SOME ASPECTS OF DE GIORGI'S BARRIERS FOR GEOMETRIC EVOLUTIONS

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1. Introduction

Motion by mean curvature of a hypersurface has been the subject of several recent papers, and it is considered an interesting example of geometric evolution. One of its main features is that it can be interpreted as the gradient flow associated with the perimeter functional, which therefore relates the evolution to minimal surfaces. From the viewpoint of applications, mean curvature flow arises in physical phenomena which exhibit surface tension effects such as phase transitions in material sciences. Also more general geometric evolutions, possibly depending on the normal vector, the principal curvatures, and explicitly on space-time, may have a geometric or a physical meaning; we recall, among others, motion by mean curvature with a forcing term (representing, for instance, the action of an exterior field) and the anisotropic mean curvature flow, which arises naturally in mathematical models of phenomena such as dendritic growth, crystal growth, and also may be used to describe the propagation of an electric stimulus in the cardiac tissue.

Motion by mean curvature can be described by a suitable non linear parabolic equation, which admits a smooth local solution starting from a smooth compact hypersurface [8,9,46,51,56,57,59,60]. This (local) evolution can be also characterized by means of an equation satisfied by the signed distance function from the evolving front [13,46]. In [60], Huisken proved that a smooth strictly convex compact hypersurface shrinks smoothly to a point, generalizing a result of Gage-Hamilton [51] (see also [6,7]). Ecker-Huisken [42] proved that if the initial smooth hypersurface is an entire graph then, under suitable assumptions on the slope and the curvature of the graph, the flow exists globally and it is smooth.

It is however well known that smooth hypersurfaces flowing by mean curvature can develop singularities at finite time. Examples of singularities (before the extinction time) can be constructed, among others, by starting from suitable tori or dumbbell shaped sets [3,28,29,58,81,85]. Results concerning the description of various types of singularities have been obtained, among others, by Angenent [9], Altschuler [2], Huisken [61], Ilmanen [63], Angenent-Ilmanen-Velasquez [12,11], and Hamilton.

The presence of singularities justifies the necessity of introducing weak definitions of motion by mean curvature and, more generally, of geometric evolutions. Clearly, any weak evolution must coincide with the classical one as long as the latter exists. Among the generalized methods to treat geometric evolutions past singularities we recall: the approach of Brakke [22], which studies the mean curvature flow in the context of varifolds theory; the approach of Angenent [8,9], concerning curves

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shortening on surfaces; the approach of Evans-Spruck [45,47,48], Chen-Giga-Goto [24], Giga-Goto-Ishii-Sato [54], which consider the level set of the solutions, in the viscosity sense, of suitable parabolic fully non linear partial differential equations, where the notion of viscosity solution was introduced by Crandall- P.-L. Lions [26], P.-L. Lions [76], Jensen [70] (see also Jasnow-Kawasaki-Ohta [69], Osher-Sethian [80], Ishii [67], Soner [82,83], and Ilmanen [63]); the solutions that can be obtained as asymptotic limits of the scaled Allen-Cahn equation [13,23,27,29,41,44,65,78] and of a nonlocal equation [36,37,38,39,40,73,74,75]; the variational approach of Almgren-Taylor-Wang [1] (see also Luckhaus-Sturzenhecker [77]) and its possible generalizations by means of the minimizing movements of De Giorgi [4,30,50]; the elliptic regularization method of Ilmanen [66]; the method of set-theoretic subsolutions of Ilmanen [64] (see also White [87]); the semigroup approach of Bence-Merriman-Osher [21] and Evans [43]; the barriers approach of De Giorgi [31,34]; the penalization method on higher derivatives of De Giorgi [32]. We remark that the relations between all these approaches (after the onset of singularities) have not been completely clarified.

The barriers method of De Giorgi, which is the argument we are concerned with in the present paper, provides a weak solution for a number of differential equations. In the geometric context, it gives a natural notion of weak evolution for a large class of flows, such as motion by mean curvature in arbitrary codimension (see [5,31]); in this geometric framework, properties such as uniqueness of the weak evolution, the comparison principle, and the coincidence of the minimal barrier with the classical flow as long as it exists, are immediate consequences of the definitions. Moreover the method is intrinsic, since it is mainly based on the distance function and on the inclusion between sets.

Let us briefly explain the concept of geometric minimal barrier in $\mathbf{R}^n$, and some of its properties. First we choose a nonempty family $\mathcal{F}$ of maps which take some time interval into the set $\mathcal{P}(\mathbf{R}^n)$ of all subsets of $\mathbf{R}^n$: for instance $\mathcal{F}$ can be the family of all smooth local evolutions with respect to a given geometric law. Then we define the class $\mathcal{B}(\mathcal{F})$ of all maps $\phi : [\overline{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ which are barriers for $\mathcal{F}$ in $[\overline{t}, +\infty[$ with respect to the inclusion of sets, that is, if $f : [a, b] \subseteq [\overline{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$, then it must hold $f(b) \subseteq \phi(b)$ (here $\overline{t} \in \mathbf{R}$ is fixed). Finally, we define the minimal barrier $\mathcal{M}(E, \mathcal{F}, \overline{t})(t)$ with origin in $E \subseteq \mathbf{R}^n$, with respect to $\mathcal{F}$, at time $t \in [\overline{t}, +\infty[$, as

$$\mathcal{M}(E, \mathcal{F}, \overline{t})(t) := \bigcap \Big\{ \phi(t) : \phi : [\overline{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n), \phi \in \mathcal{B}(\mathcal{F}), \phi(\overline{t}) \supseteq E \Big\}. \tag{1.1}$$

We stress that the minimal barrier depends on $\mathcal{F}$ and is unique and globally defined, for an arbitrary initial set E; moreover, it enjoys the comparison principle and, under minor assumptions on $\mathcal{F}$, the semigroup property.

We remark that, if we choose $\mathcal{F} = \mathcal{F}_F$ as the family of all smooth local geometric (super) solutions of an equation of the form

$$\frac{\partial u}{\partial t} + F\big(\nabla u, \nabla^2 u\big) = 0, \tag{1.2}$$

then $\mathcal{M}(E, \mathcal{F}_F, \overline{t})$ is defined under no assumptions on F and, if $f : [a, b] \subseteq [\overline{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, then $\mathcal{M}\big(f(a), \mathcal{F}_F, a\big)(t) \supseteq f(t)$ for any $t \in [a, b]$. It is not difficult to verify that the equality holds true when F is degenerate elliptic but, in general, it does not hold for a not degenerate elliptic function F, when it happens that the elements of $\mathcal{F}_F$ are not necessarily $\mathcal{F}_F$-barriers. In this respect, it turns out that, if F is lower semicontinuous and if $\mathcal{F}_F^{>}$ denotes the family of all strict local geometric supersolutions of (1.2), then

$$\mathcal{B}(\mathcal{F}_F^{>}) = \mathcal{B}(\mathcal{F}_{F^+}^{>}), \tag{1.3}$$

where F^+ is the smallest degenerate elliptic function greater than or equal to F, that is

$$F^+(p, X) := \sup\{F(p, Y) : Y \geq X\}.$$

Here $(p, X) \in (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n) =: J_0$, where $\mathrm{Sym}(n)$ denotes the space of all symmetric real $(n \times n)$-matrices.

In particular, from (1.3) we deduce that $\mathcal{M}(E, \mathcal{F}_F^{>}, \bar{t}) = \mathcal{M}(E, \mathcal{F}_{F^+}^{>}, \bar{t})$. This shows that, in presence of a non degenerate elliptic function F, the generalized evolution of any set by (1.2) is governed by the parabolic equation in which F is replaced by F^+. For instance, if we consider a geometric evolution of sets where the normal velocity of the interface is a nondecreasing function ζ of the mean curvature, then the resulting evolution is defined by means of the smallest decreasing function greater than or equal to ζ. In Proposition 3.2, under further assumptions on F (which may explicitly depend on (t, x)) we prove the analogue of (1.3) in the viscosity framework, namely that the family of all viscosity subsolutions of

$$\frac{\partial u}{\partial t} + F(t, x, \nabla u, \nabla^2 u) = 0 \tag{1.4}$$

coincides with the family of all viscosity subsolutions of

$$\frac{\partial u}{\partial t} + F^+(t, x, \nabla u, \nabla^2 u) = 0. \tag{1.5}$$

Starting from the minimal barrier, there is a natural way to construct two set-valued maps $\mathcal{M}_*(E, \mathcal{F}, \bar{t})$, $\mathcal{M}^*(E, \mathcal{F}, \bar{t})$ which play a crucial rôle both in the general theory and in the comparison between the barriers approach and other generalized evolutions. More precisely, given any set $E \subseteq \mathbf{R}^n$ and $\varrho > 0$, let

$$E_\varrho^- := \{x \in \mathbf{R}^n : \mathrm{dist}\,(x, \mathbf{R}^n \setminus E) > \varrho\}, \tag{1.6}$$

$$E_\varrho^+ := \{x \in \mathbf{R}^n : \mathrm{dist}(x, E) < \varrho\}, \tag{1.7}$$

and define the lower and upper regularizations

$$\mathcal{M}_*(E, \mathcal{F}, \bar{t}) := \bigcup_{\varrho > 0} \mathcal{M}(E_\varrho^-, \mathcal{F}, \bar{t}), \qquad \mathcal{M}^*(E, \mathcal{F}, \bar{t}) := \bigcap_{\varrho > 0} \mathcal{M}(E_\varrho^+, \mathcal{F}, \bar{t}).$$

The map $\mathcal{M}^*(E, \mathcal{F}, \bar{t})$ is always an $\mathcal{F}$-barrier, while $\mathcal{M}_*(E, \mathcal{F}, \bar{t})$ is an $\mathcal{F}$-barrier under minor assumptions on $\mathcal{F}$; moreover $\mathcal{M}_*(E, \mathcal{F}, \bar{t})$, $\mathcal{M}^*(E, \mathcal{F}, \bar{t})$ are stable with respect to topological interior part and closure respectively, and provide a lower and an upper bound for *any* generalized evolution of sets which extends the smooth evolutions, satisfies the comparison principle and the semigroup property. Also, these two maps allow to define the so called n-dimensional *fattening phenomenon*, which is a very special singularity of geometric evolutions, i.e., whenever, for some $t \in [\bar{t}, +\infty[$ it happens that

$$\mathcal{H}^n\Big(\mathcal{M}^*(E, \mathcal{F}, \bar{t})(t) \setminus \mathcal{M}_*(E, \mathcal{F}, \bar{t})(t)\Big) > 0,$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.

Other interesting properties of geometric minimal barriers are the disjoint sets property and the joint sets property with respect to $(\mathcal{F}, \mathcal{G})$, where $\mathcal{F}$, $\mathcal{G}$ are two arbitrary families of set-valued maps. Due to elementary counterexamples (to the joint sets property, for instance, in case of motion by curvature in two dimensions) we introduce the regularized versions of these two properties, which read as follows:

$$\begin{aligned} &E_1 \cap E_2 = \emptyset \ \Rightarrow\ \mathcal{M}_*\big(E_1, \mathcal{F}, \bar{t}\big)(t) \cap \mathcal{M}^*\big(E_2, \mathcal{G}, \bar{t}\big)(t) = \emptyset, \qquad t \geq t_0, \\ &E_1 \cup E_2 = \mathbf{R}^n \ \Rightarrow\ \mathcal{M}_*\big(E_1, \mathcal{F}, \bar{t}\big)(t) \cup \mathcal{M}^*\big(E_2, \mathcal{G}, \bar{t}\big)(t) = \mathbf{R}^n, \qquad t \geq t_0. \end{aligned} \tag{1.8}$$

When $\mathcal{F} = \mathcal{F}_F$ and $\mathcal{G} = \mathcal{F}_G$ for two functions $F, G : J_0 \to \mathbf{R}$, these two properties can be characterized in terms of F and G. In particular, if we let $F_c(p, X) := -F(-p, -X)$ for any $(p, X) \in J_0$, the following assertion holds. Assume that $F : J_0 \to \mathbf{R}$ is continuous, $F^+ < +\infty$ and F^+ is continuous. Then the regularized disjoint sets property and the regularized joint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_{F_c}\big)$ (resp. with respect to $\big(\mathcal{F}_F, \mathcal{F}_F\big)$) hold if and only if F is degenerate elliptic (resp. if and only if F^+ is odd).

We notice that, in general, the assertions referring to the joint sets property are more difficult to prove that the corresponding ones concerning the disjoint sets property. We remark also that the disjoint and joint sets properties, and hence their characterization, are related to the fattening phenomenon.

Concerning the comparison between barriers and viscosity solutions, it turns out that the sublevel sets of a viscosity subsolution of (1.4) are barriers and, conversely, that a function whose sublevel sets are barriers is a viscosity subsolution of (1.4). Summarizing the comparison results whenever there exists a unique uniformly continuous viscosity solution of (1.4), one obtains the following result. Let $E \subseteq \mathbf{R}^n$ be a bounded set; denote by v the unique continuous viscosity solution of (1.4) with $v(\bar{t}, x) := (-1) \vee d_E(x) \wedge 1$, where d_E is the signed distance function from ∂E negative inside E. Then for any $t \in [\bar{t}, +\infty[$ we have

$$\begin{aligned} \mathcal{M}_*(E, \mathcal{F}_F, \bar{t})(t) &= \{x \in \mathbf{R}^n : v(t, x) < 0\}, \\ \mathcal{M}^*(E, \mathcal{F}_F, \bar{t})(t) &= \{x \in \mathbf{R}^n : v(t, x) \leq 0\}. \end{aligned} \tag{1.9}$$

In particular

$$\mathcal{M}^*(E, \mathcal{F}_F, \bar{t})(t) \setminus \mathcal{M}_*(E, \mathcal{F}_F, \bar{t})(t) = \{x \in \mathbf{R}^n : v(t, x) = 0\}. \tag{1.10}$$

Equality (1.10) connects the fattening phenomenon defined through the barriers approach with the one defined through the level set approach. In case of nonuniqueness of viscosity solutions, the minimal barriers selects the maximal viscosity subsolution of (1.4).

The outline of the paper is the following. In Section 2 we give some notation. In Section 3 we recall the abstract definitions of barriers, local (in space) barriers and inner barriers, and some of their properties, such as the relations with the test family $\mathcal{F}$ (see (3.1) and Proposition 3.1), the semigroup property (see (3.3)), and a useful consequence of the translation invariance (see (3.5)). Using this latter property we prove that the function $\mathcal{M}_{u_0,\mathcal{F}} : [\bar{t}, +\infty[\times \mathbf{R}^n \to \mathbf{R}$ defined in (3.21), which is the weak evolution (as a function) of an *arbitrary* initial function u_0, preserves the Lipschitz constant (Proposition 3.3). In (3.9), (3.11), and (3.12) we list some properties of the regularizations and their connections with the minimal barrier. The section contains several examples showing the behaviour of the barriers (choice of convex test hypersurfaces in Example 3.3, inverse mean curvature flow in Example 3.4, backward mean curvature flow in Example 3.5, nonconvex anisotropic curvature flow in Example 3.6) and motivating the lower and upper regularizations (Example 3.2). Theorem 3.1 is concerned with the relations between barriers and local barriers, whereas in Theorem 3.2 and Proposition 3.2 we deepen the relations between $\mathcal{B}(\mathcal{F}_F)$ and $\mathcal{B}(\mathcal{F}_{F^+})$. In subsection 3.2 we discuss some connections between barriers and the reaction-diffusion approach, following closely some ideas of Jerrard-Soner [71,72] and Soner [84]. We conclude Section 3 by proving some results on the outer regularity of the minimal barrier (Proposition 3.4) and on the right continuity of the distance function between minimal barriers (Lemma 3.1 and Corollary 3.1). In Section 4 we introduce the notion of barrier solution (Definition 4.1) and we study existence and stability properties (Proposition 4.1 and 4.2); these two properties are reminiscent of the existence and stability of viscosity solutions [24]. In Theorems 4.2 and 4.3 we recall the connections between the barriers and the level set flow; Theorem 4.4 is concerned with the characterization of the complement of regularized barriers, and Theorem 4.5 with the connections between barriers and inner barriers. The comparison results between barriers and level set flow are generalized in Lemma 4.1 of subsection 4.2, where we introduce the notion of comparison flow (by extending a similar definition in [18]). In Section 5 we recall some results on the disjoint and joint sets properties; in Proposition 5.1 we reinterpret the disjoint sets property by means of the distance function. In Section 6 we discuss some aspects of the fattening phenomenon. Fattening for geometric evolutions in two dimensions is discussed in subsection 6.1, and fattening in dimension $n \geq 3$ is discussed in subsection 6.2. In presence of fattening, the connections between different weak approaches have not, to our knowledge, completely clarified, even in two dimensions (see Example 6.1 and below). In this section we include an explicit example of three-dimensional fattening for motion by mean curvature in codimension 2 (see Example 6.6). In the Appendix we list some assumptions used in the paper, following the notation of [54].

