Turaev-Viro representations of the mapping class groups.

Bruno Martelli
(joint with Francesco Costantino)

7 december 2010

Quantum invariants

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Quantum invariants assign an object to the pair (M, L).

- If $M = S^3$, the object is a Laurent polynomial (it is the Jones polynomial when the framing is 0 and colours are all $1/2$)
- For general manifolds M, we only get a complex number depending on q_0 when q_0 is a root of unity (Reshetikin – Turaev, Witten 1990).

This leads to a finite-dimensional representation of the mapping class group of a surface S for every root q_0 of unity. (BHMV 1995)

There are various conjetures about the asymptotic of this invariants as q_0 tends to 1 (Witten, Kashaev's volume conjecture).

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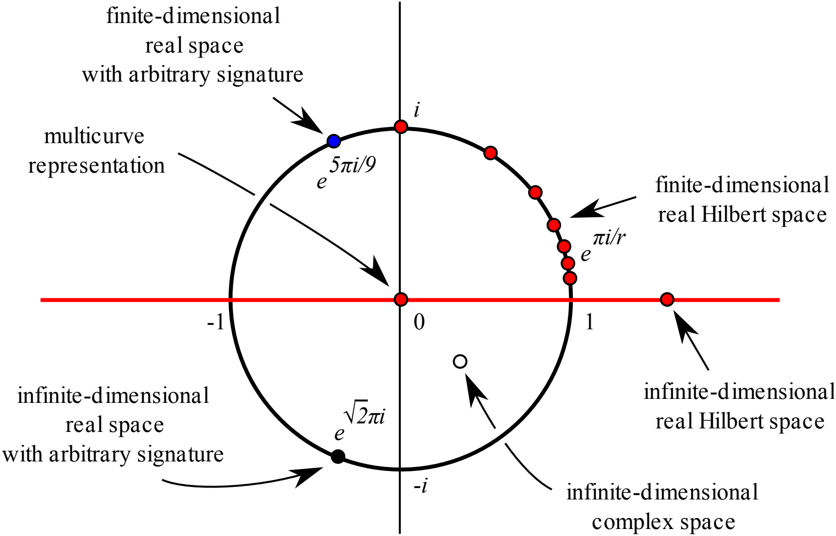
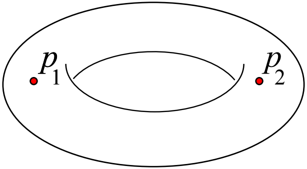
We define for a cusped S a representation for all $q \in \mathbb{C} \cup \{\infty\}$:



Mapping class group

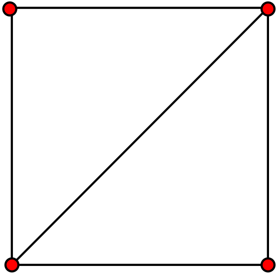
The *mapping class group* $\mathrm{Mod}(S)$ of a surface S is the group of diffeomorphisms of S seen up to homotopy. We always require that $\chi(S) < 0$ and that S is obtained from a closed orientable surface $\overline{S}$ by removing some $k \geqslant 1$ points $p_1, \ldots, p_k$ (creating *punctures*).



The group $\mathrm{Mod}(S)$ can also be defined as the group of diffeomorphisms of $\overline{S}$ that preserve the set $\{p_1, \ldots, p_k\}$, up to homotopy (that also preserves this set).

Ideal triangulation

An *ideal triangulation* for S is a triangulation of $\overline{S}$ whose vertices are $p_1, \ldots, p_k$. That is, it is a maximal collection of (pairwise non-homotopic) arcs joining the punctures.



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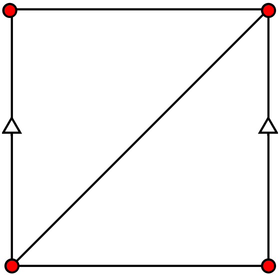

Let $\Delta_1, \ldots, \Delta_h$ be the triangulations of S, up to homeomorphisms of S. We fix on each Δ_i a *marking*: for instance, a directed edge.

