ISOPERIMETRIC PROBLEMS FOR A NONLOCAL PERIMETER OF MINKOWSKI TYPE

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Abstract. We show a quantitative version of the isoperimetric inequality for a non local perimeter of Minkowski type. We also apply this result to study isoperimetric problems with repulsive interaction terms, under volume and convexity constraints. We prove existence of minimizers, and we describe their shape as the volume tends to zero or to infinity.

Contents

1. Introduction

In a recent series of papers, [2, 7, 8] a class of variational problems which interpolate between the classical perimeter and the volume functionals have been introduced and analyzed in details, also in anisotropic contexts and in view of discretizations methods. Such nonlocal functionals are used in image processing to keep fine details and irregularities of the image while denoising additive white noise.

These objects, which we call nonlocal perimeters of Minkowski type, are modeled by an energy which resembles the usual perimeter at large scales, but presents a predominant volume contribution at small scales, giving rise to a nonlocal behavior, which may produce severe loss of regularity and compactness. A preliminary study of the main properties of such perimeters and the related Dirichlet energies has been developed recently in [5]. In that paper, the main features of sets with

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finite perimeter, in particular compactness properties, local and global isoperimetric inequalities are discussed. Moreover, some properties of minimizers of such functionals are proved, such as density properties and existence of plane-like minimizers under periodic perturbations.

In this paper our aim is to push a bit further such analysis, obtaining a quantitative version of the isoperimetric inequality for these non local perimeters of Minkowski type. The quantitative version of the isoperimetric inequality is based on recent results on a quantitative Brunn-Minkowski inequality, obtained in [3, 12], and for more general sets in [11].

We also study isoperimetric problems in presence of nonlocal repulsive interaction terms under convexity constraints. In particular, we provide existence of minimizers for every volume, we show that balls are minimizers for small volumes, and we provide a description of the asymptotic shape of minimizers in the large volume regime. Local and global minimality properties of balls with respect to the volume-constrained minimization of a free energy consisting of a the classical perimeter plus a non-local repulsive interaction term has been analyzed recently in [14, 15, 4]. In these papers existence and non-existence properties of the minimizers of the considered variational problem were established, together with a more detailed information about the shape of the minimizers in certain parameter regimes. In particular, it is proved that balls are the unique minimizers for small volume. An improvement of these results and the extension to the case of nonlocal perimeter of fractional type has been given in [10] (see also [6]).

2. Notation and preliminary definitions

We let B_r be the open ball of radius r centered at the origin, and by $B_r(x)$ the open ball of radius r centered at x. Finally, we denote with B the ball centered at the origin with volume 1.

We shall identify a measurable set $E \subseteq \mathbb{R}^n$ with its points of density one, and we let ∂E be the boundary of E in the measure theoretic sense [1]. We will also denote by $|E|$ the n-dimensional Lebesgue measure of E and

$$\begin{aligned} &E \oplus B_r := \bigcup_{x\in E} B_r(x) = (\partial E \oplus B_r) \cup E = (\partial E \oplus B_r) \cup (E \ominus B_r), \\ \text{where}\quad &E \ominus B_r := E \setminus \left(\bigcup_{x\in\partial E} B_r(x) \right) = E \setminus \big(\partial E) \oplus B_r\big). \end{aligned} \tag{1}$$

Given $r > 0$, for any measurable set $E \subseteq \mathbb{R}^n$ we consider the functional

$$\mathrm{Per}_r(E) := \frac{1}{2r}|(\partial E) \oplus B_r| = \frac{1}{2r}(|E \oplus B_r| - |E \ominus B_r|). \tag{2}$$

Notice that, since we identify a set with its points of density one, we have that

$$\mathrm{Per}_r(E) = \min_{|E'\Delta E|=0} \mathrm{Per}_r(E').$$

The definition of Per_r is inspired by the classical Minkowski content. In particular, for sets with compact and $(n-1)$-rectifiable boundaries, the functional in (2) may be seen as a nonlocal approximation of the classical perimeter functional, in the sense that

$$\lim_{r\searrow 0} \mathrm{Per}_r(E) = \mathcal{H}^{n-1}(\partial E).$$

Hence, in some sense, Per_r interpolates between the perimeter functional for small r, and the volume for large r.