Most of the results discussed in the present paper are proved in [15,16] (see also [14,18,19,64]); we will prove in details only original statements not appearing in [15,16].

2. Some notation

In the following we let $I := [t_0, +\infty[$, for a fixed $t_0 \in \mathbf{R}$. We denote by $\mathcal{P}(\mathbf{R}^n)$

(resp. $\mathcal{A}(\mathbf{R}^n)$, $\mathcal{C}(\mathbf{R}^n)$) the family of all (resp. open, closed) subsets of $\mathbf{R}^n$, $n \geq 1$, and by $\mathcal{H}^m$ we mean the m-dimensional Hausdorff measure in $\mathbf{R}^n$, for $m \in [0,n]$.

Given a set $E \subseteq \mathbf{R}^n$, we denote by $\mathrm{int}(E)$, $\overline{E}$ and ∂E the interior part, the closure and the boundary of E, respectively. We set $\mathrm{dist}(\cdot, \emptyset) \equiv +\infty$, $d_E(x) := \mathrm{dist}(x,E) - \mathrm{dist}(x, \mathbf{R}^n \setminus E)$. Given a map $\phi : L \to \mathcal{P}(\mathbf{R}^n)$, where $L \subseteq \mathbf{R}$ is a convex set, we let $d_\phi : L \times \mathbf{R}^n \to \mathbf{R}$ be the function defined as

$$d_\phi(t,x) := \mathrm{dist}\big(x, \phi(t)\big) - \mathrm{dist}\big(x, \mathbf{R}^n \setminus \phi(t)\big) = d_{\phi(t)}(x). \tag{2.1}$$

If $\phi_1, \phi_2 : L \to \mathcal{P}(\mathbf{R}^n)$, by $\phi_1 \subseteq \phi_2$ (resp. $\phi_1 = \phi_2$, $\phi_1 \cap \phi_2$) we mean $\phi_1(t) \subseteq \phi_2(t)$ (resp. $\phi_1(t) = \phi_2(t)$, $\phi_1(t) \cap \phi_2(t)$) for any $t \in L$.
Given a function $v : L \times \mathbf{R}^n \to \mathbf{R}$ we denote by v_* (resp. v^*) the lower (resp. upper) semicontinuous envelope of v.
For $x \in \mathbf{R}^n$ and $R > 0$ we set $B_R(x) := \{y \in \mathbf{R}^n : |y - x| < R\}$. We let $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$.
Given $p \in \mathbf{R}^n \setminus \{0\}$, we set $P_p := \mathrm{Id} - p \otimes p/|p|^2$, and

$$J_0 := \big(\mathbf{R}^n \setminus \{0\}\big) \times \mathrm{Sym}(n), \qquad J_1 := I \times \mathbf{R}^n \times \big(\mathbf{R}^n \setminus \{0\}\big) \times \mathrm{Sym}(n).$$

Given a function $F : J_1 \to \mathbf{R}$ we denote by F_* (resp. F^*) the lower (resp. upper) semicontinuous envelope of F, defined on $\overline{J_1}$.
We recall that F is *geometric* [53] if $F(t,x,\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(t,x,p,X)$ for any $\lambda > 0$, $\sigma \in \mathbf{R}$, $(t,x,p,X) \in J_1$, and that F is *degenerate elliptic* if

$$F(t,x,p,X) \geq F(t,x,p,Y), \qquad (t,x,p,X) \in J_1,\ Y \in \mathrm{Sym}(n), Y \geq X. \tag{2.2}$$

In the sequel we shall always assume that F is geometric.
We say that F is locally Lipschitz in X if for any $(t,x,p) \in I \times \mathbf{R}^n \times \big(\mathbf{R}^n \setminus \{0\}\big)$ the function $F(t,x,p,\cdot)$ is locally Lipschitz.
We say that F is bounded below if, for any compact set $K \subset J_0$, there exists a constant $c_K \in \mathbf{R}$ such that

$$\inf \big\{F(t,x,p,X) : t \in I, x \in \mathbf{R}^n, (p,X) \in K\big\} \geq c_K.$$

For any $(t,x,p,X) \in J_1$ we set

$$\begin{aligned} F_c(t,x,p,X) &:= -F(t,x,-p,-X), \\ F^+(t,x,p,X) &:= \sup\big\{F(t,x,p,Y) :\ Y \geq X\big\}, \\ F^-(t,x,p,X) &:= \inf\big\{F(t,x,p,Y) :\ Y \leq X\big\}. \end{aligned} \tag{2.3}$$

Note that F^+ and F^- are always degenerate elliptic, and F_c is degenerate elliptic if and only if F is degenerate elliptic. Furthermore, $\big(F_c\big)_c = F$ and $\big(F^+\big)_c = \big(F_c\big)^-$.
We give definitions similar to (2.3) if the function F is defined on J_0 (resp. $\overline{J_0}$, $\overline{J_1}$).
We say that $F : J_0 \to \mathbf{R}$ is *compatible from above* (resp. *from below*) if there exists an odd degenerate elliptic function $F_1 : J_0 \to \mathbf{R}$ such that $F_1 \geq F$ (resp. $F_1 \leq F$).
We notice that $F : J_0 \to \mathbf{R}$ is compatible from above (resp. below) if and only if $\big(F^+\big)_c \geq F^+$ (resp. $\big(F^-\big)_c \leq F^-$).

Unless otherwise specified, when we deal with the viscosity theory we mean the one developed in [54] (see also [25] and references therein).

3. Barriers, local barriers and inner barriers

Let us recall the geometric barriers and minimal barriers in the sense of De Giorgi (with respect to a family $\mathcal{F}$ of set-valued maps and to the inclusion $\subseteq$ between subsets of $\mathbf{R}^n$); we refer to the original papers [31,34] for the abstract definition of barrier and minimal barrier. We also refer to [15,16] for the proofs of the assertions which are not detailed demonstrated here.

Definition 3.1 ($\mathcal{F}$-barriers). *Let $\mathcal{F}$ be a family of functions with the following property: for any $f \in \mathcal{F}$ there exist $a, b \in \mathbf{R}$, $a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$. A function ϕ is a barrier with respect to $\mathcal{F}$ if and only if ϕ maps a convex set $L \subseteq I$ into $\mathcal{P}(\mathbf{R}^n)$ and the following property holds: if $f : [a,b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$ then $f(b) \subseteq \phi(b)$. Given such a map ϕ, we shall write $\phi \in \mathcal{B}(\mathcal{F}, L)$. When $L = I$, we simply write $\phi \in \mathcal{B}(\mathcal{F})$.*

Notice that if $\phi_i \in \mathcal{B}(\mathcal{F}, L)$ for every $i \in \Lambda$, Λ any family of indices, then $\bigcap_{i\in\Lambda} \phi_i \in \mathcal{B}(\mathcal{F}, L)$.

Definition 3.2 (minimal barrier). *Let $E \subseteq \mathbf{R}^n$ be a given set and let $\bar{t} \in I$. The minimal barrier $\mathcal{M}(E, \mathcal{F}, \bar{t}) : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time $\bar{t}$) with respect to the family $\mathcal{F}$ at any time $t \geq \bar{t}$ is defined by*

$$\mathcal{M}(E,\mathcal{F},\bar{t})(t) := \bigcap \Big\{\phi(t) : \ \phi : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n), \ \phi \in \mathcal{B}(\mathcal{F}, [\bar{t}, +\infty[), \ \phi(\bar{t}) \supseteq E\Big\}.$$

Clearly

$$\mathcal{M}\left(E, \mathcal{F}, \bar{t}\right) \in \mathcal{B}\big(\mathcal{F}, [\bar{t}, +\infty[\big).$$

Moreover the following properties are immediate:

- comparison principle: $E_1 \subseteq E_2 \Rightarrow \mathcal{M}\left(E_1, \mathcal{F}, \bar{t}\right) \subseteq \mathcal{M}\left(E_2, \mathcal{F}, \bar{t}\right)$;
- initial datum: $\mathcal{M}\left(E, \mathcal{F}, \bar{t}\right)(\bar{t}) = E$;
- relaxation of the elements of $\mathcal{F}$: if $f : [a,b] \subseteq [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, then

$$f(t) \subseteq \mathcal{M}\big(f(a), \mathcal{F}, a\big)(t), \qquad t \in [a,b]; \tag{3.1}$$

- semigroup property: assume that the family $\mathcal{F}$ satisfies the following assumption:
(3.2)
given $f : [a,b] \subseteq [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n), \ f \in \mathcal{F}, \ t \in]a,b[$, then $f_{|[a,t]}, f_{|[t,b]} \in \mathcal{F}$.

Then

$$\mathcal{M}\left(E, \mathcal{F}, \bar{t}\right)(t_2) = \mathcal{M}\left(\mathcal{M}\left(E, \mathcal{F}, \bar{t}\right)(t_1), \mathcal{F}, t_1\right)(t_2), \qquad \bar{t} \leq t_1 \leq t_2. \tag{3.3}$$

Moreover $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{B}(\mathcal{F}, [\bar{t}, +\infty[) \supseteq \mathcal{B}(\mathcal{G}, [\bar{t}, +\infty[)$, hence $\mathcal{M}(E, \mathcal{F}, \bar{t}) \subseteq \mathcal{M}(E, \mathcal{G}, \bar{t})$.

In the sequel, unless otherwise specified, we shall assume $\bar{t} = t_0$, and we often drop it in the notation.

We say that $\mathcal{F}$ is *translation invariant* (in space) if, for any $f \in \mathcal{F}$ and $y \in \mathbf{R}^n$, the map $f + y : t \to f(t) + y := \{x \in \mathbf{R}^n : (x - y) \in f(t)\}$ belongs to $\mathcal{F}$.

If $\mathcal{F}$ is translation invariant one can check that

$$\phi \in \mathcal{B}(\mathcal{F}) \Rightarrow \mathrm{int}(\phi) \in \mathcal{B}(\mathcal{F}), \tag{3.4}$$

which implies that the minimal barrier with origin in an open set remains open for any time.

Another useful property of minimal barriers, which implies (under a further assumption on $\mathcal{F}$) the preservation of the Lipschitz constant for $\mathcal{M}_{u_0,\mathcal{F}}$ (see Proposition 3.3), is the following:

- if $\mathcal{F}$ is translation invariant, for any $\varrho > 0$ and any $t \in I$ we have

$$\mathcal{M}(E^+_\varrho, \mathcal{F})(t) \supseteq \Big(\mathcal{M}(E, \mathcal{F})(t)\Big)^+_\varrho. \tag{3.5}$$

The following example clarifies the choice of $\mathcal{F}$ when dealing with geometric equations of the form (1.4).

Example 3.1. (choice of the families $\mathcal{F}_F, \mathcal{F}_F^{>}, \mathcal{F}_F^{\leq}, \mathcal{F}_F^{=}$.) Let $F : J_1 \to \mathbf{R}$, let $a, b \in \mathbf{R}$, $a < b$, $[a,b] \subseteq I$ and let $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$. We write $f \in \mathcal{F}_F$ (and we say that f is a smooth local geometric supersolution of (1.4)) if and only if the following conditions hold:

(i) $f(t)$ is closed and $\partial f(t)$ is compact for any $t \in [a,b]$;

(ii) there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a,b] \times A)$ and $\partial f(t) \subseteq A$ for any $t \in [a,b]$;

(iii) the following inequality holds

$$\frac{\partial d_f}{\partial t}(t,x) + F\big(t, x, \nabla d_f(t,x), \nabla^2 d_f(t,x)\big) \geq 0, \qquad t \in \,]a,b[\,,\ x \in \partial f(t). \tag{3.6}$$

We write $f \in \mathcal{F}_F^{>}$ (resp. $f \in \mathcal{F}_F^{\leq}$, $f \in \mathcal{F}_F^{=}$) if the strict inequality (resp. the inequality $\leq$, the equality) holds in (3.6).

The minimal barrier with respect to $\mathcal{F}_F$ starting from the set $E \subseteq \mathbf{R}^n$ will be from now on our definition of weak evolution of E, concerning equations of the form (1.4). Clearly, if F does not depend on x, then all families in Example 3.1 are translation invariant.

We recall that motion by mean curvature of hypersurfaces corresponds to the choice

$$F(t,x,p,X) = -\mathrm{tr}(P_p X P_p), \tag{3.7}$$

and that motion by mean curvature of manifolds of codimension $k \geq 1$ corresponds to the choice

$$F(p,X) = -\sum_{i=1}^{n-k} \lambda_i, \tag{3.8}$$

where $\lambda_1 \leq \ldots \leq \lambda_{n-1}$ are the eigenvalues of the matrix $P_p X P_p$ which correspond to eigenvectors orthogonal to p, see [5,31].

We remark that, when dealing with the evolution of oriented *hypersurfaces*, we prefer to think of the evolution of the *solid* set E rather than of the evolution of its boundary ∂E (see Remark 5.1 and below).

The following example [19] shows that, unless that E is open, $\mathcal{M}(E, \mathcal{F}_F)$ is very sensible to slight modifications of the original set E.

Example 3.2. Let $n = 2$, $E = \{x = (x_1, x_2) \in \mathbf{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$ and let F be as in (3.7). Then, as a consequence of the definitions and the strong maximum principle one has

$$\mathcal{M}(E, \mathcal{F}_F)(t) = \mathcal{M}\big(\mathrm{int}(E), \mathcal{F}_F\big)(t), \qquad t > t_0.$$

Similarly, if $E = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and $x^* \in \partial E$, then

$$\mathcal{M}\big(E \setminus \{x^*\}, \mathcal{F}_F\big)(t) = \mathcal{M}\big(\mathrm{int}(E), \mathcal{F}_F\big)(t), \qquad t > t_0.$$

In view of Example 3.2, the minimal barrier $\mathcal{M}(E, \mathcal{F})$ is not always "topologically stable"; on the other hand, the regularization maps $\mathcal{M}_*(E, \mathcal{F})$, $\mathcal{M}^*(E, \mathcal{F})$ defined in the Introduction, enjoy the following stability property:

- stability of the lower and upper regularizations with respect to interior part and closure: if $\mathcal{F}$ is translation invariant, then for any $t \in I$ we have

$$\begin{aligned} &\mathcal{M}_*(E, \mathcal{F})(t) = \mathcal{M}_*\big(\mathrm{int}(E), \mathcal{F}\big)(t) \in \mathcal{A}(\mathbf{R}^n), \\ &\mathcal{M}^*(E, \mathcal{F})(t) = \mathcal{M}^*\big(\overline{E}, \mathcal{F}\big)(t) \in \mathcal{C}(\mathbf{R}^n). \end{aligned} \tag{3.9}$$

We have already observed that $\mathcal{M}^*(E, \mathcal{F})$ belongs to $\mathcal{B}(\mathcal{F})$. One can ask under which conditions on $\mathcal{F}$ it holds $\mathcal{M}_*(E, \mathcal{F}) \in \mathcal{B}(\mathcal{F})$. The following result holds:

- set

$$\mathcal{F}^c := \{f : [a, b] \to \mathcal{P}(\mathbf{R}^n), f \in \mathcal{F}, f(t) \text{ is compact for any } t \in [a, b]\}. \tag{3.10}$$

  If $\mathcal{F}$ is translation invariant, then $\mathcal{M}_*(E, \mathcal{F}^c) \in \mathcal{B}(\mathcal{F}^c)$ and for any $t \in I$ there holds

$$E \in \mathcal{A}(\mathbf{R}^n) \Rightarrow \mathcal{M}_*(E, \mathcal{F}^c)(t) = \mathcal{M}(E, \mathcal{F}^c)(t) \in \mathcal{A}(\mathbf{R}^n). \tag{3.11}$$

  If additionally $\mathcal{F}$ satisfies (3.2), then $\mathcal{M}_*(E, \mathcal{F}^c)$ satisfies the semigroup property.

We notice that, if $F : J_1 \to \mathbf{R}$ is bounded below, then it turns out that $\mathcal{B}(\mathcal{F}_F) = \mathcal{B}\big((\mathcal{F}_F)^c\big)$, and therefore in this case we can ensure that $\mathcal{M}_*(E, \mathcal{F}_F) \in \mathcal{B}(\mathcal{F}_F)$.

