NONLOCAL MINIMAL CLUSTERS IN THE PLANE

ANNALISA CESARONI AND MATTEO NOVAGA

Abstract. We show existence of nonlocal minimal cluster with Dirichlet boundary data. In two dimensions we show that, if the fractional parameter s is sufficiently close to 1, the only singular minimal cone consists of three half-lines meeting at 120 degrees at a common end-point. In the case of fractional perimeter with weights, we show that there exists a unique minimal cone with three phases.

Contents

1. Introduction

A cluster is a family $\mathcal{E} = (E_i)_{i=1,\dots k}$ of disjoint measurable subsets of $\mathbb{R}^n$ such that $\cup_i E_i = \mathbb{R}^n$, up to a negligible set. We call each set E_i a phase of the cluster. Following [8], for an open set $\Omega \subset \mathbb{R}^n$ we define the fractional perimeter of $\mathcal{E}$ relative to Ω as follows:

$$\mathcal{P}_s(\mathcal{E};\Omega) = \sum_{1\le i\le k} \mathrm{Per}_s(E_i;\Omega). \tag{1}$$

The nonlocal perimeter relative to Ω for a set $E \subseteq \mathbb{R}^n$ is the nonlocal interaction between E and its complement $\mathbb{R}^n \setminus E$ relative to Ω:

$$\mathrm{Per}_s(E;\Omega) := J_s(E\cap\Omega, \mathbb{R}^n\setminus E) + J_s(\Omega\setminus E, E\setminus\Omega), \tag{2}$$

where

$$J_s(A,B) := \int_A\int_B \frac{1}{|x-y|^{n+s}}dxdy \qquad \text{for } A,B\subset\mathbb{R}^n,\ |A\cap B| = 0.$$

Energies as (1) and more generally weighted fractional perimeters such as

$$\mathcal{P}_{s,c}(\mathcal{E};\Omega) := \sum_{1\le i\le k} c_i \mathrm{Per}_s(E_i;\Omega) \tag{3}$$

with $c = (c_i)_i$ with $c_i > 0$, arise naturally in the analysis of equilibria configurations of a mixture of immiscible fluids in a container Ω, where the fluids tend to occupy disjoint regions

1991 *Mathematics Subject Classification.* 49Q05 58E12 35R11.
Key words and phrases. Fractional perimeters, partition problems, minimal cones.

in such a way to minimize the total surface tension which is measured through nonlocal interaction energies, rather then through surface area as in the classical case.

In [8] the authors proved existence of isoperimetric clusters for fractional perimeters. In particular, they showed that there exists a minimizer of the energy (1) with $\Omega = \mathbb{R}^n$, among all clusters with multiple volume constraints. They also established the regularity of minimal clusters, showing that the singular set has Hausdorff dimension less than $n-2$ (discrete in the planar case $n = 2$), that outside from the singular set the boundary of the isoperimetric cluster is a hypersurface of class $C^{1,\alpha}$ for some $\alpha > 0$, and finally that the blow-up of the cluster at a singular point is a cone.

In this short note we consider minimizers of (1) in a bounded open set $\Omega \subset \mathbb{R}^2$, with Dirichlet data. More precisely, we fix the phases E_i outside Ω, so we fix an exterior datum

$$(\bar{E}_1, \bar{E}_2, \dots, \bar{E}_k) \qquad \bar{E}_i \subseteq \mathbb{R}^n \setminus \Omega, \forall i \quad \cup_i \bar{E}_i = \mathbb{R}^n \setminus \Omega, \tag{4}$$

and we provide existence of a solution to the following Dirichlet problem

$$\inf_{\{\mathcal{E},\ E_i \setminus \Omega = \bar{E}_i\}} \mathcal{P}_{s,c}(\mathcal{E};\Omega) \tag{5}$$

for $c = (c_i)_i$, with $c_i > 0$.

We are also interested in the analysis of singularities in dimension $n = 2$, in order to characterize fractional isoperimetric clusters in some basic cases. For instance, in Theorem 3.6 w consider the energy (1) (so with all weights equal to 1), and we show that for s sufficiently near to 1, the only singular minimal cone consists of three half-lines meeting at 120 degrees at a common end-point. In particular, this implies that the unique local minimizers for the fractional perimeter on clusters, for s sufficiently near to 1, are halph-spaces and such singular cone (up to translations and rotations). We recall that in the case of a single phase, halph-spaces are the unique local minimizers of the fractional perimeter, as it has been proved in [3, 6] (see also [5, 13] for the extension to more general energies).