Groupoid

A group can be thought as a category with one object, all morphisms being invertible.

Definition

A *groupoid* is a category in which every morphism is invertible, and two objects are always connected by some morphism.

The automorphism of an object form a group. Two distinct objects yield isomorphic groups.

The modular groupoid

The *modular* (or *Ptolemy*) *groupoid* (Mosher 1995 – Harer 1986 – Penner 1987 – Checkov - Fock 1999) is defined as follows:

Objects: Ideal triangulations $\Delta_1, \ldots, \Delta_h$ up to homeomorphisms of S, each with a fixed marking.

Morphs: Pairs (Δ, Δ') of (marked) ideal triangulations of S, up to homeomorphisms of S.

That is

$$(\Delta, \Delta') = (\varphi(\Delta), \varphi(\Delta'))$$

for every diffeomorphism φ of S. The composition of (Δ, Δ') and (Δ', Δ'') is (Δ, Δ''). The inverse of (Δ, Δ') is (Δ', Δ).

The automorphism group of the modular groupoid is the mapping class group $\mathrm{Mod}(S)$.

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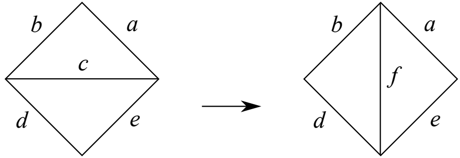
A presentation of the modular groupoid

A *flip* is a morphism (Δ, Δ') where Δ and Δ' differ only by an arc.



Theorem (Penner 1993)

Flips generate the modular groupoid. The following is a complete set of relations.

- $(\Delta, \Delta')(\Delta', \Delta) = 1$,
- *flips on edges not contained in the same triangle commute,*
- *the pentagon relation.*

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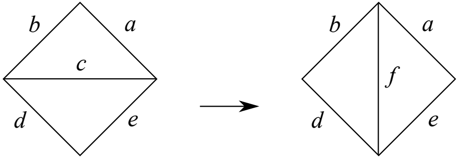

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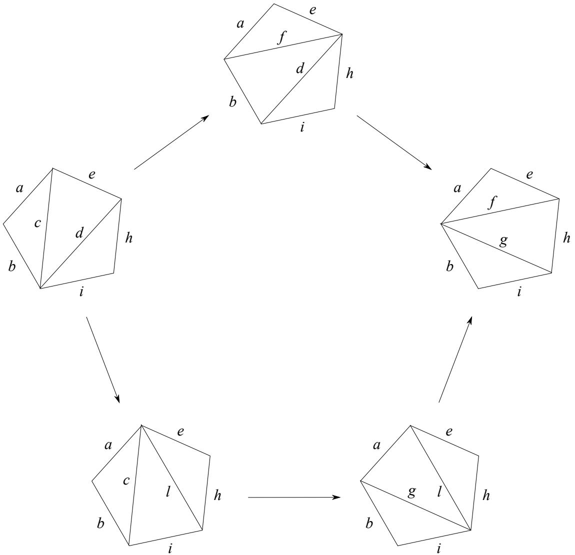
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Proof of the theorem

The *arc complex* $A(S)$ of S is the simplicial complex defined as follows. Vertices are isotopy classes of essential arcs. Whenever n distinct arcs can be isotoped to be disjoint, they span a simplex.

Ideal triangulations of S are in 1-1 correspondence with the top-dimensional simplexes of $A(S)$. Two triangulations are connected by a flip if and only if they share a codimension-1 simplex.

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Theorem (Harer 1986)

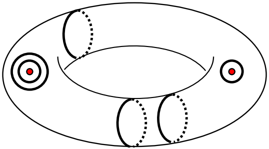
The arc complex $A(S)$ is contractible.

The link of every simplex of codimension $\geqslant 2$ is connected. Therefore every pair (Δ, Δ') of triangulations is connected by a path of flips. Two such paths are homotopy equivalent, so they can be related by passing through codimension-2 strata.