We also note that, under mild conditions on F, and possibly regularizing the minimal barrier, we can interchange $\mathcal{F}_F$ with $\mathcal{F}_F^{>}$ when defining the minimal barrier. Indeed, the following property holds:

- assume that $F : J_0 \to \mathbf{R}$ is either lower semicontinuous and locally Lipschitz in X, or is continuous and degenerate elliptic. Then, for any $E \subseteq \mathbf{R}^n$ we have

$$\mathcal{M}_*\big(E, \mathcal{F}_F\big) = \mathcal{M}_*\big(E, \mathcal{F}_F^{>}\big), \qquad \mathcal{M}^*\big(E, \mathcal{F}_F\big) = \mathcal{M}^*\big(E, \mathcal{F}_F^{>}\big). \tag{3.12}$$

The following examples show the rôle of the choice of $\mathcal{F}$ in the definition of the minimal barriers: Example 3.3 concerns motion by curvature whenever $\mathcal{F}$ consists of smooth *convex* evolutions, and Example 3.4 concerns the case of inverse mean curvature flow (see [62]).

Example 3.3. Let $n = 2$, F be as in (3.7) and

$$\mathcal{C}_F := \{f : [a,b] \to \mathcal{P}(\mathbf{R}^n), f \in \mathcal{F}_F, f(t) \text{ is convex for any } t \in [a,b]\},$$
$$\mathcal{D}_F := \{f : [a,b] \to \mathcal{P}(\mathbf{R}^n), f \in \mathcal{F}_F^{\overline{=}}, f(a) \text{ is convex}\}. \tag{3.13}$$

Then, for any $E \subseteq \mathbf{R}^2$ we have

$$\mathcal{M}_*(E, \mathcal{C}_F) = \mathcal{M}_*(E, \mathcal{D}_F) = \mathcal{M}_*(E, \mathcal{F}_{F\wedge 0}),$$
$$\mathcal{M}^*(E, \mathcal{C}_F) = \mathcal{M}^*(E, \mathcal{D}_F) = \mathcal{M}^*(E, \mathcal{F}_{F\wedge 0}).$$

Example 3.4. Let us define the family $\mathcal{G}$ as follows. A function $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{G}$ if and only if $f(t)$ is compact for any $t \in [a,b]$, there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a,b] \times A)$, $\partial f(t) \subseteq A$ for any $t \in [a,b]$, and

$$\Delta d_f > 0, \qquad \frac{\partial d_f}{\partial t} + \frac{1}{\Delta d_f} \geq 0 \qquad t \in \,]a,b[\,, x \in \partial f(t).$$

Then $\mathcal{M}(E, \mathcal{G})$ provides a definition of weak evolution of any convex set $E \subseteq \mathbf{R}^n$ by the inverse mean curvature.

Inclusion (3.1) becomes an equality whenever $\mathcal{F} = \mathcal{F}_F$ for suitable functions F (in particular for motion by mean curvature of hypersurfaces).

Proposition 3.1 (extension of classical flows). *Assume that $F : J_1 \to \mathbf{R}$ does not depend on x and is degenerate elliptic. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F^{\overline{=}}$. Then*

$$f(t) = \mathcal{M}\big(f(a), \mathcal{F}_F, a\big)(t), \qquad t \in [a,b]. \tag{3.14}$$

Proof. It is enough to show that $f(t) \supseteq \mathcal{M}\big(f(a), \mathcal{F}_F, a\big)(t)$ for any $t \in [a,b]$. Observe that the following geometric maximum principle holds: let $g, h : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $g \in \mathcal{F}_F$, $h \in \mathcal{F}_F^{\leq}$. Then

$$g(a) \subseteq h(a) \Rightarrow g(b) \subseteq h(b).$$

It follows that $f \in \mathcal{B}(\mathcal{F}_F, [a,b])$, and (3.14) follows. □

Notice that we have defined $\mathcal{M}(E, \mathcal{F}_F)$ under no assumptions on F (such as continuity and degenerate ellipticity); clearly, to end up with a nontrivial minimal barrier, we have to ensure that $\mathcal{B}(\mathcal{F}_F)$ is nonempty, which is true under minor assumptions, such as boundedness below of F. Even under these assumptions, it may happen that the minimal barrier becomes trivial for all times $t \in \,]t_0, +\infty[$. Indeed, let us consider the following example.

Example 3.5. Let $F : J_0 \to \mathbf{R}$ be the function corresponding to motion by mean curvature with the "wrong" sign (corresponding to the backward heat equation for the signed distance function), i.e.,

$$F(p, X) := \operatorname{tr}(P_p X P_p),$$

and let $A \in \mathcal{A}(\mathbf{R}^n)$. Then, as a consequence of (3.16) below, we have

$$\mathcal{M}(A, \mathcal{F}_F)(t) = \mathbf{R}^n, \qquad t > t_0.$$

Barriers are a global concept, since they are defined through sets inclusions. In order to derive differential properties of the evolution, one can look for locality properties of barriers. Following [16], we introduce the local barriers and the local minimal barrier; the localization is with respect to the space variable.

Definition 3.3 (local barriers). *Let $\mathcal{F}$ be as in Definition 3.1. A function ϕ is a local barrier with respect to $\mathcal{F}$ if and only if there exists a convex set $L \subseteq I$ such that $\phi : L \to \mathcal{P}(\mathbf{R}^n)$ and the following property holds: for any $x \in \mathbf{R}^n$ there exists $R > 0$ (depending on ϕ and x) so that if $f : [a,b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a) \cap B_R(x)$, then $f(b) \subseteq \phi(b)$. We denote by $\mathcal{B}_{\mathrm{loc}}(\mathcal{F})$ the family of all local barriers ϕ such that $L = I$ (that is, local barriers on the whole of I).*

Definition 3.4 (local minimal barrier). *Let $E \subseteq \mathbf{R}^n$ be a given set and let $\bar{t} \in I$. The local minimal barrier $\mathcal{M}_{\mathrm{loc}}(E, \mathcal{F}, \bar{t}) : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time $\bar{t}$) with respect to the family $\mathcal{F}$ at any time $t \geq \bar{t}$ is defined by*

$$\mathcal{M}_{\mathrm{loc}}(E,\mathcal{F},\bar{t})(t) := \bigcap \Big\{ \phi(t) : \ \phi : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n), \phi \in \mathcal{B}_{\mathrm{loc}}(\mathcal{F}, [\bar{t}, +\infty[), \phi(\bar{t}) \supseteq E \Big\}.$$

The definitions of regularized local barriers can be given in the obvious way. Notice also that from Definition 3.4 it does not directly follow that the local minimal barrier is a local barrier, because of the dependence of R on ϕ.

The following theorem holds [16].

Theorem 3.1 (connections between barriers and local barriers). *Assume that $F : J_0 \to \mathbf{R}$ is lower semicontinuous. Then*

$$\mathcal{B}_{\mathrm{loc}}\big(\mathcal{F}_F^{\geq}\big) = \mathcal{B}\big(\mathcal{F}_F^{\geq}\big).$$

In particular, for any $E \subseteq \mathbf{R}^n$ we have $\mathcal{M}_{\mathrm{loc}}\big(E, \mathcal{F}_F^{\geq}\big) = \mathcal{M}\big(E, \mathcal{F}_F^{\geq}\big)$.

Considering the opposite sets inclusion in Definition 3.1, we can define the inner barriers [16].

Definition 3.5 (inner barriers). *Let $\mathcal{F}$ be as in Definition 3.1. A function $\tilde{\phi}$ is an inner barrier with respect to $\mathcal{F}$ if and only if $\tilde{\phi}$ maps a convex set $L \subseteq I$ into $\mathcal{P}(\mathbf{R}^n)$ and the following property holds: if $f : [a,b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $\tilde{\phi}(a) \subseteq \mathrm{int}\big(f(a)\big)$ then $\tilde{\phi}(b) \subseteq \mathrm{int}\big(f(b)\big)$. Given such a map $\tilde{\phi}$, we shall write $\tilde{\phi} \in \widetilde{\mathcal{B}}(\mathcal{F}, L)$. When $L = I$, we simply write $\tilde{\phi} \in \widetilde{\mathcal{B}}(\mathcal{F})$.*

The definition of local inner barrier can be given in the obvious way.

Definition 3.6 (maximal inner barrier and its regularizations). *Let $E \subseteq \mathbf{R}^n$ be a given set and let $\bar{t} \in I$. The maximal inner barrier $\mathcal{N}(E, \mathcal{F}, \bar{t}\,) : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time $\bar{t}$) with respect to the family $\mathcal{F}$ at any time $t \geq \bar{t}$ is defined by*

$$\mathcal{N}(E,\mathcal{F},\bar{t})(t) := \bigcup \Big\{ \tilde{\phi}(t) : \ \tilde{\phi} : [\,\bar{t}, +\infty[\, \to \mathcal{P}(\mathbf{R}^n), \ \tilde{\phi} \in \widetilde{\mathcal{B}}(\mathcal{F}, [\bar{t}, +\infty[), \ \tilde{\phi}(\bar{t}) \subseteq E \Big\}.$$

Its lower and upper regularization are defined by

$$\mathcal{N}_*(E,\mathcal{F},\bar{t})(t) := \bigcup_{\varrho > 0} \mathcal{N}(E_\varrho^-, \mathcal{F}, \bar{t})(t), \qquad \mathcal{N}^*(E,\mathcal{F},\bar{t})(t) := \bigcap_{\varrho > 0} \mathcal{N}(E_\varrho^+, \mathcal{F}, \bar{t})(t).$$

Note that $\phi \in \mathcal{B}(\mathcal{F}_F)$ if and only if $\mathbf{R}^n \setminus \phi \in \widetilde{\mathcal{B}}(\mathcal{F}^{\leq}_{F_c})$. Consequently, for any $E \subseteq \mathbf{R}^n$, $\mathbf{R}^n \setminus \mathcal{M}(E, \mathcal{F}_F) = \mathcal{N}(\mathbf{R}^n \setminus E, \mathcal{F}^{\leq}_{F_c})$, hence

$$
\begin{aligned}
\mathbf{R}^n \setminus \mathcal{M}_*(E, \mathcal{F}_F) &= \mathcal{N}^*\big(\mathbf{R}^n \setminus E, \mathcal{F}^{\leq}_{F_c}\big),\\
\mathbf{R}^n \setminus \mathcal{M}^*(E, \mathcal{F}_F) &= \mathcal{N}_*\big(\mathbf{R}^n \setminus E, \mathcal{F}^{\leq}_{F_c}\big).
\end{aligned} \tag{3.15}
$$

If the function F is not degenerate elliptic, then the minimal barriers do not coincide, in general, with the smooth evolutions (whenever they exist). One can ask what the minimal barrier represents in this case. It turns out that the minimal barrier with respect to $\mathcal{F}_F$ coincides with the minimal barrier with respect to $\mathcal{F}_{F^+}$, where F^+ is defined in (2.3) and is degenerate elliptic. More precisely, there holds the following theorem [16], which is one of the main results on barriers.

Theorem 3.2 (representation of the minimal barrier for not degenerate elliptic functions F). *Assume that $F : J_0 \to \mathbf{R}$ is lower semicontinuous. Then*

$$
\mathcal{B}\big(\mathcal{F}^{\geq}_F\big) = \mathcal{B}\big(\mathcal{F}^{\geq}_{F^+}\big).
$$

In particular, for any $E \subseteq \mathbf{R}^n$ we have

$$
\mathcal{M}\big(E, \mathcal{F}^{\geq}_F\big) = \mathcal{M}\big(E, \mathcal{F}^{\geq}_{F^+}\big). \tag{3.16}
$$

This theorem clarifies inclusion (3.1) (when $\mathcal{F} = \mathcal{F}_F$). Under further assumptions on F, we can prove a viscosity version of Theorem 3.2.

Proposition 3.2. *Let $F : J_1 \to \mathbf{R}$ and $u : I \times \mathbf{R}^n \to \mathbf{R}$ be given functions. Assume that F is lower (resp. upper) semicontinuous, $(F^+)_* < +\infty$ (resp. $(F^-)^* > -\infty$) on $\overline{J_1}$, and $(F^+)_*(t, x, 0, X) = (F_*)^+(t, x, 0, X)$ (resp. $(F^-)^*(t, x, 0, X) = (F^*)^-(t, x, 0, X)$) for any $t \in I$, $x \in \mathbf{R}^n$ and $X \in Sym(n)$. Then u is a viscosity subsolution (resp. supersolution) of (1.4) in $]t_0, +\infty[\, \times \mathbf{R}^n$ if and only if u is a viscosity subsolution (resp. supersolution) of (1.5) (resp. of $\frac{\partial u}{\partial t} + F^-\big(t, x, \nabla u, \nabla^2 u\big) = 0$) in $]t_0, +\infty[\, \times \mathbf{R}^n$.*

Proof. As $(F^+)_c = (F_c)^-$, it is enough to show the assertion for subsolutions. As $F^+ \geq F$, we only need to show that if u is a subsolution of (1.4) then u is a subsolution of (1.5). Observe that F^+ is lower semicontinuous on J_1, hence $(F^+)_* = (F_*)^+$ on $\overline{J_1}$. Let $(\overline{t}, \overline{x}) \in\,]t_0, +\infty[\, \times \mathbf{R}^n$ and let ψ be a smooth function such that $(u^* - \psi)$ has a maximum at $(\overline{t}, \overline{x})$. Assume by contradiction that

$$
\frac{\partial \psi}{\partial t} + (F^+)_*\big(\overline{t}, \overline{x}, \nabla \psi, \nabla^2 \psi\big) = 2c > 0 \qquad \text{at } (\overline{t}, \overline{x}). \tag{3.17}
$$

By definition of $(F^+)_* = (F_*)^+$ and since $(F^+)_* < +\infty$, there exists $X \in \mathrm{Sym}(n)$, $X \geq \nabla^2 \psi(\overline{t}, \overline{x})$, such that

$$
(F^+)_*\big(\overline{t}, \overline{x}, \nabla \psi, \nabla^2 \psi\big) \leq F_*\big(\overline{t}, \overline{x}, \nabla \psi, X\big) + c \qquad \text{at } (\overline{t}, \overline{x}). \tag{3.18}
$$

Define

$$
\Psi(t, x) := \psi(t, x) + \frac{1}{2}\big\langle (x - \overline{x}), \big(X - \nabla^2 \psi(\overline{t}, \overline{x})\big)(x - \overline{x})\big\rangle,
$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product. Then $\nabla^2\Psi(\bar{t}, \bar{x}) = X$ and $(u^* - \Psi)$ has a maximum at $(\bar{t}, \bar{x})$. Therefore, using the fact that u is a subsolution of (1.4), (3.18) and (3.17), at $(\bar{t}, \bar{x})$ we have

$$\begin{aligned} 0 &\geq \frac{\partial \Psi}{\partial t} + F_*\big(\bar{t}, \bar{x}, \nabla\Psi, \nabla^2\Psi\big) = \frac{\partial \psi}{\partial t} + F_*\big(\bar{t}, \bar{x}, \nabla\psi, X\big) \\ &\geq \frac{\partial \psi}{\partial t} + (F^+)_*\big(\bar{t}, \bar{x}, \nabla\psi, \nabla^2\psi\big) - c = c > 0, \end{aligned}$$

a contradiction. □

A comment is in order about the assumptions in Proposition 3.2. In general $(F^+)_* \geq (F_*)^+$ on $\overline{J}_1$ and the equality holds if F is degenerate elliptic. If F is lower semicontinuous, then $(F^+)_* = (F_*)^+$ on J_1. Also, the equality $(F^+)_*(t, x, 0, X) = (F_*)^+(t, x, 0, X)$ holds for a geometric function $F : J_1 \to \mathbf{R}$ which coincides with F^+ outside a compact set K of J_1, and is bounded on K. Indeed, if $X \neq 0$, using the fact that F^+ is geometric, we have

$$\begin{aligned} \big(F^+\big)_*(t, x, 0, X) &= \inf \liminf_{p_n \to 0, X_n \to X} F^+(t, x, p_n, X_n) \\ &= \inf \liminf_{p_n \to 0, X_n \to X} |p_n| F^+\left(t, x, \frac{p_n}{|p_n|}, \frac{X_n}{|p_n|}\right). \end{aligned} \tag{3.19}$$

Since $\left|\frac{X_n}{|p_n|}\right| \to +\infty$ in (3.19) and $F = F^+$ outside the compact set K, we get

$$\begin{aligned} \big(F^+\big)_*(t, x, 0, X) &= \inf \liminf_{p_n \to 0, X_n \to X} |p_n| F\left(t, x, \frac{p_n}{|p_n|}, \frac{X_n}{|p_n|}\right) \\ &= (F_*)(t, x, 0, X) \leq (F_*)^+(t, x, 0, X). \end{aligned} \tag{3.20}$$

Let $X = 0$ and let $(p_n, X_n) \to (0, 0)$ be the minimizing sequence for $\big(F^+\big)_*(t, x, 0, 0)$, i.e., $\big(F^+\big)_*(t, x, 0, 0) = \lim_{n \to +\infty} F^+(t, x, p_n, X_n)$. Possibly passing to a subsequence, one of the following two cases occurs:

(i) $(t, x, \frac{p_n}{|p_n|}, \frac{X}{|p_n|}) \in K$ for any n;

(ii) $(t, x, \frac{p_n}{|p_n|}, \frac{X}{|p_n|}) \notin K$ for any n.

