge 0 \qquad t \in]a,b[, \ x \in \partial f(t). \tag{2.8}$$

We write $f \in \mathcal{F}_F^{>}$ if and only if $f \in \mathcal{F}_F$ and the strict inequality holds in (2.8).

Obviously $\mathcal{B}(\mathcal{F}_F) \subseteq \mathcal{B}(\mathcal{F}_F^{>})$, hence $\mathcal{M}(E, \mathcal{F}_F) \supseteq \mathcal{M}(E, \mathcal{F}_F^{>})$. One could equivalently replace $]a,b[$ with $[a,b]$ in (2.8).

Remark 2.2. *Definition 2.5 (and consequently the definition of minimal barrier) can be adapted to geometric flows on a riemannian manifold (V,g) by substituting $\mathcal{P}(\mathbf{R}^n)$ with the family of all subsets of V ordered by the inclusion, the euclidean distance with the geodesic distance on (V,g), and the operators ∇, ∇^2 with the corresponding intrinsic operators.*

We recall that F is *geometric* [9, (1.2)] if

$$F(t,x,\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(t,x,p,X),$$

for any $\lambda > 0$, $\sigma \in \mathbf{R}$, $(t,x,p,X) \in J_0$.

If we define $\widetilde{F}(t,x,p,X) = |p| F(t,x,\frac{p}{|p|}, \frac{P_p X P_p}{|p|})$ for $(t,x,p,X) \in J_0$, then $\widetilde{F}$ is geometric and $\widetilde{F}(t,x,\nabla d_f, \nabla^2 d_f) = F(t,x,\nabla d_f, \nabla^2 d_f)$ for $f \in \mathcal{F}_F$, so $\mathcal{F}_F = \mathcal{F}_{\widetilde{F}}$.

Hence in what follows, without loss of generality, we can always assume that the function F is geometric.

A concept close to the minimal barrier (without the regularization defined in (2.4)) in the case of motion by mean curvature (i.e., $F(t,x,p,X) = -\mathrm{tr}(P_p X P_p)$) for compact sets was introduced by Ilmanen in [20] and was called set-theoretic subsolution; in that case $\mathcal{F}_F$ is essentially the family of all local smooth evolutions by mean curvature, and (2.8) is considered with the equality instead of the inequality.

Notice that to define $\mathcal{M}(E,\mathcal{F}_F)$ we need only that $\mathcal{F}_F$ is nonempty, which is guaranteed under very mild assumptions on F.

Proposition 2.1. *Assume that there exists a function $F_1 : J_0 \to \mathbf{R}$ which is bounded on compact subsets of J_0 and $F_1 \le F$. Then the family $\mathcal{F}_{F_1}$ (hence $\mathcal{F}_F$) is nonempty.*

Proof. Let $R > 0$ and $0 < \epsilon < 1$ be such that

$$\frac{1}{\epsilon} > \sup\{|F_1(t,x,p,X)| : t \in [t_0, t_0 + R/2], |x| \in [R/2, R], |p| = 1, 0 \le |X| \le 2\sqrt{n-1}/R\}.$$

Let $R(t) := -(t-t_0)/\epsilon + R_0$ and $d(t,x) := |x| - R(t)$. When $t \in [t_0, t_0 + \epsilon R/2]$ then $R(t) \in [R/2, R]$, and therefore $\sup\limits_{t \in [t_0, t_0+\epsilon R/2], |x| = R(t)} |\nabla^2 d| \le 2\sqrt{n-1}/R$. We then have

$$\frac{\partial d}{\partial t}(t,x) = \frac{1}{\epsilon} > -F_1(t,x,\nabla d(t,x), \nabla^2 d(t,x)) \qquad t \in]t_0, t_0 + \epsilon R/2[\ , \ |x| = R(t).$$

It follows that the map $t \in [t_0, t_0 + \epsilon R/2] \to B_{R(t)}$ belongs to $\mathcal{F}^{>}_{F_1}$. □

If F is of class $\mathcal{C}^\infty$, if it does not depend explicitly on x and is geometric and uniformly elliptic then, as proved in [15], any compact boundary of class $\mathcal{C}^\infty$ has a unique smooth evolution for small times. Hence we have the following proposition, which shows in particular that the minimal barrier agrees with the smooth evolution whenever the latter exists (see (2.11)).

Proposition 2.2. *Assume that $F : J_1 \to \mathbf{R}$ does not depend on x, is geometric, uniformly elliptic and of class $\mathcal{C}^\infty$. We write $f \in \mathcal{F}^{=}_F$ if and only if $f \in \mathcal{F}_F$ and equality holds in (2.8). Then for any $E \subseteq \mathbf{R}^n$ we have*

$$\mathcal{M}(E, \mathcal{F}^{=}_F) = \mathcal{M}(E, \mathcal{F}_F) \tag{2.9}$$

and if E is open we have also

$$\mathcal{M}(E, \mathcal{F}^{>}_F) = \mathcal{M}(E, \mathcal{F}_F). \tag{2.10}$$

Moreover for any $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}^{=}_F$, we have

$$\mathcal{M}(f(a), \mathcal{F}_F, a)(t) = f(t), \qquad t \in [a,b]. \tag{2.11}$$

Proof. To prove (2.9) it is enough to show $\mathcal{M}(E, \mathcal{F}^{=}_F) \supseteq \mathcal{M}(E, \mathcal{F}_F)$. Hence we are reduced to show that $\mathcal{M}(E, \mathcal{F}^{=}_F) \in \mathcal{B}(\mathcal{F}_F)$. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $f(a) \subseteq \mathcal{M}(E, \mathcal{F}^{=}_F)(a)$. We have to prove that $f(b) \subseteq \mathcal{M}(E, \mathcal{F}^{=}_F)(b)$. The set $\partial f(s)$ is of class $\mathcal{C}^\infty$ and compact for any $s \in [a,b]$, therefore the L^∞ norm of the second fundamental form of $\partial f(s)$ (and of $\nabla^3 d_f$, if necessary) is uniformly bounded with respect to $s \in [a,b]$. Hence there is $\tau > 0$, independent of s, so that the evolution of $\partial f(s)$ by (2.8), with the equality, is of class $\mathcal{C}^\infty$ in $[s, s+\tau]$ for any $s \in [a,b]$. Write $[a,b] = \bigcup_{i=1}^{m} [t_i, t_{i+1}]$ where $a = t_1 < \cdots < t_{m+1} = b$ e $t_{i+1} - t_i \le \tau$. Let us denote by $f^i(t)$ the geometric evolution of $f(t_i)$ by means of (2.8) with the equality. Then, using the comparison principle between smooth evolutions, we have $f(t_{i+1}) \subseteq f^i(t_{i+1})$. Reasoning by induction on i, we have $f^i(t_i) \subseteq \mathcal{M}(E, \mathcal{F}_F^{\overline{=}})(t_i)$ for any $i = 1, \ldots, m+1$, hence $f(t_{i+1}) \subseteq \mathcal{M}(E, \mathcal{F}_F^{\overline{=}})(t_{i+1})$ for any $i = 1, \ldots, m$. For $i = m$ we conclude the proof of (2.9).

Let E be open. To prove (2.10) we need to show that $\mathcal{M}(E, \mathcal{F}_F^{\gtrless}) \in \mathcal{B}(\mathcal{F}_F)$. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $f(a) \subseteq \mathcal{M}(E, \mathcal{F}_F^{\gtrless})(a) =: A$. We have to show that $f(b) \subseteq \mathcal{M}(E, \mathcal{F}_F^{\gtrless})(b)$. As A is open (see (4.24)) and since $f(a)$ is compact, we have $\mathrm{dist}(f(a), \mathbf{R}^n \setminus A) > 0$. For any $t \in [a,b]$ we can find a bounded closed tubular neighbourhood of $\partial f(t)$, of thickness $c(t)$, each point of which has a unique orthogonal projection on $\partial f(t)$, and such that $c := \inf\{c(t), t \in [a,b]\}$ is strictly positive. Let L be the Lipschitz constant of $F(t, \nabla d_f(t,x), \nabla^2 d_f(t,x))$ and M be the supremum of $|\nabla^2 d_f(t,x)|^2$ when $t \in [a,b]$ and x belongs to the $c(t)$-tubular neighbourhood of $\partial f(t)$. Pick a $\mathcal{C}^\infty$ function $\varrho : [a,b] \to]0, +\infty[$ so that $\varrho(a) < \min(c, \mathrm{dist}(f(a), \mathbf{R}^n \setminus A))$ and $\dot\varrho + 2ML\varrho < 0$. The map $g : [a,b] \to \mathcal{P}(\mathbf{R}^n)$, $g(t) := \overline{f^+_{\varrho(t)}(t)} = \{x \in \mathbf{R}^n : \mathrm{dist}(x, f(t)) \le \varrho(t)\}$ is of class $\mathcal{C}^\infty$, and each point $y \in \partial g(t)$ is of the form $y = x + \varrho(t)\nabla d_f(t,x)$ for a unique $x \in \partial f(t)$. We observe that $g \in \mathcal{F}_F^{\gtrless}$. Indeed for any $y \in \partial g(t)$, $y = x + \varrho(t)\nabla d_f(t,x)$, $x \in \partial f(t)$, we have $\nabla^2 d_g(t,y) = \nabla^2 d_f(t,x)(\mathrm{Id} + \varrho(t)\nabla^2 d_f(t,x))^{-1}$, so that

$$|\nabla^2 d_g(t,y) - \nabla^2 d_f(t,x)| \le 2M\varrho(t).$$

Therefore, recalling that $f \in \mathcal{F}_F$, for any $t \in]a,b[$ we have

$$\begin{aligned} &-\frac{\partial d_g}{\partial t}(t,y) = -\frac{\partial d_f}{\partial t}(t,x) + \dot\varrho(t) \le F(t, \nabla d_f(t,x), \nabla^2 d_f(t,x)) + \dot\varrho(t) \\ &= F(t, \nabla d_g(t,y), \nabla^2 d_f(t,x)) + \dot\varrho(t) \\ &\le F(t, \nabla d_g(t,y), \nabla^2 d_g(t,y)) + 2LM\varrho(t) + \dot\varrho(t) \\ &< F(t, \nabla d_g(t,y), \nabla^2 d_g(t,y)), \end{aligned} \tag{2.12}$$

so that $g \in \mathcal{F}_F^{\gtrless}$. Hence $f(b) \subseteq g(b) \subseteq \mathcal{M}(E, \mathcal{F}_F^{\gtrless})(b)$.

Let us prove (2.11). It is enough to show that for any $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F^{\overline{=}}$, we have $\mathcal{M}(f(a), \mathcal{F}_F^{\overline{=}}, a)(t) \subseteq f(t)$ for any $t \in [a,b]$; this follows by the comparison principle between smooth evolutions, since f is a barrier on $[a,b]$ with respect to $\mathcal{F}_F^{\overline{=}}$. □

Remark 2.3. *To prove (2.10) for open sets E we need that F is locally Lipschitz in the X-variable. As we shall see in (6.5), equality (2.10) holds true under weaker assumptions on F (which may also depend on x).*

Example 2.1. Let $F(p,X) = -\mathrm{tr}(P_p X P_p)$ (i.e., motion by mean curvature) and

$$\begin{aligned} &\mathcal{C}_F := \{f : [a,b] \to \mathcal{P}(\mathbf{R}^n), f \in \mathcal{F}_F, f(t) \text{ is convex for any } t \in [a,b]\}, \\ &\mathcal{D}_F := \{f : [a,b] \to \mathcal{P}(\mathbf{R}^n), f \in \mathcal{F}_F^{\overline{=}}, f(a) \text{ is convex}\}. \end{aligned}$$

Then for any $E \subseteq \mathbf{R}^n$ we have

$$\mathcal{M}_*(E, \mathcal{C}_F) = \mathcal{M}_*(E, \mathcal{D}_F) = \mathcal{M}_*(E, \mathcal{F}_G),$$
$$\mathcal{M}^*(E, \mathcal{C}_F) = \mathcal{M}^*(E, \mathcal{D}_F) = \mathcal{M}^*(E, \mathcal{F}_G),$$

where

$$G(p, X) := \begin{cases} F(p, X) & \text{if } X \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Note that if $n = 2$ then $G = F \wedge 0$.

Proof. Let $E \subseteq \mathbf{R}^n$. Using [18] (see also [14]) we know that a smooth convex set flowing by mean curvature remains convex, hence we have that $\mathcal{M}(E, \mathcal{C}_F) \supseteq \mathcal{M}(E, \mathcal{D}_F)$. Reasoning as in the proof of (2.9) we also have $\mathcal{M}(E, \mathcal{C}_F) = \mathcal{M}(E, \mathcal{D}_F)$. Furthermore, as $\mathcal{C}_F = \mathcal{C}_G \subseteq \mathcal{F}_G$, we have $\mathcal{M}(E, \mathcal{C}_F) \subseteq \mathcal{M}(E, \mathcal{F}_G)$.