To obtain our result, we first provide the Γ-convergence of the fractional perimeter of clusters to the classical perimeter as $s \to 1$, which is a generalization of the analogous result for a single phase given in [3, 7], and the Hausdorff convergence of minimizers which is obtained by exploiting the density estimates proved in [8].

Finally, we consider the analogous problem for weighted fractional perimeters, restricted to 3-clusters. In Proposition 4.3 we show that there exists a unique minimal 3-cone, whose opening angles are uniquely determined in terms of the weights c_i.

Aknowledgement. The authors wish to thank Valerio Pagliari and Alessandra Pluda for useful discussions on the topic of this paper. The authors are members and were supported by the INDAM/GNAMPA.

2. The Dirichlet problem

We start proving existence of minimizers for (5).

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set of finite perimeter and fix an exterior datum as in* (4). *Then, there exists a solution to the Dirichlet problem* (5).

Proof. First of all note that if we consider $\mathcal{E}$ defined as follows: $E_1 = \Omega \cup \bar{E}_i$, $E_j = \bar{E}_j$ for $j \neq 1$, then we get $\mathcal{P}_s(\mathcal{E};\Omega) \leq k \max c_i \mathrm{Per}_s(\Omega) < +\infty$ for all j, since Ω is bounded of finite perimeter (see [7]). The existence result is then obtained by the direct method of the calculus of variations, using the fact that $\mathrm{Per}_s(E)$ is a Gagliardo norm of χ_E, recalling that a uniform bound on the Gagliardo norm implies compactness in L^2, and that the norm is lower semicontinuous with respect to the L^1-convergence (see [14]). □

We recall the density estimates proved in [8], which are uniform with respect to $s \to 1$. We state them for minimizers of the Dirichlet problem (5).

Theorem 2.2 (Density estimates). *Let $s_0 \in (0,1)$, and let $\mathcal{E}$ be a minimizer of (5) for some $s \in [s_0, 1)$. Then there exist $\sigma_0 = \sigma_0(n, s_0, c), \sigma_1 = \sigma_1(n, s_0, c) \in (0,1)$ such that, if $x \in \partial E_i \cap \Omega$ for some i, then*

$$\sigma_0 \omega_n r^n \le |E_i \cap B(x,r)| \le \sigma_1 \omega_n r^n \qquad \forall r < d(x, \partial\Omega).$$

Proof. The proof can be obtained as a straightforward adaptation of the proof of Lemma 3.4 in [8]. We note that if we fix $x \in \Omega$, then $\mathcal{E}$ is a $(\Lambda, d(x, \partial\Omega))$ minimizer for every $\Lambda > 0$ and observing in the proof that the constant r_1 can be chosen equal to r_0 and that σ_0 is uniform as $s \to 1$. □

Remark 2.3. Looking at the proof in [8, Lemma 3.4], we get that this estimate degenerates as $s \to 0$, in fact $\lim_{s_0 \to 0^+} \sigma_0(n, s_0) = 0$.

We recall a result about density of polyhedral clusters with respect to the (weighted) local perimeter functionals such as (3), which has been obtained in [4]. We consider an adaptation of this this result in order to apply it to Dirichlet problems.

Definition 2.4 (Polyhedral cluster). A cluster $\mathcal{K} = (K_i)_{i=1,\dots,k}$ is polyhedral in an open set Ω if for every phase K_i there is a finite number of $(n-1)$-dimensional simplexes $T_1, \dots, T_k \subseteq \mathbb{R}^n$ such that ∂E_i coincides, up to a $\mathcal{H}^{n-1}$-null set, with $\cup_j T_j \cap \Omega$.

Definition 2.5. Let Ω be an open set of class C^1. For $\delta > 0$ we define

$$\Omega^\delta := \{x \in \mathbb{R}^n \ : \ d(x, \Omega) < \delta\} \qquad \Omega_\delta := \{x \in \Omega \ : \ d(x, \mathbb{R}^n \setminus \Omega) > \delta\}.$$

We say that a measurable set F is transversal to $\partial\Omega$ if

$$\lim_{\delta \to 0^+} \mathrm{Per}(F; \Omega^\delta \setminus \Omega_\delta) = 0.$$

We say that F is transversal to $\partial\Omega^+$ if

$$\lim_{\delta \to 0^+} \mathrm{Per}(F; \Omega^\delta \setminus \Omega) = 0.$$

A cluster is transversal to $\partial\Omega$ (resp. to $\partial\Omega^+$) if every phase is transversal.