Using the fact that F^+ is bounded on K, if (i) holds we get $(F^+)_*(t, x, 0, 0) = 0$, whereas, in case (ii), one can check that $(F^+)_*(t, x, 0, 0) \leq 0$. Therefore, we can assume that the minimizing sequence satisfies (ii). Reasoning in a similar way, we can also assume that the same holds for the minimizing sequence for $F_*(t, x, 0, 0)$. Since $F^+ = F$ outside K, we deduce that (t, x, p_n, X_n) is a minimizing sequence for both $(F^+)_*(t, x, 0, 0)$ and $F_*(t, x, 0, 0)$, which implies (3.20) with $X = 0$.

Example 3.6. Let $n = 2$ and consider the anisotropic motion by mean curvature given by

$$F(p, X) := -\mathrm{tr}(P_p X P_p)\psi(\theta)\big(\psi(\theta) + \psi''(\theta)\big),$$

where $\psi : \mathbf{S}^1 \to]0, +\infty[$ is a smooth function and $p = (p_1, p_2) = (\cos\theta, \sin\theta)$ (see [20]). Then, if $\psi + \psi'' \geq 0$ on $\mathbf{S}^1$ (i.e., convex anisotropy), we have $F^+ = F$. If the anisotropy is not convex, then there exists $\overline{\theta} \in \mathbf{S}^1$ such that $\psi(\overline{\theta}) + \psi''(\overline{\theta}) < 0$, which implies $F^+(\overline{p}, X) = +\infty$ for any $X \in \mathrm{Sym}(2)$, where $\overline{p} = (\cos\overline{\theta}, \sin\overline{\theta})$. Indeed, $F(\overline{p}, \cdot)$, being linear with the "wrong sign", behaves as the backward mean curvature flow, compare Example 3.5.

3.1. The function $\mathcal{M}_{u_0,\mathcal{F}}$. In this subsection, starting from the weak evolution defined on sets as in the previous sections, we recall the definition of the weak solution as a function, and we study some of its properties. The procedure we follow is the one in [15], and is the *opposite* with respect to the one used to define the level set flow. This kind of procedure has also been used by Evans in [43] when considering the semigroup approach to motion by mean curvature.

We have seen that the minimal barrier starting from an arbitrary set E is unique and globally defined. Therefore, given any initial function $u_0 : \mathbf{R}^n \to \mathbf{R}$, there is a natural way to construct a unique global evolution function $\mathcal{M}_{u_0,\mathcal{F}}(t,x)$ (assuming u_0 as initial datum): it is indeed defined as that function which, for any $\lambda \in \mathbf{R}$, has $\mathcal{M}(\{u_0 < \lambda\}, \mathcal{F})(t)$ as λ-sublevel set at time $t \in I$.

Definition 3.7. *Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a given function. The function $\mathcal{M}_{u_0,\mathcal{F}} : I \times \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ is defined by*

$$\mathcal{M}_{u_0,\mathcal{F}}(t,x) := \inf\{\lambda \in \mathbf{R} : \mathcal{M}(\{u_0 < \lambda\}, \mathcal{F})(t) \ni x\}. \tag{3.21}$$

If $\mathcal{B}(\mathcal{F}) = \mathcal{B}(\mathcal{F}^c)$ (see (3.10)), if $\mathcal{M}(A, \mathcal{F})(t) \in \mathcal{A}(\mathbf{R}^n)$ for any $A \in \mathcal{A}(\mathbf{R}^n)$, and if $u_0 : \mathbf{R}^n \to \mathbf{R}$ is upper semicontinuous, then for any $\lambda \in \mathbf{R}^n \cup \{\pm\infty\}$ there holds

$$\mathcal{M}_{u_0,\mathcal{F}}(t,x) = \inf\{\lambda \in \mathbf{R} : \mathcal{M}_*(\{u_0 < \lambda\}, \mathcal{F})(t) \ni x\},$$

and

$$\{x \in \mathbf{R}^n : \mathcal{M}_{u_0,\mathcal{F}}(t,x) < \lambda\} = \mathcal{M}(\{u_0 < \lambda\}, \mathcal{F})(t), \qquad t \in I. \tag{3.22}$$

Hence, under these assumptions, $\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot)$ is upper semicontinuous.

Proposition 3.3 (preservation of the Lipschitz constant). *Assume that $\mathcal{F}$ is translation invariant and $\mathcal{B}(\mathcal{F}) = \mathcal{B}(\mathcal{F}^c)$. Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a Lipschitz function, and let $k > 0$ be its Lipschitz constant. Then*

$$|\mathcal{M}_{u_0,\mathcal{F}}(t,x) - \mathcal{M}_{u_0,\mathcal{F}}(t,y)| \le k|x-y|, \qquad x, y \in \mathbf{R}^n,\ t \in I,$$

where we assume that the left hand side is zero if $\mathcal{M}_{u_0,\mathcal{F}}(t,x)$ and $\mathcal{M}_{u_0,\mathcal{F}}(t,y)$ are both equal to $+\infty$ or $-\infty$.

Proof. Let $\lambda, \mu \in \mathbf{R} \cup \{\pm\infty\}$ be such that $\mu \ge \lambda$, and set $E_\lambda := \{u_0 < \lambda\}$, $E_\mu := \{u_0 < \mu\}$. Then we have

$$E_\mu \supseteq (E_\lambda)^+_{\frac{\mu-\lambda}{k}}. \tag{3.23}$$

Indeed, the inclusion is obvious if $|\lambda|$ or $|\mu|$ is equal to $+\infty$; otherwise, if $z \in (E_\lambda)^+_{\frac{\mu-\lambda}{k}}$ then $z = x + \frac{\mu-\lambda}{k} q$ for some $x \in E_\lambda$ and some $q \in \mathbf{R}^n$ with $|q| < 1$. Hence, as u_0 is k-Lipschitz,

$$u_0(z) = u_0\left(x + \frac{\mu-\lambda}{k} q\right) < u_0(x) + \frac{\mu-\lambda}{k} k < \mu,$$

so that $z \in E_\mu$. Then, by (3.22), (3.23) and (3.5) we find

$$\begin{aligned}\{x : \mathcal{M}_{u_0,\mathcal{F}}(t,x) < \mu\} &= \mathcal{M}(E_\mu, \mathcal{F})(t) \supseteq \mathcal{M}\Big((E_\lambda)^+_{\frac{\mu-\lambda}{k}}, \mathcal{F}\Big)(t) \\ &\supseteq \Big(\mathcal{M}(E_\lambda, \mathcal{F})(t)\Big)^+_{\frac{\mu-\lambda}{k}} = \Big(\{x : \mathcal{M}_{u_0,\mathcal{F}}(t,x) < \lambda\}\Big)^+_{\frac{\mu-\lambda}{k}}.\end{aligned} \tag{3.24}$$

Let $y, w \in \mathbf{R}^n$ and set $\lambda := \mathcal{M}_{u_0,\mathcal{F}}(t,y)$, $\mu := \mathcal{M}_{u_0,\mathcal{F}}(t,w)$; let us prove that

$$|\mu - \lambda| \le k|y - w|. \tag{3.25}$$

If $\lambda = \mu$ there is nothing to prove. Without loss of generality, we can assume $\lambda < \mu$. Let $\epsilon > 0$ be such that $\lambda + \epsilon < \mu$. As $w \in \{\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot) \ge \mu\}$, by (3.24) we have $w \notin \big(\{\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot) < \lambda + \epsilon\}\big)^+_{\frac{\mu-\lambda-\epsilon}{k}}$. As $y \in \{\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot) < \lambda + \epsilon\}$, we then obtain $|y - w| \ge \frac{\mu-\lambda-\epsilon}{k}$. Letting $\epsilon \downarrow 0$ we get (3.25). In particular, it follows that either $\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot) \equiv +\infty$, or $\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot) \equiv -\infty$, or $\mathcal{M}_{u_0,\mathcal{F}}(t,\cdot) : \mathbf{R}^n \to \mathbf{R}$ is a Lipschitz function with Lipschitz constant less than or equal to k. $\square$

If v_0 is smooth, bounded, constant outside a bounded subset of $\mathbf{R}^n$, and if F is as in (3.7), Evans-Spruck [47] showed that

$$\sup_{t\in I} \int_{\mathbf{R}^n} |\nabla \mathcal{M}_{v_0,\mathcal{F}_F}| \, dx \le \int_{\mathbf{R}^n} |\nabla v_0| \, dx, \tag{3.26}$$

where we have used the fact that $\mathcal{M}_{v_0,\mathcal{F}_F}$ coincides with the (Lipschitz continuous) viscosity solution v assuming v_0 as initial datum, see Theorem 4.2 below. As we have seen, we can define $\mathcal{M}_{v_0,\mathcal{F}}$ under no restriction on v_0; we do not know wether inequality (3.26) still holds for an initial datum $v_0 : \mathbf{R}^n \to \mathbf{R}$ which is upper semicontinuous and with bounded variation on $\mathbf{R}^n$ (interpreting the integrals as total variations).

We also do not know wether the two maps

$$t \to P\big(\mathcal{M}_*(E, \mathcal{F}_F)(t)\big), \qquad t \to P\big(\mathcal{M}^*(E, \mathcal{F}_F)(t)\big)$$

are nonincreasing, where P denotes the perimeter in $\mathbf{R}^n$. In this respect, we recall that Evans-Spruck [47] proved that if E has compact $(n-1)$-dimensional rectifiable boundary, and $\mathcal{H}^{n-1}(\partial E) < +\infty$, then

$$\mathcal{H}^{n-1}\big(\partial\{v(t,\cdot) = 0\}\big) \le C\mathcal{H}^{n-1}\big(\partial E\big), \qquad t \in I, \tag{3.27}$$

where $C > 0$ depends only on the dimension. Notice that, thanks to the results of [19], (3.27) can be rewritten in terms of barriers as

$$\mathcal{H}^{n-1}\left(\partial\Big(\mathcal{M}^*(E, \mathcal{F}_F)(t) \setminus \mathcal{M}_*(E, \mathcal{F}_F)(t)\Big)\right) \le C\mathcal{H}^{n-1}\big(\partial E\big), \qquad t \in I.$$

3.2. Some connections with the reaction-diffusion equation. In this subsection we recall some ideas connecting the reaction-diffusion equations and the barriers, which have been pointed out by Jerrard-Soner [71,72] and Soner [84] (see also Ilmanen [65] and Souganidis [86]). Let E be a smooth bounded open set and F be as in (3.7); let u_ϵ be the solution of the reaction-diffusion equation

$$\frac{\partial u_\epsilon}{\partial t} = \Delta u_\epsilon - \frac{1}{\epsilon^2}(u_\epsilon^3 - u_\epsilon),$$

assuming an "admissible" initial datum u_ϵ^0, which is a function which approximates the characteristic function of E as $\epsilon \to 0$, and has equibounded energy, i.e.,

$$\sup_\epsilon \mu_0^\epsilon(\mathbf{R}^n) < +\infty, \tag{3.28}$$

where

$$d\mu_t^\epsilon(x) := \epsilon E_\epsilon(t,x)\, dx, \qquad E_\epsilon := \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{\epsilon^2}(u_\epsilon^2 - 1)^2.$$

Then, it turns out that there exists a subsequence $\{\epsilon_m\}$ such that $\{\mu_t^{\epsilon_m}\}$ converges to a Radon measure μ_t, in the weak* topology, for all $t \in I$. Set

$$\Gamma(t) := \mathrm{spt}(\mu_t). \tag{3.29}$$

Then there holds

$$\Gamma(t) \subseteq \mathcal{M}^*(\partial E, \mathcal{F}_F)(t), \qquad t \in I. \tag{3.30}$$

To check (3.30), let us show that

$$\mathbf{R}^n \setminus \Gamma(t) \in \mathcal{B}(\mathcal{F}_F). \tag{3.31}$$

Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $f(a) \subseteq \mathbf{R}^n \setminus \Gamma(a)$. We have to prove that $f(b) \subseteq \mathbf{R}^n \setminus \Gamma(b)$. We recall [71,72,84] that there is a constant $C > 0$ so that

$$\int_{\mathbf{R}^n} \eta(t,x) E^\epsilon(t,x)\, dx \le e^{C(t-a)} \int_{\mathbf{R}^n} \eta(a,x) E^\epsilon(a,x)\, dx, \qquad t \in [a,b], \tag{3.32}$$

for ϵ sufficiently small, where $\eta : [a,b] \times \mathbf{R}^n \to \mathbf{R}$ is any smooth nonnegative function.

Following Soner [84], we can choose $0 < \delta < \mathrm{dist}(f(a), \Gamma(a))/2$ sufficiently small and a smooth nonnegative function $\overline{\eta}$ with the following properties: $\|\overline{\eta}\|_{C^2} < +\infty$, $\overline{\eta}(t,x) = 1$ if $\mathrm{dist}(x, f(t)) > \delta$, and $\eta(t,x) = 0$ if $\mathrm{dist}(x, f(t)) \ge 2\delta$. Recalling that $\mu_t^{\epsilon_m} \rightharpoonup \mu_t$, from (3.32) and the choice of $\overline{\eta}$ we get

$$\begin{aligned} &\lim_{m\to+\infty} \int_{\mathbf{R}^n} \overline{\eta}(b,x) E^{\epsilon_m}(b,x)\, dx = \int_{\mathbf{R}^n} \overline{\eta}(b,x) d\mu_b(x) \\ &\le e^{C(b-a)} \lim_{m\to+\infty} \int_{\mathbf{R}^n} \overline{\eta}(a,x) E^{\epsilon_m}(a,x)\, dx = \int_{\mathbf{R}^n} \overline{\eta}(a,x) d\mu_a(x) = 0. \end{aligned}$$

Therefore $\mathrm{spt}(\overline{\eta}(b,\cdot)) \cap \mathrm{spt}(\mu_b) = \emptyset$, so that $f(b) \subseteq \mathbf{R}^n \setminus \Gamma(b)$, and (3.31) is proved. Using (3.31), the fact that $\Gamma(0) = \partial E$, and [19] (see also (4.6)) we deduce

$$\mathbf{R}^n \setminus \Gamma(t) \supseteq \mathcal{M}_*(\mathbf{R}^n \setminus \partial E, \mathcal{F}_F)(t) = \mathbf{R}^n \setminus \mathcal{M}^*(\partial E, \mathcal{F}_F)(t), \qquad t \in I,$$

which proves (3.30).

3.3. Outer regularity of the minimal barrier and right continuity of the distance function between minimal barriers. In this subsection we show how a suitable notion of outer regularity of $\mathcal{F}$ reflects on the outer regularity of the minimal barrier, and we study some continuity properties of the distance function between barriers.

Definition 3.8. *We say that $\mathcal{F}$ is outer regular if for any $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, we have $f(t) = \overline{int(f(t))}$ for any $t \in [a,b]$.*

Given $E \subseteq \mathbf{R}^n$, we set

$$E^r := E \cap \overline{\mathrm{int}(E)}.$$

If $E = E^r$, we say that the set E is outer regular.

Proposition 3.4 (outer regularity of the minimal barrier). *Assume that $\mathcal{F}$ is outer regular. Let $\phi : I \to \mathcal{P}(\mathbf{R}^n)$, $\phi \in \mathcal{B}(\mathcal{F})$, and let $\phi^r : I \to \mathcal{P}(\mathbf{R}^n)$ be the map defined by $\phi^r(t) := \phi(t)^r$ for any $t \in I$. Then $\phi^r \in \mathcal{B}(\mathcal{F})$. Moreover, for any $E \subseteq \mathbf{R}^n$, we have*

$$\mathcal{M}(E,\mathcal{F})(t) = \mathcal{M}(E,\mathcal{F})(t)^r = \mathcal{M}(E^r,\mathcal{F})(t), \qquad t > t_0. \tag{3.33}$$

In particular $\mathcal{M}(E,\mathcal{F})(t) \subseteq \mathcal{M}\big(\overline{int(E)},\mathcal{F}\big)(t)$ for any $t > t_0$, and if E is closed then

$$\mathcal{M}(E,\mathcal{F})(t) = \mathcal{M}\big(\overline{int(E)},\mathcal{F}\big)(t), \qquad t > t_0. \tag{3.34}$$

Proof. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, $f(a) \subseteq \phi^r(a) \subseteq \phi(a)$. As $\phi \in \mathcal{B}(\mathcal{F})$ we have $f(b) \subseteq \phi(b)$, and then $\overline{\mathrm{int}(f(b))} = f(b) \subseteq \overline{\mathrm{int}(\phi(b))}$. Hence $f(b) \subseteq \phi^r(b)$.

