Combinatorial constructions of hyperbolic and Einstein four-manifolds

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(joint with Alexander Kolpakov)

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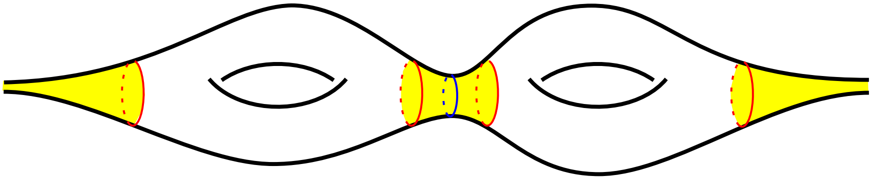
Theorem (Margulis lemma)

If M has finite volume, it is the interior of a compact manifold with boundary. Every boundary component is diffeomorphic to a flat $(n-1)$-manifold N and gives a cusp isometric to

$$N \times [0, +\infty)$$

with $N \times t$ rescaled by e^{-2t}.

Thick-thin decomposition:



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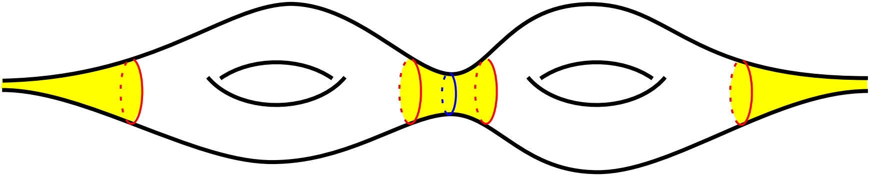
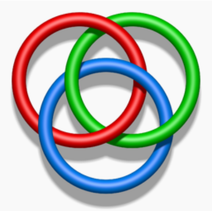


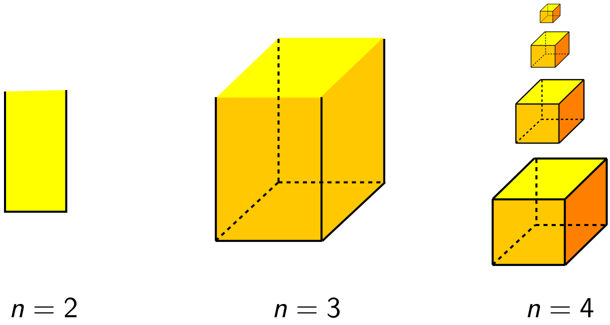
Hyperbolic link complements in S^3:



We can see a cusp using the *half-space model*

$$\mathbb{H}^n = \{x_n > 0\} \subset \mathbb{R}^n$$

with metric tensor at $x = (x_1, \ldots, x_n)$ rescaled by $\frac{1}{x_n^2}$.



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Question

For which $n \geqslant 2$ there exists a hyperbolic n-manifold with only one cusp?

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There are no arithmetic n-manifolds with one cusp for $n \geqslant 30$.

A flat $(n-1)$-manifold M *bounds geometrically* if there is a hyperbolic n-manifold with only one cusp, diffeomorphic to $M \times [0, +\infty)$.

Theorem (Long, Ried 2000)

Some flat 3-manifolds do not bound geometrically.

The η*-invariant* $\eta(M) \in \mathbb{R}$ is defined for any closed oriented 3-manifold. Long and Reid proved that if a closed flat 3-manifold M bounds geometrically a hyperbolic 4-manifold W then

$$\sigma(W) + \eta(M) = 0$$

where $\sigma(W)$ is the signature. Therefore $\eta(M) \in \mathbb{Z}$.

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There are six flat 3-manifolds up to diffeomorphism. Five are torus bundles over S^1 with monodromy:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and a sixth one has a Seifert fibration over $\mathbb{RP}^2$ with two singular fibers. They all have integral η-invariant, except the last two fiber bundles.

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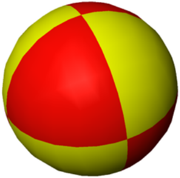
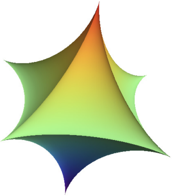
There are infinitely many hyperbolic four-manifolds M with any fixed number k of cusps. The number of such manifolds with volume $\leqslant V$ grows faster than $C^{V \ln V}$ for some $C > 0$.

These manifolds are constructed explicitly by gluing some copies of the hyperbolic right-angled ideal 24-cell.

