A counterexample to metric differentiability

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Abstract

We show that the metric version of Pansu's differentiability result for Lipschitz maps fails - this illustrates an interesting difference between Euclidean domains and domains that are nonabelian stratified groups.

Introduction

Differentiability of Lipschitz maps is a basic tool to tackle several questions in Geometric Measure Theory. In fact, Lipschitz maps allow a natural generalization of the notion of surface in a general metric space, considering subsets of the space which are parameterized by Lipschitz maps defined on some Euclidean space. These subsets are called rectifiable. When the target is another Euclidean space, by Rademacher's differentiability theorem, many of the classical properties of smooth surfaces can be extended to rectifiable sets (see [4] or [14] for a complete presentation of the subject). Recently, some properties of rectifiable sets as the existence a.e. of tangent spaces, the regularity of their Hausdorff measure, area and coarea formulae have been extended to general metric spaces, see [1], [9] and [10]. The key idea is to replace the notion of differentiability with a weaker one, namely *metric differentiability*, and to prove that any metric valued Lipschitz map defined on an Euclidean space is metrically differentiable.

On the other hand, in [15] Rademacher's theorem was generalized from the Euclidean to the setting of nonabelian stratified groups. This boosted the development of geometric measure theory methods in such groups, we quote for instance [2], [6], [7], [11], [12], [13], [16], [17], [18].

In [16] a further extension of the metric differentiability into this nonabelian framework is achieved and used as the main tool to obtain the nonexistence of quasi-isometric embeddings of nonabelian stratified group into Alexandrov metric spaces with nonnegative or nonpositive curvature in the sense of Topanogov. More precisely, in [16] a partial metric differentiability of Lipschitz maps along the so-called *horizontal directions* of the group is proved, leaving open the question of the complete metric differentiability, as posed in same paper in Remark 3.

The question whether the full metric version of Pansu's extension of Rademacher's theorem is valid arises also in another connection. In [3], Section 11.4, G.David and S.Semmes note that metric differentiabilty is the perhaps most powerful tool to find bi-Lipschitz pieces of mappings and to decide which metric spaces look down on others. In [8], Question 22, J.Heinonen and S.Semmes asked in particular whether the three-dimensional Heisenberg group looks down on all other spaces. Of course, this would be an easy consequence of the metric differentiability.

In this note we present however a counterexample showing that the metric differentiability of Lipschitz maps may fail when the domain of the map is a nonabelian stratified group, instead of an Euclidean space.

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1 Some basic definitions

In this brief section we recall the main notions we are going to use. A *stratified group* is a graded, nilpotent, simply connected Lie group $\mathbb{G}$, such that there exists a subspace of left invariant vector fields which generate all of the Lie algebra $\mathcal{G}$ with respect to the Lie product of vector fields. We have a grading $\mathcal{G} = V_1 \oplus \cdots \oplus V_n$ and a one-parameter group of dilations, setting $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$,

$$\delta_r\Big(\sum_{i=1}^{n} v_i\Big) = \sum_{i=1}^{n} r^i\, v_i\,, \qquad r > 0\,,$$

where $v_i \in V_i$, for $i = 1, \ldots, n$. The integer n is called the degree of nilpotency of the group (see [5] for more information on stratified groups). These types of groups can be endowed with a natural left invariant distance $d : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty[$ which is homogeneous with respect to the group of self-similarities, that is

$$d(\delta_r x, \delta_r y) = r\, d(x, y)\,,$$

for any $r > 0$, $x, y \in \mathbb{G}$. Note that we have identified the group with its Lie algebra, using the fact that $\mathbb{G}$ is simply connected, hence there exists a diffeomorphism between $\mathbb{G}$ and $\mathcal{G}$.

Now we introduce the definition of metric differentiability generalized to stratified groups.