The inclusion $\mathcal{M}(E,\mathcal{F}) \supseteq \mathcal{M}(E,\mathcal{F})^r$ is immediate. To prove the opposite inequality, it is enough to show that the map $\phi : I \to \mathcal{P}(\mathbf{R}^n)$ defined by

$$\phi(t) := \begin{cases} E & \text{if } t = t_0, \\ \mathcal{M}(E,\mathcal{F})(t)^r & \text{if } t > t_0 \end{cases}$$

belongs to $\mathcal{B}(\mathcal{F})$. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, $f(a) \subseteq \phi(a)$. If $a > t_0$ then $f(b) \subseteq \phi(b)$ by the previous assertion. If $a = t_0$, then $f(a) = \overline{\mathrm{int}(f(a))} \subseteq \overline{\mathrm{int}(E)}$, hence $f(a) \subseteq E^r = \mathcal{M}(E,\mathcal{F})(t_0)^r$, so that $f(b) \subseteq \phi(b)$.

In addition $\mathcal{M}(E^r,\mathcal{F}) \subseteq \mathcal{M}(E,\mathcal{F})$, and the opposite inclusion follows by observing that the map $\psi : I \to \mathcal{P}(\mathbf{R}^n)$ defined by

$$\psi(t) := \begin{cases} E & \text{if } t = t_0, \\ \mathcal{M}(E^r,\mathcal{F})(t) & \text{if } t > t_0 \end{cases}$$

belongs to $\mathcal{B}(\mathcal{F})$. □

Equality (3.33) does not hold in general for $\mathcal{M}^*(E,\mathcal{F})$; take for instance $n = 2$, $E := \{(x_1,x_2) \in \mathbf{R}^2 : x_2 = 0\}$ and F as in (3.7). Then $\mathcal{M}^*(E,\mathcal{F}_F)(t) = E$ for any $t \geq t_0$, while $\mathcal{M}^*(E,\mathcal{F}_F)(t)^r = \emptyset$ for any $t \geq t_0$.

Notice that, if $\mathcal{F}$ is outer regular and E has empty interior, then from (3.34) we deduce that $\mathcal{M}(E,\mathcal{F})(t) = \emptyset$ for any $t > t_0$.

The following lemma will be useful to prove a continuity property of the distance between minimal barriers.

Lemma 3.1. *Let $F_1, F_2 : J_0 \to \mathbf{R}$ be bounded below and let $\phi \in \mathcal{B}(\mathcal{F}_{F_1})$, $\psi \in \mathcal{B}(\mathcal{F}_{F_2})$. Set*

$$\eta(t) := dist\big(\mathbf{R}^n \setminus \phi(t), \mathbf{R}^n \setminus \psi(t)\big), \qquad t \in I. \tag{3.35}$$

Then

$$\eta(t_0) \le \liminf_{s \downarrow t_0} \eta(s), \qquad \limsup_{\sigma \uparrow t} \eta(\sigma) \le \eta(t) \le \liminf_{s \downarrow t} \eta(s), \qquad t > t_0. \tag{3.36}$$

Assume in addition that $\mathbf{R}^n \setminus \phi \in \mathcal{B}(\mathcal{F}_{F_3})$ and $\mathbf{R}^n \setminus \psi \in \mathcal{B}(\mathcal{F}_{F_4})$ for two suitable functions $F_3, F_4 : J_0 \to \mathbf{R}$ bounded below. If $\mathbf{R}^n \setminus \phi = (\mathbf{R}^n \setminus \phi)^r$ and $\mathbf{R}^n \setminus \psi = (\mathbf{R}^n \setminus \psi)^r$, then

$$\eta(t_0) = \lim_{s \downarrow t_0} \eta(s), \qquad \limsup_{\sigma \uparrow t} \eta(\sigma) \le \eta(t) = \lim_{s \downarrow t} \eta(s), \qquad t > t_0. \tag{3.37}$$

Proof. Given any $F : J_0 \to \mathbf{R}$ bounded below, there exists (see [15]) a strictly increasing function $\varrho_F : [0, +\infty[\to [0, +\infty[$, $\varrho_F \in \mathcal{C}([0, +\infty[) \cap \mathcal{C}^\infty(]0, +\infty[)$, $\varrho_F(0) = 0$, such that if we take any $t_0 \le a < b$, $\epsilon > 0$ and $x \in \mathbf{R}^n$, we have that the map $t \in [a, b] \to \overline{B_{\varrho_F(\epsilon + b - t_0 - t)}(x)}$ belongs to $\mathcal{F}_F^{>}$.

It follows that, if $\chi \in \mathcal{B}(\mathcal{F}_F)$ and $t \in I$, then

$$\big\{x \in \mathbf{R}^n : \mathrm{dist}\big(x, \mathbf{R}^n \setminus \chi(t)\big) > \varrho_F(s - t)\big\} \subseteq \mathrm{int}\big(\chi(s)\big), \qquad s > t. \tag{3.38}$$

Indeed, let $s > t$ and $x \in \chi(t)$ be such that $\mathrm{dist}(x, \mathbf{R}^n \setminus \chi(t)) > \overline{\varrho} > \varrho_F(s - t)$. Let us evolve the ball $\overline{B_{\overline{\varrho}}(x)}$ as explained above on $[t, s]$, and denote this evolution by $\sigma \in [t, s] \to \overline{B(\sigma)}$; since it belongs to $\mathcal{F}_F$, we have $x \in B(s) \subseteq \mathrm{int}(\chi(s))$ and (3.38) is proved.

Consequently, for any $s > t$ we have

$$\begin{aligned} &\mathbf{R}^n \setminus \mathrm{int}(\phi(s)) \subseteq \{x \in \mathbf{R}^n : \mathrm{dist}(x, \mathbf{R}^n \setminus \phi(t)) \le \varrho_{F_1}(s - t)\} \\ &\mathbf{R}^n \setminus \mathrm{int}(\psi(s)) \subseteq \{x \in \mathbf{R}^n : \mathrm{dist}(x, \mathbf{R}^n \setminus \psi(t)) \le \varrho_{F_2}(s - t)\}. \end{aligned} \tag{3.39}$$

For any $\epsilon > 0$ let $y \in \mathbf{R}^n \setminus \mathrm{int}(\phi(s))$, $z \in \mathbf{R}^n \setminus \mathrm{int}(\psi(s))$ be such that $|y - z| \le \eta(s) + \epsilon$. By (3.39) we have

$$\mathrm{dist}(y, \mathbf{R}^n \setminus \phi(t)) \le \varrho_{F_1}(s - t), \qquad \mathrm{dist}(z, \mathbf{R}^n \setminus \psi(t)) \le \varrho_{F_2}(s - t).$$

Using the triangular property of the distance and setting $\varrho := \varrho_{F_1} + \varrho_{F_2}$, we have

$$\eta(t) \le \eta(s) + \epsilon + \mathrm{dist}(y, \mathbf{R}^n \setminus \phi(t)) + \mathrm{dist}(z, \mathbf{R}^n \setminus \psi(t)) \le \eta(s) + \varrho(s - t) + \epsilon.$$

Letting $\epsilon \to 0^+$ we get $\eta(s) \ge \eta(t) - \varrho(s - t)$, which implies (3.36).

Let us now prove (3.37). Let $t \in I$, $\epsilon > 0$, $x_\epsilon \in \mathbf{R}^n \setminus \mathrm{int}(\phi(t))$, $y_\epsilon \in \mathbf{R}^n \setminus \mathrm{int}(\psi(t))$ be such that $|x_\epsilon - y_\epsilon| \le \eta(t) + \epsilon$. We can assume that $x_\epsilon \in \partial(\mathbf{R}^n \setminus \phi(t))$ and $y_\epsilon \in \partial(\mathbf{R}^n \setminus \psi(t))$. In particular, as $\mathbf{R}^n \setminus \phi = (\mathbf{R}^n \setminus \phi)^r$, $\mathbf{R}^n \setminus \psi = (\mathbf{R}^n \setminus \psi)^r$, we have $x_\epsilon \in \overline{\mathrm{int}(\mathbf{R}^n \setminus \phi(t))}$, $y_\epsilon \in \overline{\mathrm{int}(\mathbf{R}^n \setminus \psi(t))}$. Hence, for any $\varrho > 0$ the set

$B_\varrho(x_\epsilon) \cap \mathrm{int}(\mathbf{R}^n \setminus \phi(t))$ (resp. the set $B_\varrho(y_\epsilon) \cap \mathrm{int}(\mathbf{R}^n \setminus \psi(t))$) contains a closed ball D_1 (resp. D_2). As $\mathbf{R}^n \setminus \phi \in \mathcal{B}(\mathcal{F}_{F_3})$ and $\mathbf{R}^n \setminus \psi \in \mathcal{B}(\mathcal{F}_{F_4})$, we can find $s = s(\epsilon, x_\epsilon, y_\epsilon) > t$ so that a suitable evolution $D_1(\sigma)$ (resp. $D_2(\sigma)$) of D_1 (resp. of D_2) belongs to $\mathcal{F}_{F_3}$ (resp. to $\mathcal{F}_{F_4}$) and it is contained in $\mathbf{R}^n \setminus \phi(\sigma)$ (resp. in $\mathbf{R}^n \setminus \psi(\sigma)$) for any $\sigma \in [t, s]$.
Let $z_\sigma \in D_1(\sigma)$ and $w_\sigma \in D_2(\sigma)$. By the triangular property of η we have, for any $\sigma \in [t, s]$,

$$\eta(\sigma) \leq |z_\sigma - w_\sigma| \leq |z_\sigma - x_\epsilon| + \eta(t) + \epsilon + |y_\epsilon - w_\sigma| \leq \eta(t) + \epsilon + 2\varrho.$$

Letting $\varrho, \epsilon \to 0$, we have

$$\eta(t) \geq \limsup_{s \downarrow t} \eta(s) \qquad t \in I, \tag{3.40}$$

and (3.37) follows. $\square$

Corollary 3.1 (right continuity of the distance function between minimal barriers). *Assume that $F : J_0 \to \mathbf{R}$ is lower semicontinuous and $F^+ : J_0 \to \mathbf{R}$ is upper semicontinuous. Given $A, B \subseteq \mathbf{R}^n$, let $d : I \to \mathbf{R} \cup \{+\infty\}$ be the function defined as*

$$d(t) := dist\big(\mathcal{M}(A, \mathcal{F}_F)(t), \mathcal{M}(B, \mathcal{F}_F)(t)\big), \qquad t \in I.$$

Then d is right continuous on $]t_0, +\infty[$.

Proof. By Theorem 5.1 below (applied with $G = (F^+)_c$) we have

$$\mathbf{R}^n \setminus \mathcal{M}(E, \mathcal{F}_F) \in \mathcal{B}\big(\mathcal{F}_{(F^+)_c}\big)$$

for any $E \subseteq \mathbf{R}^n$ (see (5.6)).
Moreover, by Lemma 3.4 the minimal barrier is outer regular for $t > t_0$. We now apply Lemma 3.1 with $\phi = \mathbf{R}^n \setminus \mathcal{M}(A, \mathcal{F}_F)$, $\psi = \mathbf{R}^n \setminus \mathcal{M}(B, \mathcal{F}_F)$, $F_1 = F_2 = (F^+)_c$, $F_3 = F_4 = F$, and we get the thesis. $\square$

4. Barriers solutions, level set flow and comparison flows

In this section we study some properties of barriers, related to other generalized flows; in particular we state the comparison results between the minimal barrier and the level set flow proved in [15], and we show comparison results between minimal barriers and comparison flows.

Definition 4.1 (barrier solutions). *Given a function $F : J_1 \to \mathbf{R}$, we define the family $\mathcal{S}(F)$ of all barrier solutions of equation (1.4) as*

$$\mathcal{S}(F) := \mathcal{B}(\mathcal{F}_F) \cap \widetilde{\mathcal{B}}\big(\mathcal{F}_{\overline{F}}^{\leq}\big). \tag{4.1}$$

By Proposition 3.1, if F does not depend on x and is degenerate elliptic, then the barrier solutions coincide with the smooth evolutions, whenever the latter exist.

The next proposition shows that, under some monotonicity assumptions on F, there always exists a barrier solution of (1.4); this result is reminiscent of the proposition asserting the existence of viscosity solutions. Note that we do not still have an uniqueness result (see Theorem 4.5 below).

Proposition 4.1 (existence of barriers solutions). *Let $F : J_0 \to \mathbf{R}$ be degenerate elliptic and let $\phi, \tilde\phi : I \to \mathcal{P}(\mathbf{R}^n)$, $\phi \in \mathcal{B}(\mathcal{F}_F)$, $\tilde\phi \in \widetilde{\mathcal{B}}(\mathcal{F}_{\overline{F}}^{\leq})$ and $\tilde\phi \subseteq \phi$. Then there exists $\psi \in \mathcal{S}(F)$ such that $\tilde\phi \subseteq \psi \subseteq \phi$.*

Proof. Let

$$\psi := \bigcap \Big\{ \chi : I \to \mathcal{P}(\mathbf{R}^n),\ \chi \in \mathcal{B}(\mathcal{F}_F),\ \tilde\phi \subseteq \chi \Big\}.$$

Since $\psi \in \mathcal{B}(\mathcal{F}_F)$, we need only to show that $\psi \in \widetilde{\mathcal{B}}(\mathcal{F}_{\overline{F}}^{\leq})$. Assume by contradiction that there exists $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_{\overline{F}}^{\leq}$, such that $\psi(a) \subseteq \mathrm{int}(f(a))$ and $\psi(b)$ is not contained in $\mathrm{int}(f(b))$. Define

$$\psi_1(t) := \begin{cases} \psi(t), & t \in I \setminus [a,b], \\ \psi(t) \ \cap\ \mathrm{int}\big(f(t)\big), & t \in [a,b]. \end{cases}$$

As F is degenerate elliptic and independent of x, one can check that $\mathrm{int}(f) \in \mathcal{B}(\mathcal{F}_F, [a,b])$. Hence $\psi_1 \in \mathcal{B}(\mathcal{F}_F)$, so that $\psi_1 \supseteq \psi$. However $\psi_1(b)$ is strictly contained in $\psi(b)$, a contradiction. □

The following Proposition is reminiscent of the stability of viscosity subsolutions.

Proposition 4.2 (stability of barriers). *Let $F : J_0 \to \mathbf{R}$ be bounded below. Let $F_m : J_0 \to \mathbf{R}$ be such that $\liminf_{m\to+\infty} \inf_K \big(F_m - F\big) \geq 0$, for any compact set $K \subset J_0$. For any $m \in \mathbf{N}$, let $\phi_m \in \mathcal{B}\big(\mathcal{F}_{F_m}^{\geq}\big)$ and set $\phi := \bigcup_{h\in\mathbf{N}} \mathrm{int}\Big(\bigcap_{m\geq h} \phi_m\Big)$. Then $\phi \in \mathcal{B}(\mathcal{F}_{\overline{F}}^{\geq})$.*

Proof. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_{\overline{F}}^{\geq}$, $f(a) \subseteq \phi(a)$; we have to prove $f(b) \subseteq \phi(b)$. As we have already observed, we can assume that $f(t)$ is compact for any $t \in [a,b]$. As $f \in \mathcal{F}_{\overline{F}}^{\geq}$, there exists a constant $0 < c < +\infty$ such that

$$\frac{\partial d_f}{\partial t}(t,x) + F\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) \geq 2c, \qquad x \in \partial f(t),\ t \in \,]a,b[\ .$$

Set $K_f := \big\{\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) : x \in \partial f(t), t \in [a,b]\big\}$, and let $\overline{m} \in \mathbf{N}$ be such that $\inf_{K_f} \big(F_m - F\big) \geq -c$ for any $m \geq \overline{m}$. Then for any $x \in \partial f(t)$, $t \in \,]a,b[\ $, $m \geq \overline{m}$, we have

$$\begin{aligned} &\frac{\partial d_f}{\partial t}(t,x) + F_m\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) \\ &\geq \frac{\partial d_f}{\partial t}(t,x) + F\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) - c \geq c, \end{aligned}$$

which implies $f \in \mathcal{F}_{F_m}^{\geq}$ for any $m \geq \overline{m}$. Given $h \in \mathbf{N}$, we set $\psi_h := \mathrm{int}\Big(\bigcap_{m\geq h} \phi_m\Big)$. As $f(a) \subseteq \bigcup_h \psi_h(a)$ and $f(a)$ is compact, there exists $\overline{h}$ such that $f(a) \subseteq \psi_{\overline{h}}(a)$, which implies $f(a) \subseteq \big[\phi_m(a)\big]_\varrho^-$ for some $\varrho > 0$ and for any $m \geq \overline{h}$. Taking

$N \geq \overline{m} \vee \overline{h}$, we have $f \in \mathcal{F}^{>}_{F_m}$ and $f(a) \subseteq [\phi_m(a)]^-_\varrho$ for any $m \geq N$, therefore, as $\mathcal{F}^{>}_{F_m}$ is translation invariant $f(b) \subseteq [\phi_m(b)]^-_\varrho$, which implies

$$f(b) \subseteq \bigcap_{m \geq N} [\phi_m(b)]^-_\varrho \subseteq \Big[\bigcap_{m \geq N} \phi_m(b)\Big]^-_{\frac{\varrho}{2}} \subseteq \psi_N(b) \subseteq \phi(b).$$

This concludes the proof. □

4.1. Barriers and viscosity subsolutions. The following theorem is proved in [54, Theorem 4.9] (see the Appendix for the notation).

