STABILITY OF THE BALL UNDER VOLUME PRESERVING FRACTIONAL MEAN CURVATURE FLOW

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Abstract. We consider the volume constrained fractional mean curvature flow of a nearly spherical set, and prove long time existence and asymptotic convergence to a ball. The result applies in particular to convex initial data, under the assumption of global existence. Similarly, we show exponential convergence to a constant for the fractional mean curvature flow of a periodic graph.

Contents

1. Introduction

We recall the definition of fractional perimeter and fractional mean curvature of a set, as introduced by Caffarelli, Roquejoffre and Savin in [5]. Let $s \in (0,1)$; for a set $A \subseteq \mathbb{R}^n$, with $C^{1,1}$ boundary, we let

$$\begin{aligned} \mathrm{Per}_s(A) &:= \int_A \int_{\mathbb{R}^n \setminus A} \frac{1}{|x-y|^{n+s}} dy dx = \frac{1}{s} \int_A \int_{\partial A} \frac{(y-x)\cdot \nu(y)}{|x-y|^{n+s}} dH^{n-1}(y) dx \\ H^s_A(x) &:= \int_{\mathbb{R}^n} \frac{\chi_{A^c}(y) - \chi_A(y)}{|x-y|^{n+s}} dy = \frac{2}{s} \int_{\partial A} \frac{(y-x)\cdot \nu(y)}{|x-y|^{n+s}} dH^{n-1}(y), \end{aligned}$$

where the integrals are intended in the principal value sense, $\nu(y)$ denotes the exterior normal to A at $y \in \partial A$, and χ_A denotes the characteristic function of the set A.

Key words and phrases. Fractional mean curvature flow, nearly spherical sets, long time behavior, Alexandrov theorem.

The authors were supported by the INDAM-GNAMPA and by the PRIN Project 2019/24 *Variational methods for stationary and evolution problems with singularities and interfaces.*

The fractional mean curvature flow starting from an initial set $E_0 \subseteq \mathbb{R}^n$ is a family of sets E_t, parametrized by $t \geqslant 0$ and defined by the geometric evolution law

$$\partial_t x_t \cdot \nu(x_t) = -H^s_{E_t}(x_t) \qquad x_t \in \partial E_t. \tag{1.1}$$

Similarly, the volume preserving fractional mean curvature flow is defined by the equation

$$\partial_t x_t \cdot \nu(x_t) = -H^s_{E_t}(x_t) + \overline{H^s_{E_t}} \qquad x_t \in \partial E_t, \tag{1.2}$$

where $\overline{H^s_{E_t}}$ is the average fractional mean curvature, defined as

$$\overline{H^s_{E_t}} = \fint_{\partial E_t} H^s_{E_t}(y) dH^{n-1}(y). \tag{1.3}$$

If E_t is a smooth solution to (1.2), then the volume of E_t is constant in time, and the fractional perimeter $P_s(E_t)$ is strictly decreasing unless E_t is a ball.

A short time existence result for smooth solutions of both (1.1) and (1.2) starting from compact $C^{1,1}$ initial sets was recently provided in [21]. On the other hand, existence of weak solutions has been obtained by different authors, see [6, 9, 20]. In [10] the authors showed that the flow (1.1) is convexity preserving, also in the presence of a time dependent forcing term.

Concerning the long time behavior of solutions, in [12] it has been proved that smooth convex solutions to (1.2) converge to a ball, up to suitable translations possibly depending on time. In [8] the authors discuss the long time behavior of entire Lipschitz graphs evolving under (1.1), showing that asymptotically flat graphs and periodic graphs converge to hyperplanes uniformly in C^1 .

In this paper we shall mainly consider the long time behavior of the volume preserving flow (1.2), with initial data E_0 which are nearly spherical, according to the following definition:

Definition 1.1 (Nearly spherical set)**.** Let $B_m \subset \mathbb{R}^n$ the n-dimensional ball centered at 0 and with volume $m > 0$. A nearly spherical set E is defined as follows:

$$E := \{rx, x \in \partial B_m, r \in [0, 1 + u(x)]\} \tag{1.4}$$

where $u : \partial B_m \to \mathbb{R}$ is a $C^{1,1}$ function with $\|u\|_{C^1} < 1$. (In the sequel we may extend the function u to $\mathbb{R}^n \setminus \{0\}$ by letting $u(x) := u(x/|x|)$.)

In particular, in Theorem 3.3 we show that, if the initial set is nearly spherical with ε sufficiently small, then the volume preserving flow exists smooth for all times and converges exponentially fast in C^∞ to a translate of the reference ball. This result provides an improvement of the result in [12] discussed above, see Corollary 3.5, ruling out the possibility of indefinite translations and giving the exponential rate of convergence. Similar results in the local setting date back to [19] for the case of convex initial sets, to [1, 16] for nearly spherical initial sets, and more recently to [22] in dimension 2 for weak solutions starting from a general bounded set of finite perimeter.

The main technical tool used in the proof is a quantitative Alexandrov type estimate for nearly spherical sets, proved in Theorem 2.2. In [4, 13] a fractional analog of the classical Alexandrov theorem was established, namely that the boundary of a bounded smooth set with constant fractional mean curvature is necessarily a sphere. More generally, in [13] it is proved that the Lipschitz constant of the fractional mean curvature of a set with smooth boundary controls linearly its C^2-distance from a single sphere. On the other hand, the Alexandrov type estimate (2.4) provides a linear control on the $H^{\frac{1+s}{2}}$ distance from the reference sphere of a nearly spherical set, in terms of the L^2 deficit of the fractional curvature, so giving a stability results for nearly spherical sets.

The inequality (2.5) in Theorem 2.2 can be interpreted as a Łojasiewicz-Simon inequality (see [11]) for the energy functional $\mathrm{Per}_s(E)$. Indeed, it bounds the difference in energy between the ball, which is a critical point of the fractional perimeter, and a nearly spherical set in terms of L^2 norm of the first variation of the energy, that is, the the L^2 deficit of the fractional mean curvature. The idea of using these inequalities for proving convergence of solutions to parabolic equations goes back to the seminal paper of Simon [28], and has been used for geometric flows in different contexts, see for instance [24] and references therein.

After this work was completed, we were informed that an Alexandrov type inequality similar to the one in Theorem 2.2 has been independently established in [14], where the authors prove the existence of flat flows for the fractional volume preserving mean curvature flow, and characterize the long time behavior of its discrete-in-time approximation in low dimension (since in higher dimension it is missing an Alexandrov type theorem for non smooth sets in the fractional case). In particular, they show that the discrete flow starting from a bounded set of finite fractional perimeter converges exponentially fast to a single ball.

Finally, we observe that the case of entire periodic graphs evolving by (1.1) presents some analogies with the flow of nearly spherical sets by the volume preserving mean curvature flow. In particular, in the last section of the paper we establish a similar Łojasiewicz-Simon inequality for periodic graphs, which allows us to improve the long time convergence to hyperplanes proved in [8], getting exponential convergence in C^∞, see Corollary 4.4.

Acknowledgments. The authors are members of INDAM-GNAMPA. The second author was supported by the PRIN Project 2019/24.

2. An Alexandrov type estimate for nearly spherical sets

In this section we show that if E is a nearly spherical set, the $H^{\frac{1+s}{2}}$-distance of ∂E to the reference sphere is linearly bounded in terms of the L^2-deficit of H^s_E with respect to its average, whenever ∂E is a sufficiently small perturbation of the sphere. The analogous result of Theorem 2.2 in the case of the classical mean curvature has been proved in [23, Theorem 1.10] and [26, Theorem 1.2].

First of all we observe that, via a simple rescaling argument, we may reduce to the case of volume 1. Indeed, it is clear that the results we are going to prove can also be stated for sets parametrized over a ball B_m with volume m, with constants depending on m. The dependence of such constants on m can be made uniform as m varies on bounded intervals.

We introduce the (squared) fractional Gagliardo seminorm of u:

$$[u]^2_{\frac{1+s}{2}} := \int_{\partial B}\int_{\partial B} \frac{(u(x)-u(y))^2}{|x-y|^{n+s}} dH^{n-1}(y)dH^{n-1}(x). \tag{2.1}$$

Moreover we will indicate with $\|u\|_2^2$ the squared $L^2(\partial B)$ norm of u, that is $\int_{\partial B} u^2(x)dH^{n-1}(x)$. We will endow the fractional Sobolev space $H^{\frac{1+s}{2}}(\partial B)$ with the norm given by the square root of

$$\|u\|^2_{L^2(\partial B)} + [u]^2_{\frac{1+s}{2}}.$$

We introduce the hypersingular Riesz operator on the sphere, which is defined (up to constants depending on s and n) as

$$(-\Delta)^{\frac{1+s}{2}}u(x) = 2\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{n+s}} dH^{n-1}(y). \tag{2.2}$$

Note that

$$[u]^2_{\frac{1+s}{2}} = \int_{\partial B} u(x)(-\Delta)^{\frac{1+s}{2}}u(x)dH^{n-1}(x). \tag{2.3}$$

We recall the following results on the asymptotics of these norms (see [2], [25]) and of the curvatures ([7]), as $s \to 0, 1$.

Theorem 2.1. *Let $u \in H^1(\partial B)$. There exist dimensional constants $c(n), k(n) > 0$ such that $\lim_{s\to 1^-}(1-s)\|u\|^2_{\frac{1+s}{2}} = c(n)\|\nabla u\|_2^2$ and $\lim_{s\to 0^+} s\|u\|^2_{\frac{1+s}{2}} = k(n)\|u\|_2^2$. Let $E \subseteq \mathbb{R}^n$ be a bounded set with $C^{1,1}$ boundary. Then there exist dimensional constants $c(n), k(n) > 0$ such that $\lim_{s\to 1^-}(1-s)H^s_E(x) = c(n)H_E(x)$, uniformly in $x \in \partial E$, where $H_E(x)$ is the classical mean curvature, and $\lim_{s\to 0^+} sH^s_E(x) = k(n)|E|$ uniformly for $x \in \partial E$.*

We now state the main result of this section. For the case in which $s = 1$, we refer to [23, Theorem 1.10] and [26, Theorem 1.2].