Theorem 2.6. *Let Ω be a bounded open set with C^1 boundary, and let $\mathcal{F}$ be a cluster in Ω such that every phase F_i has finite perimeter in Ω. For every $\varepsilon > 0$ there exists a cluster $\mathcal{K}_\varepsilon$ which is polyhedral in Ω, such that $\mathcal{K}_\varepsilon \to \mathcal{F}$ in $L^1(\Omega)$ and $\mathcal{P}_c(\mathcal{K}_\varepsilon; \Omega) \to \mathcal{P}_c(\mathcal{F}; \Omega)$.*

Assume moreover that $\mathcal{F}$ is polyhedral in $\mathbb{R}^n \setminus \Omega$ and transversal to $\partial\Omega^+$. Then for every $\varepsilon > 0$ there exists a polyhedral cluster $\mathcal{K}_\varepsilon$ with the following properties:

i) $\mathcal{K}_\varepsilon \to \mathcal{F}$ *in* $L^1(\Omega)$,
ii) $\mathcal{K}_\varepsilon = \mathcal{F}$ *in* $\mathbb{R}^n \setminus \Omega$,
iii) $\mathcal{K}_\varepsilon$ *is transversal to* $\partial\Omega$,
iv) $\mathcal{P}_c(\mathcal{K}_\varepsilon; \Omega) \to \mathcal{P}_c(\mathcal{F}; \Omega)$ *as* $\varepsilon \to 0$.

Proof. The first part of the result is proved in [4, Theorem 2.1 and Corollary 2.4]. By inspecting the proof in [4] one can check that if the initial cluster is polyhedral outside Ω, then the approximating sequence of polyhedral clusters $\mathcal{K}_\varepsilon$ can be chosen in such a way that $\mathcal{K}_\varepsilon = \mathcal{F}$ in $\mathbb{R}^n \setminus \Omega^\varepsilon$.

We fix now $\delta > 0$ sufficiently small and we substitute $\mathcal{F}$ in $\Omega \setminus \overline{\Omega_\delta}$ with the reflection of $\mathcal{F}$ from $\Omega^\delta \setminus \overline{\Omega}$. The reflection is constructed as follows: We identify points in $\Omega^\delta \setminus \overline{\Omega}$ and points in $\Omega \setminus \overline{\Omega_\delta}$ by putting $x + t\hat{\nu}(x) = x - t\hat{\nu}(x)$ for $t \in (0, \delta)$, where $\hat{\nu}(x)$ is a C^1 function which coincides on $\partial\Omega$ with the outer normal at x. In this way we obtain a new cluster $\mathcal{F}_\delta$ which coincides with $\mathcal{F}$ in $(\mathbb{R}^n \setminus \Omega) \cup \Omega_\delta$, and which is the reflection of $\mathcal{F}$ in $\Omega \setminus \overline{\Omega_\delta}$. Note that, by construction, $\mathcal{F}_\delta$ is transversal to $\partial\Omega$. By using the previous result in the set Ω_δ, we construct a family of approximating polyhedral clusters $\mathcal{K}_{\varepsilon,\delta}$ for $\varepsilon \to 0$, which coincide with $\mathcal{F}_\delta$ in $\mathbb{R}^n \setminus (\Omega_\delta)^\varepsilon$. We choose now $\varepsilon = \varepsilon(\delta) < \delta$, so that $(\Omega_\delta)^{\varepsilon(\delta)} \subset \Omega$: Therefore $\mathcal{K}_{\varepsilon(\delta),\delta}$ is a polyhedral cluster which coincides with $\mathcal{F}_\delta$ in $\mathbb{R}^n \setminus (\Omega_\delta)^{\varepsilon(\delta)}$ and so in particular coincides with $\mathcal{F}$ in $\mathbb{R}^n \setminus \Omega$, and is transversal to $\partial\Omega$. Moreover $\mathcal{K}_{\varepsilon(\delta),\delta} \to \mathcal{F}$ in $L^1(\Omega)$ as $\delta \to 0$, and for every $\eta > 0$ sufficiently small, there holds $\mathcal{P}_c(\mathcal{K}_{\varepsilon(\delta),\delta}; \Omega_\eta) \to \mathcal{P}_c(\mathcal{F}; \Omega_\eta)$ as $\delta \to 0$. This implies the conclusion. □

We introduce the definition of Hausdorff convergence.

Definition 2.7. Let $E_n, E \subset \Omega$, where Ω is a open set. We say that $E_n \to E$ locally uniformly in Ω, if for any $\varepsilon > 0$ and any $\Omega' \subset\subset \Omega$, there exists $\bar{n}$ such that for all $n \geq \bar{n}$, we have that

$$\sup_{x \in E_n \cap \Omega'} d(x, E) \leq \varepsilon \quad \text{and} \quad \sup_{x \in (\Omega \setminus E_n) \cap \Omega'} d(x, \Omega \setminus E) \leq \varepsilon.$$

We provide a Γ-convergence result, which is based on the analogous result obtained for the single phase in [3, 7] and by the density of polyhedral clusters given in Theorem 2.6. From this result, by using the density estimates recalled in Theorem 2.2, we get uniform convergence of minimizers of the Dirichlet problem.