Theorem 4.1. *Assume that $F : J_1 \to \mathbf{R}$ satisfies either (F1)-(F4), (F8), or (F1), (F3), (F4), (F9), (F10). Let $v_0 : \mathbf{R}^n \to \mathbf{R}$ be a continuous function which is constant outside a bounded subset of $\mathbf{R}^n$. Then there exists a unique continuous viscosity solution (constant outside a bounded subset of $\mathbf{R}^n$) of (1.4) with $v(t_0, x) = v_0(x)$.*

Given a bounded open set $E \subseteq \mathbf{R}^n$ we define the viscosity evolutions $V(E)(t)$, $\Gamma(t)$ of $\mathrm{int}(E)$, ∂E respectively (the so-called level set flow) as

$$V(E)(t) := \{x \in \mathbf{R}^n : v(t,x) < 0\}, \qquad \Gamma(t) := \{x \in \mathbf{R}^n : v(t,x) = 0\}, \tag{4.2}$$

where v is as in Theorem 4.1 with $v_0(x) := (-1) \vee d_E(x) \wedge 1$.

The following results, proved in [15], show the connection between the minimal barriers and the viscosity solutions (notice that it applies, in particular, to the case of motion by mean curvature in arbitrary codimension).

Theorem 4.2 (comparison between barriers and level set flow). *Assume that $F : J_1 \to \mathbf{R}$ satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Let $E \subseteq \mathbf{R}^n$ be a bounded set. Then for any $t \in I$ we have*

$$\mathcal{M}_*(E, \mathcal{F}^{>}_F)(t) = \mathcal{M}_*(E, \mathcal{F}_F)(t) = V(E)(t), \tag{4.3}$$

$$\mathcal{M}^*(E, \mathcal{F}^{>}_F)(t) = \mathcal{M}^*(E, \mathcal{F}_F)(t) = V(E)(t) \cup \Gamma(t). \tag{4.4}$$

In particular $\mathcal{M}_{v_0, \mathcal{F}_F} = v$.

The difficult part of the proof of Theorem 4.2 relies in showing that, given a bounded open set $A \subseteq \mathbf{R}^n$, there holds $\mathcal{M}(A, \mathcal{F}^{>}_F) \supseteq V(A)$. To prove this, the idea is to show that the function χ, defined by

$$\chi(t,x) := -\chi_{\mathcal{M}(A, \mathcal{F}^{>}_F)(t)}(x), \qquad (t,x) \in I \times \mathbf{R}^n,$$

(where $\chi_C(x) := 1$ if $x \in C$ and $\chi_C(x) = 0$ if $x \notin C$) is a viscosity subsolution of (1.4) in $]t_0, +\infty[\times \mathbf{R}^n$. The use of characteristic functions is needed because of the explicit dependence on x of the function F; when F does not depend on x, one can equivalently reason by using the distance function.

Theorem 4.2 in the case of driven motion by mean curvature of hypersurfaces has been proved in [19], where the minimal barriers are compared with any generalized evolution of sets satisfying the semigroup property, the comparison principle, and the extension of smooth evolutions (see Corollary 4.1 below). Notice that, as a consequence of Theorem 4.2, it follows (under the same assumptions on F) that $\mathcal{M}_*(E,\mathcal{F}_F)$ and $\mathcal{M}^*(E,\mathcal{F}_F)$ verify the semigroup property.

The proof of Theorem 4.2 is based on the facts that the sublevel sets of a viscosity subsolution of (1.4) are barriers and, conversely, that a function whose sublevel sets are barriers is a viscosity subsolution of (1.4). Using these observations, one can show that the minimal barrier selects the maximal viscosity subsolution [15].

Theorem 4.3 (barriers and viscosity subsolutions). *Assume that $F : J_1 \to \mathbf{R}$ satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a given upper semicontinuous function. Define*

$$S_{u_0} := \{v\colon v \textit{ is a viscosity subsolution of } (1.4) \textit{ in }]t_0,+\infty[\times\mathbf{R}^n, v^*(t_0,x) = u_0(x)\}.$$

Then

$$\mathcal{M}_{u_0,\mathcal{F}_F} = \mathcal{M}_{u_0,\mathcal{F}_F^{\geq}} = \sup\{v : v \in S_{u_0}\}. \tag{4.5}$$

Remark 4.1. *A similar assertion of Corollary 4.3 can be given for supersolutions, see [15].*

The following results characterizes the complement of regularized barriers, and does not cover the case where E is unbounded and $\mathbf{R}^n \setminus E$ is unbounded.

Theorem 4.4 (complement set characterization). *Assume that $F : J_1 \to \mathbf{R}$ satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Then, for any bounded set $E \subseteq \mathbf{R}^n$ we have*

$$\begin{aligned} \mathcal{M}_*(E,\mathcal{F}_F) &= \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}),\\ \mathcal{M}^*(E,\mathcal{F}_F) &= \mathbf{R}^n \setminus \mathcal{M}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}). \end{aligned} \tag{4.6}$$

Moreover, if $F = F_c$ then

$$\mathcal{M}^*(E,\mathcal{F}_F) \setminus \mathcal{M}_*(E,\mathcal{F}_F) \in \mathcal{B}(\mathcal{F}_F). \tag{4.7}$$

Concerning the connections between the minimal barriers and the viscosity evolutions without growth conditions on F (see [68]) and for unbounded sets E with unbounded complement, there holds the following result. Assume that $F : J_0 \to \mathbf{R}$ is continuous and degenerate elliptic. Given any $E \subseteq \mathbf{R}^n$, let $v : I \times \mathbf{R}^n \to \mathbf{R}$ be the unique continuous viscosity solution of (1.4), in the sense of [68], with $v(t_0,x) = d_E(x)$. Then, for any $t \in I$, (4.3), (4.4), (4.6) and (4.7) hold. In particular $\mathcal{M}^*(E,\mathcal{F}_F)(t) \setminus \mathcal{M}_*(E,\mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) = 0\}$ and $\mathcal{M}_{d_E,\mathcal{F}_F} = v$.

The next theorem shows the connection between the minimal barrier and the maximal inner barrier. This property is reminiscent of the uniqueness theorem for viscosity solutions.

Theorem 4.5. *Assume that $F : J_1 \to \mathbf{R}$ satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Then, for any bounded set $E \subseteq \mathbf{R}^n$ we have*

$$\mathcal{N}_*(E,\mathcal{F}_F) = \mathcal{M}_*(E,\mathcal{F}_F), \qquad \mathcal{N}^*(E,\mathcal{F}_F) = \mathcal{M}^*(E,\mathcal{F}_F). \tag{4.8}$$

Moreover, if $F : J_0 \to \mathbf{R}$ is continuous and degenerate elliptic, then (4.8) holds for any $E \subseteq \mathbf{R}^n$.

We remark that, to prove Theorem 4.5 in [16], we need to pass through the viscosity theory, so we miss a self-contained proof based only on barriers.

4.2. Comparison flows. In this subsection we generalize the comparison results discussed above; indeed, we compare the minimal barrier with an abstract comparison flow, which is defined as follows.

Definition 4.2. *Let $F : J_1 \to \mathbf{R}$ be a given function. Let $\mathcal{Q}$ be a family of sets containing the open and the close subsets of $\mathbf{R}^n$. We say that a map $\mathcal{R}$ is a comparison flow for (1.4) if and only if, for any $E \in \mathcal{Q}$ and $\bar{t} \in I$, $\mathcal{R} = \mathcal{R}(E, \bar{t})$ maps $[\bar{t}, +\infty[$ into $\mathcal{Q}$, $\mathcal{R}(E,\bar{t})(\bar{t}) = E$, and the following properties hold:*

(i) (semigroup property) for any $E \in \mathcal{Q}$ we have

$$\mathcal{R}(E, t_1)(t) = \mathcal{R}\big(\mathcal{R}(E, t_1)(t_2), t_2\big)(t), \qquad t_1 \le t_2 \le t;$$

(ii) (relaxation of the elements of $\mathcal{F}_F$ and $\mathcal{F}_F^{\lessgtr}$) for any $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $g : [c,d] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $g \in \mathcal{F}_F^{\lessgtr}$, we have

$$f(t) \subseteq \mathcal{R}(f(a), a)(t), \qquad t \in [a,b],$$

$$int\big(g(t)\big) \supseteq \mathcal{R}\big(int\big(g(c)\big), c\big)(t), \qquad t \in [c,d];$$

(iii) (comparison principle) for any $A, B \in \mathcal{Q}$, $A \subseteq B$, and any $\bar{t} \in I$ we have

$$\mathcal{R}(A, \bar{t})(t) \subseteq \mathcal{R}(B, \bar{t})(t), \qquad t \ge \bar{t}.$$

If $\bar{t} = t_0$ we simply write $\mathcal{R}(E)$ instead of $\mathcal{R}(E, t_0)$; moreover, we define the lower and upper regularizations of $\mathcal{R}$ as

$$\mathcal{R}_*(E, \bar{t}) := \bigcup_{\varrho>0} \mathcal{R}(E_\varrho^-, \bar{t}), \qquad \mathcal{R}^*(E, \bar{t}) := \bigcap_{\varrho>0} \mathcal{R}(E_\varrho^+, \bar{t});$$

and we note that they are defined on the whole of $\mathcal{P}(\mathbf{R}^n)$.

Lemma 4.1. *Let $\mathcal{R}$ be a comparison flow and let $E \in \mathcal{Q}$. Then $\mathcal{R}(E) \in \mathcal{S}(F)$ (see (4.1)), which implies*

$$\mathcal{M}(E, \mathcal{F}_F) \subseteq \mathcal{R}(E) \subseteq \mathcal{N}\big(E, \mathcal{F}_F^{\lessgtr}\big). \tag{4.9}$$

Proof. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $f(a) \subseteq \mathcal{R}(E)(a)$. By property (ii) of Definition 4.2, we have $f(t) \subseteq \mathcal{R}(f(a), a)(t)$ for any $t \in [a,b]$. Therefore, using properties (iii) and (i) we get

$$f(b) \subseteq \mathcal{R}(f(a), a)(b) \subseteq \mathcal{R}\big(\mathcal{R}(E)(a), a\big)(b) = \mathcal{R}(E)(b).$$

Hence $\mathcal{R}(E) \in \mathcal{B}(\mathcal{F}_F)$. Reasoning in a similar way, one can check that $\mathcal{R}(E) \in \widetilde{\mathcal{B}}\big(\mathcal{F}_F^{\lessgtr}\big)$, and the thesis follows. $\square$

We are now in a position to prove that regularized minimal barriers are essentially the *only* regularized comparison flows for (1.4).

Corollary 4.1. *Assume that the function $F : J_1 \to \mathbf{R}$ satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Then, for any bounded set $E \subseteq \mathbf{R}^n$, we have*

$$\mathcal{M}_*(E, \mathcal{F}_F) = \mathcal{R}_*(E), \qquad \mathcal{M}^*(E, \mathcal{F}_F) = \mathcal{R}^*(E). \tag{4.10}$$

Moreover, if $F : J_0 \to \mathbf{R}$ is continuous and degenerate elliptic, then (4.10) holds for any $E \subseteq \mathbf{R}^n$.

Proof. The assertions follow from (4.9) and Theorem 4.5. $\square$

Problem. Implement the barrier method for geometric evolutions on an open set Ω, with suitable boundary conditions.

5. The disjoint and the joint sets properties

In this section we recall the notions of disjoint sets property and joint sets property. These properties have been introduced in [16], and we refer to that paper for all proofs omitted here.

We remark that these properties are close to uniqueness of barrier solutions and are related to the fattening phenomenon.

Definition 5.1. *We say that the disjoint sets property (resp. the regularized disjoint sets property) with respect to* $(\mathcal{F}, \mathcal{G})$ *holds if for any* $E_1, E_2 \subseteq \mathbf{R}^n$ *and* $\bar{t} \in I$

$$E_1 \cap E_2 = \emptyset \;\Rightarrow\; \mathcal{M}(E_1, \mathcal{F}, \bar{t}) \cap \mathcal{M}(E_2, \mathcal{G}, \bar{t}) = \emptyset \tag{5.1}$$

$$\big(\textit{resp.}\quad E_1 \cap E_2 = \emptyset \;\Rightarrow\; \mathcal{M}_*(E_1, \mathcal{F}, \bar{t}) \cap \mathcal{M}^*(E_2, \mathcal{G}, \bar{t}) = \emptyset\big). \tag{5.2}$$

We say that the joint sets property (resp. the regularized joint sets property) with respect to $(\mathcal{F}, \mathcal{G})$ *holds if for any* $E_1, E_2 \subseteq \mathbf{R}^n$ *and* $\bar{t} \in I$

$$E_1 \cup E_2 = \mathbf{R}^n \;\Rightarrow\; \mathcal{M}(E_1, \mathcal{F}, \bar{t}) \cup \mathcal{M}(E_2, \mathcal{G}, \bar{t}) = \mathbf{R}^n, \tag{5.3}$$

$$\big(\textit{resp.}\quad E_1 \cup E_2 = \mathbf{R}^n \;\Rightarrow\; \mathcal{M}_*(E_1, \mathcal{F}, \bar{t}) \cup \mathcal{M}^*(E_2, \mathcal{G}, \bar{t}) = \mathbf{R}^n\big). \tag{5.4}$$

Example 5.1. As proved in [19] (see also [15]), motion by mean curvature enjoys both the regularized disjoint sets property and the regularized joint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$. Notice that in this case $F = F^+$ is odd.

Example 5.2. Let

$$F(t, x, p, X) := -\mathrm{tr}(P_p X P_p) + g(t, x)|p| \tag{5.5}$$

(i.e., motion by mean curvature with a forcing term g). Then, in general the disjoint sets property and the regularized disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$ fail, compare Example 6.1. Notice that, in this case, $F = F^+$ and F^+ is not odd.

Example 5.3. Let $n = 2$, $E = \{x = (x_1, x_2) \in \mathbf{R}^2 : |x_1| \le 1, |x_2| \le 1\}$ and let F be as in (3.7). Then, recalling Example 3.2, (3.11) and (3.9) we have, for $t > \bar{t}$,

$$\mathcal{M}(E, \mathcal{F}_F, \bar{t})(t) = \mathcal{M}(\mathrm{int}(E), \mathcal{F}_F, \bar{t})(t) \in \mathcal{A}(\mathbf{R}^n),$$
$$\mathcal{M}(\mathbf{R}^n \setminus E, \mathcal{F}_F, \bar{t})(t) \in \mathcal{A}(\mathbf{R}^n),$$

and the joint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$ does not hold ("we instantly loose ∂E").

Notice that if (5.2) holds then $\mathcal{M}^*(E_1, \mathcal{F}, \bar{t}) \cap \mathcal{M}_*(E_2, \mathcal{G}, \bar{t}) = \emptyset$, and conversely. Similarly, if (5.4) holds then $\mathcal{M}^*(E_1, \mathcal{F}, \bar{t}) \cup \mathcal{M}_*(E_2, \mathcal{G}, \bar{t}) = \mathbf{R}^n$, and conversely.