Theorem 2.2. *Assume that E is a nearly spherical set such that $|E| = |B|$ and the barycenter of E is the same as B, that is, $\int_E x dx = 0$. Then there exist positive constants $C(n, s) > 0$ and $\varepsilon_0(n, s) \in (0, 1)$ such that if $\|u\|_{C^1} < \varepsilon_0$ there holds*

$$[u]^2_{\frac{1+s}{2}} + \|u\|_2^2 \leqslant C(n,s)\|H^s_E - \overline{H^s_E}\|^2_{L^2(\partial E)}. \tag{2.4}$$

Moreover there exists a positive constant $K(n, s) > 0$ depending on n, s such that

$$\mathrm{Per}_s(E) - \mathrm{Per}_s(B) \leqslant K(n,s)\|H^s_E - \overline{H^s_E}\|^2_{L^2(\partial E)}. \tag{2.5}$$

First of all we observe that it is sufficient to prove (2.4), since (2.5) is a consequence of (2.4) and of the rigidity inequality

$$\mathrm{Per}_s(E) - \mathrm{Per}_s(B) \leqslant c(n)[u]^2_{\frac{1+s}{2}} \tag{2.6}$$

which was proved in [15, Theorem 6.2].

In order to show (2.4) we need some preliminary computations. We first compute the fractional mean curvature of E in spherical coordinates. We fix a point $\bar{x}$ on ∂E. Then $\bar{x} = (1 + u(x))x$ for some $x \in \partial B$. We rewrite the curvature $H^s_E(\bar{x})$ as defined in (1.1) as an integral over ∂B, by using the area formula. Observe that if $\bar{y} = (1 + u(y))y \in \partial E$ with $y \in \partial B$, then $\nu(\bar{y}) = \frac{(1+u(y))y - \nabla u(y)}{\sqrt{|1+u(y)|^2 + |\nabla u(y)|^2}}$ where ∇u is the tangential gradient of u. Moreover the tangential jacobian (see e.g. [23, Lemma 4.1]) is given at a point $\bar{y} = (1 + u(y))y$ by $\sqrt{|1 + u(y)|^2 + |\nabla u(y)|^2}(1 + u(y))^{n-2}$.

So by the area formula the curvature $H_E^s(\bar{x})$ coincides with

$$\begin{aligned}
& H_E^s(x(1+u(x)) \\
= \; & \frac{2}{s}\int_{\partial B} \frac{(y-x+yu(y)-xu(x))\cdot[(1+u(y))y-\nabla u(y)]}{|y-x+yu(y)-xu(x)|^{n+s}}(1+u(y))^{n-2}dH^{n-1}(y) \\
= \; & \frac{2}{s}\int_{\partial B} \frac{u(y)-u(x)}{|y-x+yu(y)-xu(x)|^{n+s}}(1+u(y))^{n-1}dH^{n-1}(y) \qquad (2.7)\\
& +\frac{1}{s}\int_{\partial B} \frac{(u(x)+1)|x-y|^2}{|y-x+yu(y)-xu(x)|^{n+s}}(1+u(y))^{n-1}dH^{n-1}(y) \qquad (2.8)\\
& +\frac{2}{s}\int_{\partial B} \frac{(1+u(x))(x-y)\cdot\nabla u(y)}{|y-x+yu(y)-xu(x)|^{n+s}}(1+u(y))^{n-2}dH^{n-1}(y), \qquad (2.9)
\end{aligned}$$

where we used that $\nabla u(y)\cdot y = 0$, $(y-x)\cdot y = \frac{|y-x|^2}{2} = 1 - x\cdot y$.

We start with some integral estimates of the difference between the curvature of E (expressed in spherical coordinates) and the curvature of the ball. By using the definition (1.1) and the fact that $(y-x)\cdot y = \frac{|y-x|^2}{2}$ for all $x, y \in \partial B$, we get that the curvature of the ball H_B^s is constant and coincides with

$$H_B^s = \frac{1}{s}\int_{\partial B} \frac{1}{|x-y|^{n+s-2}}dH^{n-1}(y). \tag{2.10}$$

Lemma 2.3. *Let E be a nearly spherical set as in Definition 1.1. For every $x \in \partial B$, we denote with $H_E^s(x(1+u(x)))$ the fractional mean curvature of E at the point $x(1+u(x)) \in \partial E$. Then there holds*

$$\begin{aligned}
\int_{\partial B}(H_E^s(x(1+u(x)) - H_B^s) \;&=\; -\frac{n+s}{2}[u]^2_{\frac{1+s}{2}}(1+O(\|u\|_{C^1}) - sH_B^s\int_{\partial B}u(x) \\
&\quad +\frac{s(s+1)}{2}H_B^s\|u\|_2^2(1+O(\|u\|_{C^1}), \qquad (2.11)\\
\int_{\partial B}u(x)(H_E^s(x(1+u(x)) - H_B^s) \;&=\; \left([u]^2_{\frac{1+s}{2}} - sH_B^s\|u\|_2^2\right)(1+O(\|u\|_{C^1})), \qquad (2.12)
\end{aligned}$$

where $O(\|u\|_{C^1})$ denotes a function $f(x)$ such that $|f(x)| \leqslant C\|u\|_{C^1}$ for all $x \in \partial B$, for some $C > 0$.

Moreover, if $|E| = |B|$ we may rewrite (2.11) *as*

$$\int_{\partial B}(H_E^s(x(1+u(x))) - H_B^s) = -\frac{n+s}{2}\left([u]^2_{\frac{1+s}{2}} - sH_B^s\|u\|_2^2\right)(1+O(\|u\|_{C^1})). \tag{2.13}$$

Proof. We first notice that

$$\begin{aligned}
(1+u(y))^{n-1} \;&=\; 1+(n-1)u(y) + \frac{(n-1)(n-2)}{2}u^2(y) + O(\|u\|^3_{C^1}) \\
\frac{1}{|y-x+yu(y)-xu(x)|^{n+s}} \;&=\; \frac{1}{|y-x|^{n+s}} - \frac{n+s}{2|x-y|^{n+s}}(u(x)+u(y)) + \\
&+\; \frac{(n+s)(n+s+2)}{8|x-y|^{n+s}}(u^2(y)+u^2(x)) + \frac{(n+s)^2}{4|x-y|^{n+s}}u(x)u(y) \\
&-\; \frac{n+s}{2|x-y|^{n+s}}\frac{|u(y)-u(x)|^2}{|x-y|^2} + \frac{1}{|x-y|^{n+s}}O(\|u\|^3_{C^1}),
\end{aligned}$$

Putting together the previous expansions we conclude that

$$\begin{aligned}\frac{(1+u(y))^{n-1}}{|y-x+yu(y)-xu(x)|^{n+s}} &= \frac{1}{|x-y|^{n+s}}-\frac{n+s}{2|x-y|^{n+s}}\frac{|u(y)-u(x)|^2}{|x-y|^2}\\ &+ \frac{n-1}{|x-y|^{n+s}}u(y)-\frac{n+s}{2|x-y|^{n+s}}(u(x)+u(y))\\ &+ \frac{(n-1)(n-2)}{2|x-y|^{n+s}}u^2(y)-\frac{(n-1)(n+s)}{2|x-y|^{n+s}}u(y)(u(x)+u(y))\\ &+ \frac{(n+s)(n+s+2)}{8|x-y|^{n+s}}(u^2(y)+u^2(x))+\frac{(n+s)^2}{4|x-y|^{n+s}}u(x)u(y)\\ &+ \frac{1}{|x-y|^{n+s}}O(\|u\|^3_{C^1}).\end{aligned} \tag{2.14}$$

We shall compute $\int_{\partial B} H^s_E(x(1+u(x))dH^{n-1}(x)$ and $\int_{\partial B} u(x)H^s_E(x(1+u(x))dH^{n-1}(x)$ by considering separately the three terms in (2.7), (2.8), (2.9).

First term (2.7). We use the Taylor expansion (2.14) in the term in (2.7) and we integrate it on ∂B with respect to x: We get

$$\frac{2(n-1)}{s}\int_{\partial B}\int_{\partial B}\frac{(u(y)-u(x))u(y)}{|y-x|^{n+s}}\left[1+O(\|u\|_{C^1})\right]dH^{n-1}(y) \tag{2.15}$$

$$=\frac{(n-1)}{s}\int_{\partial B}\int_{\partial B}\frac{(u(y)-u(x))^2}{|y-x|^{n+s}}\left[1+O(\|u\|_{C^1}))\right]dH^{n-1}(y)dH^{n-1}(x)=\frac{n-1}{s}[u]^2_{\frac{1+s}{2}}(1+O(\|u\|_{C^1})).$$

We multiply now the term in (2.7) by $u(x)$ and integrate, also using the Taylor expansion and we get

$$\frac{2}{s}\int_{\partial B}\int_{\partial B}\frac{(u(y)-u(x))u(x)}{|y-x|^{n+s}}\left[1+O(\|u\|_{C^1})\right]dH^{n-1}(y)=-\frac{1}{s}[u]^2_{\frac{1+s}{2}}(1+O(\|u\|_{C^1})). \tag{2.16}$$

Second term (2.8). We use the Taylor expansion (2.14) in the term in (2.8) and we integrate it on ∂B with respect to x recalling (2.10): We get

$$\begin{aligned}&\int_{\partial B} H^s_B dH^{n-1}(x)-\frac{n+s}{2s}[u]^2_{\frac{1+s}{2}}(1+O(\|u\|_{C^1}))\\ &- sH^s_B\int_{\partial B}u(x)dH^{n-1}(x)(1O(\|\nabla u\|^2_{C^0})\\ &+ \frac{n^2-4n+s^2+2s+4}{4}H^s_B\int_{\partial B}u^2(x)dH^{n-1}(x)(1+O(\|u\|_{C^1}))\\ &+ \frac{s^2-n^2-4+4n}{4s}\int_{\partial B}\int_{\partial B}\frac{u(x)u(y)}{|x-y|^{n+s-2}}dH^{n-1}(x)dH^{n-1}(y)(1+O(\|u\|_{C^1})).\end{aligned} \tag{2.17}$$