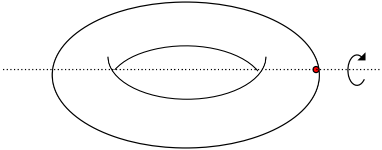
A codimension-2 simplex is a triangulation minus 2 edges: it either contains two squares or a pentagon. Passing through it translates into the quadrilateral and pentagon relation.

Multicurves

A *multicurve* on S is a collection of disjoint essential simple closed curves.



Multicurves are considered up to isotopy. The mapping class group $\mathrm{Mod}(S)$ acts multicurves, faithfully except for the *hyperelliptic* map on the punctured torus:



Admissible colourings

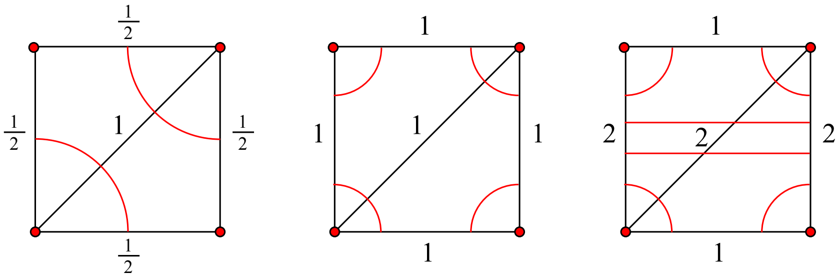
An *admissible colouring* on an ideal triangulation Δ is the assignment of a non-negative half-integer to each edge such that every a, b, c on a triangle form an *admissible triple*, that is:

- $a + b + c$ is an integer
- the triangle inequalities $a \leqslant b + c, b \leqslant c + a, c \leqslant a + b$.

Proposition

There is a natural bijection between admissible colourings and multicurves in S.

An admissible colouring defines a multicurve:



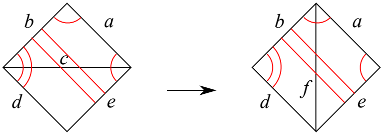
Every multicurve can be put in normal position with respect to the triangulation. The resulting colours are the geometric intersection numbers with the arcs (because there are no bigons), hence distinct colourings give distinct multicurves.

The multicurve functor

Let $\mathbb{F}$ be any field. We assign to each Δ the vector space

$$V_\Delta^{\mathbb{F}} = \mathbb{F}\{\text{admissible colourings on } \Delta\}.$$

We assign to every morphism (Δ, Δ') the isomorphism $V_\Delta^{\mathbb{F}} \to V_{\Delta'}^{\mathbb{F}}$ that sends a colouring σ to the colouring σ' representing the same multicurve as σ.



On a flip, two colouring σ, σ' of Δ, Δ' are *related* if they coincide on all edges except c, f. The isomorphism sends σ to the related σ' such that

$$c + f = \max\{a + d, b + e\}.$$

A functor from the modular groupoid to the category of $\mathbb{F}$-vector spaces induce a $\mathbb{F}$-representation of the mapping class group. The multicurve functor induces the well-known multicurve representation.

- The multicurve representation is faithful (modulo the hyperelliptic involution on the punctured torus).
- The multicurve functor is *orthogonal*: every isomorphism preserves the non-degenerate scalar product on $V_\Delta^{\mathbb{F}}$ defined on colourings as:

$$\langle \sigma, \eta \rangle = \delta_{\sigma,\eta}.$$

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Quantum integers and multinomials

Quantum integers:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} + q^{-n+3} + \ldots + q^{n-1}.$$

These Laurent polynomials are defined for any $n \geqslant 0$, by setting $[0] = 1$. Note that $[n](1) = n$. The quantum factorial is defined as

$$[n]! = [1] \cdots [n], \quad [0]! = 1.$$

The quantum multinomial is defined for any positive integers $n_1 \ldots n_k$ as follows:

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_k \end{bmatrix} = \frac{[n]!}{[n_1]! \cdots [n_k]!}$$

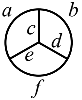
by taking $n = n_1 + \ldots + n_k$. It is a Laurent polynomial.