To complete the proof it is enough to show

$$\mathcal{M}(A, \mathcal{C}_F) \supseteq \mathcal{M}(A, \mathcal{F}_G), \tag{2.13}$$

for any open set $A \subseteq \mathbf{R}^n$. We will prove that, given $g : [a, b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $g \in \mathcal{F}_G^{\geq}$, we have

$$g(t) \subseteq \mathcal{M}(g(a), \mathcal{C}_F, a)(t), \qquad t \in [a, b], \tag{2.14}$$

which implies $\mathcal{M}(E, \mathcal{C}_F) \supseteq \mathcal{M}(E, \mathcal{F}_G^{\geq})$ for any $E \subseteq \mathbf{R}^n$, which in turn, thanks to (2.10), implies (2.13).
For any $x \in \partial g(a)$, let $C_x \subseteq g(a)$ be a convex set with smooth boundary such that $\partial C_x \cap \partial g(a) = \{x\}$, $\nabla d_{C_x}(x) = \nabla d_{g(a)}(x)$, $\nabla^2 d_{C_x}(x) = \nabla^2 d_{g(a)}(x)$ and $\sup_{y \in \partial C_x} |\nabla^2 d_{C_x}(y)| \leq 2|\nabla^2 d_{C_x}(x)|$, which implies $|\nabla^2 d_{C_x}| \leq K$, for a positive constant K independent of $x \in \partial g(a)$. By [18] we can find $\tau > 0$, independent of $x \in \partial g(a)$, such that there exists a smooth mean curvature evolution $f_x : [a, a + \tau] \to \mathcal{P}(\mathbf{R}^n)$, $f_x(a) = C_x$ and $f_x \in \mathcal{C}_F \cap \mathcal{F}_F^{=}$. Note that

$$\frac{\partial d_g}{\partial t}(a, x) > \frac{\partial d_{C_x}}{\partial t}(a, x). \tag{2.15}$$

Using (2.15) and an argument similar to the one in Lemma 5.1 (see [6]) we have $\partial g(t) \subseteq \bigcup_{x \in \partial g(a)} f_x(t)$ for any $t \in [a, a + \tau]$, that implies, as $g(t)$ is compact for any $t \in [a, b]$, $g(t) \subseteq \mathcal{M}(g(a), \mathcal{C}_F, a)(t)$ for any $t \in [a, a + \tau]$. Now (2.14) follows by an induction argument and the compactness of $[a, b]$. □

Example 2.2. Let us define the family $\mathcal{G}$ as follows. A function $f : [a, b] \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{G}$ if and only if $f(t)$ is compact for any $t \in [a, b]$, there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a, b] \times A)$, $\partial f(t) \subseteq A$ for any $t \in [a, b]$, and

$$\Delta d_f > 0, \qquad \frac{\partial d_f}{\partial t} + \frac{1}{\Delta d_f} \geq 0 \qquad t \in \,]a, b[\,, \; x \in \partial f(t).$$

Then the associated minimal barrier $\mathcal{M}(E, \mathcal{G}, t_0)$ provides a definition of weak evolution of any convex set $E \subseteq \mathbf{R}^n$ by the inverse mean curvature (see [19]).

We conclude this section by recalling that in [12] a suitable choice of $\mathcal{F}$ is suggested to obtain motion by mean curvature of manifolds of arbitrary codimension, see Remark 6.4 and [1].

3. Level sets of subsolutions are barriers

Let us begin the comparison between the minimal barriers and the viscosity evolution. From now on we take $I = [0, +\infty[$ (i.e., $t_0 = 0$) and all barriers we consider are barriers on $[0, +\infty[$. Moreover we use the word subsolution to mean viscosity subsolution (and similarly for solution and supersolution). The function F is always geometric, and we denote by F_* (resp. F^*) the lower (resp. upper) semicontinuous envelope of F.

We list here some assumptions we use in the sequel. We follow the notation of [16, pp. 462-463]; we omit those properties in [16] which are not useful in our context.

(F1) $F : J_0 \to \mathbf{R}$ is continuous;

(F2) F is degenerate elliptic, i.e.,

$$F(t, x, p, X) \geq F(t, x, p, Y)$$

for any $(t, x, p, X) \in J_0$, $Y \in \mathrm{Sym}(n)$, $Y \geq X$;

(F3) $-\infty < F_*(t, x, 0, 0) = F^*(t, x, 0, 0) < +\infty$ for all $t \in [0, +\infty[$, $x \in \mathbf{R}^n$;

(F4) for every $R > 0$, $\sup\{|F(t, x, p, X)| : |p|, |X| \leq R, (t, x, p, X) \in J_0\} < +\infty$. One can check that if F is geometric and satisfies (F4), then $F_*(t, x, 0, 0) \leq 0$, $F^*(t, x, 0, 0) \geq 0$ for any $t \in [0, +\infty[$, $x \in \mathbf{R}^n$.

(F6) For every $R > \varrho > 0$ there is a constant $c = c_{R,\varrho}$ such that

$$|F(t, x, p, X) - F(t, x, q, Y)| \leq c(|p - q| + |X - Y|)$$

for all $t \in [0, +\infty[$, $x \in \mathbf{R}^n$, $\varrho \leq |p|, |q| \leq R$, $|X|, |Y| \leq R$;

(F6') for every $R > \varrho > 0$ there is a constant $c = c_{R,\varrho}$ such that

$$|F(t, x, p, X) - F(t, x, q, X)| \leq c|p - q|$$

for any $t \in [0, +\infty[$, $x \in \mathbf{R}^n$, $\varrho \leq |p|, |q| \leq R$, $|X| \leq R$;

(F7) there are $\varrho_0 > 0$ and a modulus σ_1 such that

$$F^*(t, x, p, X) - F^*(t, x, 0, 0) \leq \sigma_1(|p| + |X|),$$
$$F_*(t, x, p, X) - F_*(t, x, 0, 0) \geq -\sigma_1(|p| + |X|),$$

provided $t \in [0, +\infty[$, $x \in \mathbf{R}^n$, $|p|, |X| \leq \varrho_0$;

(F8) there is a modulus σ_2 such that

$$|F(t, x, p, X) - F(t, y, p, X)| \leq |x - y||p|\sigma_2(1 + |x - y|)$$

for $y \in \mathbf{R}^n$, $(t, x, p, X) \in J_0$;

(F8') for any $R \geq 0$ there is a modulus σ_R such that

$$|F(t, x, p, X) - F(t, y, p, X)| \leq |x - y||p|\sigma_R(1 + |x - y|)$$

for $y \in \mathbf{R}^n$, $(t, x, p, X) \in J_0$, $|X| \leq R$;

(F9) there is a modulus σ_2 such that $F_*(t, x, 0, 0) - F^*(t, y, 0, 0) \geq -\sigma_2(|x - y|)$ for any $t \in [0, +\infty[$, $x, y \in \mathbf{R}^n$;

(F10) suppose that $-\mu \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \nu \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}$ with $\mu, \nu \geq 0$. Let $R \geq 2\nu \vee \mu$ and let $\varrho > 0$; then

$$F_*(t, x, p, X) - F^*(t, y, p, -Y) \geq -|x - y||p|\overline{\sigma}(1 + |x - y| + \nu|x - y|^2)$$

for $(t, x) \in [0, +\infty[\times\mathbf{R}^n$, $\varrho \leq |p| \leq R$, with some modulus $\overline{\sigma} = \overline{\sigma}_{R,\varrho}$ independent of t, x, y, X, Y, μ, ν.

Remark 3.1. *One can check that, if F is geometric, then condition (F6) (resp. (F6'), (F8), (F10)) is equivalent to the analogous condition in [16]. Moreover (F4) implies conditions* $(6.3_{\pm})$ *of [9] and (F10) implies (F2) (see [16, proof of Theorem 2.1, case 2]) and (F8'). Furthermore, it is proved in [16, Proposition 4.3] that (F3), (F8) imply (F9), and (F2), (F6), (F8) imply (F10).*

Let $A \subseteq \mathbf{R}^n$. We recall [9,10] that a function $u : [0,+\infty[\times A \to \mathbf{R}$ is called a viscosity sub-(super) solution of

$$\frac{\partial u}{\partial t} + F(t,x,\nabla u,\nabla^2 u) = 0 \tag{3.1}$$

in $]0,+\infty[\times A$ if $u^* < +\infty$ (resp. $u_* > -\infty$) in $[0,+\infty[\times A$ and

$$\frac{\partial \psi}{\partial t}(\bar t,\bar x) + F_*(\bar t,\bar x,\nabla\psi(\bar t,\bar x),\nabla^2\psi(\bar t,\bar x)) \le 0 \tag{3.2}$$

$$\left(\text{resp. } \frac{\partial \psi}{\partial t}(\bar t,\bar x) + F^*(\bar t,\bar x,\nabla\psi(\bar t,\bar x),\nabla^2\psi(\bar t,\bar x)) \ge 0\right) \tag{3.3}$$

for any function $\psi \in \mathcal{C}^\infty(]0,+\infty[\times A)$ such that $u^*-\psi$ (resp. $u_*-\psi$) has a maximum (resp. minimum) at $(\bar t,\bar x) \in]0,+\infty[\times A$ (one achieves an equivalent definition of viscosity sub- and supersolution by taking $\mathcal{C}^2$ test functions ψ).
Finally, we define

$$F_c(t,x,p,X) := -F(t,x,-p,-X)$$

for any $(t,x,p,X) \in J_0$. Note that if F is degenerate elliptic then F_c is degenerate elliptic.

The following theorem is proved in [16, Theorem 4.9].

Theorem 3.1. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies either (F1)-(F4), (F8), or (F1), (F3), (F4), (F9), (F10). Let $v_0 : \mathbf{R}^n \to \mathbf{R}$ be a continuous function which is constant outside a bounded subset of $\mathbf{R}^n$. Then there exists a unique continuous viscosity solution (constant outside a bounded subset of $\mathbf{R}^n$) of (3.1) in $]0,+\infty[\times\mathbf{R}^n$ with $v(0,x) = v_0(x)$.*

Given a bounded open set $E \subseteq \mathbf{R}^n$ we define the viscosity evolutions $V(E)(t)$, $\Gamma(t)$ of E, ∂E respectively (the so called level set flow) as

$$V(E)(t) := \{x \in \mathbf{R}^n : v(t,x) < 0\}, \qquad \Gamma(t) := \{x \in \mathbf{R}^n : v(t,x) = 0\}, \tag{3.4}$$

where v is as in Theorem 3.1 with $v_0(x) := (-1) \vee d_E(x) \wedge 1$. It is proved in [16] that, if u denotes the solution of (3.1) with $u(0,\cdot) = u_0(\cdot)$, where $u_0 : \mathbf{R}^n \to \mathbf{R}$ is an admissible initial function such that $\{u_0 \le 0\} = \{v_0 \le 0\}$ and $\{u_0 = 0\} = \{v_0 = 0\}$, then $\{u(t,\cdot) \le 0\} = \{v(t,\cdot) \le 0\}$ for any $t \in [0,+\infty[$. Applying the same argument to $-u, -v$, which are solutions of (3.1) with F replaced by F_c, we also have $\{u(t,\cdot) \ge 0\} = \{v(t,\cdot) \ge 0\}$ for any $t \in [0,+\infty[$. We then conclude that u_0 and v_0 give raise to the same level set flow.

When $F : J_1 \to \mathbf{R}$ does not depend on $x \in \mathbf{R}^n$ all previous definitions are consequently modified in the obvious way.

The following result can be proved reasoning as in [1, Lemma 3.11].

Lemma 3.1. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies (F2) and (F4). Let $\Omega \subseteq \mathbf{R}^n$ be an open set and let $u : [0,+\infty[\times\Omega \to]-\infty\ ,0]$ (resp. $u : [0,+\infty[\times\Omega \to\ [0,+\infty[\ $) be an upper (resp. lower) semicontinuous function satisfying the following properties:*

(i) for every $(t,x) \in]0,+\infty[\times\Omega$ with $u(t,x) = 0$, there is a sequence $\{(t_m, x_m)\}$ of points of $]0,+\infty[\times\Omega$ converging to (t,x) such that $u(t_m,x_m) = 0$ and $t_m < t$;

(ii) u is a viscosity sub (resp. super) solution of (3.1) in the set $\{(t,x) \in]0,+\infty[\times\Omega : u(t,x) < 0\}$ (resp. in the set $\{(t,x) \in]0,+\infty[\times\Omega : u(t,x) > 0\}$);

(iii) $|u(t,x) - u(t,y)| \leq |x-y|$ for any $t \in]0,+\infty[\ ,\ x,y \in \Omega$.

Then u is a viscosity subsolution (resp. supersolution) of (3.1) in $]0,+\infty[\times\Omega$.

The main result of this section reads as follows.

Theorem 3.2. *The following two statements hold.*

A) Assume that $F : J_1 \to \mathbf{R}$ does not depend on x, is geometric and satisfies (F1)-(F4), (F6), (F7). Let u and v be, respectively, a viscosity sub- and supersolution of

$$\frac{\partial u}{\partial t} + F(t, \nabla u, \nabla^2 u) = 0 \tag{3.5}$$

in $]0,+\infty[\times\mathbf{R}^n$. Then for any $\lambda \in \mathbf{R}$ we have

$$\{x \in \mathbf{R}^n : u^*(\cdot,x) < \lambda\} \in \mathcal{B}(\mathcal{F}_F), \tag{3.6}$$

$$\{x \in \mathbf{R}^n : u^*(\cdot,x) \leq \lambda\} \in \mathcal{B}(\mathcal{F}_F). \tag{3.7}$$

$$\{x \in \mathbf{R}^n : v_*(\cdot,x) > \lambda\} \in \mathcal{B}(\mathcal{F}_{F_c}), \tag{3.8}$$

$$\{x \in \mathbf{R}^n : v_*(\cdot,x) \geq \lambda\} \in \mathcal{B}(\mathcal{F}_{F_c}). \tag{3.9}$$

Let w be the unique uniformly continuous viscosity solution of (3.5) in $]0,+\infty[\times\mathbf{R}^n$ with $w(0,x) = u_0(x)$ a given continuous function which is constant outside a bounded subset of $\mathbf{R}^n$ (see Theorem 3.1). Then for any $\lambda \in \mathbf{R}$ we have (3.6), (3.7) with u^* replaced by w and (3.8), (3.9) with v_* replaced by w. If additionally $F = F_c$ then for any $\lambda \in \mathbf{R}$ we have also

$$\{x \in \mathbf{R}^n : w(\cdot,x) = \lambda\} \in \mathcal{B}(\mathcal{F}_F). \tag{3.10}$$

B) Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Then, if we substitute (3.5) with (3.1) and $F(t,p,X)$ with $F(t,x,p,X)$ all assertions of statement A) hold.

Proof. Statement A).