Theorem 2.8. *Let Ω be a C^1 bounded open set and let $\bar{\mathcal{E}}$ a cluster which is polyhedral in $\mathbb{R}^n \setminus \Omega$ and is transversal to $\partial\Omega^+$.*

For every sequence of positive numbers $c = (c_i)_i$, as $s \to 1$ there holds

$$(1-s)\mathcal{P}_{s,c}(\mathcal{E}; \Omega) \xrightarrow{\Gamma} \omega_{n-1}\mathcal{P}_c(\mathcal{E}; \Omega) = \omega_{n-1} \sum_i c_i \mathrm{Per}(E_i; \Omega)$$

with respect to the $L^1(\Omega)$-convergence, where the functionals $\mathcal{P}_{s,c}(\mathcal{E}; \Omega)$ and $\mathcal{P}_c(\mathcal{E}; \Omega)$ are defined only on clusters $\mathcal{E}$ such that $\mathcal{E} = \bar{\mathcal{E}}$ in $\mathbb{R}^n \setminus \Omega$, and extended as $+\infty$ elsewhere.

Let $\mathcal{E}_s = (E_1^s, \dots, E_k^s)$ is a sequence of minimizers for

$$\inf_{\{\mathcal{F},\ F_i \setminus \Omega = \bar{E}_i\}} \mathcal{P}_{s,c}(\mathcal{F}; \Omega)$$

then up to a subsequence $E_i^{s_n} \to E_i$ locally uniformly in Ω, where $\mathcal{E} = (E_1, \dots, E_k)$ is a minimizer of

$$\inf_{\{\mathcal{F},\ F_i \setminus \Omega = \bar{E}_i\}} \mathcal{P}_c(\mathcal{F}; \Omega).$$

Proof. Let $s \to 1$, $\mathcal{E}^s, \mathcal{E}$ clusters which coincide with $\bar{\mathcal{E}}$ outside Ω and such that $\mathcal{E}^s \to \mathcal{E}$ in $L^1(\Omega)$. Then using the $\Gamma-$liminf inequality for the single phase proved in [3, 7] we get

$$\begin{aligned}
\liminf_{s\to 1}(1-s)\mathcal{P}_{s,c}(\mathcal{E}^s;\Omega) &\geq \sum_{i=1}^{k} c_i \liminf_{s\to 1}(1-s)\mathrm{Per}_s(E_i^s;\Omega) \\
&\geq \omega_{n-1}\sum_{i=1}^{k} c_i \mathrm{Per}(E_i;\Omega) = \omega_{n-1}\mathcal{P}_c(\mathcal{E};\Omega). \qquad (6)
\end{aligned}$$

Fix now a cluster $\mathcal{E}$ which coincides with $\bar{\mathcal{E}}$ outside Ω. By the Γ-liminf inequality we can restrict to consider clusters whose phases have finite perimeter in Ω. By Theorem 2.6, for every ε, there exist polyhedral $\mathcal{K}_\varepsilon$ which are transversal to $\partial\Omega$, coincide with $\bar{\mathcal{E}}$ in $\mathbb{R}^n \setminus \Omega$, and satisfy $\mathcal{K}_\varepsilon \to \mathcal{E}$ in $L^1(\Omega)$ and $\mathcal{P}_c(\mathcal{K}_\varepsilon;\Omega) \to \mathcal{P}_c(\mathcal{E};\Omega)$ as $\varepsilon \to 0$. By [3, Lemma 8], there holds for all ε

$$\limsup_{s_n\to 1}(1-s_n)\mathcal{P}_{s_n,c}(\mathcal{K}_\varepsilon;\Omega) \leq \sum_{i=1}^{k} c_i \limsup_{s_n\to 1}(1-s_n)\mathrm{Per}_{s_n}(K_\varepsilon^i;\Omega)$$

$$\leq \omega_{n-1}\sum_{i=1}^{k} c_i \mathrm{Per}(K_\varepsilon^i;\Omega) = \omega_{n-1}\mathcal{P}_c(\mathcal{K}_\varepsilon;\Omega) \leq \omega_{n-1}\mathcal{P}_c(\mathcal{E};\Omega) + o_\varepsilon(1)$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. We conclude recalling that $\mathcal{K}_\varepsilon \to \mathcal{E}$ in $L^1(\Omega)$ as $\varepsilon \to 0$, and choosing $\varepsilon_n = \varepsilon(s_n) \to 0$ as $s_n \to 1$.