Quantum tetrahedron

More Laurent polynomials (Kauffman - Lins 1994):

$$_a = (-1)^{2a}[2a+1].$$

$$_{a,b,c} = (-1)^{a+b+c} \begin{bmatrix} a+b+c+1 \\ a+b-c, c+a-b, b+c-a, 1 \end{bmatrix}.$$



$$= \sum_{z=\max \triangle_j}^{\min \square_i} (-1)^z \begin{bmatrix} z+1 \\ z-\triangle_1, z-\triangle_2, z-\triangle_3, z-\triangle_4, \square_1-z, \square_2-z, \square_3-z, 1 \end{bmatrix}.$$

In the latter equality, triangles and squares are defined as follows:

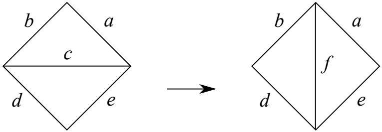
$$\triangle_1 = a+b+c,\ \triangle_2 = a+e+f,\ \triangle_3 = d+b+f,\ \triangle_4 = d+e+c,$$
$$\square_1 = a+b+d+e,\ \square_2 = a+c+d+f,\ \square_3 = b+c+e+f$$

The functor

Let $\mathbb{K}$ be the field of all rational functions on $\mathbb{C}$, *i.e.* meromorphic functions $\mathbb{CP}^1 \to \mathbb{CP}^1$. We assign to a triangulation Δ the $\mathbb{K}$-vector space

$$V_\Delta^{\mathbb{K}} = \mathbb{K}\{\text{admissible colourings on } \Delta\}.$$

Let (Δ, Δ') be a flip

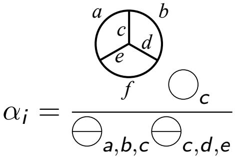


The isomorphism $F_{\Delta,\Delta'}$ sends the colouring σ on Δ to a linear combination

$$F_{\Delta,\Delta'}(\sigma) = \alpha_1\sigma'_1 + \ldots + \alpha_h\sigma'_h.$$

of colourings on Δ' related to σ.

The coefficients α_i are defined as in the *recoupling formula* (Kauffman - Lins 1994)



Theorem

This defines a functor F from the modular groupoid to $\mathbb{K}$-vector spaces.

Proof.

The pentagon relation, as the 2-3 Pachner move, is respected thanks to the *Biedenharn-Elliot identity* (Kauffman - Lins). □

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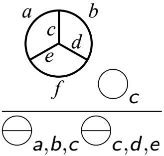
$\alpha_i =$ 

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Orthogonality

Let Δ be an ideal triangulation. Let σ be a colouring of Δ. We define the rational function

$$|\sigma| = \prod_e \bigcirc_e^{-1} \prod_f \ominus_f$$

Equip $V_\Delta^{\mathbb{K}}$ with a non-degenerate scalar product

$$\langle \sigma, \eta \rangle = \delta_{\sigma,\eta} \frac{1}{|\sigma|}.$$

Proposition

The functor F is orthogonal. Therefore it induces an orthogonal representation

$$\rho : \mathrm{Mod}(S) \to \mathrm{O}(V_\Delta^{\mathbb{K}}) \subset \mathrm{End}(V_\Delta^{\mathbb{K}})$$

Limit at $q \longrightarrow 0$.

Let U be the set of all roots of unity of order $\geqslant 3$.

Remark

The rational functions involved have poles in $U \cup \{0, \infty\}$.

Recall that $\langle \sigma, \sigma \rangle = 1/|\sigma|$. Let $\mathrm{ord}_0|\sigma|$ be the order of the pole in zero of the rational function $|\sigma|$. It is an even number.

Let $C_\Delta : V_\Delta^{\mathbb{K}} \to V_\Delta^{\mathbb{K}}$ send σ to $q^{\mathrm{ord}_0|\sigma|/2}\sigma$.

