A result on motion by mean curvature in arbitrary codimension

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Abstract

We prove a conjecture formulated by De Giorgi concerning the connections between motion by mean curvature of a k-dimensional submanifold without boundary in $\mathbb{R}^n$ and the evolution of its tubular neighbourhoods by the sum of the k smallest curvatures. The result holds also after the onset of singularities.

1 Introduction

In this paper we prove a conjecture formulated by De Giorgi in [7] concerning weak motion by mean curvature of a k-dimensional submanifold Γ without boundary in $\mathbb{R}^n$. The conjecture, partially proved by Ambrosio and Soner in [2], implies that the weak evolution of Γ can be equivalently constructed either by using smooth hypersurfaces evolving by the sum of the k smallest principal curvatures, or smooth k-dimensional submanifolds evolving by mean curvature.

The framework in which the conjecture has been formulated is the theory of barriers, introduced by De Giorgi in [6], [7], which, suitably adapted to geometric problems, provides a unique global weak evolution called minimal barrier. The minimal barrier has an intrinsic definition, verifies the comparison principle and a semigroup property and agrees with the classical evolution until the latter exists.

Let us briefly describe the content of the conjecture. Let $\mathcal{F}$ denote a family of maps taking some time interval (depending on $f \in \mathcal{F}$) into the class $\mathcal{P}(\mathbb{R}^n)$ of all subsets of $\mathbb{R}^n$. Roughly speaking, the family $\mathcal{F}$ can be considered as the class of all tests evolving manifolds, through which one constructs a barrier with respect to sets inclusion; the choice of $\mathcal{F}$ is crucial and must be adapted to the problem at hand. A map $\phi : [0,+\infty[\to \mathcal{P}(\mathbb{R}^n)$ is said to be a barrier with respect to $\mathcal{F}$, and we write $\phi \in \mathcal{B}(\mathcal{F})$, if for any $f : [a,b] \subseteq [0,+\infty[\to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, such that $f(a) \subseteq \phi(a)$, it holds $f(b) \subseteq \phi(b)$. Given any set $E \subseteq \mathbb{R}^n$, the minimal barrier $\mathcal{M}(E,\mathcal{F})$ starting at E, with respect to $\mathcal{F}$, is defined as

$$\mathcal{M}(E,\mathcal{F})(t) := \bigcap \big\{\phi(t) : \phi \in \mathcal{B}(\mathcal{F}), \phi(0) \supseteq E\big\}, \qquad t \in [0,+\infty[. \tag{1.1}$$

General properties of barriers have been studied in [4]; connections with other notions of generalized evolutions have been considered in [2], [3] (see also the papers of Ilmanen [9] and White [12], which study the properties of Ilmanen's set-theoretic subsolutions for motion by mean curvature of hypersurfaces). Other related results, also in connection with mean curvature evolution in arbitrary codimension, can be found in the papers of Jerrard and Soner [10], [11].

The original definition [7] of weak evolution by mean curvature in arbitrary codimension is obtained first by an upper regularization of (1.1) through a new barrier, denoted by $\mathcal{M}^*(E,\mathcal{F})$, and then by particularizing the choice of $\mathcal{F}$. Indeed, set

$$E_\rho^+ := \{x : \mathrm{dist}(x,E) < \rho\}, \qquad \mathcal{M}^*(E,\mathcal{F}) := \bigcap_{\rho>0} \mathcal{M}(E_\rho^+,\mathcal{F}),$$

and then choose $\mathcal{F}$ as the family $\mathcal{G}_k$ of all smooth local evolutions of k-dimensional submanifolds without boundary by mean curvature. The set valued map $t \in [0,+\infty[\to \mathcal{M}^*(E,\mathcal{G}_k)(t)$ is then the required weak evolution.

As observed in [7] and deepened in details in [2], if $t \in [a,b] \to \Gamma(t)$ is an element of $\mathcal{G}_k$, then the following system must hold on the moving manifold:

$$\frac{\partial \nabla \eta_\Gamma}{\partial t} = \Delta \nabla \eta_\Gamma \qquad \text{on } \Gamma(t),\ t \in [a,b], \tag{1.2}$$

where $\eta_\Gamma(t,x) := \big(\mathrm{dist}(x,\Gamma(t))\big)^2/2$. The squared distance function η_Γ from the flowing manifold not only describes in a simple way the evolution of $\Gamma \in \mathcal{G}_k$, but also plays a fundamental rôle in the proof of the main result (Theorem 4.1).

As pointed out in [7], there is another way of defining the weak evolution of a k-dimensional submanifold Γ. Precisely, let p be a given vector of $\mathbb{R}^n \setminus \{0\}$ and set $P_p := \mathrm{Id} - p \otimes p/|p|^2$; if $\mathrm{Sym}(n)$ stands for the space of all symmetric $(n \times n)$-matrices, denote by $F : (\mathbb{R}^n \setminus \{0\}) \times \mathrm{Sym}(n) \to \mathbb{R}$ the function defined as follows:

$$F(p,X) := -\sum_{i=1}^{k} \lambda_i(p,X), \tag{1.3}$$

where $\lambda_1(p,X) \leq \cdots \leq \lambda_{n-1}(p,X)$ are the eigenvalues of the matrix $P_p X P_p$ which correspond to eigenvectors orthogonal to p. The idea is to consider the evolution $\mathcal{M}^*(E,\mathcal{F}_F)$, where $\mathcal{F}_F$ is now the family of all maps $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ such that $\partial f(t)$ is a smooth hypersurface flowing through the geometric inequality

$$\frac{\partial d_f}{\partial t} + F(\nabla d_f, \nabla^2 d_f) \geq 0 \qquad \text{on } \partial f(t),\ t \in [a,b], \tag{1.4}$$

and $d_f(t,x) := \mathrm{dist}(x, f(t)) - \mathrm{dist}(x, \mathbb{R}^n \setminus f(t))$ is the usual signed distance from the front.
The conjecture then states that the two weak k-dimensional evolutions actually coincide, that is, for any set $E \subseteq \mathbb{R}^n$ there holds

$$\mathcal{M}^*(E, \mathcal{G}_k) = \mathcal{M}^*(E, \mathcal{F}_F).$$

A consequence of the above result is that the natural definition of weak evolution given by $\mathcal{M}^*(E, \mathcal{G}_k)$ coincides, in view of comparison results proved in [2], [3], with the zero level set of the unique solution, in the viscosity sense [5], of the equation $\frac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0$ having $u(0,x) := \mathrm{dist}(x, E)$ as initial datum. Notice that these comparison results hold also after the onset of singularities.
The inclusion $\mathcal{M}^*(E, \mathcal{G}_k) \subseteq \mathcal{M}^*(E, \mathcal{F}_F)$ has been proved in [2] and is based on the property that, if $\Gamma : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ belongs to $\mathcal{G}_k$, then the map $t \in [a,b] \to f(t) := \{x \in \mathbb{R}^n : \eta_\Gamma(t,x) \leq \sigma^2\}$, for small $\sigma > 0$, belongs to $\mathcal{F}_F$. The proof of this nontrivial fact relies on the properties of the eigenvalues of the Hessian of η_Γ and their connections with the eigenvalues of the Hessian of the distance function d_Γ from $\Gamma(t)$, out of $\Gamma(t)$.
The opposite inclusion, namely

$$\mathcal{M}^*(E, \mathcal{G}_k) \supseteq \mathcal{M}^*(E, \mathcal{F}_F), \tag{1.5}$$

