Analytic families of quantum hyperbolic invariants

Stéphane Baseilhac
Joint work with R. Benedetti

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Plan of the talk

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The quantum hyperbolic invariants (QHI)

$$\mathcal{H}_N(V, L, \rho, \omega) \in \mathbb{C}/\mu_{2N}$$

are defined for every odd integer $N \geq 3$ and (V, L, ρ, ω) such that :

- V is a compact oriented 3-manifold, $\partial V = \emptyset$ or a union of tori
- L is a link in V, and $Int(V)$ is cusped hyperbolic if $L = \emptyset$ **(2 compl. cases)**
- ρ is an augmented $PSL(2, \mathbb{C})$-character of $V \setminus L$, constrained if $L = \emptyset$
- ω is a tuple of 1-cohomology classes on V and ∂V satisfying compatibility constraints ; the pair (ρ, ω) **refines** ρ.

When V is the interior of a 1-**cusped** hyperbolic manifold M, varying (ρ, ω) the invariants $\mathcal{H}_N(V, \emptyset, \rho, \omega)$ produce **rational functions** $\mathcal{H}_N^h$ for each $h \in H^1(M; \mathbb{Z}/2\mathbb{Z})$

$$\begin{array}{ccc} \mathcal{H}_N^h : X_{0,N} & \xrightarrow{\textit{rational}} & \mathbb{C}/\mu_{2N} \\ \uparrow & & \\ \mathcal{S}^h : X_{0,\infty} & \xrightarrow{\textit{analytic}} & \mathbb{C} \end{array}$$

with :

- coverings $X_{0,\infty} \xrightarrow{/\mathbb{Z}^2} X_{0,N} \xrightarrow{/(\mathbb{Z}/N\mathbb{Z})^2} X_0$ **(the geom. cpnt)**
- The **Chern-Simons function** $\mathcal{S}^h$.

The Chern-Simons function $\mathcal{S}^h$ is an equivariant formulation of the Chern-Simons section in gauge theory :

- At the natural lift of the hyp. holonomy and $h = 0$, we have

$$\mathcal{S}^0(\tilde{\rho}_{hyp}) = \exp\left(\frac{2}{\pi}\mathrm{Vol}(M) + 2\pi i \mathrm{CS}(M)\right)$$

- The variation of $\mathcal{S}^0$ along lifted paths of characters lies over the cusp. In terms of dilation coefficients, locally we have

$$d\mathcal{S}^0 = -\frac{1}{2\pi i}\left(\log(\lambda) d\log(\mu) - \log(\mu) d\log(\lambda)\right)$$

Consider a sequence of points $\mathbf{x} = \{x_N \in X_{0,N}\}_N$. Define

$$\mathcal{H}^h_\infty(\mathbf{x}) := \limsup \frac{\log |\mathcal{H}^h_N(\mathbf{x})|}{N} \quad \in \mathbb{R} \cup \{\infty\}.$$

The $X_{0,N}$'s are curves, the $\mathcal{H}^h_N$'s are rational : what is $\mathcal{H}^h_\infty(\mathbf{x})$?

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QHI ASYMPTOTIC PROBLEM

- Study the **function** $\mathcal{H}^h_\infty(\mathbf{x})$: singularities, regularity, etc.
- Find a geometric interpretation of $\mathcal{H}^h_\infty(\mathbf{x})$.

A "volume conjecture" :

For every M there exists $\mathbf{x}$ such that $\mathcal{H}^h_\infty(\mathbf{x}) = \frac{1}{2\pi}\mathrm{Vol}(M)$.

In the simplest case of a closed manifold, the three-sphere :

Theorem

For every link L in S^3 and every odd integer $N \geq 3$ we have

$$H_N(S^3, L, \rho_{triv}, \mathbf{0}) \equiv_N \; J_N(L)(e^{2i\pi/N})$$

where $J_N(L)$ is the normalized colored Jones polynomial of L.

