CHARACTERISTIC POINTS, RECTIFIABILITY AND PERIMETER MEASURE ON STRATIFIED GROUPS

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Abstract. We establish an explicit formula between the perimeter measure of an open set E with C^1 boundary and the spherical Hausdorff measure $\mathcal{S}^{Q-1}$ restricted to ∂E, when the ambient space is a stratified group endowed with a left invariant sub-Riemannian metric and Q denotes the Hausdorff dimension of the group. Our formula implies that the perimeter measure of E is less than or equal to $\mathcal{S}^{Q-1}(\partial E)$ up to a dimensional factor. The validity of this estimate positively answers a conjecture raised by Danielli, Garofalo and Nhieu. The crucial ingredient of this result is the negligibility of "characteristic points" of the boundary. We introduce the notion of "horizontal point", which extends the notion of characteristic point to arbitrary submanifolds and we prove that the set of horizontal points of a k-codimensional submanifold is $\mathcal{S}^{Q-k}$-negligible. We propose an intrinsic notion of rectifiability for subsets of higher codimension, namely, $(\mathbb{G}, \mathbb{R}^k)$-rectifiability and we prove that Euclidean k-codimensional rectifiable sets are $(\mathbb{G}, \mathbb{R}^k)$-rectifiable.

Contents

Introduction

The project of developing analysis in metric spaces is receiving remarkable attention in the last few years with several monographs on this topic, [7], [18], [35], [36], [65]. The class of Carnot-Carathéodory spaces provides good models to be studied in this perspective. They are characterized as connected differentiable manifolds with a family of smooth vector fields $X_1, \dots, X_m$, which satisfy a Lie bracket generating condition. Under this assumption the "Carnot-Carathéodory distance" can be defined, [33].

We will focus our attention on stratified groups, also known as Carnot groups, [25], [59], which form a special class of Carnot-Carathéodory spaces. An important feature of stratified groups is the existence of a family of intrinsic dilations which scale well with the Carnot-Carathéodory distance and define the Hausdorff dimension of the group.

Stratified groups and Carnot-Carathéodory spaces have been thoroughly investigated in several different areas of Analysis as Optimal Control Theory, Differential Geometry, Harmonic Analysis and PDEs. Some relevant books are the following [10], [25], [34], [53], [67]. The systematic study of Geometric Measure Theory on stratified groups has begun only very recently with a continuously growing literature and it is now the subject of several contributions [1], [5], [8], [12], [15], [17], [26], [27], [28], [29], [30], [32], [41], [45], [46], [47], [48], [57], [54], [60], [61], [62], [64] and this list is certainly not complete.

The leading theme of this paper is the role of characteristic points in the study of intrinsic notions of rectifiability and in the expression of perimeter measure for open sets with C^1 boundary. In the area of PDEs, characteristic points have been largely studied in connection with boundary regularity of solutions to equations with nonnegative characteristic form, [11], [13], [21], [24], [42], [43], [38], [39], [44], see also the references therein. The solution to the Dirichlet problem for the Kohn Laplacian in a smooth domain may not be smooth up to the boundary due to the presence of characteristic points, [38]. Characteristic points play an important role with respect to metric properties of domains suitable for Sobolev-Poincaré inequality, Ahlfors regularity of hypersurfaces, intrinsic measure of hypersurfaces and trace theorems, [8], [12], [15], [16], [22], [31], [48], [55], [56]. In the sequel we give a detailed description of the contents of the present paper.

The first section presents all the basic materials that will be used throughout the paper. Section 2 is devoted to characteristic points of submanifolds and their negligibility. Negligibility of characteristic points with respect to the Euclidean Hausdorff measure $\mathcal{H}^{n-1}_{|\cdot|}$ was proved by Derridj for (n-1)-dimensional smooth hypersurfaces, [22]. Recently, Balogh through a remarkable construction of functions with prescribed gradient has proved that there exist C^1 hypersurfaces in the Heisenberg group such that the one codimensional Euclidean Hausdorff measure of the characteristic set is positive, [8]. However, he also shows that if we consider the Hausdorff measure $\mathcal{H}^{Q-1}$ with respect to the Carnot-Carathéodory distance, then the characteristic sets of C^1 hypersurfaces becomes negligible with respect to this measure. The covering type procedure adopted by Balogh has been extended by Franchi, Serapioni and Serra Cassano to all stratified groups of step two, [29].

Our approach to the negligibility of characteristic points in arbitrary stratified groups relies on a Sard-type theorem, [47], where characteristic points are properly interpreted

as intrinsic singular points of the submanifold. This characterization works for submanifolds of any codimension and allows us to find an analogous notion of characteristic point. In this general case, we call these points "horizontal", in that the tangent space to the submanifold at these points has a sufficiently "large" intersection with the horizontal space, see Definition 2.10. Note that in submanifolds of codimension one the notion of horizontal point coincides with the classical notion of characteristic point.

