Survey of the volume conjecture

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colored Jones polynomial

$J_N(K;q) \in \mathbb{Z}[q, q^{-1}]$: the colored Jones polynomial of a knot K associated with the N-dimensional irreducible representation of $sl(2;\mathbb{C})$.

- $J_N\Big($  $;q\Big) = 1$,
- J_2 is the original Jones polynomial.
- $qJ_2\Big($  $;q\Big) - q^{-1}J_2\Big($  $;q\Big) = (q^{1/2} - q^{-1/2})J_2\Big($  $;q\Big)$
  (skein relation)

Volume Conjecture

Conjecture (Volume Conjecture, R. Kashaev (1997), J. Murakami+H.M. (2000))

$$\lim_{N\to\infty}\frac{\log|J_N(K;\exp(2\pi\sqrt{-1}/N))|}{N}=\frac{\mathrm{Vol}(S^3\setminus K)}{2\pi}.$$

Definition (Simplicial volume (Gromov norm))

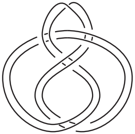
$$\mathrm{Vol}(S^3\setminus K):=\sum_{H_i:\text{hyperbolic piece}}(\text{Hyperbolic Volume of } H_i).$$

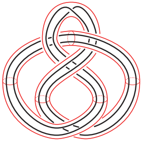
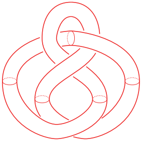
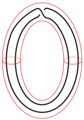
Definition (Jaco–Shalen–Johannson decomposition)

$S^3\setminus K$ can be uniquely decomposed as

$$S^3\setminus K=\left(\bigsqcup H_i\right)\sqcup\left(\bigsqcup E_j\right)$$

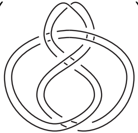
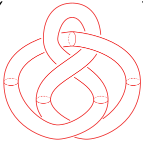
with H_i hyperbolic and E_j Seifert-fibered.

JSJ decmposition and the simplicial volume of the (2,1)-cable of the figure-eight knot 

 $=$  $\sqcup$ 

hyperbolic

Seifert fibered
(looks like surface $\times S^1$)

Vol $\Bigg($  $\Bigg) =$ Vol $\Bigg($  $\Bigg)$

Colored Jones polynomial of 

Proof of the VC for  is given by T. Ekholm (1999).

Theorem (K. Habiro, T. Lê)

$J_N\Big($  $;q\Big) = \displaystyle\sum_{j=0}^{N-1}\prod_{k=1}^{j}\left(q^{(N-k)/2}-q^{-(N-k)/2}\right)\left(q^{(N+k)/2}-q^{-(N+k)/2}\right).$

$q \mapsto \exp(2\pi\sqrt{-1}/N)$

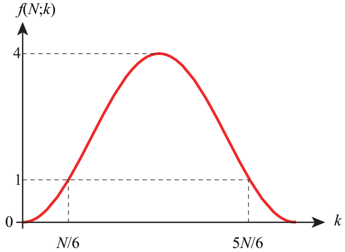
$J_N\Big($  $;\exp(2\pi\sqrt{-1}/N)\Big) = \displaystyle\sum_{j=0}^{N-1}\prod_{k=1}^{j} f(N;k)$

with $f(N;k) := 4\sin^2(k\pi/N)$.

Find the maximum of the summands

$$J_N\left(\right.$$  $$\left.; e^{2\pi\sqrt{-1}/N}\right) = \sum_{j=0}^{N-1}\prod_{k=1}^{j} f(N;k) \text{ with } f(N;k) := 4\sin^2(k\pi/N).$$

Graph of $f(N;k)$



Put $g(N;j) := \prod_{k=1}^{j} f(N;k)$.

j	0	$\cdots$	$N/6$	$\cdots$	$5N/6$	$\cdots$	1
$f(N;k)$		<1	1	>1	1	<1	
$g(N;j)$	1	$\searrow$		$\nearrow$	maximum	$\searrow$	