We show now that, by the density estimates in Theorem 2.2, we get that the convergence is locally uniform in Ω. Assume by contradiction that it is not true. Then, for some $\Omega' \subset\subset \Omega$ and for some $\varepsilon > 0$, either there exists $x_k \in E_i^{s_k} \cap \Omega'$ such that $d(x_k, E_i) > \varepsilon$ for all k or there exists $x_k \in (\Omega \setminus E_i^{s_k}) \cap \Omega'$ such that $d(x_k, \Omega \setminus E_i) > \varepsilon$. Let us consider the first case (the second is completely analogous). By the density estimates in Theorem 2.2, letting $2\delta = \min(d(\partial\Omega', \partial\Omega), \varepsilon)$ we get that $|E_i^{s_k} \cap B(x_k,\delta)| \geq \sigma_0\omega_n\delta^n$ for all k. Note that $A_k := E_i^{s_k} \cap B(x_k,\delta) \subset\subset \Omega$, $|A_k| > c > 0$ uniformly in k and $A_k \cap E_i = \emptyset$, in contradiction with the $L^1(\Omega)$-convergence of χ_{E_k} to χ_E. $\square$

3. Minimal cones

In this section we restrict to the 2-dimensional case.

Definition 3.1. A partition $\mathcal{C}$ is called a k-cone with vertex if it is invariant by dilatation, that is $\lambda\mathcal{C} = \mathcal{C}$ for every $\lambda > 0$, and it has k-phases $C_1, \dots, C_k$.

Note that a 2-cone is a half-space, more precisely is given by two phases which are both half-spaces.

We recall the definition of local minimizer (or minimizer up to compact perturbations) for (1).

Definition 3.2. We say that the cluster $\mathcal{E}$ is a local minimizer for (1) if for every $R > 0$ and every ball B_R of radius R, there holds

$$\mathcal{P}_s(\mathcal{E};B_R) \leq \mathcal{P}_s(\mathcal{F};B_R)$$

for all clusters $\mathcal{F}$, such that $F_i \setminus B_R = E_i \setminus B_R$ for all i.

Let us fix a partition $\mathcal{E}$ and a point $x \in \partial\mathcal{E}$. The blow-up of $\mathcal{E}$ at x is the cluster $\mathcal{E}_{x,r}$ defined by
$$E_i^{x,r} = \frac{1}{r}(E_i - x).$$
We recall the result in [8, Theorem 3.7], adapted to our problem.

Theorem 3.3. *Let $\mathcal{E}$ be a k-cluster which is a solution to the Dirichlet problem* (5)*, with a given boundary datum as in* (4)*, and let $x \in \partial\mathcal{E} \cap \Omega$. Then there exists a h-cone $\mathcal{C}$, with $h \le k$ such that, up to reordering eventually the phases, as $r \to 0$ there hold*
$$E_i^{x,r} \to C_i \text{ in } L^1_{loc} \text{ and locally uniformly} \qquad \forall i = 1, \dots, h.$$

The set of regular points of $\mathcal{E}$ coincide with the set of points such that there exists a 2-cone with phases H_i, H_j such that $E_i^{x,r} \to H_i, E_j^{x,r} \to H_j$, and $E_k^{x,r} \to \emptyset$ for $k \ne i, j$, in L^1_{loc}.

We now observe that there exists a unique 3-cone which is a stationary point for (1).

Lemma 3.4. *Among all* 3*-cones in $\mathbb{R}^2$, there exists a unique cone which is stationary for the functional in* (1)*, and the opening angles are equals, and coincide with* $2/3\pi$.

Proof. We consider a cone $\mathcal{C} = (C_1, C_2, C_3)$ with 3 half-lines and vertex x_0 which is stationary for the functional (1) (so, the first variation of (1) at every boundary point is 0). We denote with α_i the angle associated to the sector E_i, so $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$. Up to a translation we assume that the vertex of the cone is 0.

The stationarity condition reads
$$H_s(x, C_i) = H_s(x, C_j) \qquad \forall x \in \partial C_i \cap \partial C_j,\ x \ne 0 \tag{7}$$
where $H_s(x, C_i)$ is the fractional curvature at $x \in \partial C_i$.

It is easy to check that of $x \in \partial C_i \cap \partial C_j$, we have that
$$H_s(x, C_i) \le 0 \qquad \text{if and only if} \qquad \alpha_i \ge \pi. \tag{8}$$
Using this observation, (7), and the fact that $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, we have that $\alpha_i < \pi$.