To prove (3.6) it is enough to consider the case $\lambda = 0$. Let $f : [a,b] \subseteq [0,+\infty[\to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, and $f(a) \subseteq \{x \in \mathbf{R}^n : u^*(a,x) < 0\}$; we have to show that $f(b) \subseteq \{x \in \mathbf{R}^n : u^*(b,x) < 0\}$. Reasoning as in [1, Corollary 3.9, step 7] and using Lemma 3.1 one can check that the function $\delta := d_f \vee 0$ is a continuous supersolution of (3.5) in $]a,b[\times\mathbf{R}^n$ (see also [4, Theorem 3.1]). Then δ is a continuous supersolution of (3.5) in $]a,b]\times\mathbf{R}^n$ (see, for instance, [9, Lemma 5.7]). Moreover, since subsolutions are preserved by the composition with a continuous nondecreasing function (see [9, Theorem 5.2]), we have that $u^* \wedge 0$ is an upper semicontinuous subsolution of (3.5) in $]a,b] \times \mathbf{R}^n$. As $f(a) \subseteq \{x \in \mathbf{R}^n : u^*(a,x) < 0\}$ and $f(a)$ is compact, there is $\epsilon > 0$ such that $\delta(a,\cdot) - \epsilon \geq (u^* \wedge 0)(a,\cdot)$ on $\mathbf{R}^n$. We can apply the viscosity comparison principle in [16, Theorem 4.1] to $u^* \wedge 0$ and $\delta - \epsilon$, and we obtain

$$\delta(t,x) \geq (u^* \wedge 0)(t,x) + \epsilon, \qquad (t,x) \in [a,b] \times \mathbf{R}^n. \tag{3.11}$$

This implies

$$f(t) \subseteq \{x \in \mathbf{R}^n : u^*(t,x) \leq -\epsilon\} \subseteq \{x \in \mathbf{R}^n : u^*(t,x) < 0\}, \qquad t \in [a,b],$$

and (3.6) is proved. The relation in (3.7) follows from (3.6) by observing that $\{x \in \mathbf{R}^n : u^*(\cdot,x) \leq 0\} = \bigcap_{\epsilon>0} \{x \in \mathbf{R}^n : u^*(\cdot,x) < \epsilon\}$.

Assertions (3.8), (3.9) follow from (3.6), (3.7) by recalling that $-u^*$ is a supersolution of (3.5) with F replaced by F_c, and (3.10) follows from (3.7) and (3.9).

Statement B).

Following the proof and the notation of statement A) and using the viscosity comparison principle in [16, Theorem 4.2] and the fact that a supersolution in $]a,b[\times\mathbf{R}^n$ is a supersolution in $]a,b] \times \mathbf{R}^n$ (see [9, Lemma 5.7]), in order to show that $f(b) \subseteq \{x \in \mathbf{R}^n : u^*(b,x) < 0\}$ it is enough to prove that the function $\chi(t,x) := 1 - \chi_{f(t)}(x)$ is a supersolution of (3.1) in $]a,b[\times\mathbf{R}^n$.

Let $(\overline{t},\overline{x}) \in]a,b[\times\mathbf{R}^n$ and let ψ be a smooth function such that $(\chi - \psi)$ has a minimum at $(\overline{t},\overline{x})$. Assume first that $\overline{x} \in \mathrm{int}(f(\overline{t}))$. We can suppose that $\chi(\overline{t},\overline{x}) = \psi(\overline{t},\overline{x}) = 0$. As χ is twice differentiable at $(\overline{t},\overline{x})$ we have $\nabla\psi(\overline{t},\overline{x}) = 0$, $\nabla^2\psi(\overline{t},\overline{x}) \leq 0$. Moreover there exists two sequences $\{(t_m^{(1)},x_m)\}$, $\{(t_m^{(2)},x_m)\}$ converging to $(\overline{t},\overline{x})$, with $t_m^{(1)} < \overline{t} < t_m^{(2)}$ for any $m \in \mathbf{N}$, such that $\chi(t_m^{(i)},x_m) = 0 \geq \psi(t_m^{(i)},x_m)$, $i = 1,2$. Therefore $\frac{\partial\psi}{\partial t}(\overline{t},\overline{x}) = 0$, and we conclude $\frac{\partial\psi}{\partial t}(\overline{t},\overline{x}) + F^*(\overline{t},\overline{x},\nabla\psi(\overline{t},\overline{x}),\nabla^2\psi(\overline{t},\overline{x})) = F^*(\overline{t},\overline{x},0,\nabla^2\psi(\overline{t},\overline{x})) \geq F^*(\overline{t},\overline{x},0,0) = 0$.

The case $\overline{x} \in \mathbf{R}^n \setminus f(\overline{t})$ is similar. It remains to consider the case $\overline{x} \in \partial f(\overline{t})$. Pick $\epsilon > 0$ and $\tau > 0$ small enough so that each point of $B_{\epsilon/2}(\overline{x})$ has a unique smooth orthogonal projection on $\partial f(t)$ belonging to $B_\epsilon(\overline{x})$ for any $t \in [\overline{t}-\tau,\overline{t}+\tau] \subseteq]a,b[$. Define

$$F_\epsilon(t,x,p,X) := \begin{cases} \sup\limits_{x\in\overline{B_\epsilon(\overline{x})}} F^*(t,x,p,X) & \text{if } x \in \overline{B_\epsilon(\overline{x})}, \\ F^*(t,x,p,X) & \text{elsewhere.} \end{cases}$$

Note that $f \in \mathcal{F}_{F_\epsilon}$, F_ϵ is geometric, upper semicontinuous and satisfies (F2) and (F4). We claim that the function $\delta := d_f \vee 0$ is a supersolution of $\frac{\partial u}{\partial t} + F_\epsilon(t,x,\nabla u,\nabla^2 u) = 0$ in $]\overline{t}-\tau,\overline{t}+\tau[\times B_{\epsilon/2}(\overline{x})$. To prove the claim we follow [1, Corollary 3.9, step 7]. Thanks to Lemma 3.1 it is sufficient to prove that δ is a supersolution in $\{\delta > 0\}$. Let ξ be a smooth function and $(t_0,x_0) \in]\overline{t}-\tau,\overline{t}+\tau[\times B_{\epsilon/2}(\overline{x})$ be a minimizer of $(\delta - \xi)$, with $\delta(t_0,x_0) > 0$. Choose $y_0 \in \partial f(t_0) \cap B_\epsilon(\overline{x})$ satisfying $\delta(t_0,x_0) = |x_0 - y_0|$ and set $\zeta(t,y) := \xi(t,y+x_0-y_0)$. Then (t_0,y_0) is a minimizer of $(\delta - \zeta)$ by the triangular property of δ. Let $V := H_\sigma(\delta)$, where $\sigma > 0$ is such that d_f is smooth on $\{x \in \mathbf{R}^n : \mathrm{dist}(x,\partial f(t)) < \sigma\}$, $t \in]\overline{t}-\tau,\overline{t}+\tau[$, and $H_\sigma(r) := r \wedge \sigma/2$, $r \geq 0$. As $y_0 \in \partial f(t_0)$, (t_0,y_0) is also a minimizer of $(V - \zeta)$. Reasoning as in [1, Thorem 3.8 and Corollary 3.9], we have that V is a supersolution of $\frac{\partial u}{\partial t} + F_\epsilon(t,x,\nabla u,\nabla^2 u) = 0$ in $(]\overline{t}-\tau,\overline{t}+\tau[\times B_{\epsilon/2}(\overline{x})) \cap \{V > 0\}$; by Lemma

3.1 it is a supersolution in $]\overline{t}-\tau,\overline{t}+\tau[\times B_{\epsilon/2}(\overline{x})$. Therefore

$$\begin{aligned}0 &\leq \frac{\partial\zeta}{\partial t}(t_0,y_0)+F_\epsilon(t_0,y_0,\nabla\zeta(t_0,y_0),\nabla^2\zeta(t_0,y_0))\\ &= \frac{\partial\xi}{\partial t}(t_0,x_0)+F_\epsilon(t_0,y_0,\nabla\xi(t_0,x_0),\nabla^2\xi(t_0,x_0))\\ &= \frac{\partial\xi}{\partial t}(t_0,x_0)+F_\epsilon(t_0,x_0,\nabla\xi(t_0,x_0),\nabla^2\xi(t_0,x_0)),\end{aligned}$$

and this proves the claim.

Using the stability properties of viscosity supersolutions it then follows that also the function χ is a supersolution of $\frac{\partial u}{\partial t}+F_\epsilon(t,x,\nabla u,\nabla^2 u)=0$ on $]\overline{t}-\tau,\overline{t}+\tau[\times B_{\epsilon/2}(\overline{x})$ (see for instance [1, Lemma 4.3]). Therefore

$$\frac{\partial\psi}{\partial t}(\overline{t},\overline{x})+F_\epsilon(\overline{t},\overline{x},\nabla\psi(\overline{t},\overline{x}),\nabla^2\psi(\overline{t},\overline{x}))\geq 0.$$

Letting $\epsilon\to 0$ we get (3.3). $\square$

We recall that $F^+: J_0\to\mathbf{R}$ is defined by

$$F^+(t,x,p,X):=\sup\{F(t,x,p,Y): Y\geq X\},\qquad (t,x,p,X)\in J_0.$$

Notice that F^+ is the smallest degenerate elliptic function greater than or equal to F; moreover, if F is geometric (resp. lower semicontinuous) then F^+ is geometric (resp. lower semicontinuous).

Remark 3.2. *The theses (3.6), (3.7) still hold if we assume that F^+, in place of F, satisfies the assumptions in Theorem 3.2, statements A), B), and we replace F with F^+ in (3.1) and (3.5) (recall that $F^+\geq F$, hence $\mathcal{B}(\mathcal{F}_{F^+})\subseteq\mathcal{B}(\mathcal{F}_F)$).*

Remark 3.3. *In Theorem 3.2, if F does not depend explicitly on (t,x), then u_0 can be taken uniformly continuous (see [22, Theorem 2.2] and also [1, Theorem 2.4]).*

4. Some useful results on barriers

All results of this section will be used to prove Theorem 5.1, which is the converse of Theorem 3.2.

The next lemma shows that we can construct arbitrarily small elements of $\mathcal{F}_F^{>}$ with assigned normal and curvatures in a suitable neighbourhood of a given point. Note that we will not assume that F is degenerate elliptic.

Lemma 4.1. *Assume that $F:(\mathbf{R}^n\setminus\{0\})\times Sym(n)\to\mathbf{R}$ does not depend on (t,x), is geometric and lower semicontinuous. Let $L\subseteq\mathbf{R}^n$ be a closed set with smooth boundary. Let $\overline{x}\in\partial L$ and $\alpha\in\mathbf{R}$ be such that*

$$\alpha+F\big(\nabla d_L(\overline{x}),\nabla^2 d_L(\overline{x})\big)>0. \tag{4.1}$$

Then for any $R>0$ there exist $\tau>0$, $f:[a,a+\tau]\to\mathcal{P}(\mathbf{R}^n)$ and $\sigma>0$ such that

$$f(a)\subseteq L,\quad \partial f(a)\cap B_\sigma(\overline{x})=\partial L\cap B_\sigma(\overline{x}),\quad \alpha=\frac{\partial d_f}{\partial t}(a,\overline{x}), \tag{4.2}$$

$$f\in\mathcal{F}_F^{>},\qquad f(t)\subseteq B_R(\overline{x}),\qquad t\in[a,a+\tau]. \tag{4.3}$$

Proof. As F is lower semicontinuous, it is the pointwise supremum of a family of continuous functions, and since $F_1 \le F_2 \Rightarrow \mathcal{F}^{\geq}_{F_1} \subseteq \mathcal{F}^{\geq}_{F_2}$, we can assume that F is continuous. Fix $R > 0$ and set $(\overline{p}, \overline{X}) := \big(\nabla d_L(\overline{x}), \nabla^2 d_L(\overline{x})\big)$.

CASE 1. Assume that F is degenerate elliptic. Choose any smooth compact set, that we denote by $f(a)$, such that $f(a) \subseteq L \cap B_{2\sigma}(\overline{x})$ and $\partial f(a) \cap B_\sigma(\overline{x}) = \partial L \cap B_\sigma(\overline{x})$, for a suitable $\sigma \in \,]0, R/2[$. We claim that there exists a function $\widetilde{F} : (\{\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n) \to \mathbf{R}$ with the following properties: $\widetilde{F}$ is geometric, of class $\mathcal{C}^\infty$, uniformly elliptic and

$$\begin{aligned} &\alpha + \widetilde{F}(\overline{p}, \overline{X}) = 0, \\ &\widetilde{F}\big(\nabla d_{f(a)}(x), \nabla^2 d_{f(a)}(x)\big) < F\big(\nabla d_{f(a)}(x), \nabla^2 d_{f(a)}(x)\big), \qquad x \in \partial f(a). \end{aligned} \tag{4.4}$$

Let us prove the claim. Fix $0 < \epsilon < (\alpha + F(\overline{p}, \overline{X}))/2$; approximating F by convolution and using the compactness of $\partial f(a)$, we can find a degenerate elliptic function $G_\epsilon : (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n) \to \mathbf{R}$ of class $\mathcal{C}^\infty$ such that

$$\big|G_\epsilon\big(\nabla d_{f(a)}, \nabla^2 d_{f(a)}\big) - F\big(\nabla d_{f(a)}, \nabla^2 d_{f(a)}\big)\big| < \epsilon/2 \qquad \text{on } \partial f(a). \tag{4.5}$$

Set $F_\epsilon(p, X) := |p| G_\epsilon\Big(\frac{p}{|p|}, \frac{P_p X P_p}{|p|}\Big)$ for $(p, X) \in (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n)$; then F_ϵ is geometric, of class $\mathcal{C}^\infty$ and degenerate elliptic. In addition $G_\epsilon\big(\nabla d_{f(a)}, \nabla^2 d_{f(a)}\big) = F_\epsilon\big(\nabla d_{f(a)}, \nabla^2 d_{f(a)}\big)$ on $\partial f(a)$, hence (4.5) holds with G_ϵ replaced by F_ϵ. Let $\eta > 0$ be such that

$$\eta |\Delta d_{f(a)}(x)| < \epsilon/2, \qquad x \in \partial f(a). \tag{4.6}$$

Set $c := -\alpha - F_\epsilon(\overline{p}, \overline{X}) + \eta \Delta d_{f(a)}(\overline{x})$. Then $c < -\epsilon$, since by (4.5), (4.6) and (4.1)

$$-c = \alpha + F_\epsilon(\overline{p}, \overline{X}) - \eta \Delta d_{f(a)}(\overline{x}) > \alpha + F(\overline{p}, \overline{X}) - \epsilon > \epsilon.$$

Define $\widetilde{F} : (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n) \to \mathbf{R}$ as

$$\widetilde{F}(p, X) := F_\epsilon(p, X) - \eta \mathrm{tr}(P_p X P_p) + c.$$

Then $\widetilde{F}$ is geometric, of class $\mathcal{C}^\infty$, uniformly elliptic, $\alpha + \widetilde{F}(\overline{p}, \overline{X}) = 0$; in addition, for any $x \in \partial f(a)$, we have

$$\begin{aligned} &\widetilde{F}\big(\nabla d_{f(a)}(x), \nabla^2 d_{f(a)}(x)\big) < F\big(\nabla d_{f(a)}(x), \nabla^2 d_{f(a)}(x)\big) + \epsilon/2 - \eta \Delta d_{f(a)}(x) + c \\ &< F\big(\nabla d_{f(a)}(x), \nabla^2 d_{f(a)}(x)\big) + \epsilon + c < F\big(\nabla d_{f(a)}(x), \nabla^2 d_{f(a)}(x)\big), \end{aligned}$$

and the claim is proved.