For a set $E \subseteq \mathbb{R}^n$, we also introduce the Riesz energy

$$\Phi_\alpha(E) = \int_E \int_E \frac{1}{|x-y|^\alpha} dxdy = \int_E V_{E,\alpha}(x)dx, \tag{3}$$

where $\alpha \in (0,n)$ and the potential $V_{E,\alpha}$ is defined as

$$V_{E,\alpha}(x) = \int_E \frac{1}{|x-y|^\alpha} dy. \tag{4}$$

We recall that, by Riesz inequality [17], balls are the (unique) volume constrained maximizers of Φ_α.

3. Quantitative isoperimetric inequality

First of all we point out that a consequence of the Brunn-Minkowski inequality (see [3, 12, 11]) is that the balls are isoperimetric for the functional in (2). This has been proved in [5, Lemma 2.1]:

Proposition 3.1. *For any measurable set $E \subseteq \mathbb{R}^n$ it holds that*

$$\mathrm{Per}_r(E) \geq \mathrm{Per}_r(B_R), \tag{5}$$

where R is such that $|E| = |B_R|$. Moreover, the equality holds iff $E = B_R(x)$, for some $x \in \mathbb{R}^n$.

In this section we provide a quantitative version of this result. From now on, $C(n) > 0$ will denote a universal constant depending only on the space dimension n.

Theorem 3.2 (Quantitative isoperimetric inequality)**.** *For any measurable set $E \subseteq \mathbb{R}^n$ it holds that*

$$\frac{\mathrm{Per}_r(E) - \mathrm{Per}_r(B_R)}{\mathrm{Per}_r(B_R)} \geq C(n) \min\left(\frac{R}{r}, 1\right) \left(\inf_{x \in \mathbb{R}^N} \frac{|E \Delta B_R(x)|}{|B_R|}\right)^2. \tag{6}$$

To prove this result we will use the following quantitative versions of the Brunn-Minkowski inequality, proved in [3, Corollary 1] (see also [12, 11]).

Theorem 3.3. *Let F, K be two bounded measurable sets with $|F|, |K| < \infty$, and assume that K is convex. Then there holds*

$$|F \oplus K|^{\frac{1}{n}} - |F|^{\frac{1}{n}} - |K|^{\frac{1}{n}} \geq C(n) \min\left(|F|^{\frac{1}{n}}, |K|^{\frac{1}{n}}\right) \alpha(F,K)^2 \tag{7}$$

where

$$\alpha(F,K) = \inf_{x \in \mathbb{R}^n} \frac{|F \Delta (x + \lambda K)|}{|F|} \qquad \lambda^n = \frac{|F|}{|K|}. \tag{8}$$

Using this result we can prove Theorem 3.2.

Proof of Theorem 3.2. Without loss of generality, we can suppose that ∂E is bounded (otherwise it is easy to show that $\mathrm{Per}_r(E) = +\infty$).

Observe that, since $|E| = |B_R|$, we have $|E|^{\frac{1}{n}} + |B_r|^{\frac{1}{n}} = |B_{R+r}|^{\frac{1}{n}}$. Moreover, an easy computation shows that, if $R \geq r$, then

$$\mathrm{Per}_r(B_R) = \frac{|B_{R+r}| - |B_{R-r}|}{2r} \leq \omega_n \sum_{j=0}^{n-1} R^j r^{n-1-j} \leq n\omega_n R^{n-1}. \tag{9}$$

On the other hand, if $R < r$, then $\mathrm{Per}_r(B_R) \leq Cr^{n-1}$.

Recalling (8), we now compute

$$\alpha(E, B_r) = \inf_{x\in\mathbb{R}^n} \frac{|E\Delta B_R(x)|}{|E|} = \inf_{x\in\mathbb{R}^n} \frac{|E\Delta B_R(x)|}{|B_R|}.$$

Therefore (7), with $F = E$ and $K = B_r$, reads

$$|E \oplus B_r|^{\frac{1}{n}} - |B_{R+r}|^{\frac{1}{n}} \geq C(n)\min(r, R)\left(\inf_{x\in\mathbb{R}^N} \frac{|E\Delta B_R(x)|}{|B_R|}\right)^2.$$