Limit of the sum is the limit of the maximum

- Maximum of $\{g(N;j)\}_{0\le j\le N-1}$ is $g(N;5N/6)$, and $g(N;j)>0$.
- $J_N\Big($  $;\exp(2\pi\sqrt{-1}/N)\Big)=\sum_{j=0}^{N-1}g(N;j)$.

$$\Downarrow$$

$g(N;5N/6)\le J_N\Big($  $;\exp(2\pi\sqrt{-1}/N)\Big)\le N\times g(N;5N/6)$

$$\Downarrow$$

$$\frac{\log g(N;5N/6)}{N}\le\frac{\log J_N}{N}\le\frac{\log N}{N}+\frac{\log g(N;5N/6)}{N}$$

$$\Downarrow$$

$$\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}\le\lim_{N\to\infty}\frac{\log J_N}{N}\le\lim_{N\to\infty}\frac{\log N}{N}+\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

$$\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}\le\lim_{N\to\infty}\frac{\log J_N}{N}\le\lim_{N\to\infty}\frac{\log N}{N}+\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

$$\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}\le\lim_{N\to\infty}\frac{\log J_N}{N}\le\lim_{N\to\infty}\frac{\log N}{N}+\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

$$\Downarrow$$

$$\lim_{N\to\infty}\frac{\log J_N}{N}=\lim_{N\to\infty}\frac{\log g(N;5N/6)}{N}$$

Calculation of the limit of the maximum

$$
\lim_{N\to\infty} \frac{\log \left| J_N \left(\quad ; \exp(2\pi\sqrt{-1}/N) \right) \right|}{N} = \lim_{N\to\infty} \frac{\log g(N; 5N/6)}{N}
$$

$$
= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N;k) = 2 \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log\bigl(2\sin(k\pi/N)\bigr)
$$

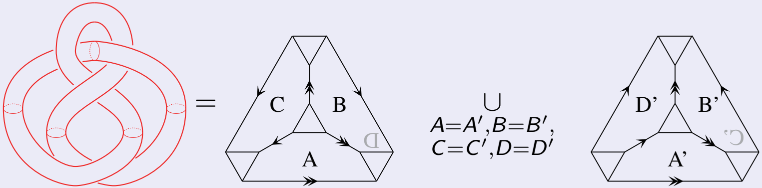
$$
= \frac{2}{\pi} \int_0^{5\pi/6} \log\bigl(2\sin x\bigr)\,dx = -\frac{2}{\pi}\Lambda(5\pi/6) = \frac{6\Lambda(\pi/3)}{2\pi} = 0.323066\ldots,
$$

where $\Lambda(\theta) := -\int_0^{\theta} \log|2\sin x|\,dx$ is the Lobachevsky function.

Decomposition of $S^3 \setminus$  into two tetrahedra

What is $6\Lambda(\pi/3)$?

Theorem (W. Thurston)



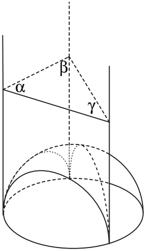
We can regard both pieces in the right hand side as regular ideal hyperbolic tetrahedra.

$\Rightarrow S^3 \setminus$  possesses a complete hyperbolic structure with finite volume.

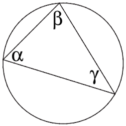
Ideal hyperbolic tetrahedron

- $\mathbb{H}^3 := \{(x, y, z) \mid z > 0\}$: with hyperbolic metric $ds := \frac{\sqrt{dx^2+dy^2+dz^2}}{z}$.
- Ideal hyperbolic tetrahedron : tetrahedron with geodesic faces with four vertices in the boundary at infinity.
- We may assume
  - One vertex is at (∞, ∞, ∞).
  - The other three are on xy-plane.