Denote by Ω an open set containing $\partial f(a)$ and such that $d_{f(a)} \in \mathcal{C}^\infty(\Omega)$. Let u be the unique smooth solution [15, Theorem 2.1] of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \widetilde{F}(\nabla u, \nabla^2 u(\mathrm{Id} - u\nabla^2 u)^{-1}) = 0, \\ u(a, x) = d_{f(a)}(x) \end{cases} \tag{4.7}$$

on $[a, a+\tau] \times \Omega$, $\tau > 0$ sufficiently small. Let $\partial f(t) := \{x : u(x,t) = 0\}$ and $f(t)$ be the closure of the bounded connected component of $\mathbf{R}^n \setminus \partial f(t)$. In [15, Lemma 2.3] it is also proven that $|\nabla u| = 1$, hence $f \in \mathcal{F}^{=}_{\widetilde{F}}$. Notice that $\frac{\partial d_f}{\partial t}(a, \overline{x}) = \alpha$ and $f(t) \subseteq B_R(\overline{x})$ for $t \in [a, a+\tau]$, provided that τ is small enough. It remains to show that $f \in \mathcal{F}^{>}_{F}$. As d_f is of class $\mathcal{C}^\infty$ in a neighbourhood of $\partial f(\cdot)$ and the inequality in (4.4) holds, possibly reducing τ we can assume

$$\widetilde{F}\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) < F\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big), \qquad t \in [a, a+\tau],\ x \in \partial f(t).$$

Therefore on $\partial f(t)$, $t \in [a, a+\tau]$, we have

$$0 = \frac{\partial d_f}{\partial t} + \widetilde{F}\big(\nabla d_f, \nabla^2 d_f\big) < \frac{\partial d_f}{\partial t} + F\big(\nabla d_f, \nabla^2 d_f\big),$$

so that $f \in \mathcal{F}^{>}_{F}$. The proof of (4.2) and (4.3) is complete.

CASE 2. Assume that F is not degenerate elliptic. The proof is divided into three steps.

STEP 1. For any $\epsilon > 0$ there exist a smooth compact set, that we denote by $f_\epsilon(a)$, and $\sigma = \sigma(\epsilon) > 0$ such that

$$f_\epsilon(a) \subseteq L, \qquad \partial f_\epsilon(a) \cap B_\sigma(\overline{x}) = \partial L \cap B_\sigma(\overline{x}), \tag{4.8}$$

$$\nabla^2 d_{f_\epsilon(a)}(x) > \overline{X} - \frac{\epsilon}{n}\mathrm{Id} \qquad \text{for any } x \in \partial f_\epsilon(a) \text{ with } \nabla d_{f_\epsilon(a)}(x) = \overline{p}; \tag{4.9}$$

moreover we can find a constant $k < 0$ independent of ϵ such that $(1+k)\mathrm{Id} < \overline{X}$ and

$$\nabla^2 d_{f_\epsilon(a)}(x) > k\mathrm{Id}, \qquad \epsilon \in\,]0,1],\ x \in \partial f_\epsilon(a). \tag{4.10}$$

Let $\epsilon > 0$; up to a rotation and a translation, we can assume that there exist a neighbourhood $U' = U'_\epsilon$ of 0 in $\mathbf{R}^{n-1}$, a smooth function $l : U' \to \mathbf{R}$ such that $\overline{x} = (0, l(0))$ and $\nabla l(0) = 0$, and a neighbourhood $U = U_\epsilon \subseteq B_R(\overline{x})$ of $\overline{x}$ in $\mathbf{R}^n$ such that

$$U \cap \partial L = \{(x', l(x')) : x' \in U'\}, \qquad U \cap L \subseteq \{(x', y) : y \geq l(x'), x' \in U'\},$$
$$\nabla^2 l(x') > \nabla^2 l(0) - \frac{\epsilon}{n}\mathrm{Id}, \qquad x' \in U'.$$

Given $\varrho > 0$, we choose a function $g = g_\varrho : [0, +\infty[\to [0, +\infty[$ with the following properties: $g \in \mathcal{C}^\infty([0, +\infty[)$, g is convex, $g = 0$ on $[0, \varrho]$, $g(s) = s^4/\varrho$ for $s \in [2\varrho, +\infty[$. We define $h = h_\varrho : U' \to \mathbf{R}$ as $h(x') := l(x') + g(|x'|)$. Notice that for $x' \in U'$ we have

$$\nabla^2 h(x') = \nabla^2 l(x') + \left[\frac{1}{|x'|} g'(|x'|)\Big(\mathrm{Id} - \frac{x' \otimes x'}{|x'|^2}\Big) + g''\big(|x'|\big)\frac{x' \otimes x'}{|x'|^2}\right] \geq \nabla^2 l(x').$$

Define $H = H_\varrho := \{(x', y) : y \geq h(x'), x' \in U'\}$. Let us observe that, at each point $(x'_0, h(x'_0))$ with $x'_0 \in U'$ and $\nabla h(x'_0) = 0$, the second fundamental form of ∂H is $\nabla^2 h(x'_0)$; therefore

$$\nabla^2 d_H(x'_0, h(x'_0)) > \overline{X} - \frac{\epsilon}{n}\mathrm{Id}.$$

To have (4.8), (4.9) it is then enough to define $f_\epsilon(a) \subseteq U$ as a $\mathcal{C}^\infty$ regularization of $H \cap \overline{B_\mu(\overline{x})}$, where $\mu = \mu(\epsilon)$ and $\varrho = \varrho(\epsilon)$ are suitable positive numbers sufficiently small.
Eventually, property (4.10) holds by construction.

STEP 2. Let $\epsilon > 0$ and $\partial f_\epsilon(a)$ be as in STEP 1. Then there exists $\delta = \delta(\epsilon, \overline{p})$ such that

$$x \in \partial f_\epsilon(a),\ \nabla d_{f_\epsilon(a)}(x) = p,\ |p - \overline{p}| < \delta \Rightarrow \nabla^2 d_{f_\epsilon(a)}(x) > \overline{X} - \frac{\epsilon}{n}\mathrm{Id}. \tag{4.11}$$

Indeed, assume by contradiction that there exists a sequence $\{p_m\} \subset \mathbf{S}^{n-1}$ such that $p_m = \nabla d_{f_\epsilon(a)}(x_m)$ for $x_m \in \partial f_\epsilon(a)$, $\lim_{m\to+\infty} p_m = \overline{p}$ and $\nabla^2 d_{f_\epsilon(a)}(x_m) \le \overline{X} - \frac{\epsilon}{n}\mathrm{Id}$. Passing to a (not relabelled) subsequence, we have

$$\lim_{m\to+\infty} x_m = \widetilde{x} \in \partial f_\epsilon(a), \quad \nabla d_{f_\epsilon(a)}(\widetilde{x}) = \overline{p}.$$

By (4.9) we have

$$\overline{X} - \frac{\epsilon}{n}\mathrm{Id} \ge \lim_{m\to+\infty} \nabla^2 d_{f_\epsilon(a)}(x_m) = \nabla^2 d_{f_\epsilon(a)}(\widetilde{x}) > \overline{X} - \frac{\epsilon}{n}\mathrm{Id},$$

a contradiction.

To continue the proof of the lemma, we introduce some notation: Y will be always an element of $\mathrm{Sym}(n)$, we set

$$\beta := \min_{q\in\mathbf{S}^{n-1}} F(q, k\mathrm{Id}), \qquad c := \mathrm{tr}\big(\overline{X} - k\mathrm{Id}\big) > 0, \tag{4.12}$$

and define $m_0, m_1 : [0,+\infty[\to [0,+\infty[$ as

$$\begin{aligned} m_0(s) &:= \max\{|F(q,k\mathrm{Id}) - F(q, k\mathrm{Id} + Y)| : q \in \mathbf{S}^{n-1}, Y \ge 0, \mathrm{tr}(Y) \le s\}, \\ m_1(s) &:= \max\Big\{|F(q,\overline{X}) - F(q,\overline{X}+Y)| : q \in \mathbf{S}^{n-1}, |Y| \le s\Big\}. \end{aligned} \tag{4.13}$$

Then $m_0(0) = m_1(0) = 0$, and m_0, m_1 are continuous and nondecreasing.

Moreover, recalling (4.1), we choose $0 < \epsilon < \min(1, c)$ in such a way that

$$F\Big(p, \overline{X} - \frac{\epsilon}{n}\mathrm{Id}\Big) + \alpha > 2m_1(\sqrt{2}\epsilon) + m_0(c) - m_0(c-\epsilon), \qquad |p - \overline{p}| \le \epsilon. \tag{4.14}$$

STEP 3. Let ϵ be chosen as in (4.14) and let $f_\epsilon(a)$ be the corresponding set given by STEP 1. Then there exists a geometric, uniformly elliptic function $\widetilde{F}$: $(\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n) \to \mathbf{R}$ of class $\mathcal{C}^\infty$ such that

$$\begin{aligned} &\alpha + \widetilde{F}(\overline{p}, \overline{X}) = 0, \\ &\widetilde{F}\big(\nabla d_{f_\epsilon(a)}(x), \nabla^2 d_{f_\epsilon(a)}(x)\big) < F\big(\nabla d_{f_\epsilon(a)}(x), \nabla^2 d_{f_\epsilon(a)}(x)\big), \qquad x \in \partial f_\epsilon(a). \end{aligned} \tag{4.15}$$

Let us define the function $m_2 : [0,+\infty[\to [0,+\infty[$ as $m_2 = m_0$ on $[0, c-\epsilon]$ and, if $s \ge c - \epsilon$,

$$m_2(s) := \max\Big(m_0(s), m_0(c-\epsilon) + \max\Big\{\Big|F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id}\Big) - F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id} + Y\Big)\Big| : \\ q \in \mathbf{S}^{n-1}, Y \ge 0, \mathrm{tr}(Y) \le s - c + \epsilon\Big\}\Big).$$

Choose $\beta' < \min(\beta, m_2(c) - \alpha)$, let $0 < \delta = \delta(\epsilon) \leq \epsilon$ be given by STEP 2 and pick a function $g = g_\delta \in \mathcal{C}^\infty(\mathbf{S}^{n-1})$ such that

$$g(\overline{p}) = \max_{q \in \mathbf{S}^{n-1}} g(q) := m_2(c) - \alpha, \qquad g(p) := \beta' \qquad \text{for } |p - \overline{p}| \geq \delta. \tag{4.16}$$

Let us prove that for any $x \in \partial f_\epsilon(a)$ we have

$$g\big(\nabla d_{f_\epsilon(a)}(x)\big) - m_2\big(\mathrm{tr}(\nabla^2 d_{f_\epsilon(a)}(x) - k\mathrm{Id})\big) < F\big(\nabla d_{f_\epsilon(a)}(x), \nabla^2 d_{f_\epsilon(a)}(x)\big). \tag{4.17}$$

Let $x \in \partial f_\epsilon(a)$ and set $(p, X) := (\nabla d_{f_\epsilon(a)}(x), \nabla^2 d_{f_\epsilon(a)}(x))$.

If $|p - \overline{p}| \geq \delta$ then, recalling (4.16) and the definition of β, β', we have

$$g(p) = \beta' < \beta \leq F(p, k\mathrm{Id}). \tag{4.18}$$

Moreover

$$\begin{aligned} &m_2(\mathrm{tr}(X - k\mathrm{Id})) \geq m_0(\mathrm{tr}(X - k\mathrm{Id})) \\ &\geq \max\big\{F(p, k\mathrm{Id} + Y) - F(p, k\mathrm{Id}) : Y \geq 0, \mathrm{tr}(Y) \leq \mathrm{tr}(X - k\mathrm{Id})\big\}. \end{aligned} \tag{4.19}$$

Taking $Y = X - k\mathrm{Id}$, by (4.18) and (4.19) we have

$$g(p) - m_2(\mathrm{tr}(X - k\mathrm{Id})) < F(p, k\mathrm{Id}) + F(p, X) - F(p, k\mathrm{Id}) = F(p, X)$$

and (4.17) is proved.

Assume now that $|p - \overline{p}| < \delta \leq \epsilon$. Then by (4.16) we have

$$g(p) \leq g(\overline{p}) = m_2(c) - \alpha. \tag{4.20}$$

Moreover, by (4.11) we also have $X \geq \overline{X} - \frac{\epsilon}{n}\mathrm{Id}$, hence $\mathrm{tr}(X - k\mathrm{Id}) \geq c - \epsilon$. Therefore, recalling the definition of m_2, we have

$$m_2(\mathrm{tr}(X - k\mathrm{Id})) \geq m_0(c - \epsilon) + F\Big(p, \overline{X} - \frac{\epsilon}{n}\mathrm{Id}\Big) - F(p, X). \tag{4.21}$$

Then, by (4.20), (4.21) and (4.14) we find

$$\begin{aligned} &g(p) - m_2\big(\mathrm{tr}(X - k\mathrm{Id})\big) \leq m_2(c) - \alpha - m_0(c - \epsilon) - F\Big(p, \overline{X} - \frac{\epsilon}{n}\mathrm{Id}\Big) + F(p, X) \\ &< F(p, X) + m_2(c) - 2m_1(\sqrt{2}\epsilon) - m_0(c). \end{aligned}$$

To prove (4.17) it is enough to show that

$$m_2(c) - 2m_1(\sqrt{2}\epsilon) - m_0(c) \leq 0. \tag{4.22}$$

If $m_2(c) = m_0(c)$ then (4.22) is obvious; otherwise

$$\begin{aligned} &m_2(c) - m_0(c) = -m_0(c) + m_0(c - \epsilon) \\ &+ \max\Big\{\Big|F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id}\Big) - F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id} + Y\Big)\Big| : q \in \mathbf{S}^{n-1}, Y \geq 0, \mathrm{tr}(Y) \leq \epsilon\Big\} \\ &\leq \max\Big\{\Big|F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id}\Big) - F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id} + Y\Big)\Big| : q \in \mathbf{S}^{n-1}, Y \geq 0, \mathrm{tr}(Y) \leq \epsilon\Big\} \\ &\leq \max\Big\{\Big|F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id}\Big) - F\Big(q, \overline{X}\Big)\Big| : q \in \mathbf{S}^{n-1}\Big\} \\ &+ \max\Big\{\Big|F\Big(q, \overline{X}\Big) - F\Big(q, \overline{X} - \frac{\epsilon}{n}\mathrm{Id} + Y\Big)\Big| : q \in \mathbf{S}^{n-1}, Y \geq 0, \mathrm{tr}(Y) \leq \epsilon\Big\} \\ &\leq m_1(\epsilon/\sqrt{n}) + m_1(\sqrt{2}\epsilon) \leq 2m_1(\sqrt{2}\epsilon). \end{aligned}$$

Therefore (4.17) is proved.