From this formula, recalling that $|E \oplus B_r| \geq |B_{R+r}|$ (again by the fomula above), and (9), we obtain that

$$\begin{aligned}
|E \oplus B_r| - |B_{R+r}| &= \left(|E \oplus B_r|^{\frac{1}{n}} - |B_{R+r}|^{\frac{1}{n}}\right)\sum_{j=0}^{n-1} |E \oplus B_r|^{\frac{n-1-j}{n}} |B_{R+r}|^{\frac{j}{n}} \\
&\geq n\left(|E \oplus B_r|^{\frac{1}{n}} - |B_{R+r}|^{\frac{1}{n}}\right)|B_{R+r}|^{\frac{n-1}{n}} \\
&\geq nC(n)(R + r)^{n-1}\min(r, R)\left(\inf_{x\in\mathbb{R}^N} \frac{|E\Delta B_R(x)|}{|B_R|}\right)^2 \\
&\geq C(n)r\mathrm{Per}_r(B_R)\min\left(1, \frac{R}{r}\right)\left(\inf_{x\in\mathbb{R}^N} \frac{|E\Delta B_R(x)|}{|B_R|}\right)^2. \qquad (10)
\end{aligned}$$

Note that if $R < r$, then $B_{R-r} = \emptyset$ and $E \ominus B_R = \emptyset$, otherwise $|B_R| = |E| \geq |B_r|$ in contradiction with the fact that $R < r$. So, if $R < r$, $|E \ominus B_r| = 0 = |B_{R-r}|$. Therefore, recalling the definition of Per_r, we get

$$\mathrm{Per}_r(E) - \mathrm{Per}_r(B_R) = \frac{1}{2r}(|E \oplus B_r| - |B_{R+r}|)$$

from which applying inequality (10) we obtain (6) for $R < r$.

Let us now assume that $R \geq r$ and take $\widetilde{R} \in [0, R]$ such that $|E \ominus B_r| = |B_{\widetilde{R}}|$. Also, recalling that $\big(E \ominus B_r\big) \oplus B_r \subseteq E$, we have that

$$|(E \ominus B_r) \oplus B_r| \leq |E| = |B_R|.$$

Accordingly, applying the Brunn-Minkowski inequality we get that

$$\begin{aligned}
|B_R|^{\frac{1}{n}} &\geq |(E \ominus B_r) \oplus B_r|^{\frac{1}{n}} \\
&\geq |E \ominus B_r|^{\frac{1}{n}} + |B_r|^{\frac{1}{n}} = |B_{\widetilde{R}+r}|^{\frac{1}{n}},
\end{aligned}$$

which implies that $\widetilde{R} \leq R - r$.

From this, we obtain that

$$|E \ominus B_r| = |B_{\widetilde{R}}| \leq |B_{R-r}|. \qquad (11)$$

Putting together (10) and (11), we finally obtain

$$\begin{aligned}
\mathrm{Per}_r(E) - \mathrm{Per}_r(B_R) &= \frac{1}{2r}\left[|E \oplus B_r| - |B_{R+r}| - (|E \ominus B_r| - |B_{R-r}|)\right] \\
&\geq \frac{1}{2}C(n)\mathrm{Per}_r(B_R)\left(\inf_{x\in\mathbb{R}^N} \frac{|E\Delta B_R(x)|}{|B_R|}\right)^2.
\end{aligned}$$

□

We conclude the section with an isodiametric estimate for convex sets. We first recall a well known result for convex sets (see [13, Lemma 2.2]):

There exists a constant $C = C(n)$ such that, up to translations and rotations, for any convex set $E \subseteq \mathbb{R}^n$ there exists $0 \le \lambda_1 \le \cdots \le \lambda_n$ such that

$$\Pi_{i=1}^n [0, \lambda_i] \subseteq E \subseteq C\Pi_{i=1}^n [0, \lambda_i]. \tag{12}$$

Notice that

$$n^{-\frac{1}{n}} \operatorname{diam}(E) \le \lambda_n \le \operatorname{diam}(E).$$

Lemma 3.4. *There exists $C = C(n) > 0$ such that*

$$\operatorname{diam}(E) \le C \operatorname{Per}_r(E)^{n-1} |E|^{-n+2}, \tag{13}$$

for any convex set $E \subseteq \mathbb{R}^n$. Moreover, there exists $C = C(n) > 0$ such that

$$\operatorname{diam}(E) \ge C\, |E|^{\frac{2}{\alpha}} \Phi_\alpha(E)^{-\frac{1}{\alpha}} \tag{14}$$

Finally, for every $i < n$, there holds

$$\lambda_i \le C \operatorname{Per}_r(E)^{n-2} |E|^{3-n-\frac{2}{\alpha}} \Phi_\alpha(E)^{\frac{1}{\alpha}}. \tag{15}$$

Proof. The argument is similar to the one in [13, Lemma 2.1], in the case of the classical perimeter.