We multiply the term (2.8) by $u(x)$ and integrate and we get, recalling (2.10),

$$\begin{aligned}H^s_B\int_{\partial B}u(x)dH^{n-1}(x) &+ [u]^2_{\frac{1+s}{2}}O(\|u\|_{C^1})+\frac{2-n-s}{2}H^s_B\int_{\partial B}u^2(x)(1+O(\|u\|_{C^1}))dH^{n-1}(x)\\ &+ \frac{n-2-s}{2s}\int_{\partial B}\int_{\partial B}\frac{u(x)u(y)}{|x-y|^{n+s-2}}dH^{n-1}(y)dH^{n-1}(x).\end{aligned} \tag{2.18}$$

Third term (2.9). Integrating (2.9) on ∂B with respect to x and using the Taylor expansion (2.14) (with $n-1$ in place of n) we get

$$\begin{aligned}(2.19)\quad &\frac{2}{s}\left[1-\frac{n+s}{2}\right]\int_{\partial B}\int_{\partial B}\frac{u(x)(x-y)\cdot\nabla u(y)}{|x-y|^{n+s}}\left(1+O(\|u\|_{C^1})\right)dH^{n-1}(y)dH^{n-1}(x)\\ &-\frac{n+s}{s}\int_{\partial B}\int_{\partial B}\frac{(x-y)\cdot\nabla u(y)}{|x-y|^{n+s}}\frac{|u(y)-u(x)|^2}{|x-y|^2}(1+O(\|u\|_{C^1}))dH^{n-1}(y)dH^{n-1}(x).\end{aligned}$$

In order to rewrite the two terms in (2.19), we are going to use the divergence theorem on ∂B. Let us fix $x\in\partial B$ and consider the map $T(y)=\frac{(u(y)-u(x))(y-x)}{|x-y|^{n+s}}$ for $y\in\partial B$. We compute the Jacobian of T:

$$JT(y)=\frac{u(y)-u(x)}{|x-y|^{n+s}}\left[\delta_{ij}-(n+s)\frac{y-x}{|y-x|}\otimes\frac{y-x}{|y-x|}\right]+\frac{\nabla u(y)\otimes(y-x)}{|x-y|^{n+s}}$$

so that

$$\mathrm{div}T(y)=-\frac{s(u(y)-u(x))}{|x-y|^{n+s}}+\frac{\nabla u(y)\cdot(y-x)}{|x-y|^{n+s}}. \tag{2.20}$$

Recalling that $\nabla u(y)\cdot y=0$, and that $y\cdot(y-x)=\frac{|y-x|^2}{2}$ we have that

$$yJT(y)\cdot y=\frac{u(y)-u(x)}{|x-y|^{n+s}}-\frac{n+s}{4}\frac{u(y)-u(x)}{|x-y|^{n+s-2}}$$

and so we get that the tangential divergence of T is given by

$$\mathrm{div}^\tau T(y)=-(s+1)\frac{u(y)-u(x)}{|x-y|^{n+s}}+\frac{(u(y)-u(x))(n+s)}{4|x-y|^{n+s-2}}+\frac{\nabla u(y)\cdot(y-x)}{|x-y|^{n+s}}.$$

By the divergence theorem on ∂B, we have that (recalling that the curvature of B is $n-1$)

$$-(s+1)\int_{\partial B}\frac{u(y)-u(x)}{|x-y|^{n+s}}+\frac{n+s}{4}\int_{\partial B}\frac{(u(y)-u(x))}{|x-y|^{n+s-2}}+\int_{\partial B}\frac{\nabla u(y)\cdot(y-x)}{|x-y|^{n+s}}=\frac{n-1}{2}\int_{\partial B}\frac{u(y)-u(x)}{|x-y|^{n+s-2}}.$$

Multiplying by $u(x)$ and integrating on the sphere we get

$$\begin{aligned}(2.21)\quad \int_{\partial B}\int_{\partial B}\frac{u(x)(x-y)\cdot\nabla u(y)}{|x-y|^{n+s}}&=\frac{s+1}{2}[u]^2_{\frac{1+s}{2}}+\frac{n-s-2}{4}sH^s_B\int_{\partial B}u^2(x)\\ &\quad+\frac{s+2-n}{4}\int_{\partial B}\int_{\partial B}\frac{u(x)u(y)}{|x-y|^{n+s-2}}.\end{aligned}$$

Repeating the same kind of computations for $S(y)=\frac{(u(y)-u(x))^3(y-x)}{|x-y|^{n+s+2}}$ we get that

$$\mathrm{div}^\tau S(y)=-(s+3)\frac{(u(y)-u(x))^3}{|x-y|^{n+s+2}}+\frac{(u(y)-u(x))^3(n+s+2)}{4|x-y|^{n+s}}+3\frac{|u(y)-u(x)|^2\nabla u(y)\cdot(y-x)}{|x-y|^{n+s+2}}.$$

Now we use the the divergence theorem on ∂B, to get

$$\int_{\partial B}\mathrm{div}^\tau S(y)=\frac{n-1}{2}\int_{\partial B}\frac{(u(y)-u(x))^3}{|x-y|^{n+s}}$$

and integrating again on ∂B, we obtain

$$\int_{\partial B}\int_{\partial B}\frac{(x-y)\cdot\nabla u(y)}{|x-y|^{n+s}}\frac{|u(y)-u(x)|^2}{|x-y|^2}dH^{n-1}(y)dH^{n-1}(x)=0. \tag{2.22}$$

We use (2.21) and (2.22) in (2.19) and we get

$$\frac{(s+1)(2-n-s)}{2s}[u]^2_{\frac{1+s}{2}}(1+O(\|u\|_{C^1}))+\frac{s^2-(n-2)^2}{4}H^s_B\int_{\partial B}u^2(x)$$
$$+\frac{(n-2)^2-s^2}{4s}\int_{\partial B}\int_{\partial B}\frac{u(x)u(y)}{|x-y|^{n+s-2}}. \tag{2.23}$$

If we multiply by $u(x)$ the term in (2.9) and integrate, recalling the Taylor expansion and using (2.21), we get

$$\frac{s+1}{s}[u]^2_{\frac{1+s}{2}}(1+O(\|u\|_{C^1}))$$
$$+\quad\frac{n-s-2}{2}H^s_B\int_{\partial B}u^2(x)+\frac{s+2-n}{2s}\int_{\partial B}\int_{\partial B}\frac{u(x)u(x)}{|x-y|^{n+s-2}}. \tag{2.24}$$

By using (2.15), (2.17) and (2.23) we conclude (2.11). By using (2.16), (2.18) and (2.24) we conclude (2.12).

Finally the volume condition reads

$$|B|=\int_E dx=\int_{\partial B}\frac{(1+u(y))^n}{n}dH^{n-1}(y).$$

So, recalling that $n|B|=\int_{\partial B}dH^{n-1}(y)$, and performing a Taylor expansion we get

$$\int_{\partial B}u(y)dH^{n-1}(y)=-\frac{n-1}{2}\int_{\partial B}u^2(y)(1+O(\|u\|_{C^1}))dH^{n-1}(y). \tag{2.25}$$

If we substitute in (2.11), we conclude (2.13).

□

Proof of Theorem 2.2. As discussed above it is sufficient to prove the validity of (2.4). The proof of this estimate is divided in two main steps. In the first step, using Lemma 2.3 and a Poincarè type inequality, we prove that there exist $\varepsilon_0=\varepsilon_0(n,s)>0$ and $C(n,s)>0$ depending on n,s such that if $\|u\|_{C^1}<\varepsilon_0$ there holds

$$[u]^2_{\frac{1+s}{2}}+\|u\|^2_2\leqslant C(n,s)\|H^s_E-H^s_B\|^2_{L^2(\partial B)}. \tag{2.26}$$

In the second step, by a rescaling argument and by the area formula we deduce (2.4) from (2.26).

Along the proof, $C(n,s)$ will indicate a constant depending on n,s which may change from line to line.

First Step: proof of (2.26).

We follow [17, Section 2], where it is provided the fractional counterpart of the classical estimates of Fuglede on nearly spherical sets, see [18]. We introduce the $L^2(\partial B)$ orthonormal basis Y^i_k of spherical harmonics of degree $k=0,1,\dots$. Of course we have that $Y_0=\frac{1}{\sqrt{n|B|}}$, $Y^i_1=\frac{x_i}{\sqrt{|B|}}$ for $i=1\dots,n$.