Ideal hyperbolic tetrahedron $\Delta(\alpha, \beta, \gamma)$



Top view



Ideal hyperbolic tetrahedron is defined (up to isometry) by the similarity class of this triangle.

$$\mathrm{Vol}(\Delta(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Proof of VC - conclusion

$$2\pi \lim_{N\to\infty} \frac{\log \left| J_N \left(\right.\right.}{}$$  $$\left.\left. ; \exp(2\pi\sqrt{-1}/N) \right) \right| \Big/ N$$

$$= 6\Lambda(\pi/3)$$

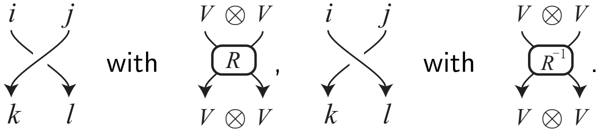
$$= 2\,\mathrm{Vol}(\text{regular ideal hyperbolic tetrahedron})$$

$$= \mathrm{Vol}\left(S^3 \setminus \right.$$  $$\left. \right)$$

$\Rightarrow$ Volume Conjecture for  .

R-matrix

The colored Jones polynomial can be calculated by using the following R-matrix $R\colon V \otimes V \to V \otimes V$ ($V := \mathbb{C}^N$).



$$R^{ij}_{kl} = \sum_m \delta_{l,i+m}\delta_{k,j-m} \frac{\pm(\text{a power of } \zeta_N) \times N^2}{(\zeta_N)_{m^+}(\zeta_N)_{i^+}(\zeta_N)_{k^+}(\zeta_N)_{j^-}(\zeta_N)_{l^-}},$$

where

- $\zeta_N := \exp(2\pi\sqrt{-1}/N)$,
- $(\zeta_N)_{k^+} := (1-\zeta_N)\cdots(1-\zeta_N^k)$, $(\zeta_N)_{k^-} := (1-\zeta_N)\cdots(1-\zeta_N^{N-1-k})$.

$\Rightarrow$

$$J_N(K;\zeta_N) \underset{\text{looks like}}{\simeq} \sum_{\substack{\text{labellings}\\ i,j,k,l\\ \text{on arcs}}} \left(\prod_{\pm\text{-crossings}} \frac{\pm(\text{a power of } \zeta_N) \times N^{\pm 2}}{(\zeta_N)_{m^+}(\zeta_N)_{i^\pm}(\zeta_N)_{k^\pm}(\zeta_N)_{j^\mp}(\zeta_N)_{l^\mp}} \right)$$

Approximation of the colored Jones polynomial by dilog

- (dilog function) $\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-y)}{y}\,dy = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$.
- $(\zeta_N)_{k^\pm} \underset{N\to\infty}{\approx} \exp\left[-\frac{N}{2\pi\sqrt{-1}}\mathrm{Li}_2(\zeta_N^{\pm k})\right]$. ($\approx$ means a very rough approximation.)

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx}$$

$$\sum_{\text{labellings}} (\text{polynomial of } N) \times (\text{power of } \zeta_N)$$

$$\exp\left[\frac{N}{2\pi\sqrt{-1}} \sum_{\text{crossings}} \left\{\mathrm{Li}_2(\zeta_N^m) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l})\right\}\right].$$

Approximation of the colored Jones polynomial by integral

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right]$$
$$\underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\,dz_1\cdots dz_c\right],$$

where

- $i_1,\dots,i_c$: labellings on arcs.
- $V(\zeta_N^{i_1},\dots,\zeta_N^{i_c}) :=$ $\displaystyle\sum_{\text{crossings}} \left\{\mathrm{Li}_2(\zeta_N^{m}) + \mathrm{Li}_2(\zeta_N^{\pm i}) + \mathrm{Li}_2(\zeta_N^{\mp j}) + \mathrm{Li}_2(\zeta_N^{\pm k}) + \mathrm{Li}_2(\zeta_N^{\mp l})\right\}.$
- $J_1,\dots,J_c$: contours.

Saddle point method

Find the maximum of $\left|\exp\left[\frac{N}{2\pi\sqrt{-1}}V(z_1,\dots,z_c)\right]\right|$.
$V(x_1,\dots,x_c)$: the maximum of $\{\operatorname{Im} V(z_1,\dots,z_c)\}_{(z_1,\dots,z_c)\in J_1\times\cdots\times J_c}$.
$\Rightarrow$

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}}V(x_1,\dots,x_c)\right],$$

modulo multiplication by a polynomial term in N.