Like in the classical case of the Chern-Simons function $\mathcal{S}^h$:

Theorem

For any sequence of closed hyperbolic Dehn fillings V_n of M with holonomies $\rho_n \to \rho_{hyp} \in X_0$ and core L_n we have

$$\lim_{n\to\infty} \mathcal{H}_N(V_n, L_n, \rho_n, \mathbf{0}) \equiv_N \mathcal{H}_N(V, \emptyset, \rho_{hyp}, \mathbf{0}).$$

Plan of the talk

I want to explain some ingredients hidden behind the diagrams :

$$\begin{array}{ccc} \mathcal{H}_N^h : X_{0,N} & \xrightarrow{\ \textit{rational}\ } & \mathbb{C}/\mu_{2N} \\ \uparrow & & \\ \mathcal{S}^h : X_{0,\infty} & \xrightarrow{\ \textit{analytic}\ } & \mathbb{C} \end{array}$$

First we need to describe simplicially X_0, and coverings of it.

Denote by X_0 the geometric component of augmented $PSL(2, \mathbb{C})$ valued characters of M, and $X(\partial \bar{M})$ the character variety of $\partial \bar{M}$.

The restriction map $res : X_0 \to X(\partial \bar{M})$ is regular.

Theorem (Dunfield)

The map $res : X_0 \to X(\partial \bar{M})$ is birational onto its image.

Fixing a cusp basis, denote the induced map and image by

$$\mathfrak{h} : X_0 \to \mathbb{C}^* \times \mathbb{C}^*$$

$$A_0 := \mathfrak{h}(X_0)$$

Let T be an ideal triangulation of M without null-homotopic edges. Then the gluing variety $G(T) \neq \emptyset$ (Segerman-Tilmann), it is a curve (Neumann-Zagier), and $\exists\ z_{hyp} \in G(T)$ with holonomy ρ_{hyp}.

Question

Is Dunfield's theorem true by replacing X_0 by compts of $G(T)$?

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Problem

- z_{hyp} may not be a regular point of $G(T)$, hence may be contained in several components
- Dunfield's proof uses the volume rigidity for closed hyperbolic Dehn fillings of M, and the variation formula of Vol.

Def. *An irreducible component of* $G(T)$ *is* **rich** *if it contains* z_{hyp} *and an infinite sequence of closed hyperbolic Dehn fillings of* M *with shape parameters* $z_n \to z_{hyp}$.

Proposition (Petronio-Porti)

The non negative ideal triangulations of M have rich components. Hence the max subdivisions of the EP cellulation of M provide a canonical finite set of rich components of gluing varieties of M.

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Corollary

For any rich component Z of a gluing variety $G(T)$ of M the (regular) map $\mathfrak{h}_Z : \ Z \xrightarrow{holonomy} X_0 \xrightarrow{\mathfrak{h}} A_0$ is birational.

We want to complete a square

$$\begin{array}{ccc} ?? & \xrightarrow{\ \mathfrak{h}_{Z,\infty}\ } & A_{0,\infty} \\ \downarrow & & \downarrow /\mathbb{Z}^2 \\ Z & \xrightarrow{\ \mathfrak{h}_{Z}\ } & A_0 \end{array}$$

where

$$A_{0,\infty} := \{(u,v) \in \mathbb{C}^2 \mid (e^u, e^v) \in A_0\}.$$

Define the analytic set (s is the number of tetrahedra of T)

$$Z_\infty = \{(l_0^1, l_1^1, l_2^1, \ldots, l_0^s, l_1^s, l_2^s) \in \mathbb{C}^{3s} \mid \forall\, j \in \{1,\ldots,s\}, r \in \{0,1,2\}, e^{l_r^j} = \pm z_r^j, (z_r^j)_{j,r} \in Z, \forall\, j \in \{1,\ldots,s\}, l_0^j + l_1^j + l_2^j = 0, \forall\, E \in E(T), \sum_{j,r} l_r^j(E) = 0\}.$$