Constructions

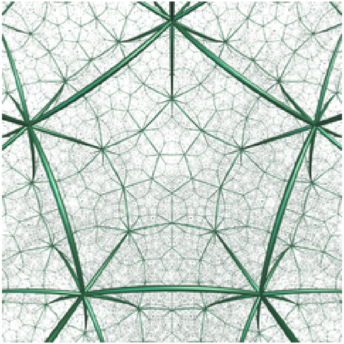
Regular polyhedra:



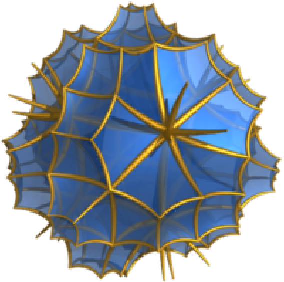


polyhedron	$\theta = \frac{\pi}{3}$	$\theta = \frac{2\pi}{5}$	$\theta = \frac{\pi}{2}$	$\theta = \frac{2\pi}{3}$
tetrahedron	ideal $\mathbb{H}^3$	S^3	S^3	S^3
cube	ideal $\mathbb{H}^3$	$\mathbb{H}^3$	$\mathbb{R}^3$	S^3
octahedron			ideal $\mathbb{H}^3$	S^3
icosahedron				$\mathbb{H}^3$
dodecahedron	ideal $\mathbb{H}^3$	$\mathbb{H}^3$	$\mathbb{H}^3$	S^3

A regular polyhedron with angle $\theta = \frac{2\pi}{n}$ yields a tessellation:



dodecahedra with $\theta = \frac{2\pi}{5}$



cubes with $\theta = \frac{2\pi}{5}$

The isometry group $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ of the tessellation is discrete. To get a finite-index subgroup $\Gamma' < \Gamma$ that acts freely we can invoke

Theorem (Selberg lemma)

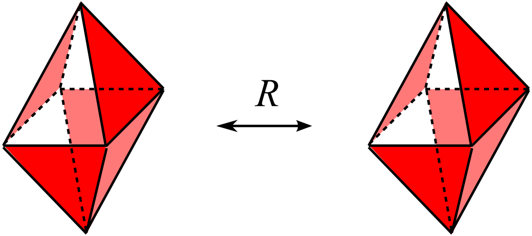
Let $\Gamma < GL(n, \mathbb{C})$ be a finitely generated group. There is a finite-index $\Gamma' < \Gamma$ without torsion.

No torsion implies that Γ' acts freely. Therefore $M = \mathbb{H}^n/_{\Gamma'}$ is a finite-volume complete hyperbolic manifold that tessellates into finitely many regular polyhedra.

Let us construct some concrete examples.

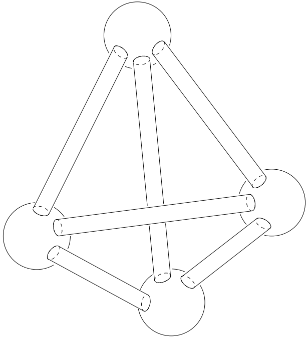
Dimension three

The *Minsky block* is obtained from two copies of an ideal regular right-angled hyperbolic octahedron:



by identifying the corresponding red faces.

The Minsky block is a complete hyperbolic manifold with geodesic boundary. Topologically it is diffeomorphic to the complement of:



It has:

- 4 geodesic thrice punctured spheres as boundary
- 6 *annular* cusps, of type $S^1 \times [0,1] \times [0,+\infty)$

The combinatorics of the Minsky block is that of a tetrahedron:

$$\{\text{faces}\} \longleftrightarrow \{\text{geodesic thrice} - \text{punctured spheres}\}$$
$$\{\text{edges}\} \longleftrightarrow \{\text{annular cusps}\}$$

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Let a *triangulation* be a face-pairing of some n tetrahedra. By replacing every tetrahedron with a Minsky block we get a finite-volume cusped orientable hyperbolic 3-manifold. The resulting map:

$$\{\text{triangulations}\} \longrightarrow \{\text{hyperbolic } 3 - \text{manifolds}\}$$

is injective.

The Minksy block appears:

- as a bulding block of the *model manifold* constructed by Minsky to prove Thurston's Ending Lamination Conjecture [2002]
- in the theory of *shadows*, by Costantino and D. Thurston [2008]
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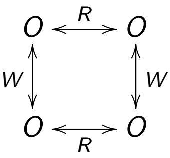
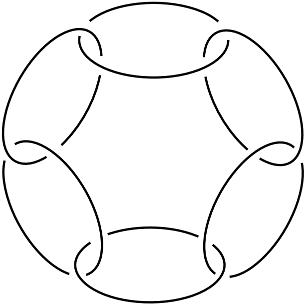
If we mirror the Minsky block we get the *octahedral manifold*:

It is made of four octahedra glued as:



and is the complement of the (minimally twisted) chain link with 6 components.