Notice also that, if $\mathcal{F}$ satisfies the (3.2), then the disjoint sets property with respect to $(\mathcal{F}, \mathcal{G})$ is equivalent to the assertion

$$\text{for any } E \subseteq \mathbf{R}^n \text{ there holds } \mathbf{R}^n \setminus \mathcal{M}(E, \mathcal{F}, \bar{t}) \in \mathcal{B}(\mathcal{G}, [\bar{t}, +\infty[); \tag{5.6}$$

also the disjoint (resp. joint) sets property with respect to $(\mathcal{F}, \mathcal{G})$ implies the regularized disjoint (resp. joint) sets property with respect to $(\mathcal{F}, \mathcal{G})$.

The following theorems characterize the disjoint and joint sets property in terms of the functions F, G describing the evolution.

Theorem 5.1 (characterization of the disjoint sets property). *Assume that $F, G : J_0 \to \mathbf{R}$ are lower semicontinuous. Then the disjoint sets property (equivalently, the regularized disjoint sets property) with respect to $(\mathcal{F}_F, \mathcal{F}_G)$ holds if and only if $(F^+)_c \geq G^+$. In particular*

(i) if $F_c = G$, then the disjoint sets property with respect to $\left(\mathcal{F}_F, \mathcal{F}_{F_c}\right)$ holds if and only if F is degenerate elliptic;

(ii) if $F = G$ then the disjoint sets property with respect to $\left(\mathcal{F}_F, \mathcal{F}_F\right)$ holds if and only if F is compatible from above.

The disjoint sets property, under suitable assumptions on an abstract family $\mathcal{F}$, can be restated by means of the distance function.

Proposition 5.1. *Assume that $\mathcal{F}$ is translation invariant and satisfies (3.2). Let $E \subseteq \mathbf{R}^n$ and $\phi \in \mathcal{B}(\mathcal{F})$. Then the function $\eta : I \to [0, +\infty]$ defined by*

$$\eta(t) := dist\big(\mathcal{M}(E, \mathcal{F})(t), \mathbf{R}^n \setminus \phi(t)\big)$$

is nondecreasing.

Moreover, assume that $F : J_0 \to \mathbf{R}$ is lower semicontinuous and compatible from above. Let $A, B \subseteq \mathbf{R}^n$. Then the function

$$t \in I \to dist\big(\mathcal{M}(A, \mathcal{F}_F)(t), \mathcal{M}(B, \mathcal{F}_F)(t)\big) \tag{5.7}$$

is nondecreasing.

Proof. Let $t_2 > t_1 \geq t_0$; we have to prove that $\eta(t_2) \geq \eta(t_1)$. We can assume that $\eta(t_1) = \delta > 0$. Notice that for any $B \subseteq \mathbf{R}^n$

$$B \subseteq \phi(t_1) \Rightarrow \mathcal{M}(B, \mathcal{F}, t_1)(t_2) \subseteq \phi(t_2). \tag{5.8}$$

By (3.3), (3.5) and (5.8) we have

$$\begin{aligned} \big(\mathcal{M}(E, \mathcal{F})(t_2)\big)^+_\delta &= \big(\mathcal{M}\big(\mathcal{M}(E, \mathcal{F})(t_1), \mathcal{F}, t_1\big)(t_2)\big)^+_\delta \\ &\subseteq \mathcal{M}\big(\big(\mathcal{M}(E, \mathcal{F})(t_1)\big)^+_\delta, \mathcal{F}, t_1\big)(t_2) \subseteq \phi(t_2), \end{aligned}$$

which proves the monotonicity of η.

Let us prove (5.7). Setting $\phi := \mathbf{R}^n \setminus \mathcal{M}(B, \mathcal{F}_F)$, we have $\phi \in \mathcal{B}(\mathcal{F}_F)$ by Theorem (5.1), hence (5.7) follows from the previous assertion. $\square$

Theorem 5.2 (characterization of the regularized joint sets property). *Assume that $F, G : J_0 \to \mathbf{R}$ are continuous, $F^+ < +\infty$, $G^+ < +\infty$ and F^+, G^+ are continuous. Then the regularized joint sets property with respect to $\left(\mathcal{F}_F, \mathcal{F}_G\right)$ holds if and only if $(F^+)_c \leq G^+$. In particular*

(i) if $F_c = G$, then the regularized joint sets property with respect to $\left(\mathcal{F}_F, \mathcal{F}_{F_c}\right)$ holds for any function F satisfying the hypotheses;

(ii) if $F = G$ then the regularized joint sets property with respect to $\left(\mathcal{F}_F, \mathcal{F}_F\right)$ holds if and only if F^+ is compatible from below.

We remark that, to prove Theorem 5.2 in [16], we need to pass through the viscosity theory, so we miss a self-contained proof based only on barriers.

Remark 5.1. *Assume that $F : J_0 \to \mathbf{R}$ is continuous, odd and degenerate elliptic. Then for any $E \subseteq \mathbf{R}^n$ we have*

$$\mathcal{M}^*(\partial E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{F}_F) \setminus \mathcal{M}_*(E, \mathcal{F}_F). \tag{5.9}$$

In [64] Ilmanen introduced a notion of weak evolution, for motion by mean curvature of hypersurfaces, called set-theoretic subsolution, which essentially coincides with $\mathbf{R}^n \setminus \mathcal{M}_*(\mathbf{R}^n \setminus \partial E, \mathcal{F}_F)$, F as in (3.7); hence, thanks to (4.6) of Theorem 4.2 and the fact that F is *odd*, the set-theoretic subsolution of Ilmanen is $\mathcal{M}^*(\partial E, \mathcal{F}_F)$ (or, more generally, $\mathcal{M}^*(E, \mathcal{F}_F)$, E any given closed set which is not necessarily a boundary). In his paper Ilmanen proved that the set-theoretic subsolution coincides with the level set flow, which is consistent with (5.9), (4.3) and (4.4), for F as in (3.7). Also, a comparison result between barriers and the level set flow for sets E with compact boundary when F is as in (5.5) has been proved in [19]; notice that in this case F is no more odd. The results of [19,64] are based on Ilmanen's interposition lemma and on Huisken's estimates [60] of the existence time for the evolution of a smooth compact hypersurface in dependence on the L^∞ norm of its second fundamental form, without requiring bounds on further derivatives of the curvatures. The above results of Ilmanen and Huisken apply basically to the case of motion by mean curvature; it seems difficult to recover the time estimates of [60] for a general evolution law of the form (1.4). This is the main reason for which the proof of Theorem 4.2 follows a completely different approach.

Solving the next problem (which asks, basically, which conditions we need to impose on a smooth elliptic function F, in order to let evolve $\mathcal{C}^{1,1}$ compact hypersurfaces) would allow, following the arguments of Ilmanen in [64], to give an alternative proof (with respect to [15]) of the comparison results between barriers and level set flows, for a class of evolutions including driven motion by mean curvature.

Problem. Assume that F does not depend on x, it is smooth and uniformly elliptic. Let $\{E_\epsilon\}$ be a sequence of sets so that there exists an open bounded set $A \subseteq \mathbf{R}^n$ such that $\partial E_\epsilon \subseteq A$ and $d_{E_\epsilon} \in \mathcal{C}^\infty(A)$ for any ϵ, and $\sup_\epsilon \sup_{x \in \partial E_\epsilon} |\nabla^2 d_{E_\epsilon}(x)| < +\infty$. Let $E_\epsilon(t)$ be the unique smooth evolution [52] of the set E_ϵ under (1.4) for small times $t \in [t_0, t_0 + \tau_\epsilon[$, that is

$$\frac{\partial d_{E_\epsilon(t)}}{\partial t} + F\big(t, \nabla d_{E_\epsilon(t)}, \nabla^2 d_{E_\epsilon(t)}\big) = 0 \qquad \text{on } \partial E_\epsilon(t),\ t \in [t_0, t_0 + \tau_\epsilon[\ .$$

Which further conditions on F are needed in such a way that τ_ϵ can be chosen *independently* of ϵ?

6. The fattening phenomenon

In this section we discuss some aspects and examples concerning the fattening phenomenon, which is considered an interesting kind of singularity in geometric evolutions.

Definition 6.1. *Let $E \subseteq \mathbf{R}^n$. We say that the set E develops m-dimensional fattening with respect to $\mathcal{F}$ at time $t \in I$ if*

$$\mathcal{H}^m\Big(\mathcal{M}^*(E, \mathcal{F})(t) \setminus \mathcal{M}_*(E, \mathcal{F})(t)\Big) > 0, \tag{6.1}$$

where $m \in]0, n]$.

Concerning motion by mean curvature of manifolds of arbitrary codimension, F has the expression in (3.8), and $\mathcal{M}_*(E, \mathcal{F}_F)$ is usually empty; Definition 6.1 then reduces to

$$\mathcal{H}^m\big(\mathcal{M}^*(E, \mathcal{F}_F)(t)\big) > 0. \tag{6.2}$$

Unless otherwise specified, throughout this section we will consider n-dimensional fattening, i.e., $m = n$.

Fattening was defined [13,45] by means of the viscosity solution as follows. Let v be the unique viscosity solution of (1.4) with $v(t_0, x) = (-1) \vee d_E(x) \wedge 1$, see Theorem 4.1, then the fattening occurs if $\{x : v(t, x) = 0\}$ has nonempty interior part.

Equalities (4.3) and (4.4) shows that definition (6.1) is consistent with the definition of fattening given by means of the (unique) viscosity solution, see [45], [19]. œ[ANotice that, adopting (6.2), fattening can be defined also in case of nonuniqueness of viscosity solutions. We use the Hausdorff measure to be slightly more general, even if we do not know meaningfull examples of fattening of intermediate dimension. In this respect, we point out the following problem.

Problem. If the initial set E is not contained in a hyperplane of $\mathbf{R}^n$, can we say that, if fattening occurs, then it is n-dimensional?

Given a function F, the main issue could be to characterize those subsets E of $\mathbf{R}^n$ which fatten under (1.4); the complete characterization is clearly a difficult problem, which is still open even for motion by mean curvature.

6.1. Fattening in two dimensions. Examples of fattening in two dimensions for pure curvature flow can be given in the following two cases:

(i) if the initial set E is not required to be smooth (Evans and Spruck [45] provided the example of the inside of the figure eight curve; similar arguments hold when the boundary of the original set is the union of two crossing straight lines in $\mathbf{R}^2$);

(ii) if the boundary ∂E is not required to be compact, see Example 6.3, which is due to Ilmanen.

On the other hand, if $E \subseteq \mathbf{R}^2$ has compact smooth boundary, fattening does not take place under motion by curvature, as a consequence of a theorem of Grayson [56]. However, if one modifies the evolution law, for instance by adding a forcing term, the situation is completely different. Barles-Soner-Souganidis [13] have given an example in two dimensions of fattening for motion by curvature with a time-dependent forcing term (see [79] for numerical evidence). Even more, one can choose the forcing term to be constant, as the following example proposed in [17] shows.

Example 6.1. Assume that we can exhibit a smooth bounded Lipschitz function $g : I \times \mathbf{R}^n \to \mathbf{R}$ and an initial smooth compact set $E \subseteq \mathbf{R}^2$, with

$$E := L \cup R, \qquad \overline{L} \cap \overline{R} = \emptyset,$$

where L and R are homeomorphic to a ball, with the following properties: if we denote by $L(t)$ (resp. $R(t)$) the evolution of $L = L(t_0)$ (resp. of $R = R(t_0)$) under the law (1.4) with the choice of F as in (5.5), then there exists $t^* > t_0$ such that:

(i) $\overline{L(t)}$ and $R(t)$ are smooth for $t \in [t_0, t^* + \delta]$, for some $\delta > 0$;
(ii) $\overline{L(t)} \cap \overline{R(t)} = \emptyset$ for any $t \in [t_0, t^*[$;
(iii) $\partial L(t^*) \cap \partial R(t^*) = \{x^*\}$;
(iv) $\partial L(t^*)$ and $\partial R(t^*)$ meet at x^* with zero relative velocity;
(v) recalling that we are considering the evolution of L and R as *independent*, $L(t)$ and $R(t)$ would smoothly "bounce back" after the collision.

Then, under the previous assumptions, fattening takes place. Notice that F is not odd and that L and R violate the disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$. We remark that one can rearrange things in such a way that g and E can be chosen as follows:

$$g \equiv 1, \qquad L := B_{r_1}(z), \quad R := B_{r_2}(w), \tag{6.3}$$

for suitable $r_1, r_2 > 0$ and $z, w \in \mathbf{R}^2$, with $r_1 + r_2 < |z - w|$.

Consider the example in the case (6.3). The heuristic idea is the following. Given a small $\varrho > 0$, the set E_ϱ^- consists of two disjoint balls which, by comparison arguments, flow smoothly remaining disjoint in $[t_0, t^*]$. Moreover, the construction is such that they flow smoothly remaining distant, independently of ϱ, after some time bigger than t^*.

On the other hand, given any small $\varrho > 0$, the evolving set starting from E_ϱ^+ becomes connected and has the shape of a "bean". The main point is to prove the following assertion: there exist a time interval $[\alpha, \beta] \subset]t^*, +\infty[$ and an open set A, *independent of* $\varrho > 0$, such that

$$A \subset \mathcal{M}\big(E_\varrho^+, \mathcal{F}_F\big)(t) \qquad \text{for all } \varrho \text{ sufficiently small and all } t \in [\alpha, \beta].$$

Notice that α is strictly larger than t^*, since the fat region increases "continuously" in time after t^*. Notice also that, being the curvature very high near the collision point, we can, heuristically, drop out the forcing term (which is bounded) in the evolution. The crucial tool to prove the above assertions are the comparison principle and a Sturmian theorem of Angenent (see [9, Theorem 3.2]) which estimates the number of intersections of two curves flowing independently by curvature (without forcing term), which reads as follows.

Theorem 6.1 (theorem on intersection points). *Let be given two families of smooth curves which evolve (independently) by their curvature for $t \in [t_0, t_0 + T]$ of which at least one is compact. Then for any $t \in]t_0, t_0 + T]$ the number of intersections of the two curves at time t is finite, and this number does not increase with time; moreover, it decreases whenever the two curves are not tranverse.*

Problem. Is it true that fattening is "generic", for instance with respect to E (or with respect to g) in Example 6.1?

Problem. Let us consider F as in (5.5), choose g and E as in (6.3), and let $\Lambda(t)$ be the Almgren-Taylor-Wang [1] evolution starting from E, constructed iteratively by minimizing the energy functional

$$P(B) + \frac{1}{\tau} \int_{(E \setminus B) \cup (B \setminus E)} \mathrm{dist}(x, \partial E)\, dx - |B|, \tag{6.4}$$

where $\tau > 0$ is the time step, $B \subseteq \mathbf{R}^2$, and $P(B)$ and $|B|$ are the perimeter and the Lebesgue measure of B, respectively. Is it true that, after the time collision t^*, there holds $\Lambda(t) = \partial M^*(E, \mathcal{F}_F)(t)$? Moreover, replace in (6.4) the quantity $|B|$ with $(1-\epsilon(\tau))|B|$; is it possible to choose $\epsilon(\tau)$ decreasing to zero as $\tau \to 0$ in such a way that the corresponding evolution in the sense of Almgren-Taylor-Wang coincides, just after the collision time, with $\partial \mathcal{M}_*(E, \mathcal{F}_F)(t)$?

Problems of this type, in a different context, have been considered by Gobbino in [55].

Problem. Let u_ϵ be the solution of the reaction-diffusion equation $\frac{\partial u_\epsilon}{\partial t} = \Delta u_\epsilon - \frac{1}{\epsilon^2}(u_\epsilon^3 - u_\epsilon) + \frac{\pi}{4\epsilon}$, assuming an admissible initial datum which depends on ϵ and approximates, as $\epsilon \to 0$, the characteristic function of the set $E := L \cup R$ (L and R as in (6.3)). How the resulting evolution (see (3.29)) obtained as the limit of u_ϵ as $\epsilon \to 0$, or as the limit of some of its subsequences, depends on the choice of the initial datum u_ϵ^0? Same questions if we replace the reaction-diffusion equation with the nonlocal reaction-diffusion equation studied in [36].

The following example shows that fattening can occur in two dimensions if the function $F(t, x, p, \cdot)$ is not Lipschitz (the dependence on (t, x) is irrelevant here).

Example 6.2. Let $n = 2$, $\zeta : \mathbf{R} \to \mathbf{R}$ be defined as

$$\zeta(s) := \begin{cases} 0 & \text{if } s \in [0,1], \\ \sqrt{1 - s^{-1}} & \text{if } s > 1, \\ -\zeta(-s) & \text{if } s \leq 0, \end{cases}$$

let $F(p, X) := -\zeta\big(\mathrm{tr}(P_p X P_p)\big)$, and $E := \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \leq 1\}$. For any $\varrho \in\,]0, 1[$ the set E_ϱ^+ stands still, while E_ϱ^- shrinks to a point at finite time $T_\varrho \leq 2$; hence E develops fattening.