(Space of Logs of $\pm$ shape parameters in Z)

and similarly the algebraic set

$$Z_N = \{(w_0^1, w_1^1, w_2^1, \ldots, w_0^s, w_1^s, w_2^s) \in \mathbb{C}^{3s} \mid \forall\, j \in \{1, \ldots, s\}, r \in \{0,1,2\}, (w_r^j)^N = z_r^j, (z_r^j)_{j,r} \in Z$$
$$\forall\, j \in \{1, \ldots, s\}, w_0^1 w_1^1 w_2^1 = -\zeta^{\frac{N-1}{2}},$$
$$\forall\, e \in E(T), \prod_{j,r} w_r^j(E) = \zeta^{-1}\}.$$

(Space of N-th roots of shape parameters in Z)

Theorem (Neumann)

(1) The natural lift $\tilde{\mathfrak{h}}_Z : Z_\infty \to A_{0,\infty}$ of $\mathfrak{h}_Z$ maps onto a Zariski open subset (no lift is missed).

(2) The fibers of the covering $Z_\infty \to Z$ are affine spaces over an abelian group C that fits in an exact sequence

$$0 \to \mathbb{Z}^{n(edges)} \to C \to H^1(\partial \bar{M}; \mathbb{Z}) \oplus H^1(\bar{M}; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{r-i^*} H^1(\partial V; \mathbb{Z}/2\mathbb{Z}) \to 0$$

We deduce a diagram

$$\begin{array}{ccc}
Z_\infty & \xrightarrow[\textbf{classical}]{\mathfrak{h}_{Z,\infty}} & A_{0,\infty} \\
\downarrow & & \downarrow {\scriptstyle /(N\mathbb{Z})^2} \\
Z_N & \xrightarrow[\textbf{quantum}]{\mathfrak{h}_{Z,N}} & A_{0,N} \\
\downarrow & & \downarrow {\scriptstyle /(\mathbb{Z}/N\mathbb{Z})^2} \\
Z & \xrightarrow{\mathfrak{h}_{Z}} & A_0
\end{array}$$

A point of Z_∞ represents a holonomy in $A_0 \approx X_0$, and for each choice of $h \in H^1(\bar{M}; \mathbb{Z}/2\mathbb{Z})$, a compatible lift by

$$\exp : H^1(\partial\bar{M}, \mathbb{C}) \to H^1(\partial\bar{M}, \mathbb{C}^*)$$

of the class associated to the dilation factors of its peripheral subgroups. **The residual** $\mathbb{Z}^{n(edges)}$ **will be irrelevant (extrinsic).**

Moreover, $H := C/\mathbb{Z}^{n(edges)}$ has a natural non degenerate skew symmetric bilinear form B making a commutative diagram

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\quad B \quad} & \mathbb{Z} \\ \downarrow & & \downarrow 2\times \\ H_1(\partial \bar{M};\mathbb{Z}) \otimes H_1(\partial \bar{M};\mathbb{Z}) & \xrightarrow{\quad \cdot \quad} & \mathbb{Z} \end{array}$$

Plan of the talk

The quantum hyperbolic invariants of M are defined by means of **state sums** over **weakly branched** ideal triangulations of M carrying the spaces Z_N defined previously.

The Chern-Simons function of M can be defined similarly, by replacing Z_N by Z_∞ and the state sums by a signed sum of classical dilogarithms.

Def. *A 3-dim. pseudo-manifold triangulation is pre-branched if each 2-face is co-oriented and two co-orientations point inwards and two outwards each tetrahedron. The triangulation is weakly branched if its tetrahedra are branched and induce compatible pre-branchings*.

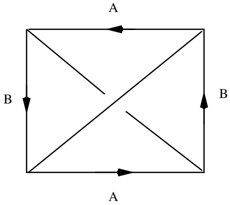


FIGURE: A pre-branched tetrahedron with its *square* edges oriented.

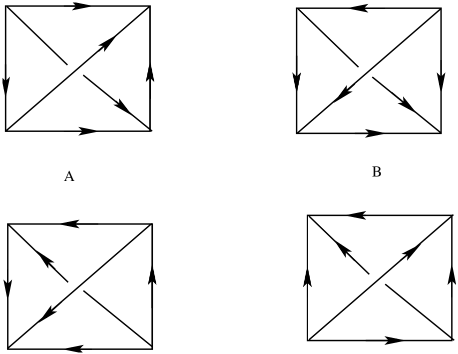


FIGURE: Branched tetrahedra inducing the same pre-branching.