By (4.17) and the continuity of F, there exists $\epsilon' > 0$ such that

$$F(\nabla d_{f_\epsilon(a)}(x), \nabla^2 d_{f_\epsilon(a)}(x) + Y) - g(\nabla d_{f_\epsilon(a)}(x)) + m_2(\mathrm{tr}\big(\nabla^2 d_{f_\epsilon(a)}(x) - k\mathrm{Id})\big) > \epsilon',$$

for any $x \in \partial f_\epsilon(a)$, $Y \geq 0$, $\mathrm{tr}(Y) \leq \epsilon'$.

Choose now an odd function $m \in \mathcal{C}^\infty(\mathbf{R})$ such that $m \geq m_2$ on $[c+\epsilon', +\infty[$, $|m - m_2| < \epsilon'$ on $[0, c+\epsilon']$, $m(c) = m_2(c)$ and $m'(x) \geq \lambda$ for any $x \in \mathbf{R}$ and for a suitable constant $\lambda > 0$. Eventually, we set

$$G(p, X) := g(p) - m(\mathrm{tr}(X - k\mathrm{Id})), \qquad (p, X) \in \mathbf{S}^{n-1} \times \mathrm{Sym}(n),$$
$$\widetilde{F}(p, X) := |p| G\Big(\frac{p}{|p|}, \frac{P_p X P_p}{|p|}\Big), \qquad (p, X) \in (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n).$$

Then one can check that $\widetilde{F}$ is smooth, geometric, uniformly elliptic and $\alpha + \widetilde{F}(\overline{p}, \overline{X}) = 0$. The inequality in (4.15) follows from (4.17) and the definition of $\widetilde{F}$. The proof of STEP 3 is concluded.

Now the thesis follows reasoning as in CASE 1 replacing $f(a)$ with $f_\epsilon(a)$ (see (4.7) and below). $\square$

Remark 4.1. *Following [24, Theorem 8.5.4], one can show that the number τ in the statement of Lemma 4.1 can be chosen depending in a continuous way on the initial datum $u(a, x)$ in (4.7) in a $\mathcal{C}^{2+\alpha}$ neighbourhood of $d_{f(a)}$, for any $\alpha \in]0, 1[$.*

Remark 4.2. *From Lemma 4.1 one can check that the following holds. Let $\alpha, \alpha_m \in \mathbf{R}$ with $\alpha_m \to \alpha$, $\overline{x}, x_m \in \mathbf{R}^n$ with $x_m \to \overline{x}$, and $\partial L, \partial L_m$ be a family of smooth closed hypersurfaces such that $\overline{x} \in \partial L$, $x_m \in \partial L_m$, $\partial L_m \to \partial L$ locally in $\mathcal{C}^\infty$, and*

$$\alpha + F(\nabla d_L(\overline{x}), \nabla^2 d_L(\overline{x})) > 0, \qquad \alpha_m + F(\nabla d_{L_m}(x_m), \nabla^2 d_{L_m}(x_m)) > 0$$

for any $m \in \mathbf{N}$. Then we can find corresponding $\tau, \tau_m > 0$, $f, f_m \in \mathcal{F}_F^{\gtrless}$ given by Lemma 4.1, such that $\partial f_m(a) \to \partial f(a)$ in $\mathcal{C}^\infty$, hence $\tau_m \to \tau$ by Remark 4.1.

Let $F : J_0 \to \mathbf{R}$ be a given function. Following [25, Section 3] and slightly changing the notation, given a map $\phi : [0, +\infty[\to \mathcal{P}(\mathbf{R}^n)$, we set

$$\phi_-(t) := \bigcup_{\epsilon > 0} \mathrm{int}\Big(\bigcap_{s \in [t-\epsilon, t+\epsilon] \cap [0, +\infty[} \phi(s) \Big), \qquad t \in [0, +\infty[. \tag{4.23}$$

Given $\varrho > 0$, we also let $\psi_\varrho^\pm : [0, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ be the map defined by $\psi_\varrho^\pm(t) := (\psi(t))_\varrho^\pm$.

Proposition 4.1. *If $F : J_1 \to \mathbf{R}$ does not depend on x, then*

$$\phi \in \mathcal{B}(\mathcal{F}_F) \ \Rightarrow \ int(\phi) \in \mathcal{B}(\mathcal{F}_F). \tag{4.24}$$

If $F : J_0 \to \mathbf{R}$ is lower semicontinuous, then

$$\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless}) \ \Rightarrow \ int(\phi) \in \mathcal{B}(\mathcal{F}_F^{\gtrless}), \quad \phi_- \in \mathcal{B}(\mathcal{F}_F^{\gtrless}). \tag{4.25}$$

Proof. Assertion (4.24) follows from the spatial translation invariance of the family $\mathcal{F}_F$ and from the definition of barrier. Assume that F depends on x and is lower semicontinuous, and let $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrsim})$. Let $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F^{\gtrsim}$, $f(a) \subseteq \mathrm{int}(\phi(a))$. As f is smooth, $f(a)$ is compact and F is lower semicontinuous, we can pick $\varrho > 0$ small enough such that the map $\overline{f_\varrho^+}$ belongs to $\mathcal{F}_F^{\gtrsim}$ and $\overline{f_\varrho^+(a)} \subseteq \phi(a)$. Then $\overline{f_\varrho^+(b)} \subseteq \phi(b)$, which implies $f(b) \subseteq \mathrm{int}(\phi(b))$, hence $\mathrm{int}(\phi) \in \mathcal{B}(\mathcal{F}_F^{\gtrsim})$. Assume now that $f(a) \subseteq \phi_-(a)$. As f is smooth, $f(a)$ is compact and $\phi_-(a)$ is open, there exists $\varrho > 0$ such that $g := \overline{f_\varrho^+}$ belongs to $\mathcal{F}_F^{\gtrsim}$ and $g(a) \subseteq \phi_-(a)$. By the definition of ϕ_-, there is $\epsilon > 0$ such that $g(a) \subseteq \phi(a+\tau)$ for any $\tau \in [-\epsilon, \epsilon]$. Let $\tau \in [-\epsilon, \epsilon]$ and define $h(t) := g(t-\tau)$ for $t \in [a+\tau, b+\tau]$. As F is lower semicontinuous, possibly reducing ϵ, we have $h \in \mathcal{F}_F^{\gtrsim}$; moreover $h(a+\tau) \subseteq \phi(a+\tau)$, hence $g(b) = h(b+\tau) \subseteq \phi(b+\tau)$. Hence $g(b) \subseteq \bigcap_{s \in [b-\epsilon, b+\epsilon] \cap [0,+\infty[} \phi(s)$, therefore

$$f(b) \subseteq \mathrm{int}\Big(\bigcap_{s \in [b-\epsilon, b+\epsilon] \cap [0,+\infty[} \phi(s) \Big) \subseteq \phi_-(b). \quad \square$$

Lemma 4.2. *Given $\phi : [0,+\infty[\to \mathcal{P}(\mathbf{R}^n)$ and $\varrho > 0$ we have $(\phi_-)_\varrho^- = (\phi_\varrho^-)_-$. In particular $(\phi_-)_\varrho^- = (\phi_{--})_\varrho^- = ((\phi_-)_\varrho^-)_-$.*

Proof. Let $\varrho > 0$ and $t \in [0,+\infty[$. Given $\epsilon > 0$ set $I(t,\epsilon) := [t-\epsilon, t+\epsilon] \cap [0,+\infty[$. Let us prove that

$$L_\varrho := \Big[\bigcup_{\epsilon>0} \mathrm{int}\Big(\bigcap_{s \in I(t,\epsilon)} \phi(s) \Big) \Big]_\varrho^- = \bigcup_{\epsilon > 0} \Big[\bigcap_{s \in I(t,\epsilon)} \phi(s) \Big]_\varrho^- =: R_\varrho. \tag{4.26}$$

It is enough to show (4.26) when, instead of $\epsilon > 0$, we take unions over a sequence $\{\epsilon_m\}_{m \in \mathbf{N}}$ of positive numbers converging to zero as $m \to +\infty$. Let us define

$$\Omega_m := \mathrm{int}\Big(\bigcap_{s \in I(t,\epsilon_m)} \phi(s) \Big).$$

We can assume that $\Omega_m \neq \mathbf{R}^n$ for any $m \in \mathbf{N}$, otherwise the result it trivial. Let $x \in L_\varrho$. Then

$$\mathrm{dist}\Big(x, \bigcap_m (\mathbf{R}^n \setminus \Omega_m) \Big) > \varrho. \tag{4.27}$$

To prove that $x \in R_\varrho$ we need to show that there exists $m_1 \in \mathbf{N}$ such that

$$\mathrm{dist}\Big(x, \mathbf{R}^n \setminus \bigcap_{s \in I(t,\epsilon_{m_1})} \phi(s) \Big) = \mathrm{dist}(x, \mathbf{R}^n \setminus \Omega_{m_1}) > \varrho.$$

Assume by contradiction that $\mathrm{dist}\,(x, \mathbf{R}^n \setminus \Omega_m) \leq \varrho$ for any $m \in \mathbf{N}$. Let $y_m \in \mathbf{R}^n \setminus \Omega_m$ be such that $|y_m - x| \leq \varrho$. Possibly passing to a subsequence (still denoted by $\{\epsilon_m\}$) we have $\lim_{m \to +\infty} y_m = y$ with $|x - y| \leq \varrho$; moreover $y \in \bigcap_m (\mathbf{R}^n \setminus \Omega_m)$, since $y_m \in \bigcap_{k=1}^m (\mathbf{R}^n \setminus \Omega_k)$. We then have a contradiction with (4.27). We have proved that $L_\varrho \subseteq R_\varrho$. The opposite inclusion follows from the fact that $\Big[\bigcap_{s \in I(t,\epsilon)} \phi(s) \Big]_\varrho^- = \Big[\mathrm{int}\Big(\bigcap_{s \in I(t,\epsilon)} \phi(s) \Big) \Big]_\varrho^- \subseteq \Big[\bigcup_{\epsilon > 0} \mathrm{int}\Big(\bigcap_{s \in I(t,\epsilon)} \phi(s) \Big) \Big]_\varrho^-$ for any $\epsilon > 0$.

Let us show now that for any $\epsilon > 0$

$$l_\varrho^\epsilon := \Big[\bigcap_{s \in I(t,\epsilon)} \phi(s) \Big]_\varrho^- = \mathrm{int}\Big(\bigcap_{s\in I(t,\epsilon)} \phi(s)_\varrho^- \Big) =: r_\varrho^\epsilon.$$

Let $x \in r_\varrho^\epsilon$. Then there is $c > 0$ so that $\mathrm{dist}\big(x, \bigcup_{s\in I(t,\epsilon)} \mathbf{R}^n \setminus \phi(s)_\varrho^-\big) = c$. Hence for any $s \in I(t,\epsilon)$ we have $\mathrm{dist}(x, \mathbf{R}^n \setminus \phi(s)_\varrho^-) \geq c$, which implies $\mathrm{dist}(x, \mathbf{R}^n \setminus \phi(s)) \geq c + \varrho$. Therefore

$$\mathrm{dist}\Big(x, \bigcup_{s\in I(t,\epsilon)} \big(\mathbf{R}^n \setminus \phi(s)\big)\Big) \geq c + \varrho > \varrho,$$

hence $x \in l_\varrho^\epsilon$. We have proved that $r_\varrho^\epsilon \subseteq l_\varrho^\epsilon$. The opposite inclusion follows from the fact that $\phi(s)_\varrho^- \supseteq l_\varrho^\epsilon$ for any $s \in I(t,\epsilon)$.

From (4.26) and (4.27) we then have

$$(\phi_-)_\varrho^-(t) = L_\varrho = R_\varrho = \bigcup_{\epsilon>0} l_\varrho^\epsilon = \bigcup_{\epsilon>0} r_\varrho^\epsilon = (\phi_\varrho^-(t))_-. \tag{4.28}$$

The last assertion of the lemma is a consequence of (4.28) and the equality $\phi_{--} = \phi_-$. □

We conclude this section with the following proposition.

Proposition 4.2. *Let $F : J_0 \to \mathbf{R}$ be a function such that for any $R > 0$*

$$C_R := \sup\{|F(t,x,p,X)| : t \in [0,+\infty[, x \in \mathbf{R}^n, |p| = 1, |X| \leq R\} < +\infty. \tag{4.29}$$

Let $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless})$ and $(\bar t, \bar x) \in]0,+\infty[\times\mathbf{R}^n$ be such that $\bar x \in \mathbf{R}^n \setminus \phi(\bar t)$. Then there exists a sequence $\{(t_m, x_m)\}$ of points of $]0,+\infty[\times\mathbf{R}^n$ with $x_m \in \mathbf{R}^n \setminus \phi(t_m)$ and $t_m < \bar t$ such that $(t_m, x_m) \to (\bar t, \bar x)$ as $m \to +\infty$.