$\Rightarrow$

$$2\pi \lim_{N\to\infty}\frac{\log|J_N(K;\zeta_N)|}{N} = \operatorname{Im} V(x_1,\dots,x_c)$$

Here $(x_1,\dots,x_c)$ satisfies the following.

$$\frac{\partial V}{\partial z_k}(x_1,\dots,x_c) = 0 \quad (k = 1,\dots,c).$$

Difficulties

Difficulties so far:

- Replacing the summation into an integral

$$\sum_{i_1,\dots,i_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(\zeta_N^{i_1},\dots,\zeta_N^{i_c})\right] \underset{N\to\infty}{\approx} \int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\, dz_1\cdots dz_c\right].$$

- How to apply the saddle point method.

$$\int_{J_1}\cdots\int_{J_c} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(z_1,\dots,z_c)\, dz_1\cdots dz_c\right] \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}} V(x_1,\dots,x_c)\right].$$

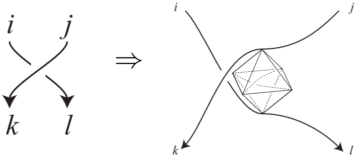
In particular, which saddle point to choose. In general, we have many solutions to the system of equations:

$$\frac{\partial V}{\partial z_k}(x_1,\dots,x_c) = 0 \quad (k = 1,\dots,c).$$

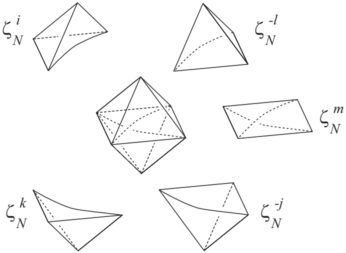
Decomposition into octahedra (by D. Thurston)

Decompose the knot complement into (topological, truncated) tetrahedra.

- Around each crossing, put an octahedron:

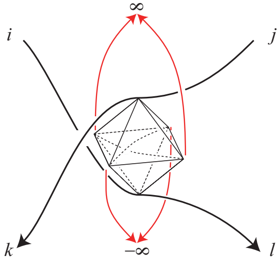


- Decompose the octahedron into five tetrahedra:



Decomposition into topological tetrahedra

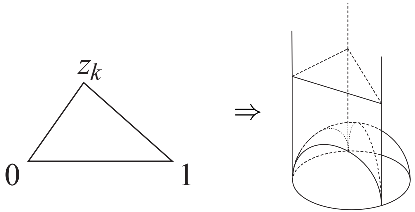
- Pull the vertices to the points at infinity:



- $S^3 \setminus K$ is now decomposed into topological, truncated tetrahedra, decorated with complex numbers $\zeta_N^{i_k}$.

Decomposition into hyperbolic tetrahedra

- Each topological, truncated tetrahedron is decorated with a complex number $\zeta_N^{i_k}$.
- We want to regard it as a hyperbolic, ideal tetrahedron.
- Recall that we have replaced a summation over i_k into an integral over z_k.
- Replace $\zeta_N^{i_k}$ with a complex variable z_k.
- Regard the tetrahedron decorated with z_k as an hyperbolic, ideal tetrahedron parametrized by z_k.



Hyperbolic structure on the knot complement

- Now the knot complement is decomposed into ideal, hyperbolic tetrahedra parametrized by $z_1, \dots, z_c$.
- Choose $z_1, \dots, z_c$ so that we can glue these tetrahedra well, that is,
  - around each edge, the sum of angles is 2π,
  - the triangles that appear in the boundary torus make the torus Euclidean.
- These conditions are the same as the system of equations that we used in the saddle point method (Y. Yokota)!

$$\frac{\partial V}{\partial z_k}(x_1, \dots, x_c) = 0 \quad (k = 1, \dots, c).$$

- $\Rightarrow (x_1, \dots, x_c)$ gives a unique complete hyperbolic structure in $S^3 \setminus K$.