Dimension four

There are six regular polytopes in dimension four:

name	facets	2-faces	edges	vertices	link of vertices
5-cell	5 tetrahedra	10	10	5	tetrahedron
8-cell	8 cubes	24	32	16	tetrahedron
16-cell	16 tetrahedra	32	24	8	octahedron
24-cell	24 octahedra	96	96	24	cube
120-cell	120 dodecahedra	720	1200	600	tetrahedron
600-cell	600 tetrahedra	1200	720	120	icosahedron

The link of the 24-cell is euclidean right-angled (a cube). Therefore the ideal 24-cell is right-angled.

The 24-cell $\mathscr{C}$ is the convex hull in $\mathbb{R}^4$ of the 24 points obtained permuting

$$(\pm 1, \pm 1, 0, 0).$$

It has 24 facets, contained in the hyperplanes

$$\{\pm x_i = 1\}, \qquad \left\{\pm \frac{x_1}{2}, \pm \frac{x_2}{2}, \pm \frac{x_3}{2}, \pm \frac{x_4}{2} = 1\right\}.$$

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Each facet is a regular octahedron. The dual polytope is hence

$$\mathscr{C}^* = \mathrm{Conv}(G \cup R \cup B)$$

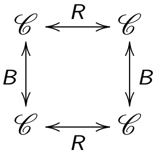
where G contains the 8 points obtained by permuting

$$(\pm 1, 0, 0, 0)$$

and $R \cup B$ contains 16 points $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2} \pm \frac{1}{2}\right)$. We let R (resp. B) be the 8 points having even (resp. odd) number of minus signs.

Facets of $\mathscr{C}$ are colored in Green, Blue, and Red.

Pick four identical ideal hyperbolic 24-cells and glue the facets as follows:



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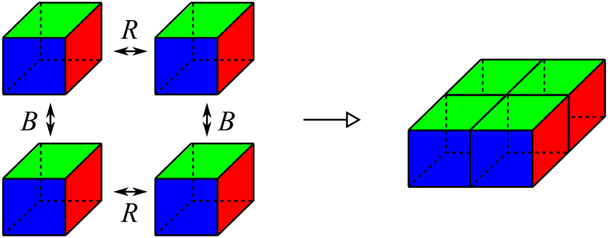
$$\begin{array}{ccc} \mathscr{C} & \stackrel{R}{\longleftrightarrow} & \mathscr{C} \\ B \updownarrow & & \updownarrow B \\ \mathscr{C} & \underset{R}{\longleftrightarrow} & \mathscr{C} \end{array}$$

A vertex is a cone over a 3-colored cube. Four cubes are glued:

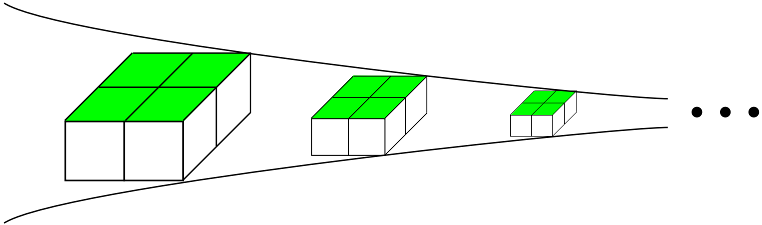


to produce a $T \times [0, 1]$.

We get a block $\mathscr{B}$. It is a hyperbolic 4-manifold with geodesic boundary. The (green) boundary has 8 components, each isometric to the octahedral 3-manifold. And it has 24 cusps of non-compact type, each isometric to

$$T \times [0,1] \times [0,+\infty)$$

where T is a 2×2 square torus and $T \times [0,1] \times t$ is shrinked by e^{-2t}:



Each cusp is adjacent to two distinct green geodesic boundary components.

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It looks like a hypercube H, with

$$\{8 \text{ facets of } H\} \longleftrightarrow \{8 \text{ geodesic boundary components of } \mathscr{B}\}$$
$$\{24 \text{ faces of } H\} \longleftrightarrow \{24 \text{ cusps of } \mathscr{B}\}$$

A facet in H is a cube (with 6 faces), which corresponds (dually) to a octahedral manifold (with 6 cusps) in $\partial\mathscr{B}$.

Vertices and edges of H have no interpretation in $\mathscr{B}$.

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In a cubulation, the 2-faces are identified in cycles:

$$Q_1 \xrightarrow{\psi_1} Q_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{h-1}} Q_h \xrightarrow{\psi_h} Q_1.$$

Every cycle has a monodromy $\psi = \psi_h \circ \cdots \circ \psi_1$ which is:

$$\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

up to conjugation.