Proof. Let $h : [0,+\infty[\to]0,+\infty[$ be any strictly increasing $\mathcal{C}^\infty$ function such that $h(R) > C_R$ for any $R \geq 0$. For any $\varrho > 0$ define $H(\varrho) := \displaystyle\int_0^\varrho \frac{1}{h(\sqrt{n-1}/r)}\,dr$. Then $H : [0,+\infty[\to [0,+\infty[$ is strictly increasing, surjective, $H(0) = 0$, $H \in \mathcal{C}^0([0,+\infty[)\cap \mathcal{C}^\infty(]0,+\infty[)$. Let $\varrho_F := H^{-1}$. Given $0 \leq a < b$, $\epsilon > 0$, $x \in \mathbf{R}^n$, one can check that the function $g : [a,b] \to \mathcal{P}(\mathbf{R}^n)$ defined by $g(t) := \{y \in \mathbf{R}^n : |y - x| \leq \varrho_F(\epsilon + b - t)\}$ belongs to $\mathcal{F}_F^{\gtrless}$. Let now $\bar x \in \mathbf{R}^n \setminus \phi(\bar t)$. To prove the proposition it is enough to show that there exists a sequence $\{t_m\}$ converging to $\bar t$ with $t_m < \bar t$, such that $\overline{B_{2\varrho_F(\bar t - t_m)}(\bar x)} \cap (\mathbf{R}^n \setminus \phi(t_m)) \neq \emptyset$. Assume by contradiction that for $t_m \uparrow \bar t$ we have $\overline{B_{2\varrho_F(\bar t - t_m)}(\bar x)} \subseteq \phi(t_m)$. Let $t^* > \bar t - t_m$ be such that $\varrho_F(t^*) = 2\varrho_F(\bar t - t_m)$. The map $t \in [t_m, \bar t\ ] \to \overline{B_{\varrho_F(t^* + t_m - t)}(\bar x)}$ belongs to $\mathcal{F}_F^{\gtrless}$. Hence, as $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless})$ and $\overline{B_{\varrho_F(t^*)}(\bar x)} = \overline{B_{2\varrho_F(\bar t - t_m)}(\bar x)} \subseteq \phi(t_m)$, we have $\bar x \in \overline{B_{\varrho_F(t^* + t_m - \bar t)}(\bar x)} \subseteq \phi(\bar t)$, a contradiction. □

5. A function whose Level sets are barriers is a subsolution

Our aim is now to prove a converse of Theorem 3.2 (Theorem 5.1). To do this we need several preliminary results.

Definition 5.1. *Let $f : [a,b] \subseteq [0,+\infty[\to \mathcal{P}(\mathbf{R}^n)$. We say that f is a smooth compact flow if and only if $f(t)$ is compact for any $t \in [a,b]$ and there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a,b] \times A)$ and $\partial f(t) \subseteq A$ for any $t \in [a,b]$.*

Lemma 5.1. *Let $f, g : [a,b] \to \mathcal{P}(\mathbf{R}^n)$ be two smooth compact flows, $x \in \mathbf{R}^n$ and $\varrho > 0$. Assume that*

$$\begin{aligned}
&\{x\} = \partial f(a) \cap \partial g(a) \cap \overline{B_\varrho(x)}, \\
&(g(a) \setminus \{x\}) \cap \overline{B_\varrho(x)} \subseteq int(f(a)) \cap \overline{B_\varrho(x)}, \\
&\frac{\partial d_f}{\partial t}(a,x) < \frac{\partial d_g}{\partial t}(a,x).
\end{aligned}$$

Then there exists $0 < \tau \le b - a$ such that

$$g(t) \cap \overline{B_\varrho(x)} \subseteq int(f(t)) \cap \overline{B_\varrho(x)} \qquad t \in]a, a+\tau]. \tag{5.1}$$

Moreover, τ depends in a continuous way on small perturbations of f and g in the $\mathcal{C}^2$ norm.

Proof. Let $c := \frac{1}{2}[\frac{\partial d_g}{\partial t}(a,x) - \frac{\partial d_f}{\partial t}(a,x)]$ and $\eta(t) := \mathrm{dist}(\partial g(t) \cap \overline{B_\varrho(x)}, \partial f(t) \cap \overline{B_\varrho(x)})$, for $t \in [a,b]$. Since f, g are smooth compact flows, using the hypotheses we can find $0 < \sigma < \varrho$ and $\tau > 0$ such that, for $t \in [a, a+\tau]$,

$$\frac{\partial d_g}{\partial t}(t,y) - \frac{\partial d_f}{\partial t}(t,z) > c \qquad y \in \partial g(t) \cap \overline{B_\sigma(x)},\ z \in \partial f(t) \cap \overline{B_\sigma(x)}, \tag{5.2}$$

and $\eta(t) = |y - z| \Rightarrow y, z \in B_\sigma(x)$. Reasoning as in [7, Lemma 4.2] one can check that for any $t \in [a, a+\tau]$ we have $\liminf_{s\to 0^+} \frac{\eta(t+s)-\eta(t)}{s} \ge c$, which in turn implies $\eta(t) \ge c(t-a)$ for any $t \in [a, a+\tau]$ and (5.1) follows.

The continuity of τ follows by construction. □

The following proposition plays a crucial rôle in the proof of Theorem 5.1 and is based on Lemma 4.1; note that we will not assume that F is degenerate elliptic.

Proposition 5.1. *Assume that $F : J_0 \to R$ is geometric and lower semicontinuous. Let $\phi \in \mathcal{B}(\mathcal{F}_F^{>})$ and let $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$ be a smooth compact flow. Assume that there exist $\theta \in]a, b[$ and $x \in \mathbf{R}^n$ such that*

$$\begin{aligned}
&\{x\} = \partial f(\theta) \cap \partial \phi(\theta), \\
&f(\theta) \setminus \{x\} \subseteq int(\phi(\theta)), \\
&f(t) \subseteq int(\phi(t)), \qquad t \in [a,b] \setminus \{\theta\}.
\end{aligned} \tag{5.3}$$

Then

$$\frac{\partial d_f}{\partial t}(\theta,x) + F\big(\theta, x, \nabla d_f(\theta,x), \nabla^2 d_f(\theta,x)\big) \le 0. \tag{5.4}$$

Proof. CASE 1. Suppose that F does not depend on (t,x).
Assume by contradiction that

$$\frac{\partial d_f}{\partial t} + F\big(\nabla d_f, \nabla^2 d_f\big) = 2c > 0 \qquad \text{at } (\theta, x).$$

As f is a smooth compact flow, there exists $\theta_1 > 0$ such that, for every $t \in [\theta - \theta_1, \theta + \theta_1] =: I(\theta)$, each point $y \in \partial f(t)$ has a unique smooth orthogonal projection $\pi(t,y)$ on $\partial f(\theta)$. From now on we restrict to the interval $I(\theta)$. Set $x(t) := \pi^{-1}(t,x)$,

$$p(t) := \nabla d_f(t, x(t)), \qquad X(t) := \nabla^2 d_f(t,x(t)), \qquad \alpha := c - F(p(\theta), X(\theta)).$$

We can assume that

$$\frac{\partial d_f}{\partial t}(t, x(t)) > \alpha \tag{5.5}$$

and, as F is lower semicontinuous, possibly taking a smaller θ_1, we can also assume

$$\alpha + F(p(t), X(t)) > 0, \qquad t \in I(\theta). \tag{5.6}$$

Choose a function $\varrho : \partial f(\theta) \to [0, +\infty[$ of class $\mathcal{C}^\infty$ verifying the following properties:

(i) $\varrho(y) = 0$ if and only if $y = x$;

(ii) $\nabla \varrho(x) = 0$, $\nabla^2 \varrho(x) = 0$;

(iii) the map $t \in I(\theta) \to f_1(t)$ is a smooth compact flow, where $\partial f_1(t) := \{z \in \mathbf{R}^n : z = y - \varrho(\pi(t,y)) \nabla d_f(t,y), y \in \partial f(t)\}$.

In particular $f_1 : I(\theta) \to \mathcal{P}(\mathbf{R}^n)$ satisfies (5.3) with $[a,b]$ replaced by $I(\theta)$, and

$$f_1(t) \subseteq f(t), \quad \partial f_1(t) \cap \partial f(t) = \{x(t)\}, \quad \nabla^2 d_{f_1}(t, x(t)) = X(t), \quad t \in I(\theta). \tag{5.7}$$

Fix $t \in]\theta - \theta_1, \theta[$. Recalling also (5.6), we apply Lemma 4.1 with $\overline{x}$ and L replaced by $x(t)$ and $f(t)$ in the order. Hence there exist $\tau_t > 0$, $\sigma_t > 0$, $g_t : [t, t+\tau_t] \to \mathcal{P}(\mathbf{R}^n)$, so that

$$\begin{aligned} & g_t \in \mathcal{F}_F^{\geqslant}, \qquad g_t(t) \subseteq f(t) \subseteq \phi(t), \qquad g_t(t) \cap \overline{B_{\sigma_t}(x)} = f(t) \cap \overline{B_{\sigma_t}(x)}, \\ & x(t) \in \partial g_t(t), \ (\alpha, p(t), X(t)) = \Big(\frac{\partial d_{g_t}}{\partial t}(t, x(t)), \nabla d_{g_t}(t, x(t)), \nabla^2 d_{g_t}(t, x(t)) \Big) \end{aligned} \tag{5.8}$$

(possibly reducing τ_t and σ_t, we can use x instead of $x(t)$ in the first equality in (5.8)). Using the first equality in (5.8) and the second relation in (5.7) we have

$$(f_1(t) \setminus \{x(t)\}) \cap \overline{B_{\sigma_t}(x)} \subseteq \mathrm{int}(g_t(t)).$$

Using Remark 4.2 we have $\tau_t \to \tau_\theta > 0$ as $t \to \theta$, so that there exists $t_1 < \theta$ such that $\tau_t > \theta - t$ for any $t \in [t_1, \theta]$. Fix $t \in]t_1, \theta]$; let us apply Lemma 5.1 to the flows f_1, g_t (recall that $\frac{\partial d_{f_1}}{\partial t}(t, x(t)) > \alpha = \frac{\partial d_{g_t}}{\partial t}(t, x(t))$ by (5.5)). Then there exists $0 < \tau_t' < \tau_t$ such that

$$f_1(s) \cap \overline{B_{\sigma_t}(x)} \subseteq \mathrm{int}(g_t(s)), \qquad s \in]t, t + \tau_t']. \tag{5.9}$$

Using Remark 4.2 we get that $\tau_t' \to \tau_\theta' > 0$ as $t \to \theta$. Choose $t_2 \in]t_1, \theta[$ such that $\tau_{t_2}' > \theta - t_2$; as $g_{t_2}(t_2) \subseteq \phi(t_2)$ by (5.8) and $g_{t_2} \in \mathcal{F}_F^{\geqslant}$, $\phi \in \mathcal{B}(\mathcal{F}_F^{\geqslant})$, by (5.9) we have, for $s = \theta$,

$$x \in \mathrm{int}(g_{t_2}(\theta)) \subseteq \mathrm{int}(\phi(\theta)),$$

which contradicts $x \in \partial\phi(\theta)$.

CASE 2. Suppose that F depends on (t,x). Assume by contradiction that

$$\frac{\partial d_f}{\partial t}(\theta,x) + F\big(\theta, x, \nabla d_f(\theta,x), \nabla^2 d_f(\theta,x)\big) > 0.$$

Let $U \subseteq [0,+\infty[\times \mathbf{R}^n$ be a compact neighbourhood of (θ,x) such that

$$\frac{\partial d_f}{\partial t}(\theta,x) + \min_{(t,y)\in U} F\big(t, y, \nabla d_f(\theta,x), \nabla^2 d_f(\theta,x)\big) > 0, \tag{5.10}$$

and define

$$G(s,z,p,X) := \begin{cases} \min_{(t,y)\in U} F(t,y,p,X) & \text{if } (s,z)\in U, \\ F(s,z,p,X) & \text{elsewhere.} \end{cases}$$

Notice that G is lower semicontinuous and that $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless})$ implies $\phi \in \mathcal{B}(\mathcal{F}_G^{\gtrless})$. Applying CASE 1 localized in U (with F replaced by G) we then get a contradiction with (5.10). □

Proposition 5.2. *Assume that $F : J_1 \to \mathbf{R}$ is geometric, lower semicontinuous and satisfies (F4). Let $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless})$. The following statements hold:*

(i) if F satisfies (F2) then the function $(t,x) \to -\chi_{\phi(t)}(x)$ is a viscosity subsolution of (3.5) in $]0,+\infty[\times\mathbf{R}^n$;

(ii) if F^+ satisfies (F4) then the function $(t,x) \to -\chi_{\phi(t)}(x)$ is a viscosity subsolution of

$$\frac{\partial u}{\partial t} + F^+(t, \nabla u, \nabla^2 u) = 0 \tag{5.11}$$

in $]0,+\infty[\times\mathbf{R}^n$.

Proof. It is enough to show (ii). Let $T := \sup\{t \geq 0 : \phi(t) \neq \emptyset, \phi(t) \neq \mathbf{R}^n\}$. To prove the thesis, it is enough to show that the function $d_\phi \wedge 0$ is a subsolution of (5.11) in $]0,T[\times\mathbf{R}^n$. Indeed, using [9, Lemma 5.7] we have that $d_\phi \wedge 0$ is a subsolution of (5.11) in $]0,T]\times\mathbf{R}^n$; moreover, using [1, Lemma 4.3] we deduce that the function $(t,x) \to -\chi_{\phi(t)}(x)$ is also a subsolution of (5.11) in $]0,T]\times\mathbf{R}^n$, hence in $]0,+\infty[\times\mathbf{R}^n$.

By [25, Lemmas 3.1,3.2] we have that $(d_\phi \wedge 0)^* = d_{\phi_-} \wedge 0$. We let $d := d_{\phi_-} \wedge 0$. Let $(\overline{t},\overline{x}) \in]0,T[\times\mathbf{R}^n$. We have to prove (3.2) (with $F^+(t,p,X)$ instead of $F(t,x,p,X)$) for any function $\psi \in \mathcal{C}^\infty(]0,T[\times\mathbf{R}^n)$ such that $(d-\psi)$ has a strict global maximum at the point $(\overline{t},\overline{x})$. Set $(\alpha,p,X) := (\frac{\partial\psi}{\partial t}(\overline{t},\overline{x}), \nabla\psi(\overline{t},\overline{x}), \nabla^2\psi(\overline{t},\overline{x}))$.