1. Global pre-branchings exist on any triangulation.
2. The global pre-branchings on a triangulation are in 1-to-1 correspondence with the sol. of the gluing equations of the form $(1, 1, -1)$ on each tetra ("$\mathbb{Z}/2$-taut angle structures").

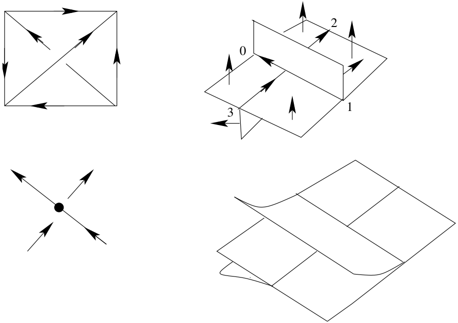


FIGURE: Graph encoding of a branched tetrahedron ($*_b = 1$).

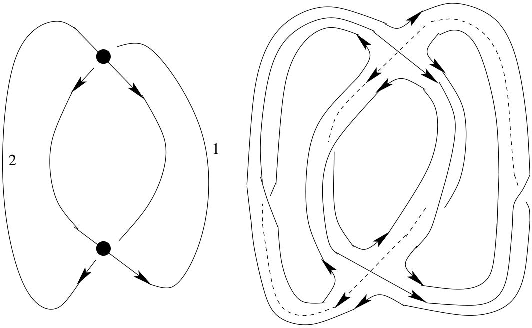


FIGURE: A graph representing a weak branching of the EP triangulation of the "figure eight sister" cusped manifold.

The gluing map $\phi : F^i(u_0^i, u_1^i, u_2^i) \to F^f(u_0^f, u_1^f, u_2^f)$ of (branched) 2-faces is determined by the permutation $\tau \in A_3$ s.t. $\phi(u_j^i) = u_{\tau(j)}^f$.

This gives **a color** $r \in \mathbb{Z}/3\mathbb{Z}$.

There is a functorial way to assign automorphisms

$$R \in \mathrm{GL}(\mathbb{C}^N \otimes \mathbb{C}^N) \quad , \quad \mathcal{Q} \in \mathrm{GL}(\mathbb{C}^N)$$

("amplitude" and "transition") to the branched tetrahedra and the co-oriented faces of a weakly branched triangulation : associate to the 2-face opposite to the j-th vertex a copy V_j of $\mathbb{C}^N$, and put

$$R = \begin{cases} (R^{i,j}_{k,l}) : V_3 \otimes V_1 \rightarrow V_2 \otimes V_0 & \text{if } *_b = +1 \\ (\bar{R}^{k,l}_{i,j}) : V_2 \otimes V_0 \rightarrow V_3 \otimes V_1 & \text{if } *_b = -1. \end{cases}$$

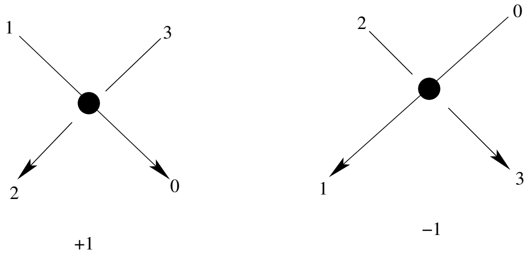


FIGURE: Assigning $R^{\pm 1}$ to the graph crossings.

Def. *Let* $(T, \tilde{b})$ *be a weakly branched triangulation of* M *having a rich component* Z*. The* **state sum** *function over* Z_N *is :*

$$\mathcal{H}_N(T, \tilde{b})(w) := \sum_{\sigma} \prod_{j} \mathrm{R}_N(\Delta_j, b_j, w^j)_\sigma \prod_{e} (\mathcal{Q}_N^{r(e)})_\sigma$$

where

- σ (a "state") runs over all maps $T^{(2)} \to \{0, 1, \ldots, N-1\}$
- $\mathrm{R}_N(\Delta, b, w) \in \mathrm{GL}(\mathbb{C}^N \otimes \mathbb{C}^N)$ is the **matrix dilogarithm**
- $\mathcal{Q}_N = T^{-1}S \in \mathrm{GL}(\mathbb{C}^N)$ has projective order 3, with S, T generating a projective representation of $SL(2, \mathbb{Z})$
- $\mathrm{R}_N(\ldots)_\sigma$ and $(\mathcal{Q}_N)_\sigma$ stand for the entries selected by σ.