We will denote with λ^s_k the k-order eigenvalue of the operator (2.2), so there holds that $(-\Delta)^{\frac{1+s}{2}}Y^i_k=\lambda^s_kY^i_k$. It is possible to show (we refer to [17, Proposition 2.3] and references therein) that $\lambda^s_k>\lambda^s_{k-1}$ for all $k\geqslant 1$ and that

$$\lambda^s_0=0,\quad \lambda^s_1=sH^s_B,\quad \lambda^s_2=\frac{2n}{n-s}\lambda^s_1\geqslant 2\lambda^s_1. \tag{2.27}$$

We write u as a Fourier serie with respect to the spherical harmonics, up to degree 2:

$$u(x) = aY_0 + \sum_{i=1}^{n} b_i \cdot Y_1^i + R(x) = \frac{1}{n|B|}\int_{\partial B} u(y)dH^{n-1}(y) + \frac{1}{|B|}\int_{\partial B} u(y)y\cdot x dH^{n-1}(y) + R(x)$$

where R is orthogonal to the harmonics of degree $0, 1$, that is $\int_{\partial B} R(y)dH^{n-1}(y) = 0$ and $\int_{\partial B} y_i R(y)dH^{n-1}(y) = 0$ for all i. We compute

(2.28)

$$\|u\|_2^2 = \int_{\partial B} u^2(x)dH^{n-1}(x) = \frac{1}{|B|}\left(\int_{\partial B} u(y)dH^{n-1}(y)\right)^2 + \frac{1}{|B|}\left|\int_{\partial B} u(y)ydH^{n-1}(y)\right|^2 + \|R\|_2^2.$$

and moreover, recalling the relation (2.3), there holds

$$[u]^2_{\frac{1+s}{2}} = \lambda_1^s \frac{1}{|B|}\left|\int_{\partial B} u(y)ydH^{n-1}(y)\right|^2 + [R]^2_{\frac{1+s}{2}}. \tag{2.29}$$

Since R is orthogonal to the harmonics of degree 0 and 1, by the monotonicity of the eigenvalues and by (2.27), there holds a fractional Poincaré type inequality

$$[R]^2_{\frac{1+s}{2}} \geqslant \lambda_2^s \|R\|_2^2 = \frac{2n}{n-s}\lambda_1^s\|R\|_2^2 = \frac{2n}{n-s} sH_B^s \|R\|_2^2. \tag{2.30}$$

Therefore

$$\lambda_1^s \|R\|_2^2 \leqslant \frac{n-s}{2n}[R]^2_{\frac{1+s}{2}} \leqslant \frac{1}{2}[R]^2_{\frac{1+s}{2}}. \tag{2.31}$$

We rewrite the $H^{\frac{1+s}{2}}$ norm of u as follows:

$$\begin{aligned}\|u\|_2^2 + [u]^2_{\frac{1+s}{2}} &= \frac{1}{|B|}\left|\int_{\partial B} u(y)dH^{n-1}(y)\right|^2 + \frac{1+\lambda_1^s}{|B|}\left|\int_{\partial B} yu(y)dH^{n-1}(y)\right|^2 + \|R\|_2^2 + [R]^2_{\frac{1+s}{2}} \\ &\leqslant \frac{1}{|B|}\left|\int_{\partial B} u(y)dH^{n-1}(y)\right|^2 + \frac{1+\lambda_1^s}{|B|}\left|\int_{\partial B} yu(y)dH^{n-1}(y)\right|^2 + \left(1+\frac{1}{2\lambda_1^s}\right)[R]^2_{\frac{1+s}{2}}. \qquad (2.32)\end{aligned}$$

We are going to estimate each term appearing on the left hand side.

First of all we observe that by exploiting the barycenter condition $\int_E x_i = 0$, rewriting the integral in polar coordinates we get for all $i = 1, \ldots, n$, $0 = \int_E x_i = \int_{\partial B} \frac{y_i(1+u(y))^{n+1}}{n+1} dH^{n-1}(y)$. Now, using a Taylor expansion and recalling that $\int_{\partial B} y_i dH^{n-1}(y) = 0$ we get

$$\left|\int_{\partial B} y_i u(y)dH^{n-1}(y)\right| = n\int_{\partial B} y_i u^2(y)(1+\varepsilon O(1))dH^{n-1}(y) \leqslant n\|u\|_2^2(1+\varepsilon O(1))$$

from which we deduce

$$\left|\int_{\partial B} yu(y)dH^{n-1}(y)\right|^2 \leqslant n\|u\|_2^4(1+\varepsilon O(1)) = \varepsilon\|u\|_2^2 O(1). \tag{2.33}$$

Now, by (2.28) and (2.29) and by (2.33), we get

$$[u]^2_{\frac{1+s}{2}} - \lambda_1^s\|u\|_2^2 + \lambda_1^s\frac{1}{|B|}\left|\int_{\partial B} u(y)dH^{n-1}(y)\right|^2 = [R]^2_{\frac{1+s}{2}} - \lambda_1^s\|R\|_2^2 \geqslant \frac{1}{2}[R]^2_{\frac{1+s}{2}}. \tag{2.34}$$

Recalling that $\lambda_1^s = sH_B^s$ and using (2.12) to substitute $[u]^2_{\frac{1+s}{2}} - \lambda_1^s\|u\|_2^2$ in the previous inequality we get

$$\int_{\partial B} u(x)(H_E^s(x(1+u(x)) - H_B^s)dH^{n-1}(x) + \frac{sH_B^s}{|B|}\left|\int_{\partial B} u(y)dH^{n-1}(y)\right|^2 \geqslant \frac{1}{2}[R]^2_{\frac{1+s}{2}}$$

from which, by Hölder inequality, we conclude that

$$\frac{1}{2}[R]^2_{\frac{1+s}{2}} \leqslant \frac{sH_B^s}{|B|}\left|\int_{\partial B} u(y)dH^{n-1}(y)\right|^2 + \|u\|_2\|H_E^s - H_B^s\|_{L^2(\partial B)}. \tag{2.35}$$

Observe that by (2.11) and Hölder inequality we get

$$\begin{aligned}\sqrt{n|B|}\|H_E^s - H_B^s\|_{L^2(\partial B)} &\geqslant \left|\int_{\partial B}(H_E^s(x(1+u(x)) - H_B^s)dH^{n-1}(x)\right| \\ &\geqslant \frac{sH_B^s}{2}\left|\int_{\partial B} u(x)dH^{n-1}(x)\right| - \frac{n+s}{2}[u]^2_{\frac{1+s}{2}}(1+\varepsilon O(1)) - sH_B^s\|u\|_2^2(1+\varepsilon O(1)).\end{aligned}$$

In particular this implies that for some constant $C(n,s) > 0$ depending on n, s there holds

$$\left|\int_{\partial B} u(x)dH^{n-1}(x)\right|^2 \leqslant C(n,s)\left[\|H_E^s - H_B^s\|^2_{L^2(\partial B)} + ([u]^2_{\frac{1+s}{2}} + \|u\|_2^2)^2(1+\varepsilon O(1))\right]. \tag{2.36}$$

By using (2.30), (2.33), (2.35), (2.36), we get that there exists a constant $C(n,s) > 0$ such that

$$\|u\|_2^2 + [u]^2_{\frac{1+s}{2}} \leqslant C(n,s)\left[\|H_E^s - H_B^s\|^2_{L^2(\partial B)} + \|H_E^s - H_B^s\|_{L^2(\partial B)}\|u\|_2\right] + \varepsilon O(1)(\|u\|_2^2 + [u]^2_{\frac{1+s}{2}}).$$

By Young inequality, we conclude (2.26).

Second step: proof of (2.4).

First of all note that by area formula and by the estimate (2.13) we get that

$$\begin{aligned}\overline{H_E^s} &= \frac{1}{\mathrm{Per}(E)}\int_{\partial B} H_E^s(x(1+u(x)x)\sqrt{(1+u(x))^2 + |\nabla u(x)|^2}(1+u(x))^{n-2}dH^{n-1}(x) \\ &= \frac{\mathrm{Per}(B)}{\mathrm{Per}(E)}H_B^s(1+\varepsilon O(1)) - \frac{n+s}{2\mathrm{Per}(E)}([u]^2_{\frac{1+s}{2}} - sH_B^s\|u\|_2^2)(1+\varepsilon O(1)) \\ &= H_B^s + \frac{\mathrm{Per}(E) - \mathrm{Per}(B)}{\mathrm{Per}(E)}H_B^s - \frac{n+s}{2\mathrm{Per}(E)}([u]^2_{\frac{1+s}{2}} - sH_B^s\|u\|_2^2)(1+\varepsilon O(1)).\end{aligned}$$

By the area formula and a linearization argument (see [26]) we get that $0 \leqslant \mathrm{Per}(E) - \mathrm{Per}(B) \leqslant C(n)\|u\|^2_{H^1(\partial B)}$. Therefore we conclude that there exists $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant C(n,s)\varepsilon$ for some constant $C(n,s)$ only depending on n, s such that

$$\overline{H_E^s} = H_B^s(1+\lambda).$$

We define $E_\lambda = (1+\lambda)^{\frac{1}{s}}E$, that is E_λ is the nearly spherical set associated with the function $u_\lambda = (1+\lambda)^{\frac{1}{s}} - 1 + (1+\lambda)^{\frac{1}{s}}u$. Note that $\overline{H^s_{E_\lambda}} = H_B^s$ and by (2.26) applied to u_λ we get

$$[u_\lambda]^2_{\frac{1+s}{2}} + \|u_\lambda\|_2^2 \leqslant C(n,s)\|H^s_{E_\lambda} - \overline{H^s_{E_\lambda}}\|^2_{L^2(\partial B)} = \frac{C(n,s)}{(1+\lambda)^2}\|H_E^s - \overline{H_{E^s}}\|^2_{L^2(\partial B)}.$$

Observing that $[u_\lambda]^2_{\frac{1+s}{2}} = (1+\lambda)^{\frac{2}{s}}[u]^2_{\frac{1+s}{2}}$ we get for ε sufficiently small

$$[u]^2_{\frac{1+s}{2}} \leqslant 2C(n,s)\|H^s_E - \overline{H_{E^s}}\|^2_{L^2(\partial B)}.$$

Finally, we recall the Poincaré type inequality (2.34)

$$[u]^2_{\frac{1+s}{2}} \geqslant \lambda^s_1\|u\|^2_2 - \lambda^s_1\frac{1}{|B|}\left|\int_{\partial B} u(y)dH^{n-1}(y)\right|^2$$

and the fact that, by the volume condition $|E| = |B|$, there holds (see (2.25))

$$\left|\int_{\partial B} u(y)dH^{n-1}(y)\right|^2 \leqslant C\|u\|^4_2.$$

Therefore we obtain

$$[u]^2_{\frac{1+s}{2}} + \|u\|^2_2 \leqslant C(n,s)\|H^s_E - \overline{H^s_E}\|^2_{L^2(\partial B)}.$$

Finally we observe that, if $\|u\|_{C^1} \leqslant \varepsilon_0$, there exists a constant depending on the dimension such that

$$C(n)^{-1}\|H^s_E - \overline{H_{E^s}}\|_{L^2(\partial E)} \leqslant \|H^s_E - \overline{H_{E^s}}\|_{L^2(\partial B)} \leqslant C(n)\|H^s_E - \overline{H_{E^s}}\|_{L^2(\partial E)}. \tag{2.37}$$

Using this estimate and the previous inequality, we obtain (2.4). $\square$

3. Volume preserving flow of nearly spherical sets

In this section we consider the long long time behavior of the volume preserving mean curvature flow (1.2) starting from nearly spherical sets. In particular we will show that if E_0 is a nearly spherical set sufficiently close to a sphere B_m, then the flow E_t exists for all times and converges exponentially fast to the reference sphere B_m, possibly translated by some vector $\bar{b}$, as $t \to +\infty$. This result will be obtained by using the short time existence result in [21], the Alexandrov theorem for the fractional mean curvature proved in [13], [4] and the quantitative inequality (2.5) obtained in Theorem 2.2, which can be regarded as a Łojasiewicz-Simon inequality for the geometric functional $\mathrm{Per}_s(E) - \mathrm{Per}_s(B_m)$. Similar arguments have been used in [28], see also [24], to provide full convergence of geometric gradient flows. We refer to [1], [16] (see also [19]) for an analogous result in the local case.