Theorem

The conjugated functor

$$\overline{F}_{\Delta,\Delta'} = C_{\Delta'}^{-1} \circ F_{\Delta,\Delta'} \circ C_\Delta$$

has no poles in $q = 0$. The evaluation in $q = 0$ is the multicurve functor.

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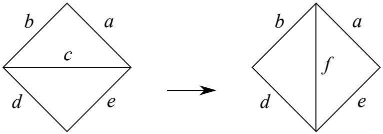

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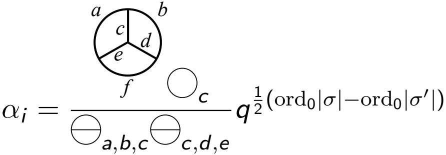

Lemma (Frohman – Kanya-Bartoszynska 2008)

$$\mathrm{ord}_0\alpha_i = -(\square_2 - \square_1)(\square_3 - \square_1) + \square_1 - a - b - d - e$$

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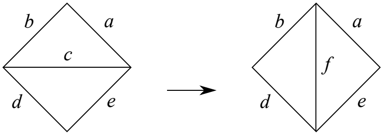

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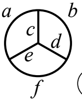

$$\alpha_i = \frac{[\text{tetrahedron } a,b,c,d,e,f] \quad \bigcirc_c}{\ominus_{a,b,c}\ \ominus_{c,d,e}}\, q^{\frac{1}{2}(\mathrm{ord}_0|\sigma| - \mathrm{ord}_0|\sigma'|)}$$

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Corollary

We have $\mathrm{ord}_0 \alpha_i \leqslant 0$ *and equality holds if and only if*

$$c + f = \max\{a + d, b + e\}$$

i.e. when α *and* α_i *represent the same multicurve.*

Cut-off at roots of unity

Let $q_0 \in U$ be a root of unity of order $\geqslant 3$. Let $r \geqslant 2$ be the smallest natural number such that $q_0^r = q_0^{-r}$. Fix a finite set of colours

$$\left\{0, \frac{1}{2}, 1, \ldots, \frac{r-2}{2}\right\}.$$

An admissible triple (a, b, c) of such colours is *r-admissible* if $a + b + c \leqslant r - 2$.

The finite vector space $V_{\Delta}^{q_0}$ is $\mathbb{R}$-generated by these colours.

Theorem

This defines a functor from the modular groupoid to finite-dimensional vector spaces for every $q_0 \in U$.