We exploit now condition (7) for $i = 1, j = 2$ (all the other cases will be analogous). We assume without loss of generality that $\alpha_1 \ge \alpha_2$ and we write $C_1 = \tilde{C}_2 \cup B$, where $\tilde{C}_2$ is the symmetric of C_2 with respect to the half-line separating C_1, C_2 and B is a sector of the cone with opening angle $\alpha_1 - \alpha_2$. Let $\tilde{B} \subseteq C_3$ be the symmetric of B with respect to the half-line separating C_1, C_2. By symmetry properties of the kernel it is easy to check that
$$\begin{aligned} H_s(x, C_1) &= \int_{C_3} \frac{1}{|x-y|^{2+s}} dy - \int_B \frac{1}{|x-y|^{n+s}} dy = \int_{(C_3 \setminus \tilde{B}) - x} \frac{1}{|y|^{2+s}} dy, \\ H_s(x, C_2) &= \int_{C_3} \frac{1}{|x-y|^{2+s}} dy + \int_B \frac{1}{|x-y|^{2+s}} dy = \int_{(C_3 \cup B) - x} \frac{1}{|y|^{2+s}} dy. \end{aligned} \tag{9}$$
Note that $C_3 \setminus \tilde{B}$ is a sector of the cone with opening angle $\alpha_3 - \alpha_1 + \alpha_2 = 2\pi - 2\alpha_1 > 0$, whereas $C_3 \cup B$ is a sector of the cone with opening angle $\alpha_3 + \alpha_1 - \alpha_2 = 2\pi - 2\alpha_2 > 0$, and both are symmetric with respect to the half-line separating C_1, C_2. Therefore condition (7) implies that $2\pi - 2\alpha_1 = 2\pi - 2\alpha_2$. Repeating the argument we get that $\alpha_1 = \alpha_2 = \alpha_3$. ☐

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set containing the origin, let $k = 3$ and let $\bar{E}_i$ be the exterior datum defined as*
$$\bar{E}_i := \left\{ x \in \mathbb{R}^2 : x \cdot n_i > \frac{1}{2} \right\}, \qquad n_i := \left(\cos\left(\frac{2}{3}\pi i\right), \sin\left(\frac{2}{3}\pi i\right) \right).$$

Then there exists $s_0 \in (0,1)$ *such that every minimizer of the Dirichlet problem* (1) *for* $s > s_0$ *has a nonempty singular set in* Ω*.*

Proof. Let $\mathcal{E}_s = (E_1^s, E_2^s, E_3^s)$ be a solution to the Dirichlet problem (1). By Theorem 2.8, up to a subsequence we get that $E_i^s \to \bar{E}_i$ locally uniformly in Ω as $s \to 1$, for $i \in \{1,2,3\}$. Let $R > 0$ be such that $B(0,R) \subset \Omega$.

Assume by contradiction that there is a sequence $s_n \to 1$ such that $\partial E_i^{s_n} \cap \Omega$ is of class C^1 for all n's. There exists $r \in (0,R)$ such that, for $i \neq j$, the set $\gamma_{ij}^n := \partial E_i^{s_n} \cap \partial E_j^{s_n} \cap B(0,r)$ is a finite number of C^1 curves with endpoints on $\partial B(0,r)$, converging to the segment $\partial \bar{E}_i \cap \partial \bar{E}_j \cap B(0,r)$ as $n \to +\infty$ in the Hausdorff distance. In particular, given $\varepsilon > 0$, for n large enough the set γ_{ij}^n divides the circle $B(0,r)$ into a finite number of small connected components and one large connected component of area greater than $|B(0,r)| - \varepsilon$. As a consequence either the set $E_i^{s_n} \cap B(0,r)$ or $E_j^{s_n} \cap B(0,r)$ is contained in the union of such small connected components, so that either $|E_i^{s_n} \cap B(0,r)| \leq \varepsilon$ or $|E_j^{s_n} \cap B(0,r)| \leq \varepsilon$ for n large enough, contradicting the convergence of $E_k^{s_n} \cap B(0,r)$ to $\bar{E}_k \cap B(0,r)$, for all $k \in \{1,2,3\}$. □

Theorem 3.6. *There exists* $s_0 \in (0,1)$ *such that the following holds: Among all cones, the unique local minimizers for* $\mathcal{P}_s$*, for* $s > s_0$*, are* 2*-cones (half-spaces) and* 3*-cones. In the case of* 3*-cones, the opening angles of all phases are equal, and coincide with* $2/3\pi$*.*

Proof. Let $s_n \to 1$ and let $\mathcal{C}_n$ be a sequence of minimal cones for $\mathcal{P}_s$. By Theorem 2.8 there exists a minimal cone $\mathcal{C}$ for the classical perimeter such that $\mathcal{C}_n \to \mathcal{C}$ locally uniformly as $n \to \infty$. Since the only minimal cones in $\mathbb{R}^2$ are half-planes or 3-cones with angles of $2/3\pi$ [1],it follows by the uniform convergence that also the $\mathcal{C}_n$'s are a half-spaces or 3-cones for n large enough. By Lemma 3.4, if $\mathcal{C}_n$ is a minimal 3-cone then necessarily it has equal angles of $2/3\pi$.