is more delicate, and consists in proving that, if A is an open set, then $\mathcal{M}(A, \mathcal{G}_k)$ is a barrier with respect to $\mathcal{F}_F$. This is proved arguing by contradiction, and is the main result of the present paper. A sketch of the proof, omitting technical details, runs as follows. Choose a map $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ belonging to $\mathcal{F}_F$ such that $f(a) \subseteq \mathcal{M}(A, \mathcal{G}_k)(a)$ and such that, at some $t^* \in]a,b[$, $f(t^*)$ "crosses" $\partial M(A, \mathcal{G}_k)(t^*)$ at a certain $x^* \in \partial f(t^*) \cap \partial M(A, \mathcal{G}_k)(t^*)$. The main point is to contruct a flow $\Gamma_\epsilon : [t^*, t^* + \tau_\epsilon] \to \mathcal{P}(\mathbb{R}^n)$ belonging to $\mathcal{G}_k$ such that its ϵ-tubular neighbourhood $\Gamma_\epsilon(t^*)^+_\epsilon$ at time t^* contains x^* in its boundary, is contained in $f(t^*)$ and, at the point x^*, has the k smallest principal curvatures equal to those of $f(t^*)$ and has an expanding velocity strictly larger than the expanding velocity of $f(t^*)$. Then $\Gamma_\epsilon(t)^+_\epsilon$ must cross $f(t)$ at later times $t > t^*$ at points close to x^*, and therefore $\Gamma_\epsilon(t)^+_\epsilon$ cannot be contained in $\mathcal{M}(A, \mathcal{G}_k)(t)$. However, this inclusion must be verified, due to the spatially translation invariance of the involved equations, and we reach a contradiction.
The content of the paper is the following. In Section 2 we give the notation, all needed properties of the distance function and of the squared distance function, and the definitions of barriers used throughout the paper. In Theorem 2.7 we point out a comparison

result between hypersurfaces and k-dimensional submanifolds. Section 3 is devoted to preliminary lemmas, used to prove (1.5). In Section 4 we prove the conjecture. We conclude the paper with some observations and generalizations.

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2 Notation and main definitions

In the following for simplicity we let $I := [0, +\infty[$, even if all results still hold if I is replaced by $[0, T[$ for some $T > 0$. We denote by $\mathcal{P}(\mathbb{R}^n)$ the family of all subsets of $\mathbb{R}^n$, $n \geq 2$; k will be an integer with $1 \leq k \leq n-1$. If $x \in \mathbb{R}^n$ and $\rho > 0$, we set $B_\rho(x) := \{y \in \mathbb{R}^n : |y - x| < \rho\}$.

Given a set $E \subseteq \mathbb{R}^n$, we denote by $\mathrm{int}(E)$, $\overline{E}$ and ∂E the interior, the closure and the boundary of E, respectively. We set $\mathrm{dist}(\cdot, \emptyset) \equiv +\infty$,

$$d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathbb{R}^n \setminus E), \qquad \eta_E(x) := \frac{1}{2}\big(\mathrm{dist}(x, E)\big)^2.$$

It is well known that, if E has smooth compact boundary, then d_E is smooth in a suitable tubular neighbourhood U of ∂E, ∇d_E is, on ∂E, the exterior unit normal to ∂E, and the restriction of $\nabla^2 d_E$ to the tangent space to ∂E coincides with the second fundamental form of ∂E. In addition, if $y \in U$ and denoting by $\pi(y) := y - d_E(y)\nabla d_E(y)$ the (unique) orthogonal projection of y on ∂E, then

$$\lambda_i(y) = \frac{\kappa_i(\pi(y))}{1 + d_E(y)\kappa_i(\pi(y))}, \qquad i = 1, \ldots, n-1, \tag{2.1}$$

where $\lambda_1(y), \ldots, \lambda_{n-1}(y)$ are the eigenvalues of $\nabla^2 d_E(y)$ corresponding to eigenvectors orthogonal to $\nabla d_E(\pi(y))$, and $\kappa_1(\pi(y)), \ldots, \kappa_{n-1}(\pi(y))$ are the principal curvatures of ∂E at $\pi(y)$.

Notice that Δd_E, evaluated on ∂E, is nonnegative for smooth convex sets.

The following results on the square distance function have been proved in [1], [2]. Let Γ be a smooth compact submanifold of dimension k without boundary; then η_Γ is smooth in a suitable tubular neighbourhood Ω of Γ. On Γ the matrix $\nabla^2\eta_\Gamma$ represents the orthogonal projection on the normal space to Γ; if $y \in \Omega$, $\nabla^2\eta_\Gamma(y)$ has exactly $n-k$ eigenvalues equal to one, and the remaining k eigenvalues are strictly smaller than one. Precisely, if $\pi(y) := y - \nabla\eta_\Gamma(y)$ is the (unique) orthogonal projection of y on Γ, then

$$\mu_i(y) = \begin{cases} \dfrac{d_\Gamma(y)\kappa_i(\pi(y))}{1 + d_\Gamma(y)\kappa_i(\pi(y))} & \text{if } 1 \leq i \leq k, \\ 1 & \text{if } k < i \leq n, \end{cases} \tag{2.2}$$

where $\mu_1(y), \ldots, \mu_n(y)$ are the eigenvalues of $\nabla^2\eta_\Gamma(y)$ and $\kappa_1(\pi(y)), \ldots, \kappa_k(\pi(y))$ are the principal curvatures of Γ at $\pi(y)$ along $\nabla d_\Gamma(y)$.
Notice that, if $y \in \Omega \setminus \Gamma$, then

$$\nabla^2 d_\Gamma(y) = \frac{1}{d_\Gamma(y)} \left(\nabla^2\eta_\Gamma(y) - \nabla d_\Gamma(y) \otimes \nabla d_\Gamma(y)\right).$$

Therefore,

$$\lambda_i(y) = \begin{cases} \dfrac{\kappa_i(\pi(y))}{1 + d_\Gamma(y)\kappa_i(\pi(y))} & \text{if } 1 \le i \le k, \\ \\ \dfrac{1}{d_\Gamma(y)} & \text{if } k < i \le n-1, \\ \\ 0 & \text{if } i = n, \end{cases} \tag{2.3}$$

where $\lambda_1(y), \ldots, \lambda_{n-1}(y)$ are the eigenvalues of $\nabla^2 d_\Gamma(y)$ corresponding to eigenvectors orthogonal to $\nabla d_\Gamma(y)$, and $\lambda_n(y)$ is the eigenvalue corresponding to $\nabla d_\Gamma(y)$, which vanishes.
Finally, $-\Delta\nabla\eta_\Gamma$ coincides, on Γ, with the mean curvature vector of Γ.
Given a map $\phi : L \to \mathcal{P}(\mathbb{R}^n)$, where $L \subseteq \mathbb{R}$ is a convex set, we denote by $d_\phi, \eta_\phi : L \times \mathbb{R}^n \to \mathbb{R}$ the functions defined as

$$d_\phi(t,x) := \operatorname{dist}\big(x, \phi(t)\big) - \operatorname{dist}\big(x, \mathbb{R}^n \setminus \phi(t)\big) = d_{\phi(t)}(x),$$
$$\eta_\phi(t,x) := \frac{1}{2}\big(\operatorname{dist}(x, \phi(t)\big)^2 = \eta_{\phi(t)}(x).$$

If $\phi_1, \phi_2 : L \to \mathcal{P}(\mathbb{R}^n)$, by $\phi_1 \subseteq \phi_2$ (resp. $\phi_1 = \phi_2$, $\phi_1 \cap \phi_2$) we mean $\phi_1(t) \subseteq \phi_2(t)$ (resp. $\phi_1(t) = \phi_2(t)$, $\phi_1(t) \cap \phi_2(t)$) for any $t \in L$.

Geometric barriers. The families $\mathcal{F}_F$ and $\mathcal{G}_k$

Let us recall the definitions of geometric barriers and minimal barriers in the sense of De Giorgi [7].