The main result of this paper is given in Theorem 2.16, where we show that the set of horizontal points in a k-codimensional submanifold is $\mathcal{H}^{Q-k}$-negligible. The key observation is to look at horizontal points as those points where the defining map of the surface is "intrinsically singular", namely, it has a nonsurjective P-differential (Definition 2.2). Once horizontal points are regarded as singular points we construct an *ad hoc* argument in order to apply the weak Sard-type theorem proved in [47], due to which a.e. level set of a vector valued Lipschitz map contains an $\mathcal{H}^{Q-k}$-negligible subset of singular points. The negligibility of horizontal points in stratified groups immediately extends the validity of Theorem 2.3 in [48] to any C^1 hypersurface. Let Σ be a C^1 hypersurface Σ and let $\mathcal{S}^{Q-1}$ be the spherical Hausdorff measure with respect to a homogeneous distance. Then Theorem 2.20 gives an explicit formula for the density of $\mathcal{S}^{Q-1}\llcorner\Sigma$ with respect to the Riemannian measure restricted to the hypersurface.

In Section 3 we present other consequences of Theorem 2.16. We first introduce the notions of $(\mathbb{G},\mathbb{R}^k)$-regularity and of $(\mathbb{G},\mathbb{R}^k)$-rectifiability (Definition 3.1 and Definition 3.2). These notions for $k=1$ have been first introduced and studied by Franchi, Serapioni and Serra Cassano in the cycle of papers [27], [28] and [29]. As the classical notion of differentiability is used to define the smoothness of a manifold, we observe that P-differentiability naturally defines $(\mathbb{G},\mathbb{R}^k)$-regularity for every codimension, introducing a vast class of new geometrical objects to be studied, where $(\mathbb{G},\mathbb{R}^k)$-regular sets play the role of "intrinsic regular submanifolds". However, those groups $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ does not contain any k-dimensional commutative subalgebra in the first layer cannot have $(\mathbb{G},\mathbb{R}^k)$-regular sets. Clearly, an $(\mathbb{R}^q,\mathbb{R}^k)$-regular set is exactly a C^1 submanifold of codimension k. On the other hand, as soon as we consider a noncommutative stratified group, things can dramatically change. A recent paper by Kirchheim and Serra Cassano, [41], shows a remarkable example of an $(\mathbb{H}^1,\mathbb{R})$-regular set S such that $\mathcal{H}^{\alpha}_{|\cdot|}(S)>0$, where $|\cdot|$ is the Euclidean norm and $2<\alpha<5/2$, where $\mathbb{H}^1$ is the three dimensional Heisenberg group. This interesting hypersurface cannot be 2-rectifiable in the Euclidean sense, although there exists a $\frac{1}{2}$-Hölder continuous parametrization which makes it a topological submanifold of topological dimension two. Recently, $(\mathbb{H}^n,\mathbb{R}^k)$-regular sets, with $1\le k\le n$, have been studied by Franchi, Serapioni and Serra Cassano, [30]. In the terminology of [30], these sets are called k-codimensional $\mathbb{H}$-regular surfaces. Note that there are no $(\mathbb{H}^n,\mathbb{R}^k)$-regular sets whenever $k>n$.

Our negligibility result applies in comparing $(\mathbb{G},\mathbb{R}^k)$-rectifiability with Euclidean rectifiability in the Federer sense, 3.2.14 of [23]. In the papers [27] and [29] Franchi, Serapioni and Serra Cassano have proved that Euclidean rectifiable sets of codimension one in stratified groups of step two are always $(\mathbb{G},\mathbb{R})$-rectifiable. As application of Theorem 2.16, in Theorem 3.8 we extend this result to any Euclidean rectifiable set of any stratified group. Note that horizontal curves or more general Legendrian submanifolds cannot be $(\mathbb{G},\mathbb{R}^k)$-regular. In fact, from the notion of horizontal point,