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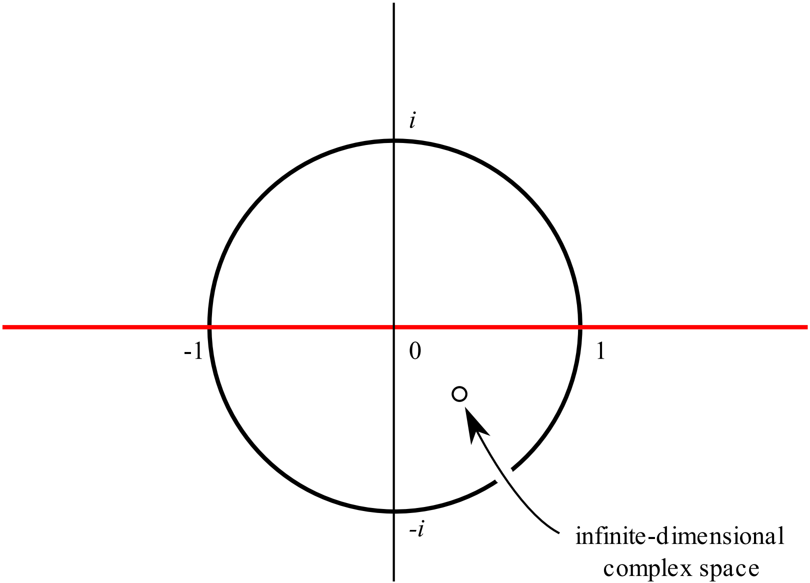
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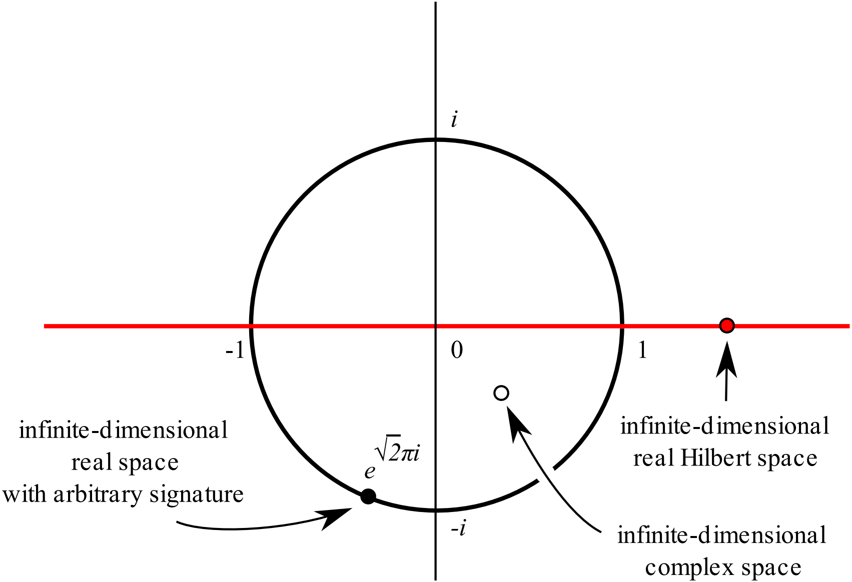
-i
infinite-dimensional
complex space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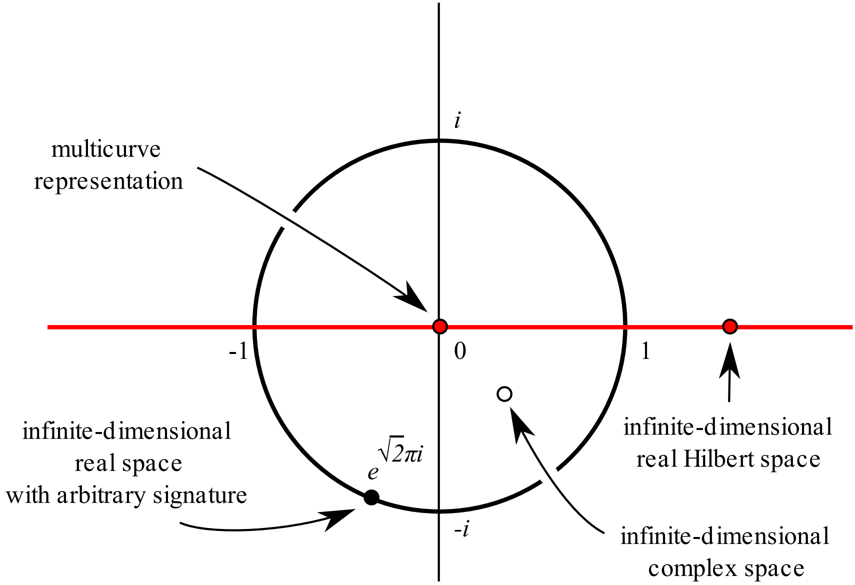
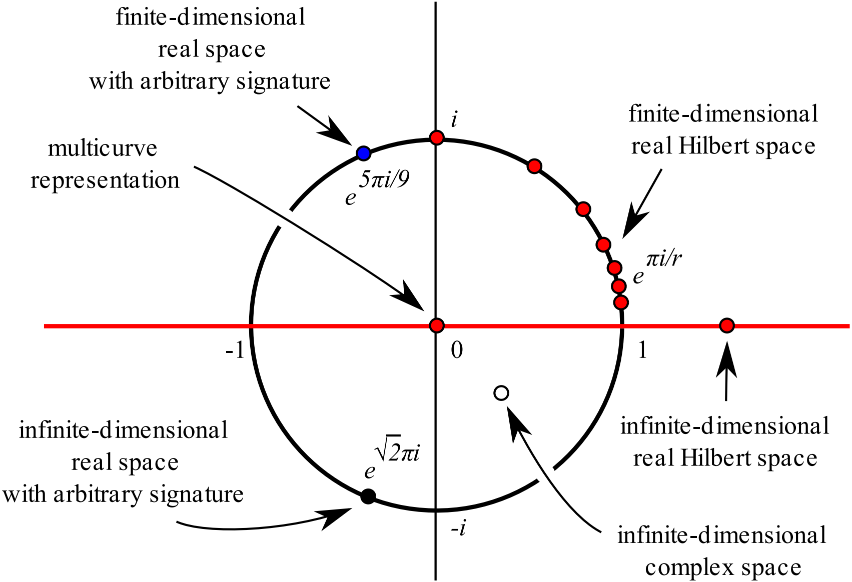
$e^{\sqrt{2}\pi i}$
infinite-dimensional
real space
with arbitrary signature
infinite-dimensional
real Hilbert space
infinite-dimensional
complex space