We observe that by (12) we have

$$\Pi_{i=1}^n \lambda_i \le |E| \le C^n \Pi_{i=1}^n \lambda_i,$$

and there exist constants C_1, C_2 depending only on n such that

$$C_1 \Pi_{i=2}^n \max(\lambda_i, r) \le \operatorname{Per}_r(E) \le C_2 \Pi_{i=2}^n \max(\lambda_i, r). \tag{16}$$

Therefore, taking the ratio between the terms above, we get

$$\frac{|E|}{\operatorname{Per}_r(E)} \le \frac{C^n \Pi_{i=1}^n \lambda_i}{C_1 \Pi_{i=2}^n \max(\lambda_i, r)} \le C\lambda_1. \tag{17}$$

Recalling that the λ_i are decreasing, we compute

$$\operatorname{Per}_r(E) \ge C_1 \Pi_{i=2}^n \max(\lambda_i, r) \ge C_1 \max(\lambda_n, r)(\max(\lambda_1, r))^{n-2} \ge C_1 \lambda_n \lambda_1^{n-2}. \tag{18}$$

By putting togheter (17) and (18) and we obtain the thesis (13).

We observe now that

$$\Phi_\alpha(E) \ge \frac{|E|^2}{\operatorname{diam}(E)^\alpha},$$

which immediately gives (14).

Using now (16), (17), (14) and the fact that $\lambda_j \ge \lambda_1$ for every j, we get that

$$\begin{aligned}
\operatorname{Per}_r(E) &\ge C_1 \Pi_{i=2}^n \max(\lambda_i, r) \ge C_1 \lambda_n \Pi_{i=2}^{n-1} \lambda_i \\
&\ge C|E|^{2/\alpha} \Phi_\alpha(E)^{-1/\alpha} (\lambda_1)^{n-3} \lambda_i \\
&\ge C|E|^{2/\alpha} \Phi_\alpha(E)^{-1/\alpha} |E|^{n-3} \operatorname{Per}_r(E)^{-n+3} \lambda_i,
\end{aligned}$$

for all $i < n$, from which we deduce (15). □

4. Isoperimetric problems with repulsive interaction terms

Given $m > 0$, we consider the following problem:

$$\min_{|E|=m,\ E \text{ convex}} \mathrm{Per}_{rm^{1/n}}(E) + \Phi_\alpha(E). \tag{19}$$

Note that the we consider the Minkowski perimeter at a scale $rm^{1/n}$, depending on the volume of the sets. This is a natural scale if we want to analyze the behavior of minimizers for small volumes. Indeed, if E_m is a minimizer for problem (19), then it is easy to check that $\tilde{E}_m = m^{-1/n}E_m$ is a minimizer for the rescaled problem

$$\min_{|E|=1,\ E \text{ convex}} \mathrm{Per}_r(E) + m^{\frac{n+1-\alpha}{n}}\Phi_\alpha(E). \tag{20}$$

By minimizing (19), we observe a competition between the perimeter term, which favors round shapes by the isoperimetric inequality, and the Riesz energy Φ_α, for which balls are actually maximizers, due to the Riesz inequality [17].

We point out that the convexity constraint is quite restrictive and could be removed, leading to the more general problem

$$\min_{|E|=m} \mathrm{Per}_{rm^{1/n}}(E) + \Phi_\alpha(E). \tag{21}$$

However, the known strategy to attack these problems is based on regularity theory for (almost) minimizers of the perimeter functionals, and in the case of the r-perimeter Per_r such theory is currently not available.

Theorem 4.1. *For every $m > 0$ there exists a minimizer E_m of the problem* (19).

Moreover, letting $\tilde{E}_m = m^{-\frac{1}{n}}E_m$, as $m \to 0$ we have that, up to translations, $|\tilde{E}_m \Delta B| \to 0$ and $d_H(\tilde{E}_m, B) \to 0$, where B is the ball with $|B| = 1$ and d_H denotes the Hausdorff distance.