By Proposition 3.5 we know that there exist minimal cones which are not half-planes, and this concludes the proof. □

Remark 3.7. An interesting issue which is left open is whether Theorem 3.6 is true for all $s \in (0,1)$. We conjecture this is the case, but in order to prove this result it would be necessary to develop some new technical argument. A related problem is about the possibility of extending the nonlocal calibrations recently introduced in [5, 13] to clusters, in the same spirit of the paired calibrations used in [11].

4. Weighted perimeters

Let us fix a sequence c_i with $i \in \mathbb{N}$, such that $c_i > 0$ for all i and consider the energy associated to a k-cluster $\mathcal{E}$ and to the sequence c_i as

$$\mathcal{P}_{s,c}(\mathcal{E};\Omega) = \sum_{1 \leq i \leq k} c_i \mathrm{Per}_s(E_i;\Omega). \tag{10}$$

First of all we consider the generalization of Lemma 3.4.

Lemma 4.1. *Among all* 3*-cones in* $\mathbb{R}^2$ *there exists a unique cone which is stationary for the functional in* (10)*, and the opening angles are uniquely determined as functions of* c_i*.*

Proof. The proof is analogous to that of Lemma 3.4. The stationarity condition reads

$$c_i H_s(x, C_i) = c_j H_s(x, C_j) \qquad \forall x \in \partial C_i \cap \partial C_j,\ x \neq 0, \tag{11}$$

and since $c_i > 0$ for all i, we get $\alpha_i < \pi$.

Proceeding as in (9) in the proof of Lemma 3.4 and using the same notation, we note that for all $\lambda > 0$, $\lambda((C_3 \setminus \tilde{B}) - x) = (C_3 \setminus \tilde{B}) - \lambda x$ and $\lambda((C_3 \cup B) - x) = (C_3 \cup B) - \lambda x$. Therefore $H_s(x, C_i) = \lambda^s H_s(\lambda x, C_i)$. This implies that it is sufficient to verify condition (11) just for one $x \neq 0$. We fix from now on x, with $|x| = 1$.

We introduce the function $F : [0, \pi) \to \mathbb{R}$ as

$$F(\alpha) = 2 \int_0^\alpha \int_0^{+\infty} \frac{\rho}{(1 + \rho^2 + 2\rho \cos\theta)^{1+s/2}} d\rho d\theta. \tag{12}$$

Note that if K is a sector of the cone with opening angle 2α and which is symmetric with respect to the half-line separating C_1, C_2, then $F(\alpha) = \int_K \frac{1}{|x-y|^{2+s}} dy$. Note that $F(0) = 0$ and

$$F'(\alpha) = 2 \int_0^{+\infty} \frac{\rho}{(1 + \rho^2 + 2\rho \cos\alpha)^{1+s/2}} d\rho > 0.$$

Therefore F is invertible.

Recalling the definition of F and (9), we may restate (11) as

$$c_2 F(\pi - \alpha_2) = c_1 F(\pi - \alpha_1). \tag{13}$$

With the same argument we conclude that the cone $\mathcal{C}$ is stationary iff

$$c_2 F(\pi - \alpha_2) = c_1 F(\pi - \alpha_1) = c_3 F(\pi - \alpha_3). \tag{14}$$

Let $k > 0$ be the solution to the equation

$$F^{-1}(k/c_1) + F^{-1}(k/c_2) + F^{-1}(k/c_3) = \pi,$$

which exists and is unique due to the fact that $F^{-1} : [0, +\infty) \to \mathbb{R}$ is monotone increasing. Then the angles α_i are uniquely determined as

$$\alpha_i = \pi - F^{-1}(k/c_i).$$

□

Remark 4.2. In the case of standard perimeter, it has been proved in [10] that the unique 3-cone which is a local minimizer for the functional $\sum_{1 \le i \le 3} c_i \mathrm{Per}(E_i)$ has opening angles α_i which satisfies the following relation

$$\frac{\sin\alpha_1}{c_2 + c_3} = \frac{\sin\alpha_2}{c_1 + c_3} = \frac{\sin\alpha_3}{c_1 + c_2}.$$

For general k-clusters, with $k > 3$, in general there could be singular cones with more than 3 phases which are local minimizers. However, in [9] it is proved that if the weights c_i are sufficiently close to 1, it is possible to recover the triple-point property: Only 3-cones are local minimizers.