Definition 2.1. *Let $\mathcal{F}$ be a family of maps with the following property: $f \in \mathcal{F}$ if there exist $a, b \in \mathbb{R}$, $a < b$, $[a,b] \subseteq I$, such that $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$. We say that a map $\phi : I \to \mathcal{P}(\mathbb{R}^n)$ is a* barrier *with respect to $\mathcal{F}$, and we write $\phi \in \mathcal{B}(\mathcal{F})$, provided that the following property holds: if $f : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$, then $f(b) \subseteq \phi(b)$.*

It is clear that the intersection of an arbitrary family of barriers is a barrier.

Definition 2.2. *Let $E \subseteq \mathbb{R}^n$ be a given set. The* minimal barrier $\mathcal{M}(E,\mathcal{F}) : I \to \mathcal{P}(\mathbb{R}^n)$ *with respect to $\mathcal{F}$, with origin at $E \subseteq \mathbb{R}^n$ (at time 0) is defined as*

$$\mathcal{M}(E,\mathcal{F})(t) := \bigcap \Big\{\phi(t) : \ \phi \in \mathcal{B}(\mathcal{F}), \phi(0) \supseteq E\Big\}. \tag{2.4}$$

If $\rho > 0$, we also set

$$E^+_\rho := \big\{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < \rho\big\}, \qquad E^-_\rho := \big\{x \in \mathbb{R}^n : \mathrm{dist}(x, \mathbb{R}^n \setminus E) > \rho\big\},$$

$$\mathcal{M}_*\big(E,\mathcal{F}\big) := \bigcup_{\rho>0} \mathcal{M}\big(E^-_\rho, \mathcal{F}\big), \qquad \mathcal{M}^*\big(E,\mathcal{F}\big) := \bigcap_{\rho>0} \mathcal{M}\big(E^+_\rho, \mathcal{F}\big).$$

Recalling the definition of F in (1.3), we now define the family $\mathcal{F}_F$ of all smooth evolutions of compact hypersurfaces without boundary, evolving with inward normal velocity bigger than or equal to the sum of the smallest k principal curvatures.

Definition 2.3. *Let $a, b \in \mathbb{R}$, $a < b$, $[a,b] \subseteq I$ and let $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$. We write $f \in \mathcal{F}_F$ if and only if the following conditions hold:*

(i) $f(t)$ is closed and $\partial f(t)$ is compact for any $t \in [a,b]$;

(ii) there exists an open set $A \subseteq \mathbb{R}^n$ such that $d_f \in \mathbf{C}^\infty([a,b] \times A)$ and $\partial f(t) \subseteq A$ for any $t \in [a,b]$;

(iii) the following inequality holds on $\partial f(t)$:

$$\frac{\partial d_f}{\partial t}(t,x) + F\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) \geq 0, \quad t \in [a,b], \ x \in \partial f(t). \tag{2.5}$$

We write $f \in \mathcal{F}^>_F$ (resp. $f \in \mathcal{F}^=_F$) if the inequality $>$ (resp. the equality) holds in (2.5).

One can prove that the families $\mathcal{F}_F$, $\mathcal{F}^>_F$ and $\mathcal{F}^=_F$ are nonempty. We recall that $-\frac{\partial d_f}{\partial t}$ is positive for expanding sets. Notice that $F(p, \cdot)$ is 1-Lipschitz.
We now define the family $\mathcal{G}_k$ of all smooth local mean curvature evolutions of compact submanifolds of dimension k without boundary.

Definition 2.4. *Let $a, b \in \mathbb{R}$, $a < b$, $[a,b] \subseteq I$, and let $\Gamma : [a,b] \to \mathcal{P}(\mathbb{R}^n)$. We write $\Gamma \in \mathcal{G}_k$ if and only if the following conditions hold:*

(i) $\Gamma(t)$ is compact for any $t \in [a,b]$;

(ii) there exists an open set $A \subseteq \mathbb{R}^n$ such that $\eta_\Gamma \in \mathbf{C}^\infty\big([a,b] \times A\big)$, $\Gamma(t) \subseteq A$ for any $t \in [a,b]$, and $\mathrm{rank}(\nabla^2 \eta_\Gamma) = n - k$ for any $t \in [a,b]$, $x \in \Gamma(t)$;

(iii) the following system holds on $\Gamma(t)$:

$$\frac{\partial \nabla \eta_\Gamma}{\partial t}(t,x) - \Delta \nabla \eta_\Gamma(t,x) = 0, \qquad t \in [a,b], \ x \in \Gamma(t). \tag{2.6}$$

The properties of η_Γ listed at the beginning of this section, together with the observation that $-\frac{\partial \nabla \eta_\Gamma}{\partial t}$ represents the projection of the velocity on the normal space (see [2]), motivates (iii) of Definition 2.4.

In [1] it is proved that the elements of $\mathcal{G}_k$ can be, surprisingly, characterized by an equation involving only second derivatives of η_Γ, valid in a tubular neighbourhood of Γ. More precisely, let Γ be a map satisfying properties (i) and (ii) of Definition 2.4; then $\Gamma \in \mathcal{G}_k$ if and only if, in a tubular neighbourhood of $\Gamma(t)$ where η_Γ is smooth, $t \in [a,b]$, there holds

$$\frac{\partial \eta_\Gamma}{\partial t} - \sum_{\mu<1} \frac{\mu}{1-\mu} = 0, \tag{2.7}$$

where μ varies among the eigenvalues of $\nabla^2 \eta_\Gamma$.

Remark 2.5. *The families $\mathcal{F}_F$ and $\mathcal{G}_k$ are* translation invariant in space*, that is, if $f \in \mathcal{F}_F$ (resp. $\Gamma \in \mathcal{G}_k$) then $f + y \in \mathcal{F}_F$ (resp. $\Gamma + y \in \mathcal{G}_k$) for any $y \in \mathbb{R}^n$. Using this fact one can check that, if $A \subseteq \mathbb{R}^n$ is an open set, then $\mathcal{M}(A, \mathcal{F}_F)(t)$ and $\mathcal{M}(A, \mathcal{G}_k)(t)$ are open for any $t \geq 0$.*

The following result concerns short time existence of k-dimensional smooth mean curvature flows.

Theorem 2.6. *Let $a \geq 0$ and let $M \subseteq \mathbb{R}^n$ be a smooth compact k-dimensional submanifold without boundary. Then there exist $\tau > 0$ and a map $\Gamma : [a, a+\tau] \to \mathcal{P}(\mathbb{R}^n)$ such that $\Gamma(a) = M$ and $\Gamma \in \mathcal{G}_k$.*

Proof. See [8, Section 2]. □

We conclude this section with the following observation, which is a geometric maximum principle between smooth manifolds of different codimension evolving by mean curvature, and is an interesting result by itself. We give a proof based only on properties of barriers.

Theorem 2.7. *Let $\Gamma, f : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$, $\Gamma \in \mathcal{G}_k$, $f \in \mathcal{F}_F^{=}$, and assume that $\Gamma(a) \subseteq f(a)$. Then $\Gamma(b) \subseteq f(b)$.*

Proof. As $\Gamma(a) \subseteq f(a)$, we have $\mathcal{M}^*(\Gamma(a), \mathcal{F}_F, a)(b) \subseteq \mathcal{M}^*(f(a), \mathcal{F}_F, a)(b)$, where the two barriers are defined starting at time a (i.e., replace 0 with a in (2.4)). We recall now that in [2] it is proved that, if u denotes the unique solution, in the viscosity sense [5], of the equation $\dfrac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0$ with initial datum $u(a,x) = \mathrm{dist}(x, \Gamma(a))$, then $\{x \in \mathbb{R}^n : u(t,x) = 0\} = \Gamma(t)$ for any $t \in [a,b]$. Moreover, by a comparison result between minimal barriers and viscosity solutions [3, Corollary 6.1], there holds

$$\mathcal{M}^*(\Gamma(a), \mathcal{F}_F, a)(t) = \{x \in \mathbb{R}^n : u(t,x) = 0\}.$$

We deduce that $\mathcal{M}^*(\Gamma(a), \mathcal{F}_F, a)(b) = \Gamma(b)$. Finally, $\mathcal{M}^*(f(a), \mathcal{F}_F, a)(b) = f(b)$ (see [3]). We then have

$$\Gamma(b) = \mathcal{M}^*(\Gamma(a), \mathcal{F}_F, a)(b) \subseteq \mathcal{M}^*(f(a), \mathcal{F}_F, a)(b) = f(b).$$

□

3 Some useful lemmas

In this section we show some preliminary results needed in the proof of Theorem 4.1.