First of all we observe that we may restrict without loss of generality to the case in which the reference ball has volume $m = 1$. Indeed the general case can be obtained via a simple rescaling argument.

We start recalling the fractional analogue of the classical Alexandrov theorem:

Theorem 3.1 ([4],[13]). *If Ω is a bounded open set with boundary of class $C^{1,s}$ and H^s_Ω is constant on $\partial\Omega$, then $\partial\Omega$ is a sphere.*

Short time existence of a smooth solution to (1.2) has been proved in [21] for compact initial data with $C^{1,1}$ boundary, by parametrizing the flow using the height function over a smooth reference surface and by exploiting a fixed point argument.

Theorem 3.2 ([21]). *Let $\alpha \in (0, 1-s)$, and Σ be a smooth compact surface and assume that the initial datum ∂E_0 can be written as the graph of a function u_0 on Σ, which is called the height function.*

Then there exists $\delta_0 > 0$, $T_0 > 0$ and $C_k > 0$ for $k \geqslant 2$ such that if $\|u_0\|_{C^{1,s+\alpha}(\Sigma)} \leqslant \delta_0$ then (1.2) has a unique classical solution E_t for $t \in [0, T_0)$ starting from E_0, moreover ∂E_t is the graph of a smooth function $u(x,t)$ on the surface Σ which satisfies

$$\sup_{t\in[0,T_0)} \|u\|_{C^{1,\alpha+s}} \leqslant 2\delta_0$$

and for every $k > 1$

$$\sup_{t\in[0,T_0)} t^{k!}\|u\|_{C^k} \leqslant C_k.$$

Since for every compact set E_0 with $C^{1,1}$ boundary there exists a reference smooth surface Σ such that ∂E_0 can be written as a graph on Σ of a function with $\|u_0\|_{C^{1,s+\alpha}(\Sigma)} \leqslant \delta$ and $\|u\|_{C^0(\Sigma)} \leqslant \varepsilon(\delta) < \delta$ the theorem implies short time existence of smooth solutions to (1.2) with compact $C^{1,1}$ initial datum.

We restrict now the class of initial data of the flow to nearly spherical sets, in which therefore the reference surface is given by ∂B. We parametrize the flow in terms of the height function on the reference surfaces ∂B, see [27] and [21]. Let E_0 be a nearly spherical set according to Definition 1.1. Then the external normal of E_0 at a point $p = (1 + u_0(x))x$, with $x \in \partial B$ can be expressed as

$$\nu(p) = \frac{(1 + u_0(x))x - \nabla u_0(x)}{\sqrt{(1 + u_0(x))^2 + |\nabla u_0(x)|^2}} \tag{3.1}$$

and as long as $E_t = \{p = rx \ : \ x \in \partial B, r \in [0, 1 + u(x,t)]\}$, then $u(x,t)$ satisfies

$$\begin{cases} u_t(x,t) = -\left[H^s_{E_t}(x(1 + u(x,t)) - \overline{H^s_{E_t}}\right] \frac{\sqrt{(1+u(x,t))^2+|\nabla u(x,t)|^2}}{1+u(x,t)} & t \in (0,T) \\ u(x,0) = u_0(x). \end{cases} \tag{3.2}$$

Also viceversa, if u is a solution to (3.2) in $[0,T)$, then the set defined as $E_t = \{p = rx \ : \ x \in \partial B, r \in [0, 1 + u(x,t)]\}$ is a solution to the flow (1.2) in $[0,T)$.

We state now our main result, which gives that the volume preserving flow starting from a set which is sufficiently close to the ball in C^1 norm smoothly converges to the ball, possibly translated by a fixed vector.

Theorem 3.3. *Let E_0 be a nearly spherical set on a given ball B, according to Definition 1.1, with $|E_0| = |B|$, and let $u_0 : \partial B \to \mathbb{R}$ be the corresponding function. For any $C > 0$ there exists $\varepsilon = \varepsilon(C) > 0$ such that if $\|u_0\|_{C^{1,1}} \leqslant C$ and $\|u_0\|_{C^1} < \varepsilon$, then the flow E_t of (1.2) starting from E_0 exists smooth for every time t and moreover $E_t - \bar{b} \to B$ in C^∞, for some $\bar{b} \in \mathbb{R}^n$. Moreover, there exists a constant $C(n,s)$ depending on n, s such that*

$$\mathrm{Per}_s(E_t) - \mathrm{Per}_s(B) \leqslant C(n,s)(\mathrm{Per}_s(E_0) - \mathrm{Per}_s(B))e^{-C(n,s)t} \qquad \forall t \geqslant 0$$

and for all $m \geqslant 1$ there exists a constant $C(m,n,s) > 0$

$$\|u(x,t) - (\bar{b}\cdot x)x\|_{C^m(\partial B)} \leqslant C(m,n,s)(\mathrm{Per}_s(E_0) - \mathrm{Per}_s(B))e^{-C(m,n,s)t} \qquad \forall t \geqslant 0.$$

Proof. Along the proof, $C(n)$ will indicate a dimensional constant which may change from line to line, and $C(n,s)$ will indicate a constant depending on n, s which may change from line to line.

By interpolation inequality, we get that for $\alpha \in (0, 1-s)$, $\|u_0\|_{C^{1,s+\alpha}} \leqslant C'\varepsilon^{1-s-\alpha}$, where $C' > 0$ depends on C, s, α. So, if ε is sufficiently small such that $C'\varepsilon^{1-s-\alpha} \leqslant \delta_0$, where δ_0 is as in Theorem 3.2, then the flow E_t exists smooth for $t \in [0, T_0)$, and we have that E_t is a nearly spherical set on ∂B with height function $u(\cdot,t)$ with $\sup_{t\in[0,T_0)}\|u\|_{C^{1+\alpha+s}}\leqslant 2\delta_0$ and for every $k>1$, $\sup_{t\in[0,T_0)} t^{k!}\|u\|_{C^k}\leqslant C_k$.

Note that in particular

$$\sup_{t\in[0,T_0)}\|u(\cdot,t)\|_{C^1}\leqslant 2\delta_0. \tag{3.3}$$

Moreover u is a solution to (3.2) in $[0,T_0)$ and in particular, recalling also (2.37), we get

$$\begin{aligned}\|u_t\|^2_{L^2(\partial B)} &= \int_{\partial B} u_t^2(x,t) = \int_{\partial B}\left[H^s_{E_t}(x(1+u(x,t)) - \overline{H^s_{E_t}}\right]^2\left(1+\frac{|\nabla u(x,t)|^2}{(1+u(x,t))^2}\right) \\ &\leqslant \|H^s_{E_t}-\overline{H^s_{E_t}}\|^2_{L^2(\partial B)}(1+\delta_0 O(1)) \leqslant C(n,s)\|H^s_{E_t}-\overline{H^s_{E_t}}\|^2_{L^2(\partial E_t)}.\end{aligned} \tag{3.4}$$

We recall the evolution law of some geometric quantities associated with the flow (1.2) (see [12, Section 2]). First of all it is easy to check that the flow preserves the volume of the set E_t since $\frac{d}{dt}|E_t| = -\int_{\partial E_t}(H^s_{E_t}(y)-\overline{H^s_{E_t}})dH^{n-1}(y)=0$, moreover

$$\frac{d}{dt}\mathrm{Per}_s(E_t) = -\int_{\partial E_t}(H^s_{E_t}(y)-\overline{H^s_{E_t}})H^s_{E_t}(y)dH^{n-1}(y) = -\|H^s_{E_t}-\overline{H^s_{E_t}}\|^2_{L^2(\partial E_t)}. \tag{3.5}$$

In particular this implies that $\mathrm{Per}_s(E_t)\leqslant \mathrm{Per}_s(E_0)$, for all $t\in(0,T_0)$.