one can check that any k-codimensional Legendrian submanifold coincides with its horizontal set, then Theorem 2.16 proves that it is $\mathcal{H}^{Q-k}$-negligible. In other words, a k-codimensional Legendrian submanifold is invisible in a $(\mathbb{G},\mathbb{R}^k)$-rectifiable set in the same way as Euclidean $(q\text{-}k\text{-}s)$-rectifiable sets are invisible in an Euclidean $(q\text{-}k)$-rectifiable set, where $s>0$. It is certainly interesting to investigate which couples $(\mathbb{G},\mathbb{R}^k)$ possess nontrivial $(\mathbb{G},\mathbb{R}^k)$-regular sets and to check whether these sets have Hausdorff dimension $Q-k$ and topological dimension $q-k$, where q is the topological dimension of $\mathbb{G}$. Presently, this question has been positively answered for $k=1$ and arbitrary $\mathbb{G}$ in [28] and for $k\leq n$ and $\mathbb{G}=\mathbb{H}^n$ in [30]. Expanding our perspective, it is natural replacing $\mathbb{R}^k$ with another stratified group $\mathbb{M}$, getting the general notion of $(\mathbb{G},\mathbb{M})$-regularity (Definition 3.5). We introduce this notion of regularity to illustrate its potential in connection with future developments of Geometric Measure Theory on stratified groups. These notions of intrinsic regularity provide several types of rectifiable objects modeled with respect to different geometries. This shows how the geometry of stratified groups is rich and is still far from being well understood.

Our negligibility result can be also applied to study the relationship between the perimeter measure of a set with C^1 boundary and the spherical Hausdorff measure of its boundary. In the more general case when a portion $\partial E\cap U$ of ∂E is a $(\mathbb{G},\mathbb{R})$-regular set and U is an open subset, the following formula

$$|\partial E|_H \llcorner U=\theta^g_{Q-1}(\nu_H)\,\mathcal{S}^{Q-1}\llcorner(U\cap\partial E)$$

can be found in [28] and [49]. Due to $\mathcal{H}^{Q-1}$-negligibility of characteristic points the previous formula holds for any set E with C^1 boundary and any measurable set of $\mathbb{G}$. As a consequence, we can positively answer a conjecture raised by Danielli, Garofalo and Nhieu in [15]. A detailed discussion of this application will be given in the last part of Section 3.

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1. Preliminaries

We will mean by *measure* on a metric space X a set function $\mu:\mathcal{P}(X)\longrightarrow[0,+\infty]$ with the properties $\mu(\emptyset)=0$ and $\mu(E)\leq\sum_{j=1}^\infty\mu(E_j)$ whenever $E\subset\bigcup_{j=1}^\infty E_j$. Every measure μ naturally induces a σ-algebra $\mathcal{A}_\mu\subset\mathcal{P}(X)$ where it is additive on countable disjoint unions. Recall that $\mathcal{B}(X)$ is the smallest σ-algebra containing open sets of X. Elements of $\mathcal{B}(X)$ are called *Borel sets* and a measure μ such that $\mathcal{B}(X)\subset\mathcal{A}_\mu$ is called *Borel measure*. The push-forward of a measure μ under the map $F:X\longrightarrow Y$ is defined by $F_\sharp\mu(E)=\mu\left(F^{-1}(E)\right)$ for every $E\subset Y$. The restriction of a measure μ to a subset $A\subset X$ is defined by $\mu\llcorner A(E)=\mu(A\cap E)$ for every $E\subset X$. The following elementary change of variable formula will be useful in the sequel. Its proof can be obtained by approximation of measurable functions with measurable step functions.

Proposition 1.1 (Change of variable)**.** *Let X,Y be two metric spaces and let N be either $\overline{\mathbb{R}}$ or a finite dimensional space. Suppose that $F:X\longrightarrow Y$ and $u:Y\longrightarrow N$ are Borel maps, where μ is a Borel measure over X and $u\circ F$ is μ-summable. Then*

for every $B \in \mathcal{B}(X)$ *we have*

$$\int_{F^{-1}(B)} u \circ F \, d\mu = \int_B u \, dF_\sharp \mu \,. \tag{1}$$

Definition 1.2 (Stratified group)**.** Let $\mathbb{G}$ be a simply connected Lie group and let $\mathcal{G}$ be its Lie algebra of left invariant vector fields. Let H^1 be a subspace of $\mathcal{G}$ and suppose that the inductively defined sequence of subspaces

$$H^{j+1} = [H^j, H^1] = \text{span}\{[Z, Y] \mid Z \in H^j,\, Y \in H^1\}$$

has the properties

(1) $H^j \cap H^k = \{0\}$ whenever $j \neq k$ and $j, k \in \mathbb{N} \setminus \{0\}$
(2) there exists $\iota \geq 1$ such that $H^\iota \neq \{0\}$ and $H^{\iota+1} = \{0\}$
(3) $\mathcal{G}$ is spanned by all the subspaces $\{H^j \mid j = 1, \dots, \iota\}$.