The following example is a particular case of an example due to Ilmanen [63], and concerns the case of motion by curvature of an initial smooth non compact set with non compact complement.

Example 6.3. Let $n = 2$, F be as in (3.7), and

$$v_0(x_1, x_2) := x_2^2(1 + x_1^2)^2.$$

For any $\lambda > 0$ the set $E_\lambda := \{v_0 \leq \lambda\}$ is smooth, non compact, has non compact boundary, and has finite Lebesgue measure. It turns out that E_λ develops fattening instantly. Intuitively, since in two dimensions the shrinking time of a connected closed smooth bounded curve flowing by curvature depends on the enclosed area and since E_λ has finite Lebesgue measure, the set $\mathcal{M}_*(E_\lambda, \mathcal{F}_F)(t)$ becomes bounded for times arbitrarily close to t_0 (note that, for any $\varrho > 0$, $(E_\lambda)_\varrho^-$ is bounded). On the other hand, for any $\varrho > 0$, the boundary of each set $(E_\lambda)_\varrho^+$ is composed by two entire graphs, that smoothly evolve by curvature remaining graphs for all times [42].

Clearly v_0 is not uniformly continuous and Ilmanen proved nonuniqueness of continuous viscosity solutions of (1.4) with $v(t_0, x) = v_0(x)$; we point out that Ilmanen selected a special viscosity solution for this problem, see [63, Definition

7.1]. Notice that $\mathcal{M}_{v_0,\mathcal{F}_F}$ is, by Corollary 4.3, the maximal viscosity (sub) solution. One can check, following [63], that there exist $t \in I$ and $x \in \mathbf{R}^n$ such that $\mathcal{M}_{v_0,\mathcal{F}_F}(t,x) > -\mathcal{M}_{-v_0,\mathcal{F}_F}(t,x)$, where $-\mathcal{M}_{-v_0,\mathcal{F}_F}$ represents the minimal viscosity (super) solution. We conclude the discussion of this example with two further observations. Assume that we are interested in the evolution of a special $E_{\overline{\lambda}}$: then, if we choose $v(t_0,x) := (-1) \vee d_{E_{\overline{\lambda}}}(x) \wedge 1$ as (Lipschitz continuous) initial datum, equation (1.4) has a unique viscosity solution; nevertheless, $E_{\overline{\lambda}}$ develops fattening. Finally, we remark that in this case $\mathcal{M}^*(E_\lambda, \mathcal{F}_F)$ does not coincide with $\bigcap\{\mathcal{M}(A,\mathcal{F}_F) : A \in \mathcal{A}(\mathbf{R}^n), A \supseteq E_\lambda\}$.

To conclude this subsection, we recall that, as a consequence of a theorem of Angenent (see [9, Theorem 8.1] for a precise statement) it results that if $V = V(\tau,\kappa)$ is an odd function (τ the unit tangent vector, κ the curvature) of class $\mathcal{C}^{2,1}$ such that $\lambda^{-1} \leq \frac{\partial V}{\partial \kappa} \leq \lambda$, $|V(\tau,0)| \leq \mu$, $|\kappa||\frac{\partial V}{\partial \tau}| \leq \nu(1+\kappa^2)$ for two positive constants λ, ν, then, setting $F(p,X) := -|p| V\left(\frac{p^\perp}{|p^\perp|}, \frac{\mathrm{tr}(P_p X P_p)}{|p|}\right)$, any smooth compact set E evolving by (1.2) does not develop fattening.

6.2. The n-dimensional case. In $n \geq 3$ dimensions the situation is much more complicated than in two dimensions. First of all, as a consequence of a result of Huisken [60], a smooth bounded strictly convex set $E \subseteq \mathbf{R}^n$ flowing by mean curvature does not develop fattening, and the same holds if the set is bounded and convex [46].

A few years ago De Giorgi [29], [28, Conjecture 11] conjectured that a torus of the form

$$\left\{(x_1,\dots,x_n) \in \mathbf{R}^n : \left(\left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} - 1\right)^2 + x_n^2 \leq \lambda\right\} \tag{6.5}$$

flowing by mean curvature should develop fattening at finite time, for a suitable choice of the parameter λ. The conjecture was disproved by Soner-Souganidis in [85], who showed in particular the result below (confirmed by the numerical simulations of Paolini-Verdi in [81]). We refer also to the paper [3] of Altschuler-Angenent-Giga, where singularities of a smooth, compact, rotationally symmetric hypersurface flowing by mean curvature are studied.

Theorem 6.2. *Let $n \geq 3$. Then the torus defined in (6.5) does not fatten under motion by mean curvature. Moreover, up to a parabolic scaling, at the singularity the torus converges to a cylinder.*

Later on, Evans-Spruck [45] conjectured that a smooth open set can not fatten under motion by mean curvature. The negative answer to this conjecture was given by Angenent-Ilmanen-Velasquez in [12,11] (see the discussion below).

As already remarked, the complete characterization of those sets which fatten is still open. In this respect, Barles-Soner-Souganidis [13, Theorem 4.3] gave the following sufficient condition for an initial set E of class $\mathcal{C}^2$ to not develop fattening.

Theorem 6.3. *Suppose that $F : J_0 \to \mathbf{R}$ satisfies the assumptions of Theorem 4.1, and moreover*

$$F(\mu Q^t p, \mu^2 Q^t X Q) = \mu^2 F(p,X),$$

for all $\mu > 0$, $p \in \mathbf{R}^n \setminus \{0\}$, $X \in Sym(n)$, and any orthogonal $(n \times n)$-matrix Q. Assume that there exist nonnegative constants c_1, c_2, c_3, a skewsymmetric matrix M and $x_0 \in \mathbf{R}^n$ such that

$$c_1(x - x_0) \cdot \nabla d_E + c_2 M(x - x_0) \cdot \nabla d_E - c_3 F(\nabla d_E, \nabla^2 d_E) \neq 0 \qquad \text{on } \partial E.$$

Then E does not develop fattening.

A particular case of such a geometric condition corresponds to surfaces of positive mean curvature everywhere. However, this condition does not cover general rotationally symmetric hypersurfaces even in three dimensions.

To our knowledge, in three dimensions there are no examples of smooth *compact* sets which develop fattening at finite time under mean curvature flow. Also, there is no rigorous proof of the existence of a smooth set with non compact boundary developping fattening at finite time; in this direction, Angenent-Chopp-Ilmanen [10] have exhibited an example which we briefly recall (the construction given in [10] is not completely rigorous; it is however supported by numerical evidence).

Example 6.4. Let $n = 3$ and F be as in (3.7). In [10] it is numerically computed a complete, smooth, non compact surface ∂E_0 of genus three (invariant under certain symmetries) asymptotic to a suitable double cone at infinity, which shrinks self-similarly and, at a certain time, becomes a (not rotationally symmetric) double cone with a unique singularity at the origin, and aperture of approximately 72.3°. Using the evolutions of rotationally symmetric cones barriers (see Theorem 6.4), it is then rigorously proved that the evolving set develops fattening.

Still in the three-dimensional case, we recall the following example, which is studied by White in [88] and answers to some questions raised by De Giorgi (see [33] for further conjectures related to this example and to the fattening phenomenon in dimension $n \geq 3$).

Example 6.5. Let $n = 3$, F be as in (3.7),

$$v_0(x_1, x_2, x_3) := \sin x_1 + \sin x_2 + \sin x_3,$$

and for any $\lambda \in [-3, 0]$ set $E_\lambda := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : v_0(x_1, x_2, x_3) \leq \lambda\}$. Notice that E_{-3} consists of isolated points, E_{-1} is not smooth, and for any $\lambda \in]-3, -1[\cup]-1, 0]$, the set E_λ is smooth. Notice also that $E_0 \cap \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \sin x_1 = \sin x_2 = 1\} = \emptyset$. It turns out that, if $\lambda \in]-3, 0[$, then there exists a time $T(\lambda) \in]t_0, +\infty[$ such that $\mathcal{M}(E_\lambda, \mathcal{F}_F)(t) = \emptyset$ for any $t > T(\lambda)$. Moreover $\sup_{\lambda \in [-3,0[} T(\lambda) = +\infty$. Finally, E_0 evolves smoothly by mean curvature for any $t \geq t_0$ and converges smoothly as $t \to +\infty$ to a triply periodic minimal surface (in particular E_0 does not fatten).

The following result is proved in [10, Theorem 4] and generalizes the behaviour of the two-dimensional cross under motion by curvature.

Theorem 6.4. *Let $n \geq 3$, F be as in (3.7), and let E_α be the double rotationally symmetric cone of aperture $\alpha \in]0, \pi/2[$. Then there exists $\alpha(n) \in]0, \pi/2[$ such that E_α develops fattening if and only if $\alpha \in [\alpha(n), \pi/2[$.*

In [49] Fierro-Paolini showed numerical evidence of fattening for mean curvature flow of the smooth initial torus in $\mathbf{R}^4$

$$\left\{(x_1,x_2,x_3,x_4)\in\mathbf{R}^4:\left(\left(x_1^2+x_2^2\right)^{1/2}-1\right)^2+x_3^2+x_4^2\le\lambda\right\}.$$

It seems that in this case there are two critical choices $\lambda_* < \lambda_{**}$ of the parameter λ, corresponding to different singularities. If $\lambda = \lambda_{**}$, the singularity is similar to that of the dumbbell, and fattening is not expected. On the other hand, the shape of the singularity corresponding to λ_* seems to be the one of a cone with the proper aperture, in such a way that fattening is expected.

The following is the first explicit example, to our knowledge, of three-dimensional fattening for motion by mean curvature in codimension 2.

Example 6.6. Let $n = 3$ and F be as in (3.8). Then the set

$$E := \left\{(x_1,x_2,x_3)\in\mathbf{R}^3 : (x_1^2+x_2^2)(x_2^2+x_3^2)(x_3^2+x_1^2)=0\right\}$$

develops 3-dimensional fattening for any t arbitrarily close to t_0.

Proof. As $\mathcal{M}_*(E,\mathcal{F}_F)=\emptyset$, the thesis reduces to check that

$$\mathcal{H}^3\big(\mathcal{M}^*(E,\mathcal{F}_F)(t)\big) > 0. \tag{6.6}$$

In particular, it is enough to prove that for any $R>0$ there exists $T(R)>t_0$, with $T(R)\downarrow t_0$ as $R\downarrow 0$, such that $\mathcal{M}^*(E,\mathcal{F}_F)(T)\supseteq B_R(0)$.
Fix $R>0$. We recall that there exists $T_1>t_0$ such that the generalized evolution by curvature of the two-dimensional cross $\{(x,y)\in\mathbf{R}^2 : xy=0\}$ contains the ball $B_R(0)$ at time T_1. This result implies that $\mathcal{M}^*(E,\mathcal{F}_F)(T_1)$ contains the boundary of any triangle with sides lying on the coordinate planes and which is contained in $B_R(0)$. We recall now that, if an initial curve is contained in a plane, then its evolution coincides with the usual evolution by curvature in that plane. Hence, the evolution of the boundary of the above triangles can be regarded, after the initial time, as a classical curvature flow [46] of codimension one in the plane containing the triangle, and this evolution exists for a time controlled by $R^2/2$. Following [5, Remark 6.2], it follows that, for any $\varrho>0$ we have that $\mathcal{M}\big(E_\varrho^+,\mathcal{F}_F\big)$ is a barrier for such flows, and the same is true for $\mathcal{M}^*(E,\mathcal{F}_F)$. Considering now the evolutions of the boundaries of equilateral triangles, one gets the following estimate:

$$\mathcal{M}^*(E,\mathcal{F}_F)(t)\supseteq B_{\frac{R}{\sqrt{3}}}(0), \qquad t > T_1+\frac{R^2}{3}\ ,$$

which implies (6.6). $\square$

If the following question has a positive answer, if fattening occurs, it occurs at the same time everywhere in the connected component.

Problem. Let $E\subseteq\mathbf{R}^n$ and F be as in (3.7). Assume that $\mathcal{M}^*(E,\mathcal{F}_F)$ is connected with nonempty interior in $[t_0,t_0+\tau[$, for some $\tau>0$. Then $\mathcal{M}^*(E,\mathcal{F}_F)$ is outer regular in $]t_0,t_0+\tau[$.

Problem. In [31,35] De Giorgi suggested to consider the evolution of the two knotted circles in $\mathbf{R}^3$

$$E := \{(x_1, x_2, x_3) : x_3 = 0, x_1^2 + x_2^2 = 1\} \cup \{(x_1, x_2, x_3) : x_1 = 0, (x_2 - 1)^2 + x_3^2 = 1\},$$

with F as in (3.8), and to study the behaviour of the minimal barrier after the collision time. It would be interesting, in particular, to check wether E develops two-dimensional (or even three-dimensional) fattening .

7. Appendix

We list here some assumptions used in this paper. We follow the notation of [54, pp. 462-463]; we omit those properties in [54] which are not useful in our context.

(F1) $F : J_1 \to \mathbf{R}$ is continuous;

(F2) F is degenerate elliptic;

(F3) $-\infty < F_*(t, x, 0, 0) = F^*(t, x, 0, 0) < +\infty$ for all $t \in I$, $x \in \mathbf{R}^n$;

(F4) for every $R > 0$, $\sup\{|F(t, x, p, X)| : |p|, |X| \le R, (t, x, p, X) \in J_1\} < +\infty$;

(F6') for every $R > \varrho > 0$ there is a constant $c = c_{R,\varrho}$ such that

$$|F(t, x, p, X) - F(t, x, q, X)| \le c|p - q|$$

for any $t \in I$, $x \in \mathbf{R}^n$, $\varrho \le |p|, |q| \le R$, $|X| \le R$;

(F7) there are $\varrho_0 > 0$ and a modulus σ_1 such that

$$F^*(t, x, p, X) - F^*(t, x, 0, 0) \le \sigma_1(|p| + |X|),$$
$$F_*(t, x, p, X) - F_*(t, x, 0, 0) \ge -\sigma_1(|p| + |X|),$$

provided $t \in I$, $x \in \mathbf{R}^n$, $|p|, |X| \le \varrho_0$.

The following example shows that, if $F(t, \cdot, p, X)$ is not Lipschitz (the dependence on X is irrelevant here), then the viscosity solution of (1.4) is not necessarily continuous, and motivates assumption (F8).

Example 7.1. Let $n = 2$, $F(x, p) := -g(x)|p|$, where

$$g(x) := \begin{cases} 0 & \text{if } |x| \le 1, \\ \sqrt{|x| - 1} & \text{if } |x| > 1. \end{cases}$$

Then g is uniformly continous and is not Lipschitz. Let $v_0(x) := \min(1, |x| - 1)$ and $E_\lambda := \{v_0 \le \lambda\}$ for $\lambda \in \mathbf{R}$. Then for any $\lambda \in]-1, 0]$ the set E_λ stands still, while, if $\lambda \in]0, 1[$, E_λ shrinks to E_0 at time $T_\lambda = 2\sqrt{\lambda}$. The function v having as sublevels the evolution of all sets E_λ (corresponding to a viscosity solution of (1.4) with $v(t_0, x) = v_0(x)$) is therefore not continuous.

(F8) There is a modulus σ_2 such that

$$|F(t, x, p, X) - F(t, y, p, X)| \le |x - y||p|\sigma_2(1 + |x - y|)$$

for $y \in \mathbf{R}^n$, $(t, x, p, X) \in J_1$;

(F9) there is a modulus σ_2 such that $F_*(t,x,0,0) - F^*(t,y,0,0) \geq -\sigma_2(|x-y|)$ for any $t \in I$, $x, y \in \mathbf{R}^n$;

(F10) suppose that $-\mu \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \nu \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}$ with $\mu, \nu \geq 0$. Let $R \geq 2\nu \vee \mu$ and let $\varrho > 0$; then

$$F_*(t,x,p,X) - F^*(t,y,p,-Y) \geq -|x-y||p|\overline{\sigma}(1 + |x-y| + \nu|x-y|^2)$$

for $(t,x) \in I \times \mathbf{R}^n$, $\varrho \leq |p| \leq R$, with some modulus $\overline{\sigma} = \overline{\sigma}_{R,\varrho}$ independent of t, x, y, X, Y, μ, ν.

ACKNOWLEDGEMENTS

We wish to thank Ennio De Giorgi for the encouragement he gave us and for many useful suggestions and advices.

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