Definition 1.1 We say that a map $\nu : \mathbb{G} \longrightarrow [0, +\infty[$ is a *homogeneous seminorm* if for each $x, y \in \mathbb{G}$ and $r > 0$ we have

1. $\nu(\delta_r x) = r\, \nu(x)\,,$
2. $\nu(xy) \leq \nu(x) + \nu(y)\,.$

Definition 1.2 Let (Y, ρ) and $(\mathbb{G}, d)$ be a metric space and a stratified group, respectively. We say that a map $f : A \longrightarrow Y$, where A is an open subset of $\mathbb{G}$, is *metrically differentiable* at $x \in A$, if there exists a homogeneous seminorm ν_x such that

$$\frac{\rho\left(f(x\delta_t v), f(x)\right)}{t} \longrightarrow \nu_x(v) \quad \text{as} \quad t \to 0^+\,,$$

uniformly in v which varies in a compact neighbourhood of the unit element.

Remark 1.3 We point out that if $\mathbb{G} = \mathbb{R}^n$, then any Lipschitz map is metrically differentiable a.e. as it is proved in [1], [9] and [10]. Furthermore, in [16] it is shown that bi-Lipschitz maps are a.e. metric differentiable on stratified groups if one allows the direction v to vary only on the elements of V_1, namely the *horizontal directions*. The latter result of course directly applies also to Lipschitz maps, as they can be easily turned into bi-Lipschitz ones by adding a suitable (pushforward of the domain's) metric to the one given on the image of the function. So, it is clear that we will consider nonhorizontal direction in order to show that the metric differentiability does not hold in general.

We choose as a stratified group to build our counterexample the 3-dimensional Heisenberg group $\mathbb{H}$, which can be linearly identified with $\mathbb{R}^3$. We denote the elements $\eta, \xi \in \mathbb{H}$ as $\xi = (z,t)$, $\eta = (w,\tau)$, where $z = (z_1, z_2)$, $w = (w_1, w_2)$ belong to $\mathbb{R}^2$. As usual, on $\mathbb{H}$ we have the nonabelian group operation

$$(z,t)(w,\tau) = (z + w, t + \tau + 2(z_1 w_2 - z_2 w_1)) .$$

In this case the nonhorizontal directions are of the type $(0,0,s)$, with $s \neq 0$. We consider $G : \mathbb{H} \longrightarrow \mathbb{R}$, defined as $G(z,t) = |z| \vee \sqrt{|t|}$, where the symbol $\vee$ denotes the "maximum" operation. It is known that $d(\xi,\eta) = G(\xi^{-1}\eta)$, for $\xi, \eta \in \mathbb{H}$, yields a left invariant distance on the Heisenberg group, see for instance [6]. The dilations $\delta_r : \mathbb{H} \longrightarrow \mathbb{H}$ are defined as $\delta_r((z,t)) = (rz, r^2 t)$. It is clear that these dilations scale homogeneously with the distance d.

2 The counterexample

In this section we build a new metric ρ on $\mathbb{H}$ such that the identity map $I : \mathbb{H} \longrightarrow \mathbb{H}$ is a Lipschitz function with respect to the homogeneous distance d on the domain and the metric ρ on the codomain, more precisely a 1-Lipschitz function. We will show that with this distance the metric differentiability fails. We have seen that a homogeneous distance in the Heisenberg group can be defined as $d(\xi,\eta) = G(\xi^{-1}\eta)$, where $G(z,t) = |z| \vee \sqrt{|t|}$. We obtain our counterexample replacing the square root function in the definition of G with a concave map $g : [0,+\infty[\longrightarrow [0,+\infty[$ such that the function $S : \mathbb{H} \longrightarrow \mathbb{R}$, $S(z,t) = |z| \vee g(|t|)$ satisfies the following three claims:

1. the function $S : \mathbb{H} \longrightarrow \mathbb{R}$ yields a left invariant metric on $\mathbb{H}$ which is defined as $\rho(\xi,\eta) = S(\xi^{-1}\eta)$, $\xi, \eta \in \mathbb{H}$.
2. the map $I : (\mathbb{H}, d) \longrightarrow (\mathbb{H}, \rho)$ is 1-Lipschitz,
3. if we consider the nonhorizontal direction $v = (0,0,1) \in \mathbb{H}$, then for any $\zeta \in \mathbb{H}$ there does not exist the limit of
$$\frac{\rho(I(\zeta\delta_t v), I(\zeta))}{t} = \frac{\rho(\delta_t v, 0)}{t} \qquad \text{as} \qquad t \to 0_+ ,$$
in fact, we reach the maximal possible oscillation of the quotient
$$\limsup_{t\to 0_+} \frac{\rho(I(\zeta\delta_t v), I(\zeta))}{t} = 1 , \qquad \liminf_{t\to 0_+} \frac{\rho(I(\zeta\delta_t v), I(\zeta))}{t} = 0 .$$

Claim 3 says in particular that the 1-Lipschitz map $I : (\mathbb{H}, d) \longrightarrow (\mathbb{H}, \rho)$ is not metrically differentiable at any point of $\mathbb{H}$. The following two theorems will prove the existence of a map $g : [0,+\infty[\longrightarrow [0,+\infty[$ such that our claims are satisfied and in this way establish the counterexample.

Theorem 2.1 *Let $\kappa : [0,+\infty[\longrightarrow [0,+\infty[$ be a convex, strictly increasing function, which is continuous at the origin and satisfies $\kappa(0) = 0$. Then, defining $h(t) = \kappa(t) + t^2$, the concave map $g = h^{-1}$ yields a function $S(z,t) = |z| \vee g(|t|)$ which satisfies claims 1 and 2.*

PROOF. The convexity and the continuity at the origin of κ imply $\kappa(t)+\kappa(s) \leq \kappa(t+s)$ for any $t, s \geq 0$, hence

$$h(t+s) \geq h(t) + h(s) + 2ts \quad \text{for} \quad t, s \geq 0\,. \tag{1}$$

The function $h(t) = \kappa(t) + t^2$ is strictly monotone, thus $g = h^{-1}$ is well defined and $S(z,t) = |z| \vee g(|t|)$ also. The triangle inequality for the function $\rho(\xi,\eta) = S(\xi^{-1}\eta)$ is equivalent to $S(\xi\eta) \leq S(\xi) + S(\eta)$, for every $\xi, \eta \in \mathbb{H}$. We denote $\xi = (z,t)$, $\eta = (w,\tau)$, where $z = (z_1, z_2)$ and $w = (w_1, w_2)$, then

$$S(\xi\eta) = |z+w| \vee g(|t + \tau + 2(z_1 w_2 - z_2 w_1)|)\,.$$

If $|z+w| \geq g(|t + \tau + 2(z_1 w_2 - z_2 w_1)|)$, then we clearly have

$$S(\xi\eta) = |z+w| \leq |z| + |w| \leq S(\xi) + S(\eta)\,.$$

So, our inequality holds if we prove that

$$g(|t + \tau + 2(z_1 w_2 - z_2 w_1)|) \leq S(\xi) + S(\eta)\,. \tag{2}$$

We have

$$|t + \tau + 2(z_1 w_2 - z_2 w_1)| \leq |t| + |\tau| + 2|(z_1, z_2)\cdot(w_2, -w_1)| \leq |t| + |\tau| + 2|z||w|$$

and $|t| = h(g(|t|)) \leq h(S(\xi))$, $|\tau| = h(g(|\tau|)) \leq h(S(\eta))$, hence

$$|t + \tau + 2(z_1 w_2 - z_2 w_1)| \leq h(S(\xi)) + h(S(\eta)) + 2S(\xi)S(\eta)\,.$$