Lemma 3.1. *For any $E \subseteq \mathbb{R}^n$ we have*

$$\mathcal{M}_*(E, \mathcal{F}_F) = \mathcal{M}_*(E, \mathcal{F}_F^{>}), \qquad \mathcal{M}^*(E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{F}_F^{>}). \tag{3.1}$$

Proof. Equalities (3.1) are proved if we show that $\mathcal{M}(A, \mathcal{F}_F) = \mathcal{M}(A, \mathcal{F}_F^{>})$ for any open set $A \subseteq \mathbb{R}^n$. This, in turn, thanks to Remark 2.5, will follow if we show that given any $\phi : I \to \mathbb{R}^n$, if $\phi(t)$ is open for any $t \in I$, then

$$\phi \in \mathcal{B}(\mathcal{F}_F) \iff \phi \in \mathcal{B}(\mathcal{F}_F^{>}). \tag{3.2}$$

It is clear that $\mathcal{B}(\mathcal{F}_F^{>}) \supseteq \mathcal{B}(\mathcal{F}_F)$. In order to prove (3.2), we need to show that if $\phi \in \mathcal{B}(\mathcal{F}_F^{>})$, then $\phi \in \mathcal{B}(\mathcal{F}_F)$. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}_F$, $f(a) \subseteq \phi(a)$. For any $t \in [a,b]$ we can find a bounded tubular neighbourhood $\big(\partial f(t)\big)^+_{c(t)}$ of $\partial f(t)$, of thickness $c(t)$, each point of which has a unique orthogonal projection on $\partial f(t)$. We set $2c := \inf\{c(t), t \in [a,b]\} > 0$. Let L be the Lipschitz constant of $F(\nabla d_f, \nabla^2 d_f)$ and M be the supremum of $|\nabla^2 d_f|^2$ when $t \in [a,b]$ and $x \in \big(\partial f(t)\big)^+_c$. Pick a $\mathbf{C}^\infty$ function $\rho : [a,b] \to]0, +\infty[$ such that $\rho(a) < \min\big(c, \mathrm{dist}(\partial f(a), \mathbb{R}^n \setminus \phi(a))\big)$ and $\dot\rho + 2ML\rho < 0$. The map $g : [a,b] \to \mathcal{P}(\mathbb{R}^n)$, $g(t) := \overline{f(t)^+_{\rho(t)}} = \{x \in \mathbb{R}^n : \mathrm{dist}(x, f(t)) \le \rho(t)\}$ is of class $\mathbf{C}^\infty$, and each point $y \in \partial g(t)$ is of the form $y = x + \rho(t) \nabla d_f(t,x)$ for a unique $x \in \partial f(t)$. Moreover $g \in \mathcal{F}_F^{>}$. Indeed for any $t \in [a,b]$ and any $y \in \partial g(t)$, $y = x + \rho(t)\nabla d_f(t,x)$, $x \in \partial f(t)$, we have $\nabla^2 d_g(t,y) = \nabla^2 d_f(t,x)(\mathrm{Id} + \rho(t)\nabla^2 d_f(t,x))^{-1}$, so that

$$|\nabla^2 d_g(t,y) - \nabla^2 d_f(t,x)| \le 2M\rho(t).$$

Therefore, recalling that $f \in \mathcal{F}_F$, we have

$$\begin{aligned}
&-\frac{\partial d_g}{\partial t}(t,y) = -\frac{\partial d_f}{\partial t}(t,x) + \dot\rho(t) \\
&\le F(\nabla d_f(t,x), \nabla^2 d_f(t,x)) + \dot\rho(t) = F(\nabla d_g(t,y), \nabla^2 d_f(t,x)) + \dot\rho(t) \\
&\le F(\nabla d_g(t,y), \nabla^2 d_g(t,y)) + 2LM\rho(t) + \dot\rho(t) < F(\nabla d_g(t,y), \nabla^2 d_g(t,y)),
\end{aligned} \tag{3.3}$$

so that $g \in \mathcal{F}_F^{>}$. Hence $f(b) \subseteq g(b) \subseteq \phi(b)$, and therefore $\phi \in \mathcal{B}(\mathcal{F}_F)$. □

Given $f : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$, we say that f is a $(n-1)$-dimensional smooth compact flow if and only if conditions (i) and (ii) of Definition 2.3 hold.

The next lemma is a kinematic result on $(n-1)$-dimensional smooth compact flows, and concerns the case in which two initial smooth sets $g(a) \subseteq f(a)$ cross each other during the subsequent evolution (which happens when, at $x \in \partial g(a) \cap \partial f(a)$, the expanding velocity of $g(a)$ is strictly larger than the expanding velocity of $f(a)$). We omit the proof which is based on continuity and compactness arguments.

Lemma 3.2. *Let $f, g : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be two $(n-1)$-dimensional smooth compact flows, $x \in \mathbb{R}^n$ and $\rho > 0$. Assume that*

$$\begin{aligned} & x \in \partial f(a) \cap \partial g(a), \\ & g(a) \subseteq f(a), \\ & \frac{\partial d_g}{\partial t}(a,x) < \frac{\partial d_f}{\partial t}(a,x). \end{aligned}$$

Let $0 < \delta \leq b-a$ be such that each point of $\partial f(t)$, for $t \in [a, a+\delta]$, has a unique smooth orthogonal projection $\pi(t,\cdot)$ on $\partial f(a)$. Set $x(t) := \pi^{-1}(t,x)$. Then there exists $\tau \in]0,\delta]$ such that the following holds: for any $t \in]a, a+\tau]$ there exists $R(t) > 0$ such that

$$f(t) \cap \overline{B_{R(t)}(x(t))} \subseteq int(g(t)) \cap \overline{B_{R(t)}(x(t))}.$$

Moreover τ depends in a continuous way on small perturbations of f, g in the $\mathbf{C}^2$-norm.

Remark 3.3. *Since the statement of Lemma 3.2 is local, the regularity of f and g on open sets containing $\partial f(t)$, $\partial g(t)$, $t \in [a,b]$ (see (ii) of Definition 2.3) is not necessary. More precisely, it is enough to assume that the functions d_f, d_g are of class $\mathbf{C}^\infty$ in $[a, a+\delta'] \times B_{2\rho}(x)$, for suitable $\rho > 0$ and $0 < \delta' \leq b-a$ such that each point of $\partial f(t) \cap \overline{B_\rho(x)}$ has a unique smooth orthogonal projection $\pi(t,\cdot)$ on $\partial f(a) \cap \overline{B_{2\rho}(x)}$, for $t \in [a, a+\delta']$.*

We conclude this section with the following result.

Lemma 3.4. *Let $\Gamma : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$, $\Gamma \in \mathcal{G}_k$ and choose $\sigma > 0$ such that η_Γ is smooth on $\Omega := \{(t,x) : t \in [a,b], \eta_\Gamma(t,x) < \sigma^2\}$. Then*

$$\frac{\partial d_\Gamma}{\partial t}(t,x) + F\big(\nabla d_\Gamma(t,x), \nabla^2 d_\Gamma(t,x)\big) \geq 0, \qquad (t,x) \in \Omega,\ x \notin \Gamma(t).$$

As a consequence, the map $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ defined by $f(t) := \{x \in \mathbb{R}^n : dist(x, \Gamma(t)) < \sigma\}$ belongs to $\mathcal{F}_F$.