A cubulation defines an orientable cusped finite-volume hyperbolic 4-manifold. A cycle with monodromy ψ gives a cusp isometric to

$$(T \times [0, h])/_{\psi} \times \mathbb{R}_{>0}.$$

The first factor $M = T \times [0, h]/_{\psi}$ is a flat 3-manifold. The slice $M \times t$ is scaled by e^{-2t} as usual.

The resulting map

$$\{\text{cubulations}\} \longrightarrow \{\text{hyperbolic } 4-\text{manifolds}\}$$

is injective on cubulations with $\geqslant 3$ hypercubes. The proof uses that the decomposition into 24-cells is the Epstein-Penner canonical one (w. r. to some intrinsically defined sections).

For a finite-volume complete hyperbolic four-manifold M, the generalized Gauss-Bonnet formula gives

$$\mathrm{Vol}(M) = \frac{4\pi^2}{3}\chi(M) = \frac{16n}{3}\pi^2.$$

We have $\mathrm{Vol}(\mathscr{C}) = \frac{4\pi^2}{3}$ and $\chi(\mathscr{B}) = 4$, $\chi(M) = 4n$.
Here n is the number of hypercubes, and hence blocks.

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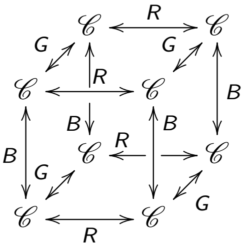
We get 24 cycles, each of 2 square faces, with trivial monodromy.

The hyperbolic manifold M has 24 toric cusps, each a $2 \times 2 \times 2$ cubic three-torus. It tessellates into eight 24-cells glued along a cubic diagram:



This manifold has many symmetries.

Proposition

Fix $k > 0$. The number of cubulations with n hypercubes and k cycles of squares grows faster than $C^{n \ln n}$, for some $C > 0$.

This implies immediately:

Corollary

Fix $k > 0$. The number of hyperbolic 4-manifolds M with $\chi(M) = 4n$ and k cusps grows faster than $C^{n \ln n}$, for some $C > 0$.

In particular, there are hyperbolic manifolds with any number of cusps. We may require all cusp sections being 3-tori.

Dehn filling

Let $M = \mathrm{Int}(N)$ be a cusped hyperbolic n-manifold, with ∂N consisting of tori T^{n-1}. A *Dehn filling* of N is the operation of attaching a

$$T^{n-2} \times D^2$$

to some of these boundary tori. The filled manifold depends on

$$\gamma = \{x\} \times S^1 \in \pi_1(T^{n-1}) = \mathbb{Z}^{n-1}$$

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Fix disjoint flat cusp sections. Now γ has a geodesic representative of some length $l(\gamma)$. If M has k cusps we can fill them along curves $\gamma_1, \ldots, \gamma_k$ and get a closed filled manifold N.

Theorem (Gromov-Thurston 2π)

If $l(\gamma_i) > 2\pi$ for all i then N admits a metric of non-positive sectional curvature (and is hence aspherical by Cartan-Hadamard).

Theorem (Anderson 2003)

If $l(\gamma_i)$ is sufficiently large for all i then N admits an Einstein metric.

When $n = 3$ this is Thurston's Dehn filling theorem (Einstein $\Longrightarrow$ constant curvature when $n = 3$).

We can construct an Einstein four-manifold via:

- a cubulation with trivial monodromies on cycles of squares,
- a sufficiently complicate primitive triple $(p, q, r) \in \mathbb{Z}^3$ at each cycle.

Geodesic boundary

Similarly, a hyperbolic 3-manifold M *bounds geometrically* if it is the geodesic boundary of a finite-volume complete 4-manifold.

Theorem (Long-Reid 2000, 2001)

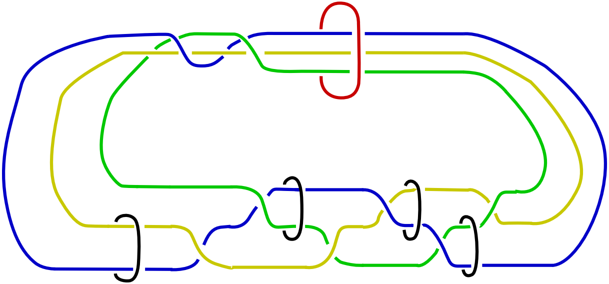
Infinitely many closed hyperbolic 3-manifolds bound geometrically, infinitely many do not bound geometrically.

Concrete low-volume examples may be constructed using right-angled 120-cells and dodecahedra [Kolpakov, M., Tschantz 2013].

Cusped 3-manifolds can also bound:

Theorem (Slavich 2014)

The following link complement bounds geometrically:



It tessellates into eight regular ideal octahedra and bounds a four-manifold that tessellates into two regular ideal 24-cells.