Proof. First we prove existence of minimizers. Let $R > 0$ such that $|B_R| = \omega_n R^n = m$. Let E_n be a minimizing sequence such that $|E_n| = |B_R| = m$ and we assume that

$$\mathrm{Per}_r(E_n) \le \mathrm{Per}_r(E_n) + \Phi_\alpha(E_n) \le \mathrm{Per}_r(B_R) + \Phi_\alpha(B_R).$$

By the estimate (13), we conclude that there exists a constant C depending on m, n such that $\mathrm{diam}(E_n) \le C$. Therefore, up to a translation, using the compactness in BV (see [1]), we can extract a sequence E_n which converges in L^1 to a convex set E with volume m. Since $\mathrm{diam}(E_n) \le C$, up to translation we can assume that $E_n \subseteq K$, where K is a compact set. Then the sequence E_n is also precompact in the Hausdorff topology, and so $E_n \to E$ also in Hausdorff sense, since the sets are all convex.

Since the r-perimeter functional is lower semicontinuous with respect to the L^1 convergence, and the Riesz potential is lower semicontinuous with respect to the Hausdorff convergence, we conclude that E is a minimizer of (19).

We now rescale the problem as follows. We define

$$\tilde{E} = m^{-\frac{1}{n}}E,$$

so that $|\tilde{E}| = 1$. An easy computation shows that

$$\Phi_\alpha(\tilde{E}) = m^{\frac{\alpha-2n}{n}}\Phi_\alpha(E) \qquad \mathrm{Per}_r(\tilde{E}) = m^{\frac{1-n}{n}}\mathrm{Per}_{rm^{1/n}}(E).$$

Therefore, as observed above, if E_m is a minimizer of (19), then $\tilde{E}_m = m^{-\frac{1}{n}}E_m$ is a minimizer of (20).

For all $z \in \mathbb{R}^n$ and $s > 0$, we denote with $B(z)$ the ball $B_{\omega_n^{1/n}}(z)$.

$$
\begin{aligned}
|\Phi_\alpha(B(z)) - \Phi_\alpha(\tilde{E}_m)| &= \left| \int_{B(z)} \int_{B(z)} \frac{1}{|x-y|^\alpha} dxdy - \int_{\tilde{E}_m} \int_{\tilde{E}_m} \frac{1}{|x-y|^\alpha} dxdy \right| \\
\leq\ & 2 \int_{\tilde{E}_m \Delta B(z)} \int_{\tilde{E}_m \cap B(z)} \frac{1}{|x-y|^\alpha} dxdy + 2 \int_{\tilde{E}_m \Delta B(z)} \int_{\tilde{E}_m \Delta B(z)} \frac{1}{|x-y|^\alpha} dxdy \\
=\ & 2 \int_{\tilde{E}_m \Delta B(z)} \int_{\tilde{E}_m \cup B(z)} \frac{1}{|x-y|^\alpha} dxdy \leq \int_{\tilde{E}_m \Delta B(z)} \int_{\tilde{E}_m \cup B(z)} \sup_{t \in \mathbb{R}^n} \frac{1}{|t-y|^\alpha} dxdy \\
\leq\ & 2 |\tilde{E}_m \Delta B(z)| \sup_{t \in \mathbb{R}^n} \left(\int_{B_s(t)} \frac{1}{|t-y|^\alpha} dy + \int_{(\tilde{E}_m \cup B(z)) \setminus B_s(t)} \frac{1}{|t-y|^\alpha} dy \right) \\
\leq\ & 2 |\tilde{E}_m \Delta B(z)| \left(n\omega_n \frac{1}{n-\alpha} s^{n-\alpha} + 2 s^{-\alpha} \right).
\end{aligned}
$$

The previous term is minimal when $s = \left(\frac{2(n-\alpha)}{n\omega_n}\right)^{\frac{1}{n}}$ giving that there exists a constant $C(n, \alpha)$ such that

$$
|\Phi_\alpha(B) - \Phi_\alpha(\tilde{E}_m)| \leq C(n, \alpha) \inf_{x \in \mathbb{R}^n} |\tilde{E}_m \Delta B(x)|. \tag{22}
$$