Geometric meaning of the limit

So far we have "proved"

$$J_N(K;\zeta_N) \underset{N\to\infty}{\approx} \exp\left[\frac{N}{2\pi\sqrt{-1}}V(x_1,\dots,x_c)\right].$$

• $V(x_1,\dots,x_c)$ is the sum of $\mathrm{Li}_2(x_k)$ (and log), where the x_k define complete hyperbolic structure in $S^3 \setminus K$.
• Vol(tetrahedron parametrized by z) $= \operatorname{Im}\mathrm{Li}_2(z) - \log|z|\arg(1-z)$.
$\Rightarrow$

$$2\pi \lim_{N\to\infty} \frac{\log|J_N(K,\zeta_N)|}{N} = \mathrm{Vol}(S^3 \setminus K),$$

which is the Volume Conjecture.

First Generalization of Volume Conjecture

Volume Conjecture $\Rightarrow$

$$J_N\left(K;\exp\left(\frac{2\pi\sqrt{-1}}{N}\right)\right) \underset{N\to\infty}{\sim} \exp\left[\left(\sqrt{-1}\,\mathrm{Vol}(S^3\setminus K)+\text{something}\right)\left(\frac{N}{2\pi\sqrt{-1}}\right)\right] \times (\text{polynomial of } N).$$

Conjecture (Gukov + HM (2008))

K: hyperbolic knot. u: small complex parameter. Define $S(u)$ so that

$$J_N\left(K;\exp\left(\frac{2\pi\sqrt{-1}+u}{N}\right)\right) \underset{N\to\infty}{\sim} \exp\left[S(u)\left(\frac{N}{2\pi\sqrt{-1}+u}\right)\right] \times (\text{polynomial of } N).$$

$\Rightarrow$ *$S(u)$ would determine the volume and the $(SO(3))$ Chern–Simons invariant of the three-manifold K_u associated with a representation $\rho_u\colon \pi_1(S^3\setminus K)\to SL(2;\mathbb{C})$.*

Parameter u in the generalized VC

$$\rho_u \colon \pi_1(S^3 \setminus K) \to SL(2;\mathbb{C})$$

$$\rho_u(\text{meridian}) \mapsto \begin{pmatrix} \exp(u/2) & * \\ 0 & \exp(-u/2) \end{pmatrix}$$
$$\rho_u(\text{longitude}) \mapsto \begin{pmatrix} \exp(v(u)/2) & * \\ 0 & \exp(-v(u)/2) \end{pmatrix}$$

with

$$v(u) := 2\frac{d\,S(u)}{d\,u} - 2\pi\sqrt{-1}$$

- $u = 0 \Rightarrow \rho_0$: the complete hyperbolic structure.
- $u \neq 0 \Rightarrow$ incomplete hyperbolic structure.
  completion $\Rightarrow$ closed 3-manifold K_u
  ((p, q)-Dehn surgery if $pu + qv(u) = 2\pi\sqrt{-1}$).

Topological Interpretation of $S(u)$

Put

$$\mathrm{CS}(u) := S(u) - \pi\sqrt{-1}u - \frac{uv(u)}{4}.$$

$\Rightarrow$ $\mathrm{CS}(u)$ is the $SL(2;\mathbb{C})$-Chern–Simons invariant of $S^3 \setminus K$ associated with ρ_u, u and $v(u)$.

- Note that we need lifts of $\exp(u/2)$ and $\exp(v(u)/2)$ to define $\mathrm{CS}(u)$.

$\Rightarrow$ $S(u)$ determines

$$(SO(3) \text{ Chern–Simons invariant}) + \sqrt{-1}\,\mathrm{Vol}$$

of the closed hyperbolic three-manifold K_u, where $SO(3)$ Chern–Simons invariant is defined by using the Levi-Civita connection.

Further Generalization

Conjecture (Dimofte + Gukov (2010), HM (2011))

K: hyperbolic knot. $u \neq 0$: small complex parameter. Define $T(u)$ so that

$$J_N(K; \exp((2\pi\sqrt{-1}+u)/N)) \underset{N\to\infty}{\sim} \frac{\sqrt{-\pi}}{2\sinh(u/2)} T(u)^{1/2} \left(\frac{N}{2\pi\sqrt{-1}+u}\right)^{1/2} \exp\left[S(u)\left(\frac{N}{2\pi\sqrt{-1}+u}\right)\right],$$

$\Rightarrow$ *$T(u)$ would be the $SL(2;\mathbb{C})$-Reidemeister torsion associated with ρ_u.*

Figure-eight knot – colored Jones for $u = 0$

E: the figure-eight knot.