In view of Proposition 4.2 and Lemma 3.1, it is enough to consider the case $d(\overline{t},\overline{x}) < 0$, i.e., $\overline{x} \in \phi_-(\overline{t})$. Let $\overline{x} \in \phi_-(\overline{t})$; then $|p| = 1$ (we use the fact that d is locally semiconvex in $\phi_-(\overline{t})$, see [23]). Since F is geometric, possibly replacing X with $P_p X P_p$, we can suppose $Xp = 0$. Let $\overline{y} \in \partial\phi_-(\overline{t})$ be such that $|\overline{x}-\overline{y}| = -d(\overline{t},\overline{x})$, and

$$\Psi(t,x) := \psi(t, x+\overline{x}-\overline{y}) - \psi(\overline{t},\overline{x}), \qquad (t,x) \in]0,T[\times\mathbf{R}^n.$$

Clearly $(\alpha,p,X) = (\frac{\partial\Psi}{\partial t}(\overline{t},\overline{y}), \nabla\Psi(\overline{t},\overline{y}), \nabla^2\Psi(\overline{t},\overline{y}))$. Moreover, using the triangular property of the distance, we have $\mathrm{dist}(x+\overline{x}-\overline{y}, \mathbf{R}^n\setminus\phi_-(t)) \leq \mathrm{dist}(x, \mathbf{R}^n\setminus\phi_-(t)) + |\overline{x}-\overline{y}|$, hence

$$d(t,x) \leq d(t,\overline{y}) + d(t, x+\overline{x}-\overline{y}) + |\overline{x}-\overline{y}|.$$

Therefore

$$\begin{aligned}&d(t,x)-\Psi(t,x)-\psi(\bar t,\bar x)=d(t,x)-\psi(t,x+\bar x-\bar y)\le d(\bar t,\bar y)+d(t,x+\bar x-\bar y)\\&+|\bar x-\bar y|-\psi(t,x+\bar x-\bar y)<d(\bar t,\bar y)+d(\bar t,\bar x)-\psi(\bar t,\bar x)+|\bar x-\bar y|\\&=d(\bar t,\bar y)-\psi(\bar t,\bar x)=d(\bar t,\bar y)-\Psi(\bar t,\bar y)-\psi(\bar t,\bar x),\end{aligned}$$

which implies that $(d-\Psi)$ has a strict global maximum at the point $(\bar t,\bar y)$.

Pick $\tau>0$ with $[\bar t-\tau,\bar t+\tau]\subseteq\,]0,T[$ and a smooth function $\zeta:[\bar t-\tau,\bar t+\tau]\times\mathbf{R}^n\to\mathbf{R}$ with the following properties: $\zeta\ge\Psi$, $\zeta(\bar t,\bar y)=\Psi(\bar t,\bar y)=0$,

$$(\alpha,p,X)=\left(\frac{\partial\zeta}{\partial t}(\bar t,\bar y),\nabla\zeta(\bar t,\bar y),\nabla^2\zeta(\bar t,\bar y)\right),$$

$\zeta^2+|\nabla\zeta|^2>0$ (recall that $|\nabla\zeta(\bar t,\bar y)|=|\nabla\psi(\bar t,\bar x)|=1$) and ζ is positive outside a compact subset of $\mathbf{R}^n$.

Let us define $f:[\bar t-\tau,\bar t+\tau]\to\mathcal{P}(\mathbf{R}^n)$ as $f(t):=\{x\in\mathbf{R}^n:\zeta(t,x)\le 0\}$. Then f is a smooth compact flow and

$$(\alpha,p,X)=\left(\frac{\partial d_f}{\partial t}(\bar t,\bar y),\nabla d_f(\bar t,\bar y),\nabla^2 d_f(\bar t,\bar y)\right)$$

(recall that $|p|=1$ and $Xp=0$). Then assumptions (5.3) (with θ, $[a,b]$ and ϕ replaced by $\bar t$, $[\bar t-\tau,\bar t+\tau]$ and ϕ_- in the order) of Proposition 5.1 are fulfilled (recall that $\phi_-\in\mathcal{B}(\mathcal{F}_F^{\geq})$ by (4.25)). Then, from (5.4) it follows

$$\alpha+F(\bar t,p,X)\le 0$$

and therefore d is a subsolution of (3.5) in $\{(t,x)\in\,]0,T[\times\mathbf{R}^n:d(t,x)<0\}$.

Assume now by contradiction that there exists $0<c<+\infty$ such that

$$\alpha+F^+(\bar t,p,X)=2c.$$

Let $Y\in\mathrm{Sym}(n)$ be such that $Y\ge X$ and

$$F^+(\bar t,p,X)\le F(\bar t,p,Y)+c.$$

Define

$$\Phi(t,x):=\psi(t,x)+\frac12\langle(x-\bar x),(Y-X)(x-\bar x)\rangle.$$

Then $\nabla^2\Phi(\bar t,\bar x)=Y$ and $(d-\Phi)$ has a maximum at $(\bar t,\bar x)$. Therefore, as d is a subsolution of (3.5), at $(\bar t,\bar x)$ we have

$$0\ge\frac{\partial\Phi}{\partial t}+F(\bar t,\nabla\Phi,\nabla^2\Phi)=\alpha+F(\bar t,p,Y)\ge\alpha+F^+(\bar t,p,X)-c=c>0,$$

a contradiction. $\square$

We are now in a position to prove the converse of Theorem 3.2.

Theorem 5.1. *Let $u : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ be a function such that $u^* < +\infty$. The following statements hold.*

A) Assume that $F : J_1 \to \mathbf{R}$ does not depend on x, is geometric, lower semicontinuous and satisfies (F4). Suppose that for any $\lambda \in \mathbf{R}$

$$\{x \in \mathbf{R}^n : u^*(\cdot, x) < \lambda\} \in \mathcal{B}(\mathcal{F}_F^{\gtrless}). \tag{5.12}$$

(i) If F satisfies (F2) then u is a viscosity subsolution of (3.5) in $]0,+\infty[\times\mathbf{R}^n$;

(ii) if F^+ satisfies (F4) then u is a viscosity subsolution of (5.11) in $]0,+\infty[\times\mathbf{R}^n$.

B) Assume that $F : J_0 \to \mathbf{R}$ is geometric, lower semicontinuous and satisfies (F4). Assume that for any $\lambda \in \mathbf{R}$ relation (5.12) holds.

(iii) If F satisfies (F2),(F8') then u is a viscosity subsolution of (3.1) in $]0,+\infty[\times\mathbf{R}^n$;

(iv) if F^+ satisfies (F4),(F8') then u is a viscosity subsolution of

$$\frac{\partial u}{\partial t} + F^+(t, x, \nabla u, \nabla^2 u) = 0 \tag{5.13}$$

in $]0,+\infty[\times\mathbf{R}^n$.

Proof. Statement A). It is enough to prove (ii). Let $(\overline{t}, \overline{x}) \in]0,+\infty[\times\mathbf{R}^n$; we have to prove (3.2) (with $F^+(t,p,X)$ instead of $F(t,x,p,X)$) for any smooth function ψ such that $(u^* - \psi)$ has a maximum at $(\overline{t}, \overline{x})$. Let $\overline{\lambda} := u^*(\overline{t}, \overline{x})$; we define the function $z : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ as

$$z(t,x) := \begin{cases} \overline{\lambda} & \text{if } u^*(t,x) \geq \overline{\lambda}, \\ \overline{\lambda} - 1 & \text{elsewhere.} \end{cases}$$

By (5.12) and Proposition 5.2 (ii), setting $\phi(t) := \{x \in \mathbf{R}^n : u^*(t,x) < \overline{\lambda}\}$, it follows that the function $(t,x) \to -\chi_{\phi(t)}(x)$ is a subsolution of (5.11) in $]0,+\infty[\times\mathbf{R}^n$. Therefore also z is a subsolution of (5.11) in $]0,+\infty[\times\mathbf{R}^n$. Since $(z - \psi)$ has a maximum at $(\overline{t}, \overline{x})$, (3.2) follows.

Statement B). It is enough to prove (iv). Following the arguments of the proof of statement A), it is sufficient to show the following assertion: given $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless})$, the function

$$\chi(t,x) := -\chi_{\phi_-(t)}(x)$$

is a subsolution of (5.13) in $]0,+\infty[\times\mathbf{R}^n$. For any $0 < \epsilon < 1$ we define $d_\epsilon(t,x) := (-\epsilon) \vee d_{\phi_-}(t,x) \wedge 0$.

We shall prove that d_ϵ is a subsolution of

$$\frac{\partial u}{\partial t} + F_\epsilon(t, x, \nabla u, \nabla^2 u) = 0 \tag{5.14}$$

in $]0,+\infty[\times\mathbf{R}^n$, where

$$F_\epsilon(t,x,p,X) := F^+(t,x,p,X) - \epsilon\sigma_{|X|}(1+\epsilon), \tag{5.15}$$

and $\sigma_{|X|}$ is the modulus of continuity defined in (F8'). In view of Lemma 3.1 it is enough to check that d_ϵ is a subsolution in $\{d_\epsilon < 0\}$.

Let $(\bar{t},\bar{x}) \in \,]0,+\infty[\times\mathbf{R}^n$. Let ψ be a smooth function such that $(d_\epsilon - \psi)$ has a strict global maximum at $(\bar{t},\bar{x})$ and $d_\epsilon(\bar{t},\bar{x}) = \psi(\bar{t},\bar{x})$. Set

$$(\alpha, p, X) := \left(\frac{\partial\psi}{\partial t}(\bar{t},\bar{x}), \nabla\psi(\bar{t},\bar{x}), \nabla^2\psi(\bar{t},\bar{x})\right).$$

CASE 1. $\bar{x} \in \phi_-(\bar{t})$ and $\mathrm{dist}(\bar{x}, \mathbf{R}^n \setminus \phi_-(\bar{t})) > \epsilon$. Then d_ϵ is twice differentiable at $(\bar{t},\bar{x})$ with respect to x, therefore $p = 0$, $X \geq 0$. Moreover by Lemma 4.2 we have $(\phi_-)^-_\epsilon = ((\phi_-)^-_\epsilon)_-$, hence, as $\bar{x} \in (\phi_-(\bar{t}))^-_\epsilon$, there exists a sequence $t_m \uparrow \bar{t}$ such that $d_\epsilon(t_m,\bar{x}) = -\epsilon \leq \psi(t_m,\bar{x})$, which yields $\alpha \leq 0$. Therefore

$$\alpha + (F_\epsilon)_*(\bar{t},\bar{x},p,X) \leq (F^+)_*(\bar{t},\bar{x},0,X) \leq (F^+)_*(\bar{t},\bar{x},0,0) \leq 0.$$

CASE 2. $\bar{x} \in \phi_-(\bar{t})$ and $\mathrm{dist}(x, \mathbf{R}^n \setminus \phi(\bar{t})) \leq \epsilon$. As d_ϵ is locally semiconvex in $\phi_-(\bar{t})$, we have $|p| = 1$. Let $\bar{y} \in \partial\phi(\bar{t})$ be such that $|\bar{x}-\bar{y}| = -d_\epsilon(\bar{t},\bar{x}) \leq \epsilon$. Following the proof of Proposition 5.2 and applying Proposition 5.1 we get

$$\alpha + F^+(\bar{t},\bar{y},p,X) \leq 0.$$

Therefore, using (F8') and recalling (5.15), we have

$$\alpha + (F_\epsilon)_*(\bar{t},\bar{x},p,X) \leq \alpha + F^+(\bar{t},\bar{y},p,X) \leq 0.$$

We have proved that d_ϵ is a subsolution of (5.14) in $]0,+\infty[\times\mathbf{R}^n$. Reasoning as in [1, Lemma 4.3] we then obtain that $\epsilon\chi$ is a subsolution of (5.14) in $]0,+\infty[\times\mathbf{R}^n$, hence $\frac{1}{\epsilon}(\epsilon\chi) = \chi$ is also a subsolution of (5.14) in $]0,+\infty[\times\mathbf{R}^n$. Letting $\epsilon \to 0$ and using [9, Proposition 2.4], we get that χ is a subsolution of (5.13) in $]0,+\infty[\times\mathbf{R}^n$. $\square$

We also have a similar statement of Theorem 5.1 for supersolutions.

Remark 5.2. *Assume that $F : J_0 \to \mathbf{R}$ is geometric, upper semicontinuous and satisfies (F4). Let $v : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ be a function such that $v_* > -\infty$ and $\{x \in \mathbf{R}^n : v_*(\cdot,x) > \lambda\} \in \mathcal{B}(\mathcal{F}^{\geq}_{F_c})$ for any $\lambda \in \mathbf{R}$. If F^- satisfies (F4), (F8') then v is a viscosity supersolution of*

$$\frac{\partial v}{\partial t} + F^-(t,x,\nabla v,\nabla^2 v) = 0$$

in $]0,+\infty[\times\mathbf{R}^n$, where $F^-(t,x,p,X) := \inf\{F(t,x,p,Y) : Y \leq X\}$.

6. Conclusions

The following result shows the connection between the minimal barrier and the continuous viscosity solution whenever the latter exists and is unique, see Theorem 3.1.