Proof. See [2, Theorem 3.8 and Remark 6.2]. □

4 Main result

We are now in a position to prove De Giorgi's conjecture.

Theorem 4.1. *Let $E \subseteq \mathbb{R}^n$ be a given set. Then*

$$\mathcal{M}_*(E, \mathcal{G}_k) = \mathcal{M}_*(E, \mathcal{F}_F), \qquad \mathcal{M}^*(E, \mathcal{G}_k) = \mathcal{M}^*(E, \mathcal{F}_F). \tag{4.1}$$

Proof. In order to prove (4.1), it is enough to show that, given any open set $A \subseteq \mathbb{R}^n$ there holds

$$\mathcal{M}(A, \mathcal{G}_k) = \mathcal{M}(A, \mathcal{F}_F).$$

As already remarked in the Introduction, the inclusion $\mathcal{M}(A, \mathcal{G}_k) \subseteq \mathcal{M}(A, \mathcal{F}_F)$ has been proved in [2, Remark 6.2]. Indeed, let $\Gamma : [a, b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$, $\Gamma \in \mathcal{G}_k$, with $\Gamma(a) \subseteq \mathcal{M}(A, \mathcal{F}_F)(a)$. Let us show that $\Gamma(b) \subseteq \mathcal{M}(A, \mathcal{F}_F)(b)$. Choose $\sigma > 0$ and f as in Lemma 3.4; then $f \in \mathcal{F}_F$. Moreover, as $\mathcal{M}(A, \mathcal{F}_F)(a)$ is open (Remark 2.5) and $\Gamma(a)$ is compact, possibly reducing σ we can assume that $f(a) \subseteq \mathcal{M}(A, \mathcal{F}_F)(a)$, which implies $\Gamma(t) \subseteq f(t) \subseteq \mathcal{M}(A, \mathcal{F}_F)(t)$ for any $t \in [a, b]$. Therefore $\mathcal{M}(A, \mathcal{F}_F) \in \mathcal{B}(\mathcal{G}_k)$, which implies $\mathcal{M}(A, \mathcal{G}_k) \subseteq \mathcal{M}(A, \mathcal{F}_F)$.

Let us prove that $\mathcal{M}(A, \mathcal{G}_k) \supseteq \mathcal{M}(A, \mathcal{F}_F)$.
Define $\phi := \mathcal{M}(A, \mathcal{G}_k)$, and recall that $\phi(t)$ is open for any $t \in I$ (Remark 2.5). We shall first prove that ϕ is a barrier for suitable spherical evolutions belonging to $\mathcal{F}_F$; precisely, given $x \in \mathbb{R}^n$, $R > 0$ and $T > 0$,

$$\overline{B_R(x)} \subseteq \phi(T) \implies \overline{B_{R(t)}(x)} \subseteq \phi(T + t), \qquad t \in \big[0, R^2/(2k)\big[\ , \tag{4.2}$$

where $R(t) := \sqrt{R^2 - 2kt}$. Indeed, (4.2) follows from the fact that ϕ is a barrier for all evolutions $\partial B_{R(t)}(x) \cap V$, where V is a generic affine subspace of $\mathbb{R}^n$ of dimension k passing through x.
By (3.2) of Lemma 3.1, it is enough to prove that

$$\mathcal{M}(A, \mathcal{G}_k) \in \mathcal{B}\big(\mathcal{F}_F^{>}\big).$$

Let $f : [a, b] \subseteq I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}_F^{>}$, with $f(a) \subseteq \phi(a)$. We have to show that $f(b) \subseteq \phi(b)$. We preliminarly prove that

$$\mathrm{int}\big(f(b)\big) \subseteq \phi(b). \tag{4.3}$$

Suppose by contradiction that (4.3) does not hold. Set

$$t^* := \sup\big\{t \in [a, b] : \mathrm{int}\big(f(s)\big) \subseteq \phi(s), \quad s \in [a, t]\big\}. \tag{4.4}$$

It is not difficult to show that $\mathrm{int}(f(t^*)) \subseteq \phi(t^*)$, so that $t^* < b$. Indeed, assume by contradiction $t^* > a$ and $\mathrm{int}(f(t^*)) \not\subseteq \phi(t^*)$. As f is a $(n-1)$-dimensional smooth compact flow, given $x \in \mathrm{int}(f(t^*)) \setminus \phi(t^*)$, we can find $0 < \tau_1 < t^* - a$ and $R > 0$ so that $B_R(x) \subseteq \mathrm{int}(f(t))$ for any $t \in [t^* - \tau_1, t^*]$. Therefore $B_R(x) \subseteq \phi(t)$ for any $t \in [t^* - \tau_1, t^*[$ and $x \notin \phi(t^*)$, which contradicts (4.2).

We now divide the proof of (4.3) into five steps; step 6 then concludes the proof.

Step 1. There exist $x^* \in \partial f(t^*) \cap \partial\phi(t^*)$, a decreasing sequence $\{t_m\}$ of points of $]t^*, b]$ and a sequence $\{R_m\}$ of positive numbers, with $\lim_{m\to+\infty} t_m = t^*$, $\lim_{m\to+\infty} R_m = 0$, such that for any $m \in \mathbb{N}$

$$\Big(\mathrm{int}\big(f(t_m)\big) \setminus \phi(t_m)\Big) \cap B_{R_m}(x^*) \neq \emptyset. \tag{4.5}$$

The proof of this step, as well as the proof of step 2, follows closely a part of the proof of Theorem 5.1 of [4]. Let us first prove that $\partial f(t^*) \cap \partial\phi(t^*) \neq \emptyset$. Assume by contradiction that $\partial f(t^*) \cap \partial\phi(t^*) = \emptyset$, and set $\eta(t) := \mathrm{dist}(f(t), \mathbb{R}^n \setminus \phi(t))$ for $t \in [a, b]$. As $\partial f(t^*)$ is compact and $\phi(t^*)$ is open, we have $\eta(t^*) > 0$. Let us prove that $\eta(t^*) \leq \liminf_{s\downarrow t^*} \eta(s)$. Indeed, if not, there exists a sequence $\{s_m\}$, $s_m > t^*$, $s_m \downarrow t^*$, such that $\eta(t^*) > \lim_{m\to+\infty} \eta(s_m)$. Then $\eta(s_m) = |y_m - p_m|$, for some $y_m \in f(s_m)$, $p_m \in \mathbb{R}^n \setminus \phi(s_m)$; possibly passing to a subsequence, we have $y_m \to y \in f(t^*)$, $p_m \to p \notin \mathbb{R}^n \setminus \phi(t^*)$ as $m \to +\infty$. Let $\rho > 0$ be such that $B_\rho(p) \subseteq \phi(t^*)$. Then $B_{\rho/2}(p) \cap (\mathbb{R}^n \setminus \phi(s_m)) \neq \emptyset$ definitively in m, which contradicts (4.2). Then $0 < \eta(t^*) \leq \liminf_{s\downarrow t^*} \eta(s) = 0$, a contradiction. Then $K := \partial f(t^*) \cap \partial\phi(t^*) \neq \emptyset$.