Theorem (Andersen + Hansen (2006))

$$J_N(E;\exp(2\pi\sqrt{-1}/N))$$
$$\underset{N\to\infty}{\sim} 2\pi^{3/2}\left(\frac{2}{\sqrt{-3}}\right)^{1/2}\left(\frac{N}{2\pi\sqrt{-1}}\right)^{3/2}\exp\left(\frac{N}{2\pi\sqrt{-1}}\times\sqrt{-1}\,\mathrm{Vol}(E)\right).$$

- $\frac{2}{\sqrt{-3}}$: *Reidemeister torsion.*
- $\sqrt{-1}\,\mathrm{Vol}(E)$: *Chern–Simons invariant.*
- *both associated with the holonomy representation (that defines the complete hyperbolic structure for* $S^3 \setminus E$*).*

Figure-eight knot – colored Jones for $u \neq 0$

E: the figure-eight knot.

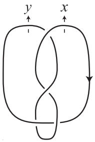
Theorem (Yokota + HM (2007), HM (2011))

$0 < u < \log\left(\frac{3+\sqrt{5}}{2}\right)$

$$J_N\bigl(E;\exp((u+2\pi\sqrt{-1})/N)\bigr) \underset{N\to\infty}{\sim} \frac{\sqrt{-\pi}}{2\sinh(u/2)}T(u)^{1/2}\sqrt{\frac{N}{2\pi\sqrt{-1}+u}}\exp\left(\frac{N}{2\pi\sqrt{-1}+u}S(u)\right).$$

- $S(u) := \mathrm{Li}_2(e^{u-\varphi(u)}) - \mathrm{Li}_2(e^{u+\varphi(u)}) - u\varphi(u)$.
- $T(u) := \frac{2}{\sqrt{(e^u+e^{-u}+1)(e^u+e^{-u}-3)}}$.
- $\varphi(u) := \operatorname{arccosh}(\cosh(u) - 1/2)$.

Figure-eight knot – representation ρ_u



$$\pi_1(S^3 \setminus K) = \langle x, y \mid (xy^{-1}x^{-1}y)x = y(xy^{-1}x^{-1}y)\rangle$$

$$\rho_u : \begin{cases} x & \mapsto \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix} \\ y & \mapsto \begin{pmatrix} e^{u/2} & 0 \\ -d(u) & e^{-u/2} \end{pmatrix}, \end{cases}$$

where $d(u) := \cosh u - 3/2 + \sqrt{(2\cosh u + 1)(2\cosh u - 3)}/2$ (Riley).

$$\rho_u(\text{longitude}) = \begin{pmatrix} \ell(u) & * \\ 0 & \ell(u)^{-1} \end{pmatrix}$$

with $\ell(u) := \cosh(2u) - \cosh u - 1 + \sinh u\sqrt{(2\cosh u + 1)(2\cosh u - 3)}$.

- ρ_u is a deformation of the holonomy representation ρ_0.

Figure-eight knot – colored Jones, $S(u)$, $T(u)$, and ρ_u

Put

$$v(u) := 2\frac{d\,S(u)}{d\,u} - 2\pi\sqrt{-1},$$
$$CS(u) := S(u) - \pi\sqrt{-1}u - uv(u)/4.$$

Then $e^{v(u)/2} = -\ell(u)$ and so

$$\rho_u(\text{longitude}) = \begin{pmatrix} -e^{v(u)/2} & * \\ 0 & -e^{-v(u)/2} \end{pmatrix}.$$

- $CS(u)$ is the $SL(2;\mathbb{C})$ Chern–Simons invariant associated with ρ_u, u and $v(u)$
- $T(u)$ is the Reidemeister torsion associated with ρ_u.

Torus knot – colored Jones for $u = 0$

$T(a, b)$: torus knot of type (a, b) $(a, b > 0)$.