Corollary 6.1. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Let $E \subseteq \mathbf{R}^n$ be a bounded set and denote with $v : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ the unique uniformly continuous viscosity solution of (3.1) with $v(0,x) = v_0(x) := (-1) \vee d_E(x) \wedge 1$. Then for any $t \in [0,+\infty[$ we have*

$$\mathcal{M}_*(E,\mathcal{F}^{\geq}_F)(t) = \mathcal{M}_*(E,\mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) < 0\}, \tag{6.1}$$

$$\mathcal{M}^*(E,\mathcal{F}^{\geq}_F)(t) = \mathcal{M}^*(E,\mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) \leq 0\}. \tag{6.2}$$

In particular (1.6) holds true and $\mathcal{M}_{v_0,\mathcal{F}_F} = v$. *Moreover if* $F = F_c$ *then*

$$\mathcal{M}^*(E, \mathcal{F}_F) \setminus \mathcal{M}_*(E, \mathcal{F}_F) \in \mathcal{B}(\mathcal{F}_F). \tag{6.3}$$

Proof. It is enough to show that for any bounded open set $A \subseteq \mathbf{R}^n$

$$\mathcal{M}(A, \mathcal{F}_F^{\gtrless}) = \mathcal{M}(A, \mathcal{F}_F) = V(A), \tag{6.4}$$

where $V(A)$ is defined in (3.4). By statement B) of Theorem 3.2 we have $V(A) \in \mathcal{B}(\mathcal{F}_F)$, hence $V(A) \supseteq \mathcal{M}(A, \mathcal{F}_F)$.
Let

$$\chi(t, x) := -\chi_{\mathcal{M}(A,\mathcal{F}_F^{\gtrless})(t)}(x), \qquad (t, x) \in [0, +\infty[\times \mathbf{R}^n.$$

By statement B) of Theorem 5.1, χ is a subsolution of (3.1) in $]0, +\infty[\times \mathbf{R}^n$ (note that $\chi(\cdot, x)$ is upper semicontinuous by (4.25) and [25, Lemma 3.1]). Applying the viscosity comparison theorem [16, Theorem 2.1] we get $\chi(t, x) \leq v(t, x)$ for any $(t, x) \in [0, +\infty[\times \mathbf{R}^n$, hence $V(A) \subseteq \mathcal{M}(A, \mathcal{F}_F^{\gtrless})$.
We conclude that

$$\mathcal{M}(A, \mathcal{F}_F^{\gtrless}) \supseteq V(A) \supseteq \mathcal{M}(A, \mathcal{F}_F) \supseteq \mathcal{M}(A, \mathcal{F}_F^{\gtrless}), \tag{6.5}$$

and the proof is (6.1), (6.2), (1.6) is complete. Finally, (6.3) follows from (3.10). □

Remark 6.1. *Equality (1.6) proved in Corollary 6.1 shows that definition (2.5) is consistent with the definition of fattening given by means of the (unique) viscosity solution, see [13], [7]. Notice that, if we adopt definition (2.5), fattening can be defined also when there is non uniqueness of viscosity solutions, see Example 6.1 below.*

Remark 6.2. *From (6.5) it follows that, under the assumptions of Corollary 6.1, if* $E \subseteq \mathbf{R}^n$ *is bounded and open then* $\mathcal{M}(E, \mathcal{F}_F)(t)$ *is open for any* $t \in [0, +\infty[$.

Remark 6.3. *Corollary 6.1 in the case of driven motion by mean curvature in codimension one has been proved in [7], where the minimal barriers are compared with any generalized evolution of sets satifying the semigroup property, the comparison principle, and the extension of smooth evolutions.*

Remark 6.4. *Corollary 6.1 also applies to the case of motion by mean curvature in arbitrary codimension, i.e., when* F *has the form* $F(p, X) = -\sum_{i=1}^{n-k} \lambda_i$, *where* $1 \leq k \leq n-1$ *is the codimension and* $\lambda_1 \leq \ldots \leq \lambda_{n-1}$ *are the eigenvalues of the matrix* $P_p X P_p$ *corresponding to eigenvectors orthogonal to* p. *In [1] it has been proved that for such a function* F *there holds* $V(A) \supseteq \mathcal{M}(A, \mathcal{F}_F)$ *for any bounded open set* $A \subseteq \mathbf{R}^n$.

The following results generalize Corollary 6.1.

Corollary 6.2. *Assume that* $F : J_0 \to \mathbf{R}$ *is geometric, lower semicontinuous and satisfies (F4). Assume that* F^+ *satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Then for any bounded set* $E \subseteq \mathbf{R}^n$ *and any* $t \in [0, +\infty[$ *we have*

$$\mathcal{M}_*(E, \mathcal{F}_F)(t) = \mathcal{M}_*(E, \mathcal{F}_F^{\gtrless})(t) = \{x \in \mathbf{R}^n : v(t, x) < 0\},$$
$$\mathcal{M}^*(E, \mathcal{F}_F)(t) = \mathcal{M}^*(E, \mathcal{F}_F^{\gtrless})(t) = \{x \in \mathbf{R}^n : v(t, x) \leq 0\},$$

where v *is the unique uniformly continuous viscosity solution of (5.13) and* $v(0, x) = v_0(x) := (-1) \vee d_E(x) \wedge 1$. *In particular, thanks to Corollary 6.1, we have*

$$\mathcal{M}_*(E, \mathcal{F}_F) = \mathcal{M}_*(E, \mathcal{F}_{F^+}), \qquad \mathcal{M}^*(E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{F}_{F^+}).$$

Corollary 6.3. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a given function such that $u_0^* < +\infty$. Define*

$$S_{u_0} := \{v : v \textit{ is a viscosity subsolution of } (3.1) \textit{ in }]0,+\infty[\ \times \mathbf{R}^n, v^*(0,x) = u_0^*(x)\}.$$

If u_0 is upper semicontinuous then

$$\mathcal{M}_{u_0,\mathcal{F}_F} = \mathcal{M}_{u_0,\mathcal{F}_F^{\gtrless}} = \sup\{v : v \in S_{u_0}\}. \tag{6.6}$$

In the general case we have

$$\overline{\mathcal{M}}_{u_0,\mathcal{F}_F} = \overline{\mathcal{M}}_{u_0,\mathcal{F}_F^{\gtrless}} = \sup\{v : v \in S_{u_0}\}. \tag{6.7}$$

Proof. Let $w_{u_0} := \sup\{v : v \in S_{u_0}\}$. Let u_0 be upper semicontinuous. Given any set $E \subseteq \mathbf{R}^n$ one can verify that $\mathcal{M}(E, \mathcal{F}_F^{\gtrless})(0) = E$. Moreover, given $\lambda \in \mathbf{R}$, as $\{u_0 < \lambda\}$ is open, by (4.25) the set $\mathcal{M}(\{u_0 < \lambda\}, \mathcal{F}_F^{\gtrless})(t)$ is open. Then, by (4.25) and (2.7) we have

$$\{x \in \mathbf{R}^n : \mathcal{M}_{u_0,\mathcal{F}_F^{\gtrless}}(\cdot, x) < \lambda\} = \mathcal{M}(\{u_0 < \lambda\}, \mathcal{F}_F^{\gtrless})(\cdot) \in \mathcal{B}(\mathcal{F}_F^{\gtrless}). \tag{6.8}$$

In particular $\{x \in \mathbf{R}^n : \mathcal{M}_{u_0,\mathcal{F}_F^{\gtrless}}(0,x) < \lambda\} = \{u_0 < \lambda\}$, hence $\mathcal{M}_{u_0,\mathcal{F}_F^{\gtrless}}(0,x) = u_0(x)$ for any $x \in \mathbf{R}^n$. Moreover, by (6.8) and statement B) of Theorem 5.1 it follows that $\mathcal{M}_{u_0,\mathcal{F}_F^{\gtrless}}$ is a subsolution of (3.1). Hence $\mathcal{M}_{u_0,\mathcal{F}_F^{\gtrless}} \le w_{u_0}$.

Let now v be any subsolution of (3.1) such that $v^*(0,x) = u_0(x)$. Then, given $\lambda \in \mathbf{R}$, by statement B) of Theorem 3.2 we have $\{x \in \mathbf{R}^n : v^*(\cdot, x) < \lambda\} \in \mathcal{B}(\mathcal{F}_F)$. Therefore

$$\{x \in \mathbf{R}^n : v^*(\cdot, x) < \lambda\} \supseteq \mathcal{M}(\{u_0 < \lambda\}, \mathcal{F}_F)(\cdot) = \{x \in \mathbf{R}^n : \mathcal{M}_{u_0,\mathcal{F}_F}(\cdot, x) < \lambda\},$$

which implies $v^* \le \mathcal{M}_{u_0,\mathcal{F}_F}$. Hence $\mathcal{M}_{u_0,\mathcal{F}_F} \ge w_{u_0}$. Since $\mathcal{M}_{u_0,\mathcal{F}_F^{\gtrless}} \ge \mathcal{M}_{u_0,\mathcal{F}_F}$, (6.6) follows.

Let now u_0 be arbitrary. It is not difficult to show [6] that given any set $E \subseteq \mathbf{R}^n$ we have

$$\mathcal{M}_*(E, \mathcal{F}_F^{\gtrless}) = \mathcal{M}_*(\mathrm{int}(E), \mathcal{F}_F^{\gtrless}) = \mathcal{M}(\mathrm{int}(E), \mathcal{F}_F^{\gtrless}),$$

and that $\mathcal{M}_*(E, \mathcal{F}_F^{\gtrless})(t)$ is open for any $t \in [0, +\infty[$. Therefore, given $\lambda \in \mathbf{R}$, we have

$$\begin{aligned}\mathcal{M}_*(\{u_0 < \lambda\}, \mathcal{F}_F^{\gtrless}) &= \mathcal{M}_*(\mathrm{int}(\{u_0 < \lambda\}), \mathcal{F}_F^{\gtrless})\\ &= \mathcal{M}_*(\{u_0^* < \lambda\}, \mathcal{F}_F^{\gtrless}) = \mathcal{M}(\{u_0^* < \lambda\}, \mathcal{F}_F^{\gtrless}).\end{aligned}$$

Then (6.7) follows from (6.6). □

Remark 6.5. *A similar assertion of Corollary 6.3 (under the same hypotheses) holds for supersolutions. Precisely, if u_0 is lower semicontinuous (resp. arbitrary) such that $u_{0*} > -\infty$ we have that, for any $(t,x) \in [0,+\infty[\times \mathbf{R}^n$, the function*

$$\sup\{\mu : \mathcal{M}(\{u_0 > \mu\}, \mathcal{F}_F)(t) \ni x\}$$
$$(\textit{resp. } \sup\{\mu : \mathcal{M}_*(\{u_0 > \mu\}, \mathcal{F}_F)(t) \ni x\})$$

coincides with the infimum of $u(t,x)$, where u varies over all viscosity supersolutions of (3.1) in $]0,+\infty[\ \times \mathbf{R}^n$ such that $u_*(0,x) = u_0(x)$ (resp. $u_*(0,x) = u_{0*}(x)$) and same assertions with $\mathcal{F}_F$ replaced by $\mathcal{F}_F^{\gtrless}$.

The following remark shows the connections between the barriers and the viscosity evolution without growth conditions on F (see [22,17]) and for unbounded sets E.

Remark 6.6. *Assume that* $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ *does not depend on* (t,x), *is geometric and satisfies (F1), (F2). Let* u *and* v *be, respectively, a viscosity sub- and supersolution of*

$$\frac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0 \tag{6.9}$$

in $]0,+\infty[\,\times\mathbf{R}^n$, *in the sense of [22, Definition 1.2]. Then, reasoning as in Theorem 3.2 and using [22, Proposition 1.6, Theorem 1.7], one can check that (3.6)-(3.9) hold. Moreover, using also [22, Proposition 1.3], it turns out that Lemma 3.1 is still true and that, given* $\phi \in \mathcal{B}(\mathcal{F}_F^{\geq})$, $d_\phi \wedge 0$ *is a viscosity subsolution of (6.9). Therefore, as [1, Lemma 4.3] still holds, if* $u : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ *is a function such that* $u^* < +\infty$ *and satisfies (5.12) for any* $\lambda \in \mathbf{R}$, *then* u *is a viscosity subsolution of (6.9) in* $]0,+\infty[\,\times\mathbf{R}^n$. *Finally, in view of Remark 3.3, Corollary 6.1 still holds, even if* E *is unbounded.*

In particular we have the following result.

Corollary 6.4. *Assume that* $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ *does not depend on* (t,x), *is geometric and satisfies (F1),(F2). Let* $E \subseteq \mathbf{R}^n$ *and let* $v : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ *be the unique uniformly continuous viscosity solution of (6.9) with* $v(0,x) = v_0(x) := d_E(x)$. *Then for any* $t \in [0,+\infty[$ *we have (6.1) and (6.2). In particular* $\mathcal{M}^*(E,\mathcal{F}_F)(t) \setminus \mathcal{M}_*(E,\mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) = 0\}$ *and* $\mathcal{M}_{v_0,\mathcal{F}_F} = v$.

Example 6.1. Let $n = 2$, $F(p,X) := -\mathrm{tr}(P_p X P_p)$ (i.e., motion by mean curvature) and let

$$v_0(x_1,x_2) := x_2^2(1+x_1^2)^2.$$

Then u_0 is not uniformly continuous and we have nonuniqueness of continuous viscosity solutions of (6.9) with $v(0,x) = v_0(x)$, see [21]. In this case $\mathcal{M}_{v_0,\mathcal{F}_F}$ is, by Corollary 6.3, the maximal viscosity (sub) solution. One can check, following [21], that there exist $t \in [0,+\infty[$ and $x \in \mathbf{R}^n$ such that $\mathcal{M}_{v_0,\mathcal{F}_F}(t,x) > -\mathcal{M}_{-v_0,\mathcal{F}_F}(t,x)$, where $-\mathcal{M}_{-v_0,\mathcal{F}_F}$ represents the minimal viscosity (super) solution.

Note that for any $\lambda > 0$ the set $\{v_0 < \lambda\}$ develops fattening (with respect to $\mathcal{F}_F$).

Remark 6.7. *Let* $n = 2$ *and consider the anisotropic motion by mean curvature given by*

$$F(p,X) = -tr(P_p X P_p)\psi(\theta)(\psi(\theta) + \psi''(\theta)),$$

where $\psi : \mathbf{S}^1 \to \mathbf{R}$ *is a smooth function and* $p = (p_1,p_2) = (\cos\theta, \sin\theta)$ *(see [8]). Then, if* $\psi + \psi'' \geq 0$ *on* $\mathbf{S}^1$ *(i.e., convex anisotropy), we have* $F^+ = F$. *If the anisotropy is not convex, then there exists* $\overline{\theta} \in \mathbf{S}^1$ *such that* $\psi(\overline{\theta}) + \psi''(\overline{\theta}) < 0$, *which implies* $F^+(\overline{p}, X) = +\infty$ *for any* $X \in Sym(2)$, *where* $\overline{p} = (\cos\overline{\theta}, \sin\overline{\theta})$.

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