Assume now by contradiction that for any $x \in K$ there exists $R(x) > 0$ and $0 < t(x) < b - t^*$ so that

$$\Big(\mathrm{int}\big(f(s)\big) \setminus \phi(s)\Big) \cap B_R(x) = \emptyset, \qquad R \in\,]0, R(x)],\ s \in\,]t^*, t^* + t(x)]. \tag{4.6}$$

As K is compact, we can find $x_1, \ldots, x_h \in K$ (and corresponding $t(x_1), \ldots, t(x_h)$) so that each $R(x_i)$ satisfies (4.6) and $\bigcup_{i=1}^h B_{R(x_i)} \supseteq K$. Let $\overline{R} > 0$ be such that $H := \bigcup_{x\in K} B_{\overline{R}}(x) \subseteq \bigcup_{i=1}^h B_{R(x_i)}$, and let $\overline{t} := \min_{i=1,\ldots,h} t(x_i)$. Then for any $x \in K$ we have

$$\Big(\mathrm{int}\big(f(s)\big) \setminus \phi(s)\Big) \cap B_{\overline{R}}(x) = \emptyset, \qquad s \in\,]t^*, t^* + \overline{t}]. \tag{4.7}$$

Let $c > 0$ be such that $\mathrm{dist}(f(t^*) \setminus H, \mathbb{R}^n \setminus \phi(t^*)) \geq c$. Then using (4.2), (4.7) and the fact that f is a $(n-1)$-dimensional smooth compact flow, we contradict the definition of t^*. This concludes the proof of step 1.

Since $f \in \mathcal{F}_F^{>}$, there exists $c > 0$ such that

$$\frac{\partial d_f}{\partial t}(t^*, x^*) + F\big(\nabla d_f(t^*, x^*), \nabla^2 d_f(t^*, x^*)\big) = 2c > 0. \tag{4.8}$$

Let $R > 0$ be such that

$$\frac{\partial d_f}{\partial t}(t^*, x) + F\big(\nabla d_f(t^*, x), \nabla^2 d_f(t^*, x)\big) \geq c \qquad \forall x \in S, \tag{4.9}$$

where

$$S := \partial f(t^*) \cap \overline{B_R(x^*)}.$$

Step 2. Let x^* be as in step 1. We can assume that

$$\{x^*\} = \partial f(t^*) \cap \partial\phi(t^*), \qquad f(t^*) \setminus \{x^*\} \subseteq \mathrm{int}\,(\phi(t^*))\,. \tag{4.10}$$

Indeed, let $0 < \tau_1 < b - t^*$ be such that each point $x \in \partial f(t)$ has a unique smooth orthogonal projection $\pi(t,x)$ on $\partial f(t^*)$ for any $t \in [t^*, t^* + \tau_1]$. Choose a function $\rho : \partial f(t^*) \to [0, +\infty[$ of class $\mathbf{C}^\infty$ verifying the following properties:

(i) $\rho(x) = 0$ if and only if $x = x^*$;

(ii) the map $t \in [t^*, t^* + \tau_1] \to \zeta(t)$ belongs to $\mathcal{F}_F^{>}$, where $\zeta(t) := \overline{f^-_{\rho(\cdot)}(t)} \subseteq f(t)$ and $\partial\zeta(t) := \{y \in \mathbb{R}^n : y = x - \rho(\pi(t,x))\nabla d_f(t,x), x \in \partial f(t)\}$;

(iii) ϕ is not a barrier for ζ on $[t^*, t^* + \tau_1]$.

Property (ii) can be achieved by taking $\rho(\cdot)$ sufficiently small in the $\mathbf{C}^2$ norm, since there exists $c_1 > 0$ so that $\frac{\partial d_f}{\partial t} + F(\nabla d_f, \nabla^2 d_f) \geq c_1$ for any $x \in \partial f(t)$, $t \in [a,b]$, and F is continuous.
Property (iii) can be achieved by observing that, by (4.5), for any $m \in \mathbb{N}$ there exist a point $x_m \in \mathrm{int}(f(t_m)) \setminus \phi(t_m)$ and $\sigma_m > 0$ such that $B_{\sigma_m}(x_m) \subseteq \mathrm{int}(f(t_m)) \cap B_{R_m}(x^*)$. Then, if we impose $\rho(x) < \sigma_m$ for any $x \in \partial f(t^*)$ such that $|\pi^{-1}(t_m, x) - x^*| < R_m$, we get $x_m \in \mathrm{int}(\zeta(t_m))$. Therefore, possibly replacing f by ζ, we can assume that (4.10) holds, and the proof of step 2 is concluded.
Step 3. Construction of the k-dimensional flow $\Gamma_\epsilon \in \mathcal{G}_k$.
Fix now a point $x \in S$, let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of eigenvectors for $\nabla^2 d_f(t^*, x)$, ordered in such a way that $e_1, \ldots, e_k$ are the directions corresponding to the *smallest* k principal curvatures of S at x (if there exist principal directions of S at x such that the corresponding principal curvatures have the same value, we arbitrarily choose a set of k directions corresponding to k smallest curvatures), and $e_n := \nabla d_f(t^*, x)$. For any $y \in \mathbb{R}^n$, set $P(y) := \{y + w : \ w \in P\}$, where $P := \mathrm{span}(e_1, \ldots, e_k, e_n)$ is the vector space generated by $e_1, \ldots, e_k$ and the normal to S at x. Notice that, as $\partial f(t^*)$ and $P(x)$ are transverse at x, there exist $\rho_0, \epsilon_0 > 0$ such that

$$M_\epsilon := \partial\big(f(t^*)^-_\epsilon\big) \cap P(x) \cap \overline{B_{\rho_0}(x)}$$

is a smooth submanifold of dimension k with boundary for any $0 < \epsilon < \epsilon_0$. We arbitrarily extend M_ϵ out of $\overline{B_{\rho_0}(x)}$ to a smooth compact k-dimensional submanifold without boundary, and we denote it by $\Gamma_\epsilon(t^*)$, with the constraints

$$\Gamma_\epsilon(t^*) \cap \overline{B_{\rho_0}(x)} = M_\epsilon, \qquad \Gamma_\epsilon(t^*) \subseteq \overline{f(t^*)^-_\epsilon}, \tag{4.11}$$

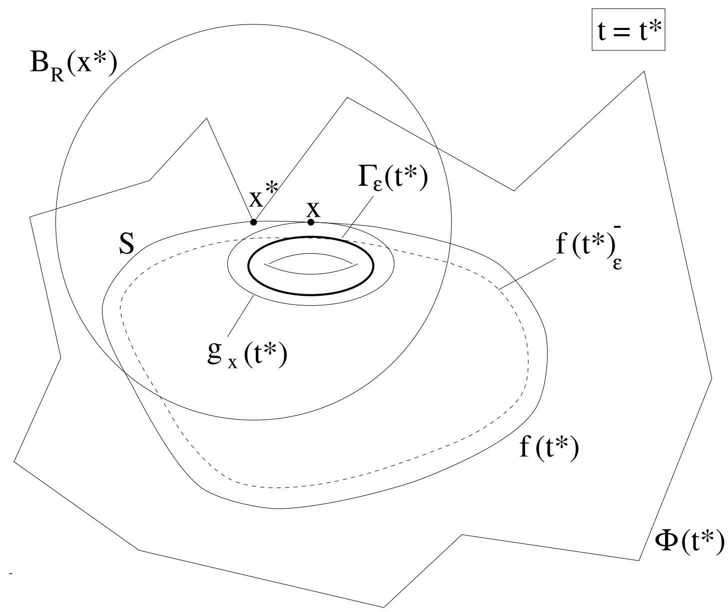

Figure 1: Construction of $\Gamma_\epsilon \in \mathcal{G}_k$ and of its tubular neighbourhood g_x.

see Figure 1.
By Theorem 2.6, for $\epsilon \in \,]0, \epsilon_0[$ we can find $\tau_\epsilon > 0$ such that there exists a smooth evolution $\Gamma_\epsilon(t)$ of $\Gamma_\epsilon(t^*)$ by mean curvature, defined in $[t^*, t^* + \tau_\epsilon]$. This concludes step 3.