The latter inequality and property (1) give $|t + \tau + 2(z_1 w_2 - z_2 w_1)| \leq h(S(\xi) + S(\eta))$, which corresponds to $g(|t + \tau + 2(z_1 w_2 - z_2 w_1)|) \leq S(\xi) + S(\eta)$. It remains to prove $I : (\mathbb{H}, d) \longrightarrow (\mathbb{H}, \rho)$ is 1-Lipschitz. This fact is equivalent to show that $S \leq G$ which is true if $g(|t|) \leq \sqrt{t}$, that is $|t| \leq h(\sqrt{|t|}) = \kappa(\sqrt{t}) + |t|$. So the proof is complete. □

Now, among all the maps κ which enjoy the properties assumed in the preceding lemma, we want to find a particular one which produces the oscillation required in Claim 3. We notice that if $v = (0,0,1) \in \mathbb{H}$, then $\rho(I(\zeta\delta_t v), I(\zeta)) = \rho(\delta_t v, 0) = g(t^2)$, so Claim 3 is equivalent to require the following

$$\limsup_{t\to 0_+} \frac{g(t^2)}{t} = 1\,, \qquad \liminf_{t\to 0_+} \frac{g(t^2)}{t} = 0\,, \tag{3}$$

where $g = h^{-1}$ and $h(t) = \kappa(t) + t^2$.

Theorem 2.2 *There exists a strictly increasing convex map $\kappa : [0,+\infty[\longrightarrow [0,+\infty[$, continuous at the origin, with $\kappa(0) = 0$, such that, defining $g = h^{-1}$, with $h(t) = \kappa(t) + t^2$, $t \geq 0$, the upper and lower limits as given in (3) hold.*

Proof. It is easy to see that the requirement (3) for g is equivalent to the condition

$$\limsup_{t\to 0_+} \frac{\kappa(t)}{t^2} = +\infty \text{ and } \liminf_{t\to 0_+} \frac{\kappa(t)}{t^2} = 0\,, \tag{4}$$

on the corresponding function κ. To find such a κ, we use the following simple observation. If we are given an affine, increasing function κ that vanishes at some positive number t' very close to zero, then the quotient $\kappa(t)/t^2$ oscillates a lot. Indeed, if t declines from 1 towards t' then the quotient first gets very large and then approaches zero. Stopping shortly before t', we can connect κ to another affine function with smaller but still positive slope that vanishes much closer to zero. Thus, the quotient considered oscillates along the new function even more and the combined function is convex.

To make this argument precise, we fix two positive sequences $(\varepsilon_l) \subset]0,1[$, $(m_l) \subset]0,+\infty[$, with $\varepsilon_l \to 0$ and $m_l \to +\infty$ as $l \to \infty$. We consider an arbitrary number $b_0 > 0$ and choose $t_0, a_0 > 0$ such that $t_0\varepsilon_0 < b_0$, $a_0 < \varepsilon_0 t_0^2$. Then, we define $\kappa_0(t) = a_0 + b_0(t-t_0)$, observing that $\kappa_0(t_0)/t_0^2 < \varepsilon_0$. We consider $\beta_1 = a_0/t_0 < t_0\varepsilon_0 < b_0$ and fix $\tau_1 \in]0, t_0[$ such that $\beta_1/\tau_1 > m_1$. We observe that

$$\lim_{b\to\beta_1^+} \frac{b}{\tau_1} + \frac{(\beta_1 - b)t_0}{\tau_1^2} = \frac{\beta_1}{\tau_1} > m_1\,, \quad \lim_{b\to\beta_1^+} \frac{t_0(b-\beta_1)}{b^2} = 0$$

hence we can choose $b_1 \in]\beta_1, b_0[$ such that

$$\frac{b_1}{\tau_1} + \frac{(\beta_1 - b_1)t_0}{\tau_1^2} > m_1 \quad \text{and} \quad \frac{t_0(b_1-\beta_1)}{b_1^2} < \frac{1}{2}\,. \tag{5}$$