We compute the barycenter of E_t, by using polar coordinates:

$$b_{E_t} = \frac{1}{|E_t|}\int_{E_t} y\,dy = \frac{1}{|B|(n+1)}\int_{\partial B} x(1+u(x,t))^{n+1}dH^{n-1}(x).$$

Assuming that δ is sufficiently small, and recalling (3.3) we may perform a Taylor expansion getting

$$b_{E_t} = \frac{1}{|B|}\int_{\partial B} x\left[u(x,t)+\frac{n}{2}u^2(x,t)(1+\delta O(1))\right]dH^{n-1}(x)$$

where $O(1)$ is a generic bounded function on ∂B, so that $|b_{E_t}|\leqslant C(n)\|u\|_{L^2(\partial B)}$ for some dimensional constant $C(n)$. In particular this implies that $E_t - b_t$ is a nearly spherical set on ∂B with height function $\tilde u(x,t) := u(x,t)-(b_{E_t}\cdot x)x$ which still satisfies

$$\sup_{t\in[0,T_0)}\|\tilde u\|_{C^1}\leqslant C(n)\delta_0$$

for some $C(n)>0$ dimensional constant. This implies that, eventually choosing a smaller δ_0, we may apply the quantitative Alexandrov inequality (2.5) obtained in Theorem 2.2 to the set $E_t - b_{E_t}$. Recalling that the fractional perimeter and the fractional mean curvatures are independent by spatial translations, this inequality reads as follows:

$$\mathrm{Per}_s(E_t)-\mathrm{Per}_s(B)\leqslant K(n,s)\|H^s_{E_t}-\overline{H^s_{E_t}}\|^2_{L^2(\partial E_t)}. \tag{3.6}$$

We define the function

$$H(t) = \left[\mathrm{Per}_s(E_t)-\mathrm{Per}_s(B)\right]^{\frac12}.$$

Obvioulsy, by (3.5), $H(t)$ is decreasing in time and $H(t)\leqslant\sqrt{\mathrm{Per}_s(E_0)-\mathrm{Per}_s(B)}$. Using (3.5) and (3.6), and recalling (3.4), we get the following

$$\begin{aligned}\frac{d}{dt}H(t) &= -\frac12\frac{1}{H(t)}\|H^s_{E_t}-\overline{H^s_{E_t}}\|^2_{L^2(\partial E_t)} \leqslant -\frac{1}{2\sqrt{K(n,s)}}\|H^s_{E_t}-\overline{H^s_{E_t}}\|_{L^2(\partial E_t)} \\ &\leqslant -\frac{1}{2\sqrt{K(n,s)C(n,s)}}\|u_t\|_{L^2(\partial B)}.\end{aligned} \tag{3.7}$$

Let us fix $0 \leqslant t_1 < t_2 < T_0$, and integrate (3.7) between t_1 and t_2. For some constant $C(n,s) > 0$ depending on n, s, we have that

$$H(0) \geqslant H(t_1) - H(t_2) \geqslant C(n,s) \int_{t_1}^{t_2} \|u_t\|_{L^2(\partial B)} \geqslant C(n,s) \|u(\cdot,t_2) - u(\cdot,t_1)\|_{L^2(\partial B)} \tag{3.8}$$

where we used that for a smooth function $f : \partial B \times [t_1, t_2] \to \mathbb{R}$ there holds $\int_{t_1}^{t_2} \|f\|_{L^2(\partial B)} \geqslant \left\| \int_{t_1}^{t_2} f \right\|_{L^2(\partial B)}$, see e.g. [24, proof of Theorem 1.2].

We recall now the Fuglede type inequality (2.6) provided in [15, Theorem 6.2] which gives

$$H(0) = \sqrt{\mathrm{Per}_s(E_0) - \mathrm{Per}_s(B)} \leqslant \sqrt{K(n,s)[u_0]^2_{\frac{1+s}{2}}} \leqslant C(n,s)\varepsilon.$$

So, (3.8) implies for all $t < T_0$,

$$\|u(\cdot,t)\|_{L^2(\partial B)} \leqslant C(n,s)\|u(\cdot,t) - u_0(\cdot)\|_{L^2(\partial B)} + \|u_0\|_{L^2(\partial B)} \leqslant (C(n,s)+1)\varepsilon.$$

By Gagliardo-Nirenberg-Sobolev inequality, for $k > m \geqslant 2$ we get

$$\|u(\cdot,t)\|_{H^m(\partial B)} \leqslant C(n) \|u(\cdot,t)\|^{1-\frac{m}{k}}_{L^2(\partial B)} \|u(\cdot,t)\|^{\frac{m}{k}}_{H^k(\partial B)}$$

for all $t \in (T_0/2, T_0)$. Since $\|u(\cdot,t)\|_{H^k(\partial B)} \leqslant C(n)\|u(\cdot,t)\|_{C^k(\partial B)}$, by the previous estimate and by Theorem 3.2 we get

$$\|u(\cdot,t)\|_{H^m(\partial B)} \leqslant C(n,s) T_0^{-m(k-1)!} C_k^{\frac{m}{k}} \varepsilon^{1-\frac{m}{k}},$$

where the constants C_k depend only on n, s, C and, in particular, do not depend on higher derivatives of u_0. By Sobolev embedding, taking m sufficiently large, we conclude that

$$\|u(\cdot,t)\|_{C^{1,s+\alpha}} \leqslant C(n,s) T_0^{-m(k-1)!} C_k^{\frac{m}{k}} \varepsilon^{1-\frac{m}{k}}.$$

Observe that we can choose $\varepsilon > 0$ sufficiently small in order to have that $\|u(\cdot,t)\|_{C^{1,s+\alpha}} < \delta_0$ and so, we may apply again Theorem 3.2, to extend the solution on a time interval $[0, 2T_0)$. By iterating the argument, we conclude that the solution exists smooth for all $t \geqslant 0$.

Note that by using (3.6) and (3.7), we have also that for all $t \geqslant 0$

$$\frac{d}{dt} H(t) \leqslant -\frac{C(n,s)}{2K(n,s)} H(t),$$

which implies that $H(t) \leqslant H(0)e^{-C(n,s)t}$ and so in particular

$$0 \leqslant \mathrm{Per}_s(E_t) - \mathrm{Per}_s(B) \leqslant (\mathrm{Per}_s(E_0) - \mathrm{Per}_s(B))e^{-2C(n,s)t}.$$

Moreover, the estimate on $H(t)$ implies, through (3.8), that $u(\cdot,t)$ is a Cauchy sequence in $L^2(\partial B)$ as $t \to +\infty$, that is for all $t_2 > t_1$

$$\|u(\cdot,t_2) - u(\cdot,t_1)\|_{L^2(\partial B)} \leqslant C(n,s) H(0) e^{-C(n,s)t_1}$$

and by the same argument as before based on Gagliardo-Nirenberg-Sobolev inequality, and Sobolev embedding, it is also a Cauchy sequence in $C^m(\partial B)$ as $t \to +\infty$ for all $m \geqslant 1$. This implies that u converges to some limit function $\bar{u} : \partial B \to \mathbb{R}$ as $t \to +\infty$ in $C^m(\partial B)$. Therefore, $\bar{E} = \{rx,\ r \in [0, 1 + \bar{u}(x)], x \in \partial B\}$ is a regular set which solves $H^s_{\bar{E}}(y) = \overline{H^s_{\bar{E}}}$ for all $y \in \partial \bar{E}$. So, by Theorem 3.1 we conclude that $\bar{E} = B + \bar{b}$ for some $\bar{b} \in \mathbb{R}^n$. $\square$

We conclude observing that the previous argument gives also an improvement of a result on the long time behavior of the flow (1.2) obtained in [12] for convex initial sets E_0 under the assumption that the flow exists smooth for all times. More precisely in [12] it is assumed the following regularity condition:

Assumption 3.1. *If $H^s_{E_t}$ is bounded on E_t for all $t \in [0, T_0)$ for some $T_0 \leqslant T$, where T is the maximal time of existence of the flow* (1.2)*, then the $C^{2,\beta}$ norm of ∂E_t, up to translations, is also bounded for some $\beta > s$ by a constant only depending on the supremum of H_s. In addition either $T = T_0 = +\infty$ or $T_0 < T$.*

A particular case in which assumption (3.1) has been proven to hold for the fractional mean curvature flow (1.1) is the case the initial set is the subgraph of a Lipschitz continuous function and has bounded fractional curvature, see [8].

We recall the result in [12].

Theorem 3.4 ([12])**.** *Let E_0 be a smooth compact convex set. Let E_t be a solution to* (1.2) *in $[0, T)$, where T is the maximal time of existence, and assume that* (3.1) *holds. Then the flow E_t is defined for all times $t \in [0, +\infty)$, E_t is smooth and convex, and there exist $b_t \in \mathbb{R}^n$ such that $E_t - b_t$ converges in C^2 as $t \to +\infty$ to a ball with volume $|E_0|$.*

Our argument provide a refinement of the previous result, ruling out the translations in time:

Corollary 3.5. *Under the assumption of Theorem 3.4, then E_t converges exponentially fast in C^∞ as $t \to +\infty$ to a ball with volume $|E_0|$.*

Proof. Without loss of generality we assume that $|E_0| = |B|$. By Theorem 3.4, we have that for $\varepsilon > 0$, there exists t_ε, b_{t_ε} such that $E_{t_\varepsilon} - b_{t_\varepsilon}$ is a C^2 set with $\sup_{x \in ((E_{t_\varepsilon} - b_{t_\varepsilon}) \Delta B} d(x, \partial B) \leqslant C\varepsilon$ and $|\nu_{E_{t_\varepsilon} - b_{t_\varepsilon}}(y) - y| \leqslant C\varepsilon$ for all $y \in \partial(E_{t_\varepsilon} - b_{t_\varepsilon})$. Then $E_{t_\varepsilon} - b_{t_\varepsilon}$ can be written as a nearly spherical set on B, with height function u_ε with $\|u_\varepsilon\|_{C^1} \leqslant C\varepsilon$. We apply now Theorem 3.3 to the flow starting from $E_{t_\varepsilon} - b_{t_\varepsilon}$ and we conclude that if we choose $\varepsilon > 0$ sufficiently small, we obtain that $E_t - b_{t_\varepsilon} - \bar{b} \to B$ in C^∞, as $t \to +\infty$, for some $\bar{b} \in \mathbb{R}^n$ with exponential rate of convergence. $\square$

4. Evolution of periodic graphs

In this last section we show that similar arguments as for nearly spherical sets can be used also to provide the exponential convergence to an hyperplane of entire periodic graphs in $\mathbb{R}^n$, evolving by the fractional mean curvature flow (1.1). More precisely we consider the geometric flow (1.1) under the additional assumption that the boundary of the initial datum E_0 can be written as a periodic entire graph on an hyperplane, that is, there exists $e \in \mathbb{R}^n$, such that $\nu(x) \cdot e > 0$ for every $x \in \partial E_0$. By monotonicity of the flow it is possible to show that the evolution E_t maintains this property for all positive times $t > 0$, that is, $\nu(x_t) \cdot e > 0$ for every $x_t \in \partial E_t$.