multicurve
representation
i
-1
0
1
-i
$e^{\sqrt{2}\pi i}$
infinite-dimensional
real space
with arbitrary signature
infinite-dimensional
real Hilbert space
infinite-dimensional
complex space


finite-dimensional real space with arbitrary signature
multicurve representation
$e^{5\pi i/9}$
i
finite-dimensional real Hilbert space
$e^{\pi i/r}$
-1
0
1
infinite-dimensional real Hilbert space
infinite-dimensional real space with arbitrary signature
$e^{\sqrt{2}\pi i}$
$-i$
infinite-dimensional complex space

Asymptotic faithfullness

A sequence ρ_i of finite representations of some group G is *asymptotically faithful* if for every $g \in G$ there is a i_0 such that $\rho_i(g) \neq \mathrm{id}$ for all $i > i_0$.

Theorem (Andersen, M. Freedman – Walker – Wang 2002)

The representations of the mapping class groups arising from quantum invariants are asymptotically faithful (modulo hyper-elliptic involutions).

We can give an alternative proof for punctured surfaces.

Proposition

Let $f \in \mathrm{Mod}(S)$ be a non-hyperelliptic element. The automorphism $\rho^q(f)$ is the identity only for finitely many choices of $q \in \mathbb{CP}^1$.

Proof.

It is not the identity at $q = 0$ on multicurves. Therefore $\rho(f)$ is not the identity on $V_\Delta^{\mathbb{K}}$. Since it is expressed via rational functions on q, it can be the identity only on finitely many values of q. □

Corollary

The finite representations ρ^q at roots of unity are asymptotically faithful.

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The Fell topology on representations

Let G be a (discrete) group. The set of all orthogonal (unitary) representations of G into some Hilbert spaces can be equipped with a topology.

In the *Fell topology* a sequence $\rho_i \colon G \to V_i$ of representations converge to a representation $\rho \colon G \to V$ if for any unit vector $v \in V$ and any finite subset $S \subset G$ there is a sequence of unit vectors $v_i \in V_i$ such that for all $g \in S$ we have

$$\langle \rho_i(g) v_i, v_i \rangle_i \to \langle \rho(g) v, v \rangle.$$

Remark

The representation ρ_q varies continuously with q in the Fell topology.

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Theorem (Marché – Narimannejad 2006)

A version of the quantum representation ρ^q for $q = -e^{\frac{\pi i}{r}}$ for closed surfaces S converges to the unitary representation

$$\rho\colon \mathrm{Mod}(S) \to U(\mathcal{H}(S))$$

as $q \to -1$. Here $\mathcal{H}(S)$ is the ring of regular functions on the character variety

$$\mathrm{Hom}\big(\pi_1(S), \mathrm{SL}(2,\mathbb{C})\big)/_{\mathrm{SL}(2,\mathbb{C})}.$$

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