Step 4. Let $x \in S$ be fixed as in step 3. For ϵ small enough, we can assume that d_{Γ_ϵ} is smooth on $[t^*, t^* + \delta] \times B_{\rho_0/2}(x)$, for a suitable $0 < \delta \leq \tau_\epsilon$, and we claim that

$$\frac{\partial d_{\Gamma_\epsilon}}{\partial t}\big(t^*, x\big) \leq \frac{c}{2} - F\big(\nabla d_f(t^*, x), \nabla^2 d_f(t^*, x)\big) \tag{4.12}$$

(notice the presence of Γ_ϵ at the left hand side and of f at the right hand side).
Indeed, using (2.7) there holds

$$\frac{\partial d_{\Gamma_\epsilon}(t^*, x)}{\partial t} = \frac{1}{d_{\Gamma_\epsilon}(t^*, x)} \frac{\partial \eta_{\Gamma_\epsilon}(t^*, x)}{\partial t} = \frac{1}{d_{\Gamma_\epsilon}(t^*, x)} \sum_{i=1}^{k} \frac{\mu_i(t^*, x)}{1 - \mu_i(t^*, x)}, \tag{4.13}$$

where $\mu_1(t^*, x) \leq \ldots \leq \mu_k(t^*, x) < 1$ are the eigenvalues of $\nabla^2 \eta_{\Gamma_\epsilon}(t^*, x)$ which are smaller than one, and precisely by (2.2)

$$\mu_i(t^*, x) = \frac{d_{\Gamma_\epsilon}(t^*, x)\kappa_i(\pi(t^*, x))}{1 + d_{\Gamma_\epsilon}(t^*, x)\kappa_i(\pi(t^*, x))},$$

where $\kappa_1(\pi(t^*,x)),\ldots,\kappa_k(\pi(t^*,x))$ are the principal curvatures of $\Gamma_\epsilon(t^*)$ along $\nabla d_{\Gamma_\epsilon}(t^*,x)$, at the projected point $\pi(t^*,x) \in \Gamma_\epsilon(t^*)$. Using (2.3) we then obtain

$$\begin{aligned} \sum_{i=1}^{k} \frac{\mu_i(t^*,x)}{1-\mu_i(t^*,x)} &= d_{\Gamma_\epsilon}(t^*,x) \sum_{i=1}^{k} \kappa_i(\pi(t^*,x)) \\ &= d_{\Gamma_\epsilon}(t^*,x) \sum_{i=1}^{k} \frac{\lambda_i(t^*,x)}{1-d_{\Gamma_\epsilon}(t^*,x)\lambda_i(t^*,x)}, \end{aligned} \tag{4.14}$$

where $\lambda_1(t^*,x) \leq \ldots \leq \lambda_{n-1}(t^*,x)$ are the eigenvalues of $\nabla^2 d_{\Gamma_\epsilon}(t^*,x)$ corresponding to eigenvectors orthogonal to $\nabla d_{\Gamma_\epsilon}(t^*,x)$.
Putting together (4.13) and (4.14) we get

$$\frac{\partial d_{\Gamma_\epsilon}(t^*,x)}{\partial t} = \sum_{i=1}^{k} \frac{\lambda_i(t^*,x)}{1-d_{\Gamma_\epsilon}(t^*,x)\lambda_i(t^*,x)}. \tag{4.15}$$

We notice that $d_{\Gamma_\epsilon}(t^*,x) = \epsilon$ and we claim that, by construction, the smallest k eigenvalues of $\nabla^2 d_{\Gamma_\epsilon}(t^*,x)$ corresponding to eigenvectors orthogonal to $\nabla d_{\Gamma_\epsilon}(t^*,x)$ are also the smallest k eigenvalues of $\nabla^2 d_f(t^*,x)$ corresponding to eigenvectors orthogonal to $\nabla d_f(t^*,x) = \nabla d_{\Gamma_\epsilon}(t^*,x)$, i.e.,

$$\langle e_i, \nabla^2 d_{\Gamma_\epsilon}(t^*,x) e_i\rangle = \langle e_i, \nabla^2 d_f(t^*,x) e_i\rangle, \qquad i = 1,\ldots,k. \tag{4.16}$$

Geometrically, (4.16) means that, at the point x, being $d_{\Gamma_\epsilon(t^*)^+_\epsilon} = d_{\Gamma_\epsilon(t^*)} - \epsilon$, the smallest k principal curvatures of the ϵ-tubular neighbourhood $\Gamma_\epsilon(t^*)^+_\epsilon$ are the same as the smallest k principal curvatures of $\partial f(t^*)$. Let us prove the claim: using (2.3) we have

$$\langle e_i, \nabla^2 d_{\Gamma_\epsilon(t^*)}(t^*,x) e_i\rangle = \frac{\kappa_i(\pi(t^*,x))}{1+\epsilon\kappa_i(\pi(t^*,x))}, \qquad i = 1,\ldots,k. \tag{4.17}$$

In addition, as the second fundamental form of $\partial\big(f(t^*)^-_\epsilon\big)$ restricted to the tangent space to $\Gamma_\epsilon(t^*)$ at $\pi(t^*,x)$ concides with the contraction of the second fundamental form of $\Gamma_\epsilon(t^*)$ at $\pi(t^*,x)$ in the normal direction given by $\nabla d_{f(t^*)}(t^*,\pi(t^*,x))$, we then have

$$\langle e_i, \nabla^2 d_{f(t^*)}(t^*,\pi(t^*,x)) e_i\rangle = \kappa_i(\pi(t^*,x)), \qquad i = 1,\ldots,k,$$

and using (2.1)

$$\langle e_i, \nabla^2 d_{f(t^*)}(t^*,x) e_i\rangle = \frac{\kappa_i(\pi(t^*,x))}{1+\epsilon\kappa_i(\pi(t^*,x))}, \qquad i = 1,\ldots,k. \tag{4.18}$$

Using (4.17) and (4.18), equalities (4.16) follow.

Recalling the definition of F, using (4.15), (4.16) and letting $\epsilon > 0$ small enough, we conclude the proof of step 4.
Step 5. Conclusion of the proof of (4.3).
Let $\tau > 0$ be small enough in such a way that there exists a unique smooth orthogonal projection $\pi(t, \cdot)$ of $\partial f(t)$ on $\partial f(t^*)$ for any $t \in [t^*, t^* + \tau] \subseteq [a, b]$.
Let $\tau' := \min\{\delta, \tau\}$ where δ is defined in step 4, and let $g_x : [t^*, t^* + \tau'] \to \mathcal{P}(\mathbb{R}^n)$ be defined as

$$g_x(t) := \overline{\Gamma_\epsilon(t)^+_\epsilon}.$$

The set $g_x(t)$ is the ϵ-tubular neighbourhood around the smoothly evolving k-dimensional submanifold Γ_ϵ, see Figure 1; notice that, since we are not imposing other conditions on $\Gamma_\epsilon(t^*)$ besides (4.11), the boundary of the set $\Gamma_\epsilon(t)^+_\epsilon$, for $t \in [t^*, t^* + \tau_\epsilon]$ is not necessarily smooth out of a small ball centered at x. Notice also that $g_x(t^*) \subseteq f(t^*)$ by the inclusion in (4.11).
Since $\mathcal{G}_k$ is translation invariant in space and $\phi \in \mathcal{B}(\mathcal{G}_k)$, we deduce that the function $t \to \mathrm{dist}(\Gamma_\epsilon(t), \mathbb{R}^n \setminus \phi(t))$ is nondecreasing on $[t^*, t^* + \tau_\epsilon]$. Therefore, recalling the definition of g_x, we find that the function $t \to \mathrm{dist}(g_x(t), \mathbb{R}^n \setminus \phi(t))$ is nondecreasing on $[t^*, t^* + \tau_\epsilon]$. Hence, as $\mathrm{int}(g_x(t^*)) \subseteq \phi(t^*)$, we have $\mathrm{int}(g_x(t)) \subseteq \phi(t)$, $t \in [t^*, t^* + \tau_\epsilon]$.
Observe also that, in view of (4.12) and (4.9), we have that, at (t^*, x),

$$\frac{\partial d_{g_x}}{\partial t} = \frac{\partial d_{\Gamma_\epsilon}}{\partial t} \leq \frac{c}{2} - F(\nabla d_f, \nabla^2 d_f) < c - F(\nabla d_f, \nabla^2 d_f) \leq \frac{\partial d_f}{\partial t},$$

so that $\frac{\partial d_{g_x}}{\partial t} < \frac{\partial d_f}{\partial t}$ at (t^*, x). Applying Lemma 3.2 and Remark 3.3 to f and g_x (with $[a, b]$ replaced by $[t^*, t^* + \tau']$), it follows that there exist $\tilde{\tau}_x \in]0, \tau']$ and $R(t, x) > 0$ such that

$$f(t) \cap \overline{B_{R(t,x)}\big(x(t)\big)} \subseteq \mathrm{int}\big(g_x(t)\big) \subseteq \phi(t), \qquad t \in [t^*, t^* + \tilde{\tau}_x]. \tag{4.19}$$