We get in this case the following analogous of Theorem 3.6 for the case of 3 cones. We state it in this form since for the functional $\sum_i c_i \mathrm{Per}(E_i)$ it is not known if the unique local minimizers among cones are just 2-cones (half-spaces) and 3-cones, see [11].

Proposition 4.3. *There exists $s_0 \in (0, 1)$ depending on $(c_i)_i$ such that the following holds: Among all 2-cones and 3-cones, the unique local minimizers for $\mathcal{P}_{s,c}$, for $s > s_0$, are 2-cones (half-spaces) and the 3-cone obtained in Lemma 4.1.*

Proof. Arguing as in the proof of Theorem 3.6, we consider $s_n \to 1$ and $\mathcal{C}_n$ to be a sequence of minimal cones for $\sum_{i=1}^3 c_i \mathrm{Per}_{s_n}(\cdot)$. By Theorem 2.8 there exists a minimal cone $\mathcal{C}$ for $\sum_{i=1}^3 c_i \mathrm{Per}(\cdot)$ such that $\mathcal{C}_n \to \mathcal{C}$ locally uniformly as $n \to \infty$. Since the only minimal cones in $\mathbb{R}^2$ are half-planes or 3-cones with angles given in Remark 4.2, it follows by the uniform convergence that also the $\mathcal{C}_n$'s are a half-spaces or 3-cones for n large enough. By Lemma 4.1, if $\mathcal{C}_n$ is a minimal 3-cone then necessarily it coincides with the 3-cone computed in the Lemma. Arguing as in Proposition 3.5, and recalling Remark 4.2, we get that there exist minimal cones which are not half-planes, and this concludes the proof. $\square$

References

[1] F. Almgren, Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure, *Ann. of Math. (2)*, **87** (1968), 321–391.

[2] L. Ambrosio, A. Braides, Functionals defined on partitions in sets of finite perimeter. II. Semicontinuity, relaxation and homogenization, *J. Math. Pures Appl.* **69** (1990), no. 3, 307–333.

[3] L. Ambrosio, G. De Philippis, L. Martinazzi, Gamma-convergence of nonlocal perimeter functionals, *Manuscripta Math.*, **134** (2011), no. 3-4, 377–403.

[4] A. Braides, S. Conti, G. Garroni, Density of polyhedral partitions, *Calc. Var. Partial Differential Equations*, **56** (2017), no. 2, Art. 28, 10 pp.

[5] X. Cabré, Calibrations and null-Lagrangians for nonlocal perimeters and an application to the viscosity theory, *Arxiv Preprint 2019*, https://arxiv.org/abs/1905.10790.

[6] L. A. Caffarelli, J.-M. Roquejoffre and O. Savin, Nonlocal minimal surfaces, *Comm. Pure Appl. Math.*, **63** (2010), 1111–1144.

[7] L. A. Caffarelli, E. Valdinoci, Uniform estimates and limiting arguments for nonlocal minimal surfaces, *Calc. Var. Partial Differential Equations*, **41** (2011), 1111–1144.

[8] M. Colombo, F. Maggi, Existence and almost everywhere regularity of isoperimetric clusters for fractional perimeters, *Nonlinear Anal.* 153 (2017), 243–274.

[9] G. Del Nin, Sticky-disk limit of planar N-bubbles, to appear in *Adv. Calc. Var.*

[10] G.R. Lawlor, Double bubbles for immiscible fluids in $\mathbb{R}^n$, *J. Geom. Anal.* 24 (2014), no. 1, 190–204.

[11] G.R. Lawlor, F. Morgan, Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms, *Pacific J. Math.* 166 (1994), no. 1, 55–83.

[12] V. Maz'ya, *Lectures on Isoperimetric and Isocapacitary Inequalities in the Theory of Sobolev Spaces,* Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[13] V. Pagliari, Halfspaces minimise nonlocal perimeter: a proof via calibrations, *Arxiv Preprint 2019*, https://arxiv.org/abs/1905.00623.

[14] O. Savin, E. Valdinoci, Γ-convergence for nonlocal phase transitions, *Ann. Inst. H. Poincar Anal. Non Linaire* 29 (2012), no. 4, 479–500.

Dipartimento di Scienze Statistiche, Università di Padova, Via Cesare Battisti 241/243, 35121 Padova, Italy

E-mail address: annalisa.cesaroni@unipd.it

Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

E-mail address: matteo.novaga@unipi.it