Note that, without loss of generality, we may assume that $e = e_n$, up to a rotation of coordinates.

So, let us consider a set E_0 which is given by the subgraph of a Lipschitz continuous, periodic function u_0. Without loss of generality, we will assume that the periodicity cell of u_0 is $[0, 1]^{n-1}$, so u_0 is $\mathbb{Z}^{n-1}$ periodic.

It is standard to show (see [8]) that if $E_0 = \{(x,z) \ : z \leqslant u_0(x), x \in \mathbb{R}^{n-1}\}$, then the solution E_t of (1.1) coincides with $\{(x,z) \ : z \leqslant u(x,t), x \in \mathbb{R}^{n-1}\}$, where u solves the following nonlocal quasilinear system:

$$\begin{cases} u_t(x,t) = & -H^s_{E_t}(x,u(x,t))\sqrt{1+|\nabla u(x,t)|^2} \\ u(x,0) = & u_0(x). \end{cases} \tag{4.1}$$

Moreover if the initial datum u_0 is Lipschitz continuous, and $\mathbb{Z}^{n-1}$ periodic, then by comparison arguments we get that $u(\cdot,t)$ is Lipschitz continuous (with Lipschitz norm bounded by the Lipschitz norm of u_0) and $\mathbb{Z}^{n-1}$ periodic.

We recall a result about existence of a smooth solution to (1.1) and long time behavior obtained in [8, Theorem 3.3, Proposition 6.1].

Theorem 4.1 ([8])**.** *Assume that $E_0 = \{(x,z) \ : z \leqslant u_0(x), x \in \mathbb{R}^{n-1}\}$, and that the function u_0 is $\mathbb{Z}^n$ periodic and satisfies $u_0 \in C^{1,s+\alpha}(\mathbb{R}^{n-1})$ with $\|u\|_{C^{1,s+\alpha}(\mathbb{R}^{n-1})} \leqslant C$ for some $\alpha > 0$. Then the flow $E_t = \{(x,z) \ : z \leqslant u(x,t), x \in \mathbb{R}^{n-1}\}$ of (1.1) starting from E_0 exists smooth for every time $t > 0$, with norms bounded in $\mathbb{R}^{n-1} \times [t_0, +\infty)$ by constants depending on t_0, C. Moreover there exists $c \in \mathbb{R}$ such that $u(x,t) \to c$ as $t \to +\infty$ uniformly in $C^1(\mathbb{R}^{n-1})$.*

In order to improve this convergence result to exponential convergence in C^∞, we need first of all to derive an analog of the inequality (2.5) obtained in Theorem (2.2) for a $C^{1,1}$ function $u_0 : R^{n-1} \to \mathbb{R}$, which is $\mathbb{Z}^{n-1}$ periodic and satisfies $\|u_0\|_{C^1} \leqslant \varepsilon$, for $\varepsilon > 0$ sufficiently small.

For a set E which is given by the subgraph of a periodic function u, we define the periodic fractional perimeter as follows, by considering the localized version of (1.1) on the set $[0,1]^{n-1} \times \mathbb{R}$:

$$\begin{aligned} \mathrm{Per}^p_s(E) \ &:= \int_{E\cap([0,1]^{n-1}\times\mathbb{R})} \int_{\mathbb{R}^n\setminus E} \frac{1}{|x-y|^{n+s}} dydx \\ &= \int_{[0,1]^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{u(y)}^{+\infty} \int_{-\infty}^{u(x)} \frac{1}{(|x-y|^2+(w-z)^2)^{\frac{n+s}{2}}} dwdzdydx. \end{aligned} \tag{4.2}$$

We recall that $\partial E = \{(x,u(x)), \ : \ x \in \mathbb{R}^{n-1}\}$, and for $p \in \partial E$, where $p = (x,u(x))$ for some $x \in \mathbb{R}^{n-1}$, the exterior normal to E is given by $\nu(p) = \frac{(-\nabla u(x),1)}{\sqrt{1+|\nabla u(x)|^2}}$. So, for $p = (x,u(x)) \in \partial E$ we have that $H^s_E(x,u(x))$ as defined in (1.1) coincides with

$$\begin{aligned} H^s_E(x,u(x)) \ &= \ \frac{2}{s}\int_{\mathbb{R}^{n-1}} \frac{(y-x, u(y)-u(x))\cdot(-\nabla u(y),1)}{(|y-x|^2+(u(x)-u(y))^2)^{\frac{n+s}{2}}} dy \\ &= \ \frac{2}{s}\int_{\mathbb{R}^{n-1}} \frac{u(y)-u(x)+(x-y)\cdot\nabla u(y)}{(|y-x|^2+(u(x)-u(y))^2)^{\frac{n+s}{2}}} dy. \end{aligned}$$

We introduce the (squared) fractional Gagliardo seminorm of u (recalling that the periodicity cell of u is $[0,1]^{n-1}$) which is defined as

$$[u]^2_{\frac{1+s}{2}} := \int_{[0,1]^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{(u(x)-u(y))^2}{|x-y|^{n+s}} dxdy. \tag{4.3}$$

Moreover we will indicate with $\|u\|_2^2$ the squared L^2 norm of u on its periodicity cell, that is $\int_{[0,1]^{n-1}} u^2(x)dx$. We recall the following Poincarè type inequality, see [3]: there exists a dimensional constant such that if u is a periodic function with $\int_{[0,1]^{n-1}} u(x)dx = 0$, there holds

$$[u]^2_{\frac{1+s}{2}} \geqslant \frac{C(n)}{1-s}\|u\|_2^2. \tag{4.4}$$

We prove now a rigidity type result in the same spirit of (2.6).

Lemma 4.2. *Assume that E is the subgraph E of a periodic function $u \in C^{1,1}(\mathbb{R}^{n-1})$. Let H_c be the hyperplane $\{(x,z)\ x \in \mathbb{R}^{n-1}, z \leqslant c\}$. Then there holds*

$$\frac{1}{2(1+4\|\nabla u\|_\infty^2)^{\frac{n+s}{2}}}[u]^2_{\frac{1+s}{2}} \leqslant \mathrm{Per}_s^p(E) - \mathrm{Per}_s^p(H_c) \leqslant \frac{1}{2}[u]^2_{\frac{1+s}{2}}.$$

Proof. Recalling the definition (4.2) and using the periodicity of u, we observe that

$$\begin{aligned}\mathrm{Per}_s^p(E) - \mathrm{Per}_s^p(H_c) &= \frac{1}{2}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{u(y)}^{+\infty}\int_{-\infty}^{u(x)} \frac{1}{(|x-y|^2+(w-z)^2)^{\frac{n+s}{2}}}dwdzdydx \\ &+ \frac{1}{2}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{u(x)}^{+\infty}\int_{-\infty}^{u(y)} \frac{1}{(|x-y|^2+(w-z)^2)^{\frac{n+s}{2}}}dwdzdydx \\ &- \int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{c}^{+\infty}\int_{-\infty}^{c} \frac{1}{(|x-y|^2+(w-z)^2)^{\frac{n+s}{2}}}dwdzdydx.\end{aligned}$$

Observe that

$$\begin{aligned}\int_{-\infty}^{a}\int_{b}^{+\infty} + \int_{-\infty}^{b}\int_{a}^{+\infty} &= \int_{-\infty}^{b}\int_{b}^{+\infty} + \int_{b}^{a}\int_{b}^{+\infty} + \int_{-\infty}^{a}\int_{a}^{+\infty} + \int_{a}^{b}\int_{a}^{+\infty} \\ &= \int_{-\infty}^{b}\int_{b}^{+\infty} + \int_{-\infty}^{a}\int_{a}^{+\infty} + \int_{a}^{b}\int_{a}^{b}.\end{aligned}$$

So, substituting in the previous formula with $u(x) = a$, $u(y) = b$, we get

$$\mathrm{Per}_s^p(E) - \mathrm{Per}_s^p(H_c) = \frac{1}{2}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{u(x)}^{u(y)}\int_{u(x)}^{u(y)} \frac{1}{(|x-y|^2+(w-z)^2)^{\frac{n+s}{2}}}dwdzdydx.$$

So, it is immediate to check that

$$\begin{aligned}\mathrm{Per}_s^p(E) - \mathrm{Per}_s^p(H_c) &\leqslant \frac{1}{2}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\int_{u(x)}^{u(y)}\int_{u(x)}^{u(y)} \frac{1}{|x-y|^{n+s}}dwdzdydx \\ &= \frac{1}{2}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{(u(x)-u(y))^2}{|x-y|^{n+s}}dxdy = \frac{1}{2}[u]^2_{\frac{1+s}{2}}.\end{aligned}$$

On the other hand, by a simple change of variables

$$\begin{aligned}\mathrm{Per}_s^p(E) - \mathrm{Per}_s^p(H_c) &= \frac{1}{2}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^{\frac{u(y)-u(x)}{|x-y|}}\int_0^{\frac{u(y)-u(x)}{|x-y|}} \frac{|x-y|^2}{|x-y|^{n+s}(1+(w-z)^2)^{\frac{n+s}{2}}}dwdzdydx \\ &\geqslant \frac{1}{2(1+4\|\nabla u\|_\infty^2)^{\frac{n+s}{2}}}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^{\frac{u(y)-u(x)}{|x-y|}}\int_0^{\frac{u(y)-u(x)}{|x-y|}} \frac{|x-y|^2}{|x-y|^{n+s}}dwdzdydx \\ &= \frac{1}{2(1+4\|\nabla u\|_\infty^2)^{\frac{n+s}{2}}}\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{(u(x)-u(y))^2}{|x-y|^{n+s}}dxdy.\end{aligned}$$