Theorem (Kashaev + Tirkkonen, Dubois + Kashaev)

$$
\begin{aligned}
&J_N\left(T(a,b); \exp(2\pi\sqrt{-1}/N)\right) \\
\underset{N\to\infty}{\sim}\ & \frac{\pi^{3/2}}{2ab}\left(\frac{N}{2\pi\sqrt{-1}}\right)^{3/2} \\
&\times\left(\sum_{k=1}^{ab-1} T_k^{1/2}(-1)^{k+1}k^2 \exp\left[S_k(0)\left(\frac{N}{2\pi\sqrt{-1}}\right)\right]\right),
\end{aligned}
$$

where

$$
S_k(u) := \frac{-\left(2k\pi\sqrt{-1} - ab(2\pi\sqrt{-1}+u)\right)^2}{4ab},
$$

$$
T_k := \frac{16\sin^2(k\pi/a)\sin^2(k\pi/b)}{ab}.
$$

Torus knot – colored Jones for $u \neq 0$

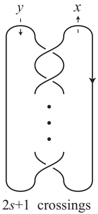
Theorem (Hikami + HM)

$u \neq 0$: small

$$
J_N\left(T(a,b);\exp((u+2\pi\sqrt{-1})/N)\right)
\underset{N\to\infty}{\sim}
\begin{cases}
\frac{1}{\Delta(T(a,b);e^{u/2})} & (\operatorname{Re} u > 0), \\
\frac{1}{\Delta(T(a,b);e^{u/2})} \\
\quad + \frac{1}{2\sinh(u/2)} \sum_k (-1)^{k+1}\sqrt{-1}\,T_k^{1/2}\left(\frac{N}{2\pi\sqrt{-1}}\right)^{1/2} \\
\qquad \times \exp\left[S_k(u)\left(\frac{N}{2\pi\sqrt{-1}+u}\right)\right] & (\operatorname{Re} u \leq 0),
\end{cases}
$$

where $\Delta(K;t)$ is the Alexander polynomial of a knot K.

Torus knot $T(2, 2s+1)$ – representation ρ_u^k



$$\pi_1(S^3 \setminus [\text{diagram}]) = \langle x, y \mid (xy)^s x = y(xy)^s \rangle$$

$$\rho_u^k : \begin{cases} x & \mapsto \begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix} \\ y & \mapsto \begin{pmatrix} e^{u/2} & 0 \\ -d_k(u) & e^{-u/2} \end{pmatrix}, \end{cases}$$

where $d_k(u) := 2\cosh u - 2\cos\left(\frac{(2k+1)\pi}{2s+1}\right)$ (Riley).

$$\rho_u^k(\text{longitude}) = \begin{pmatrix} -e^{-(2s+1)u} & * \\ 0 & -e^{(2s+1)u} \end{pmatrix}.$$

Torus knot – colored Jones, $S_k(u)$, $T_k(u)$, and ρ_u^k

$$v_k(u) := 2\frac{d\,S_k(u)}{d\,u} - 2\pi\sqrt{-1}.$$

$$CS_k(u) := S_k(u) - \pi\sqrt{-1}u - uv_k(u)/4.$$

$\Rightarrow e^{v_k(u)/2} = (-1)^k e^{-(2s+1)u}$ and so

$$\rho_u^k(\text{longitude}) = \begin{pmatrix} \pm e^{v_k(u)/2} & * \\ 0 & \pm e^{-v_k(u)/2} \end{pmatrix}.$$

- $CS_k(u)$: Chern–Simons invariant associated with ρ_u^k, u and $v_k(u)$
- $T_k(u)$: Reidemeister torsion associated with ρ_u^k.

$\Rightarrow J_N$ splits into terms corresponding to representations.

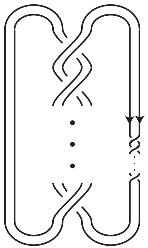
Theorem (Garoufalidis + Lê (2011))

For any knot K, if $|\theta|$ is small, then

$$\lim_{N\to\infty} J_N(K;\exp(\theta/N)) = 1/\Delta(K;\exp\theta).$$

$\Rightarrow$ the term $1/\Delta(T(a,b);e^{u/2})$ corresponds to an Abelian representation.