Now, we define $\kappa_1(t) = t_0(\beta_1 - b_1) + b_1 t$, so by the first inequality (5) we have $\kappa_1(\tau_1)/\tau_1^2 > m_1$ and $\kappa_1(t_0) = \beta_1 t_0 = a_0 = \kappa_0(t_0)$. We note that $\kappa_1(\bar{t}) = 0$ if and only if $\bar{t} = t_0(b_1 - \beta_1)/b_1 > 0$. By the second inequality of (4) we get $\bar{t} < b_1/2$ and since $\kappa_1(\tau_1) > 0$ we infer that $\bar{t} < \tau_1$. Thus, we can choose $t_1 \in]\bar{t}, \min(\tau_1, b_1/2)[$ such that $\kappa_1(t_1) < \varepsilon_1 t_1^2$ and $t_1\varepsilon_1 < b_1$. Defining $a_1 = \kappa_1(t_1)$, we see that $\kappa_1(t) = a_1 + b_1(t-t_1)$ and we have shown that for every $b_0, a_0, t_0, m_1 > 0$, with $a_0/t_0 < b_0$, for each $\varepsilon_1 > 0$ and $m_1 \in \mathbb{R}$ there exist $t_1 < \tau_1$ in $]0, t_0[$ and $a_1 > 0$, $b_1 \in]0, b_0[$ such that

$$\begin{cases} \kappa_1(t_0) = \kappa_0(t_0)\,, & \kappa_1(\tau_1)/\tau_1^2 > m_1\,, \\ \kappa_1(t_1)/t_1^2 < \varepsilon_1\,, & \kappa_1(t_1)/t_1 < b_1 < b_0\,, \quad t_1 < b_1/2\,. \end{cases}$$

This procedure can be iterated by induction, obtaining for each $j \geq 1$ that there exists $\tau_j, t_j > 0$, $\tau_j \in]t_j, t_{j-1}[$, and $a_j, b_j > 0$ such that the map $\kappa_j(t) = a_j + b_j(t-t_j)$ satisfies

$$\begin{cases} \kappa_j(t_{j-1}) = \kappa_{j-1}(t_{j-1}) & b_j < b_{j-1} \\ \kappa_j(\tau_j)/\tau_j^2 > m_j & \kappa_j(t_j)/t_j^2 < \varepsilon_j\,, \quad t_j < 2^{-j} b_j\,. \end{cases} \tag{6}$$

We define

$$\kappa(t) = \kappa_0(t)\, \mathbf{1}_{[t_0,+\infty[}(t) + \sum_{j=1}^{\infty} \kappa_j(t)\, \mathbf{1}_{[t_j, t_{j-1}[}(t)\,,$$

observing that $t_j < b_j/2^j < b_0/2^j \to 0$ as $j \to \infty$, so by conditions (6) κ is a strictly increasing convex map defined on $]0, +\infty[$. The convexity follows from the continuity and from the fact that the sequence of slopes (b_j) decreases as the intervals get close to the origin. By the construction of κ we have that

$$\liminf_{t\to 0_+} \frac{\kappa(t)}{t^2} \le \limsup_{j\to\infty} \frac{\kappa(t_j)}{t_j^2} \le \lim_{j\to\infty} \varepsilon_j = 0\,, \tag{7}$$

$$\limsup_{t\to 0_+} \frac{\kappa(t)}{t^2} \ge \liminf_{j\to\infty} \frac{\kappa(\tau_j)}{\tau_j^2} \ge \lim_{j\to\infty} m_j = +\infty\,. \tag{8}$$

The sequence $(\kappa(t_j))$ converges to zero as $j \to \infty$ and κ is monotone, so $\kappa(t) \to 0$ as $t \to 0_+$ and κ is continuous at the origin. Thus, we have proved the existence of a strictly increasing convex map $\kappa : [0, +\infty[\longrightarrow [0, +\infty[$ which is continuous at the origin with $\kappa(0) = 0$ and which satisfies (7) and (8). These two conditions are of course just (4), so our proof is finished. □

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