□

We introduce the fractional Laplacian of order $1+s$, which can be defined (up to constants depending on s and n) as

$$(-\Delta)^{\frac{1+s}{2}}u(x)=2\int_{\mathbb{R}^{n-1}}\frac{u(x)-u(y)}{|x-y|^{n+s}}dy. \tag{4.5}$$

By periodicity of u there holds $\int_{[0,1]^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{u(x)(u(x)-u(y))}{|x-y|^{n+s}}dydx=\int_{\mathbb{R}^{n-1}}\int_{[0,1]^{n-1}}\frac{u(x)(u(x)-u(y))}{|x-y|^{n+s}}dydx$, therefore we get

$$[u]^2_{\frac{1+s}{2}}=\int_{[0,1]^{n-1}}u(x)(-\Delta)^{\frac{1+s}{2}}u(x)dx. \tag{4.6}$$

Lemma 4.3. *Let E be the subgraph of a periodic function $u\in C^{1,1}(\mathbb{R}^{n-1})$, with $\|\nabla u\|_{C^0}\leqslant 1$. Then there holds*

$$\int_{[0,1]^{n-1}}u(x)H^s_E(x,u(x))dx=[u]^2_{\frac{1+s}{2}}(1+O(\|\nabla u\|_{C^0})). \tag{4.7}$$

Moreover there exists $\varepsilon_0=\varepsilon_0(n)\in(0,1)$ and $C(n)>0$ such that if $\|\nabla u\|_{C^0}<\varepsilon_0$, and $\int_{[0,1]^{n-1}}u(x)=0$, there holds

$$\|H^s_E\|^2_{L^2([0,1]^{n-1})}\geqslant C(n)[u]^2_{\frac{1+s}{2}}.$$

Finally by Lemma 4.2 we also have

$$\|H^s_E\|^2_{L^2([0,1]^{n-1})}\geqslant 2C(n)(\mathrm{Per}^p_s(E)-\mathrm{Per}^p_s(H_c)) \tag{4.8}$$

for every hyperplane $H_c=\{(x,z)\ x\in\mathbb{R}^{n-1},z\leqslant c\}$.

Proof. We write the following Taylor expansions, where $O(\|\nabla u\|_{C^0})$ is any function f such that $|f(x)|\leqslant C\|\nabla u\|_{C^0}$ for all $x\in\mathbb{R}^{n-1}$:

$$\frac{1}{(|y-x|^2+(u(x)-u(y))^2)^{\frac{n+s}{2}}}=\frac{1}{|x-y|^{n+s}}-\frac{n+s}{|x-y|^{n+s}}\frac{(u(x)-u(y))^2}{|x-y|^2}(1+O(\|\nabla u\|^2_{C^0}),$$

and so, substituting in the previous formula for H^s_E we get

$$H^s_E(x,u(x))=\frac{2}{s}\int_{\mathbb{R}^{n-1}}\frac{u(y)-u(x)+(x-y)\cdot\nabla u(y)}{|x-y|^{n+s}}(1+O(\|\nabla u\|^2_{C^0})dy.$$

Now, we fix $x\in[0,1]^{n-1}$ and we define $T(y)=\frac{(u(y)-u(x))(y-x)}{|x-y|^{n+s}}$ for $y\in\mathbb{R}^{n-1}$. We observe that, by the same computations as in (2.20), we have

$$\mathrm{div}T(y)=-\frac{(s+1)(u(y)-u(x))}{|x-y|^{n+s}}+\frac{\nabla u(y)\cdot(y-x)}{|x-y|^{n+s}}.$$

It is easy to check that $\int_{\mathbb{R}^{n-1}}\mathrm{div}T(y)dy=\lim_{R\to+\infty}\int_{B(x,R)}\mathrm{div}T(y)dy=0$ and so we get that

$$\int_{\mathbb{R}^{n-1}}\frac{(x-y)\cdot\nabla u(y)}{|x-y|^{n+s}}dy=-(s+1)\int_{\mathbb{R}^{n-1}}\frac{u(y)-u(x)}{|x-y|^{n+s}}dy.$$

Therefore, we conclude that the fractional mean curvature is given by H^s_E we get

$$H^s_E(x,u(x))=-2\int_{\mathbb{R}^{n-1}}\frac{u(y)-u(x)}{|x-y|^{n+s}}(1+O(\|\nabla u\|^2_{C^0})dy.$$

This implies the conclusion (4.7), by recalling the periodicity of u.

Now, assume that u has zero average. We apply Hölder inequality to (4.7) and we get, recalling also the Poincarè inequality (4.4) and choosing ε_0 sufficiently small

$$\begin{aligned} \|u\|_2\|H^s_E\|_2 &\geqslant \int_{[0,1]^{n-1}} u(x)H^s_E(x,u(x))dx \\ &\geqslant \frac{1}{2}[u]^2_{\frac{1+s}{2}} \geqslant \frac{1}{4}[u]^2_{\frac{1+s}{4}} + \frac{C(n)}{4(1-s)}\|u\|_2^2. \end{aligned}$$

We fix $\delta = C(n)/2$ and we conclude by Young inequality that

$$\begin{aligned} \frac{1}{C(n)}\|H^s_E\|_2^2 &= \frac{1}{2\delta}\|H^s_E\|_2^2 \geqslant \frac{1}{4}[u]^2_{\frac{1+s}{2}} + \left(\frac{C(n)}{4(1-s)} - \frac{\delta}{2}\right)\|u\|_2^2 \\ &= \frac{1}{4}[u]^2_{\frac{1+s}{2}} + \frac{C(n)s}{4(1-s)}\|u\|_2^2 \end{aligned}$$

which implies the conclusion. □

We are ready to prove the exponential convergence which improve Theorem 4.1.

Corollary 4.4. *Assume that $E_0 = \{(x,z) \ : z \leqslant u_0(x), x \in \mathbb{R}^{n-1}\}$, where $u_0 \in C^{1,s}(\mathbb{R}^{n-1})$ is $\mathbb{Z}^{n-1}$ periodic with $\|u\|_{C^{1,s}(\mathbb{R}^{n-1})} \leqslant C$. Then the flow $E_t = \{(x,z) \ : z \leqslant u(x,t), x \in \mathbb{R}^{n-1}\}$ of (1.1) starting from E_0 exists smooth for every time $t > 0$, and moreover there exists $\bar{c} \in \mathbb{R}$ such that for all $m \geqslant 1$ there exists a constant $C(m,n,s) > 0$ such that*

$$\|u(x,t) - \bar{c}\|_{C^m} \leqslant C(m,n,s)\sqrt{[u_0]^2_{\frac{1+s}{2}}}\, e^{-C(m,n,s)t} \qquad \forall t \geqslant 0.$$

Proof. By Theorem 4.1, we know that the solution exists smooth for all times, and moreover $u \to \bar{c}$ uniformly in C^1 as $t \to +\infty$. So, for every $\varepsilon > 0$ there exists t_ε such that $\|\nabla u(\cdot,t)\|_{C^0} \leqslant \varepsilon$ for all $t > t_\varepsilon$. Let $m_t = \int_{[0,1]^{n-1}}(u(x,t) - \bar{c})dx$. If $\varepsilon > 0$ is sufficiently small, I may apply to $u(x,t) - \bar{c} - m_t$ the results in Lemma 4.3 for all $t > t_\varepsilon$. In particular (4.8) reads as follows, observing that the fractional mean curvature and the fractional perimeter are independent by translations,

$$\|H^s_{E_t}\|^2_{L^2([0,1]^{n-1})} \geqslant 2C(n)(\mathrm{Per}^p_s(E_t) - \mathrm{Per}^p_s(H_c)).$$

We proceed as in the proof of Theorem 3.3. First of all by using (4.1) we have that, for $\varepsilon > 0$ small, and $t > t_\varepsilon$,

$$\|u_t\|^2_{L^2([0,1]^{n-1})} = \int_{[0,1]^{n-1}} (H^s_{E_t}(x,u(x,t)))^2(1 + |\nabla u(x,t)|^2)dx \leqslant 2\|H^s_{E_t}\|^2_{L^2([0,1]^{n-1})}.$$

We define $H(t) = \sqrt{\mathrm{Per}^p_s(E_t) - \mathrm{Per}^p_s(H_c)}$, so that, by using the previous inequalities and denoting $C(n)$ a dimensional constant, which may change from line to line

$$\begin{aligned} \frac{d}{dt}H(t) &= \frac{1}{2H(t)}\frac{d}{dt}\mathrm{Per}^p_s(E_t) = -\frac{1}{2H(t)}\int_{\partial E_t}(H^s_{E_t}(x))^2 dH^{n-1}(x) \\ &\leqslant -\frac{1}{2H(t)}\|H^s_{E_t}\|^2_{L^2([0,1]^{n-1})} \leqslant -\frac{\sqrt{2C(n)}}{2}\|H^s_{E_t}\|_{L^2([0,1]^{n-1})} \leqslant -C(n)\|u_t\|_{L^2([0,1]^{n-1})}. \end{aligned}$$

Moreover by the same computation we have that $\frac{d}{dt}H(t) \leqslant -C(n)H(t)$, so $H(t) \leqslant H(t_\varepsilon)e^{-C(n)(t-t_\varepsilon)}$ for all $t > t_\varepsilon$.

As in the proof of Theorem 3.3, we deduce that $\|u(\cdot,t) - \bar{c}\|_{L^2([0,1]^{n-1})}$ is a Cauchy sequence as $t \to +\infty$, and moreover, by the estimate in Theorem 4.1 and Sobolev embedding, $\|u(\cdot,t) -$

$\bar{c}\|_{C^m([0,1]^{n-1})}$ is a Cauchy sequence as $t \to +\infty$ for all $m \geqslant 1$, with exponential rate of convergence. This gives the thesis. $\square$

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