The matrix dilogarithm $\mathrm{R}_N(\Delta, b, w) \in \mathrm{GL}(\mathbb{C}^N \otimes \mathbb{C}^N)$ is derived from the 6j-symbols of "generic" representations of $U_q sl_2$.

It satisfies highly non trivial tensor/functional "5-term" identities, the pentagon relations, and equivariance under tetra. symmetries.

Alg. identities $\leftrightarrow$ "moves" of triangulations $(T, \tilde{b}, w)$

A pentagon identity (a non Abelian 3-**cocycle relation**) :

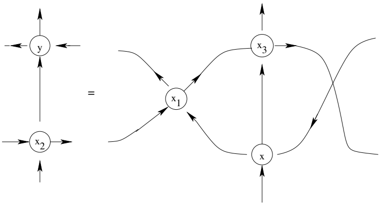
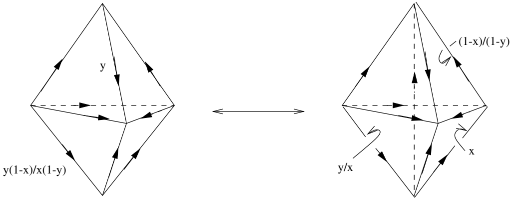


FIGURE: $x_1 = y/x$, $x_2 = y(1-x)/x(1-y)$, $x_3 = (1-x)/(1-y)$).



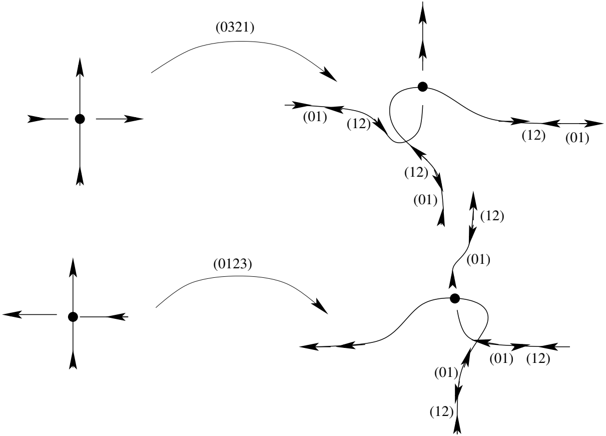
Theorem

$\mathcal{H}_N(T,\tilde{b})(w)$, $w \in Z_N$, is an invariant of the represented tuple (V,ρ,ω) up to multiplication by $2N$-th roots of 1, and varying w with fixed $h \in H^1(M;\mathbb{Z}/2\mathbb{Z})$ yields a rational function $\mathcal{H}_N^h$ on $A_{0,N}$.

Strategy :

- The state sums are invariant under enhancements of triangulation moves carrying all the extra structures
- The state sums are invariant under weak branching changes.
- Two $(T,\tilde{b},w)$'s are related by such transformations.
- A factorization result removing additional cohomological datas.

Example : invariance under moves preserving the pre-branching



Plan of the talk

- Lift $\mathcal{H}_N^h : X_{0,N} \to \mathbb{C}/\mu_{2N}$ to a $\mathbb{C}$-valued function
- $\mathcal{H}_N^h$ is a rational function : expression in terms of augmented characters of meridian/longitude ? Poles, periods ?
- Find a skein theoretic construction of $\mathcal{H}_N^h$
- Find a geometric quantization construction of $\mathcal{H}_N^h$ from the Chern-Simons function and line bundle
- Relate $\mathcal{H}_N^h$ of **link complements** to Jones and ADO invariants of **links in S^3**
- ...Study the QHI asymptotic problem.