Using the minimality of $\tilde{E}_m$, (6) and (22), and the invariance by translation of the energy, we get

$$
\mathrm{Per}_r(B) \min \left(\frac{\omega_n^{1/n}}{r}, 1 \right) \left(\inf_{x \in \mathbb{R}^n} \frac{|\tilde{E}_m \Delta B(x)|}{|B|} \right)^2 - m^{\frac{n+1-\alpha}{n}} C(n, \alpha) \inf_{x \in \mathbb{R}^n} |\tilde{E}_m \Delta B(x)| \leq 0,
$$

which implies

$$
\inf_{x \in \mathbb{R}^n} |\tilde{E}_m \Delta B(x)| \leq C(n, m, \alpha)\, m^{\frac{n+1-\alpha}{n}}.
$$

So, letting $m \to 0$, we conclude that the sets $\tilde{E}_m$ converge to B in L^1, up to translations. Finally, by Lemma 3.4 we have that $\mathrm{diam}(\tilde{E}_m) \leq C$, so that $\tilde{E}_m \to B$ also in the Hausdorff distance. □

We now show that the rescaled minimizers $\tilde{E}_m$ given by Theorem 4.1 are indeed balls for m small enough. An analogous result when Per_r is replaced by the usual perimeter has been proved in [14, 15] (see also [10] for a generalization to fractional perimeters).

Theorem 4.2. *Let $\alpha \in (0, n-1)$ and let $r \in (0,1)$. Then there exists $m_0 = m_0(n, \alpha) > 0$ such that, up to translations, $\tilde{E}_m = B$ for all $m \in (0, m_0)$.*

Proof. Since the sets $\tilde{E}_m$ are all convex, uniformly bounded, with volume 1, they have uniformly Lipschitz boundaries. Moreover, by Theorem 4.1, up to suitable translations we can write

$$
\partial \tilde{E}_m = \{ (\omega_n^{1/n} + u_m(x)) x \mid x \in \partial B \},
$$

where $u_m \to 0$ in $W^{1,\infty}(\partial B)$ as $m \to 0$. This holds also in the case $\alpha \in [n-1, n)$.

Using the fact that $\alpha < n-1$, recalling [15, Eq. (6.8)], for m sufficiently small we have that

$$
\Phi_\alpha(B) - \Phi_\alpha(\tilde{E}_m) \leq C(n, \alpha) \|u_m\|^2_{L^2(\partial B)} \leq C'(n, \alpha) |\tilde{E}_m \Delta B|^2. \tag{23}
$$

By the minimality of $\tilde{E}_m$, (6) and (23) we then get

$$\begin{aligned} C(n)|\tilde{E}_m \Delta B|^2 &\le \mathrm{Per}_r(\tilde{E}_m) - \mathrm{Per}_r(B) \\ &\le m^{\frac{n+1-\alpha}{n}}(\Phi_\alpha(B) - \Phi_\alpha(\tilde{E}_m)) \le m^{\frac{n+1-\alpha}{n}} C'(n,\alpha)|\tilde{E}_m \Delta B|^2, \end{aligned}$$

which implies that $\tilde{E}_m = B$ if m is small enough. □

Finally, we give a description of the asymptotic shape of minimizers of (19) as $m \to +\infty$.

Theorem 4.3. *Let $\alpha \in (0, n-1)$ and let E_m be minimizers of* (19)*. Then, as $m \to +\infty$,*

$$\widehat{E}_m = m^{-\left(\frac{n-1}{n}\right)\left(\frac{n+1-\alpha}{\alpha(n-1)+1}\right)-\frac{1}{n}} E_m \to [0, \widehat{L}] \times \{0\}^{n-1}, \tag{24}$$

in the Haudorff distance, up to rotations, translations and subsequences.

Proof. We consider $\tilde{E}_m = m^{-1/n} E_m$, so that $\tilde{E}_m$ is a minimizer of (20). Let us compute the energy of the cylinder $C_L = B_R \times [0, L]$ such that $|C_L| = 1$, so that

$$R = \left(\frac{1}{\omega_{n-1} L}\right)^{\frac{1}{n-1}}. \tag{25}$$

Eventually we will choose L in dependance on m, such that $L \to +\infty$ as $m \to +\infty$. So, without loss of generality we may assume that $L \ge \max(R, r, 1)$. Therefore there exists a constant $C = C(n)$ such that

$$\mathrm{Per}_r(C_L) \le CL \max(r, R)^{n-2} + 2\max(R, r)^{n-1} \le CLL^{\frac{n-2}{1-n}} = CL^{\frac{1}{n-1}}. \tag{26}$$