Since $\tilde{\tau}_x$ depends in a continuous way on $x \in S$ and S is compact, we have $\tau^* := \min_{x \in S} \tilde{\tau}_x > 0$. Possibly reducing τ^* and using (4.19), we get

$$\partial f(t) \cap \overline{B_{R/2}(x^*)} \subseteq \bigcup_{x \in S} \mathrm{int}\big(g_x(t)\big) \subseteq \phi(t), \qquad t \in]t^*, t^* + \tau^*].$$

Furthermore, we can find $\eta > 0$ such that

$$\big(\partial f(t)\big)^+_\eta \cap f(t) \cap \overline{B_{R/2}(x^*)} \subseteq \bigcup_{x \in S} \mathrm{int}\big(g_x(t)\big) \subseteq \phi(t), \qquad t \in]t^*, t^* + \tau^*].$$

Possibly reducing τ^* and using (4.2), we then have

$$f(t) \cap \overline{B_{R/4}(x^*)} \subseteq \phi(t), \qquad t \in]t^*, t^* + \tau^*].$$

Moreover, by using (4.2) and (4.10), it follows that there exists $\tau_1 > 0$ such that

$$f(t) \setminus \overline{B_{R/4}(x^*)} \subseteq \phi(t), \qquad t \in]t^*, t^* + \tau_1].$$

Hence, for any $t \in]t^*, t^* + \min\{\tau^*, \tau_1\}]$, we have $f(t) \subseteq \phi(t)$, which contradicts (4.5). The proof of (4.3) is complete.

Step 6. Proof of $f(b) \subseteq \phi(b)$.

Let $\omega > 0$ be such that $\frac{\partial d_f}{\partial t} + F(\nabla d_f, \nabla^2 d_f) \geq 2\omega$ for any $x \in \partial f(t)$ and $t \in [a, b]$. Pick a $\mathbf{C}^\infty$ function $\rho : [a, b] \to [0, +\infty[$ such that $\rho(a) = 0$, $\rho(b) < c$ and $0 < \dot{\rho} < \omega(1 + 2ML(b-a))^{-1}$, where c, L and M are as in the proof of Lemma 3.1. Then

$$\dot{\rho} + 2ML\rho - 2\omega < \frac{\omega}{1 + 2ML(b-a)} + \frac{2ML\omega(b-a)}{1 + 2ML(b-a)} - 2\omega < 0,$$

so that, reasoning as in (3.3), it follows that the map taking $t \in [a, b]$ into $\overline{f(t)^+_{\rho(t)}} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, f(t)) \leq \rho(t)\}$ belongs to $\mathcal{F}_F^>$. Therefore, from (4.3) (applied with $\overline{f^+_{\rho(\cdot)}}$ in place of f) we have

$$f(b) \subseteq \operatorname{int}\left(\overline{f(b)^+_{\rho(b)}}\right) \subseteq \phi(b).$$

□

The following remark shows that, to compute the weak evolution $\mathcal{M}^*(\cdot, \mathcal{G}_k)$ in arbitrary codimension, it is enough to construct the minimal barrier $\mathcal{M}^*(\cdot, \mathcal{G}_F)$ with respect to a family of tests $\mathcal{G}_F$ consisting only of tubular neighbourhoods of evolving k-dimensional submanifolds.

Corollary 4.2. *Let $a, b \in \mathbb{R}$, $a < b$, $[a, b] \subseteq I$, and let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$. We write $f \in \mathcal{G}_F$ if and only if f verifies (i) and (ii) of Definition 2.3, and there exist $\Gamma \in \mathcal{G}_k$ and $\rho > 0$ such that $f(t) = \overline{\Gamma(t)^+_\rho}$ for any $t \in [a, b]$. Then for any $E \subseteq \mathbb{R}^n$ we have*

$$\mathcal{M}^*(E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{G}_k) = \mathcal{M}^*(E, \mathcal{G}_F).$$

Proof. By Lemma 3.4 we have $\mathcal{G}_F \subseteq \mathcal{F}_F$, hence $\mathcal{M}^*(E, \mathcal{G}_F) \subseteq \mathcal{M}^*(E, \mathcal{F}_F)$. In addition, if A is an open set, then $\mathcal{M}(A, \mathcal{G}_F) \in \mathcal{B}(\mathcal{G}_k)$, hence $\mathcal{M}(A, \mathcal{G}_F) \supseteq \mathcal{M}(A, \mathcal{G}_k)$. Therefore the assertion follows from the equality $\mathcal{M}(A, \mathcal{G}_k) = \mathcal{M}(A, \mathcal{F}_F)$ proved in Theorem 4.1.

□

Remark 4.3. *From Theorem 4.1 and [3, Corollary 6.1], it follows that that, for any set $E \subseteq \mathbb{R}^n$, there holds*

$$\mathcal{M}^*(E, \mathcal{G}_k) = \mathcal{M}^*(E, \mathcal{F}_F) = V(E),$$

where $V(E)$ is the zero sublevel set of the viscosity solution of $\dfrac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0$, in the sense of [5].

Remark 4.4. *The proof of Theorem 4.1 actually shows that a slightly stronger result holds, namely that, if* $\phi : I \to \mathcal{P}(\mathbb{R}^n)$ *is a map such that* $\phi(t)$ *is an open set for any* $t \in I$*, then*

$$\phi \in \mathcal{B}(\mathcal{G}_k) \iff \phi \in \mathcal{B}(\mathcal{F}_F).$$

With slight modifications of the proof, one can extend Theorem 4.1 to the case of mean curvature motion with a time dependent "forcing term" $G : [0, +\infty[\to \mathbb{R}^n$, $G \in \mathbf{C}^\infty([0, +\infty[; \mathbb{R}^n)$; i.e., for k-dimensional smooth compact flows evolving by the law

$$V^\perp = H + G^\perp(t),$$

where $V^\perp$ and H are, respectively, the normal velocity and the mean curvature vector, while $G^\perp$ is the projection of G onto the normal space.
In particular (4.1) still holds, if we define $\mathcal{F}_F$ and $\mathcal{G}_k$ by substituting (2.5) and (2.6), in Definitions 2.3 and 2.4, respectively with

$$\frac{\partial d_f}{\partial t}(t,x) + F\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) + \langle \nabla d_f(t,x), G(t)\rangle \geq 0, \quad t \in [a,b],\ x \in \partial f(t),$$

and

$$\frac{\partial \nabla \eta_\Gamma}{\partial t}(t,x) - \Delta \nabla \eta_\Gamma(t,x) + \nabla^2 \eta_\Gamma(t,x) G(t) = 0, \qquad t \in [a,b],\ x \in \Gamma(t).$$

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