Cable of Torus knot – colored Jones for $u = 0$



Theorem (van der Veen)

$$\lim_{N\to\infty} \frac{\log\left|J_N\left(\text{iterated torus knot}; \exp(2\pi\sqrt{-1}/N)\right)\right|}{N} = 0.$$

- $\boxed{\mathrm{Vol}(K) = 0} \Leftrightarrow \boxed{K \text{ is an iterated torus knot}}$

Cable of Torus knot – colored Jones for $u \neq 0$

$T(a,b)^{(2,2m+1)}$: $(2,2m+1)$-cable of $T(a,b)$ with $2m+1 > 2ab$.
u: small, $\operatorname{Re} u > 0$. $\theta := u + 2\pi\sqrt{-1}$.

Theorem (HM)

$$
\begin{aligned}
& J_N\left(T(a,b)^{(2,2m+1)}; \exp(\theta/N)\right) \\
\underset{N\to\infty}{\sim} & \frac{1}{\Delta\left(T(a,b)^{(2,2m+1)}; \exp(\theta)\right)} \\
& - \sum_{0<j<\frac{ab\operatorname{Im}(\theta)}{\pi}} \frac{\sqrt{-\pi}}{\theta\sqrt{ab}\sinh\left(\frac{\theta}{2}\right)} \sqrt{\frac{N}{\theta}}\tau_1(j)\exp\left[\frac{N}{\theta}S_1(\theta;j)\right] \\
& + \sum_{0\le k<\frac{(2m+1)\operatorname{Im}(\theta)}{2\pi}-\frac{1}{2}} (-1)^k \frac{\sqrt{-2\pi}}{\sqrt{(2m+1)}\sinh\left(\frac{\theta}{2}\right)} \sqrt{\frac{N}{\theta}}\tau_2(k)\exp\left[\frac{N}{\theta}S_2(\theta;k)\right] \\
& + \sum_{(j,k)} (-1)^{k+l+1} \frac{4\pi\sqrt{-1}}{\sqrt{2ab(2m+1-2ab)}\sinh\left(\frac{\theta}{2}\right)} \frac{N}{\theta}\tau_3(j,k)\exp\left[\frac{N}{\theta}S_3(\theta;j,k)\right],
\end{aligned}
$$

Cable of Torus knot – colored Jones for $u \neq 0$, continued

$$S_1(\theta; j) := -ab\theta^2 + 2j\theta\pi\sqrt{-1} + \frac{j^2\pi^2}{ab},$$

$$\tau_1(j) := \frac{\sin(j\pi/a)\sin(j\pi/b)}{\cosh(\theta(2m+1-2ab)/2)},$$

$$S_2(\theta; k) := -\frac{(2m+1)}{2}\theta^2 + (2k+1)\theta\pi\sqrt{-1} + \frac{(2k+1)^2\pi^2}{2(2m+1)},$$

$$\tau_2(k) := \frac{\sin(b(2k+1)\pi/(2m+1))\sin(a(2k+1)\pi/(2m+1))}{\sin(ab(2k+1)\pi/(2m+1))},$$

$$S_3(\theta; j, k) := -\frac{(2m+1)}{2}\theta^2 + (2k+1)\pi\theta\sqrt{-1} + \frac{(ab(2k+1)^2 + 2j^2(2m+1) - 4abj(2k+1))\pi^2}{2ab(2m+1-2ab)},$$

$$\tau_3(j) := \sin(j\pi/a)\sin(j\pi/b).$$

Observation and speculation

- $S_1(\theta; j)$ and $S_2(\theta; k)$ define the Chern–Simons invariant as in the case of torus knots.
- We expect that for large N, $J_N\Big(K; \exp\big((2\pi\sqrt{-1} + u)/N\big)\Big)$ splits into terms corresponding to representations. Moreover each term determines the Chern-Simons invariant and the Reidemeister torsion associated with the representation.
- We hope a similar formula holds for any knot.