Moreover, recalling also (25), there exists a constant $C = C(n)$ such that

$$\begin{aligned} &\Phi_\alpha(C_L) \ge \\ &L\left(\sum_{i=1}^{L-1} \int_{B_R\times[0,1]} \int_{B_R\times[i,i+1]} \frac{1}{|x-y|^\alpha} dxdy + \int_{B_R\times[0,1]} \int_{B_R\times[0,1]} \frac{1}{|x-y|^\alpha} dxdy\right) \\ &\qquad \ge CL^{2-\alpha} R^{2(n-1)} + CLR^{2(n-1)} = CL^{-\alpha}, \end{aligned} \tag{27}$$

where the second integral is bounded by $L|B_R \times [0,1]|^2$ due to the fact that $\alpha < n-1$.

Using the minimality of $\tilde{E}_m$ together with (27) and (26), we get that there exists a constant $C = C(n)$ such that

$$\Phi_\alpha(\tilde{E}_m) \le CL^{-\alpha} + m^{-\frac{n+1-\alpha}{n}} CL^{\frac{1}{n-1}} \le CL^{-\alpha}.$$

From this we deduce, recalling the inequality (14), that there exists a constant $C = C(n)$

$$\mathrm{diam}(\tilde{E}_m) \ge \Phi_\alpha(\tilde{E}_m)^{-1/\alpha} \ge CL. \tag{28}$$

Using again the minimality of $\tilde{E}_m$ and the inequalities (13), (26) and (27), we then obtain

$$\begin{aligned} \mathrm{diam}(\tilde{E}_m) &\le C\mathrm{Per}_r(\tilde{E}_m)^{n-1} \\ &\le C\left[\mathrm{Per}_r(C_L) + m^{\frac{n+1-\alpha}{n}} \Phi_\alpha(C_L)\right]^{n-1} \\ &\le C\left[L^{\frac{1}{n-1}} + m^{\frac{n+1-\alpha}{n}} L^{-\alpha}\right]^{n-1}. \end{aligned} \tag{29}$$

The previous term is minimal for

$$L = m^{\left(\frac{n-1}{n}\right)\left(\frac{n+1-\alpha}{\alpha(n-1)+1}\right)}. \tag{30}$$

For this choice of L, we get, putting together (28) and (29) we get that there exist $C = C(n)$ and $C' = C'(n)$ such that

$$Cm^{\left(\frac{n-1}{n}\right)\left(\frac{n+1-\alpha}{\alpha(n-1)+1}\right)} \leq \mathrm{diam}(\tilde{E}_m) \leq C' m^{\left(\frac{n-1}{n}\right)\left(\frac{n+1-\alpha}{\alpha(n-1)+1}\right)}. \tag{31}$$

Let now $\tilde{\lambda}_1 \leq \ldots \tilde{\lambda}_n$ such that (12) holds for $E = \tilde{E}_m$. Then, using (15) and proceeding as in the estimates (28) and (31), we obtain that

$$\tilde{\lambda}_i \leq C\mathrm{Per}_r(\tilde{E}_m)^{n-2}\Phi_\alpha(\tilde{E}_m)^{1/\alpha} \leq CL^{\frac{n-2}{n-1}}L^{-1} \leq Cm^{-\left(\frac{1}{n}\right)\left(\frac{n+1-\alpha}{\alpha(n-1)+1}\right)} \tag{32}$$

for every $i < n$. As a consequence, if we define $\widehat{E}_m$ as in (24), from (31) we obtain that there exist C, C' depending only on n such that

$$C \leq \mathrm{diam}(\widehat{E}_m) \leq C'.$$

Moreover, if $\hat{\lambda}_i$ are such that (12) holds for $\widehat{E}_m$, then from (32) we get that, for all $i < n$,

$$\hat{\lambda}_i \leq Cm^{-\frac{n+1-\alpha}{\alpha(n-1)+1}}.$$

Letting $m \to +\infty$, and eventually extracting a subsequence, we conclude that $\hat{\lambda}_i \to 0$ for every $i < n$, whereas $\mathrm{diam}(\widehat{E}_m) \to \widehat{L}$, for some $\widehat{L} \in [C, C']$, which gives the thesis. □

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