We say that $\mathcal{G}$ is a *stratified algebra* and that $\mathbb{G}$ is a stratified group. The integer ι is called the *step* of the group. A *left translation* of the group is a map $l_p : \mathbb{G} \longrightarrow \mathbb{G}$ defined by $s \longrightarrow l_p(s) = ps$, where $p, s \in \mathbb{G}$. As a consequence of (1) and (3) every vector $Y \in \mathcal{G}$ can be written in a unique way as the sum $Y = \sum_{j=1}^{\iota} Y_j$, where $Y_j \in H^j$. This enables us to introduce canonical projections $\mathfrak{p}_{H^j} : \mathcal{G} \longrightarrow H^j$ defined by $\mathfrak{p}_{H^j}(Y) = Y_j$, where Y is written as above. In the case of projection on the first layer we also define $\mathfrak{p}_H = \mathfrak{p}_{H^1}$.

Conditions (1) and (3) of the previous definition can be shortly stated writing

$$\mathcal{G} = H^1 \oplus H^2 \oplus \cdots \oplus H^\iota,$$

where the symbol $\oplus$ indicates the direct sum of vector spaces.

Recall that the exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ associates to any left invariant vector field $W \in \mathcal{G}$ the value $\gamma_W(1) \in \mathbb{G}$ of the curve $\gamma_W : \mathbb{R} \longrightarrow \mathbb{G}$ which solves the Cauchy problem

$$\begin{cases} \gamma'(t) = W(\gamma(t)) \\ \gamma(0) = e \end{cases}$$

In the sequel we will use the fact that the exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ of simply connected nilpotent Lie groups is a diffeomorphism, [14].

Definition 1.3 (Graded metric)**.** We say that a left invariant Riemannian metric g on $\mathbb{G}$ is a *graded metric* if all subspaces $\{H^j \mid j = 1, \dots, \iota\}$ are orthogonal each other. The left invariant scalar product between two vectors $V, W \in T_p\mathbb{G}$ will be denoted by $\langle V, W \rangle_p$ or $\langle V(p), W(p) \rangle$. In the case $X, Y \in \mathcal{G}$ the scalar product $\langle X, Y \rangle_p$ is independent of the point p and can be simply denoted by $\langle X, Y \rangle$.

Definition 1.4 (Graded coordinates)**.** We define the numbers $n_j = \dim H^j$ for any $j = 1, \dots, \iota$, $m_0 = 0$ and $m_i = \sum_{j=1}^{i} n_j$ for any $i = 1, \dots \iota$. We say that a basis $(W_1, \dots, W_q)$ of $\mathcal{G}$ is an *adapted basis* if $(W_{m_{j-1}+1}, W_{m_{j-1}+2}, \dots, W_{m_j})$ is a basis of H^j for any $j = 1, \dots \iota$. We say that $(W_1, \dots, W_q)$ is a *graded basis* if it is an adapted and

orthonormal basis with respect to a graded metric. The *graded coordinates* with respect to the basis $(W_1, \dots, W_q)$ are given by the diffeomorphism $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ defined by

$$F(x) = \exp\Big(\sum_{j=1}^{q} x_j W_j\Big).$$

The *degree* of the coordinate x_j is the unique integer d_j such that $W_j \in H^{d_j}$.

Let $X, Y \in \mathcal{G}$. The Baker-Campbell-Hausdorff formula (shortly BCH-formula) allows us to obtain an explicit polynomial $P(X,Y)$ with respect to the nonassociative Lie product of $\mathcal{G}$, such that $\exp(P(X,Y)) = \exp X \exp Y$, see for instance [37]. The BCH-formula and graded coordinates allow us to read the group operation in $\mathbb{R}^q$ as a polynomial operation (which depends on the fixed system graded coordinates). We will denote this polynomial operation by $x \cdot y$, where $x, y \in \mathbb{R}^q$. In the case the Lie algebra $\mathcal{G}$ is commutative, namely $[X,Y] = 0$ for every $X, Y \in \mathcal{G}$, we simply have $x \cdot y = x + y$ and all coordinates have degree one (see also Chapter XIII, Section 5 of [66]).

Proposition 1.5. *Let $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ be a system of graded coordinates with respect to a graded metric g. Then the equality $F_\sharp \mathcal{L}^q = v_g$ holds, where $\mathcal{L}^q$ is the q-dimensional Lebesgue measure.*

Proof. We know that $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ is a smooth diffeomorphism. Let A be a measurable set of $\mathbb{R}^q$. By the classical area formula and the left invariance of both v_g and $F_\sharp \mathcal{L}^q$ we have

$$c\, \mathcal{L}^q(A) = v_g(F(A)) = \int_A J_q F(\xi)\, d\xi$$

for some constant $c > 0$. We have denoted by $J_q F$ the q-dimensional Jacobian of F. Then $\int_A J_q F = c$ for any measurable A. By continuity of $\xi \longrightarrow J_q F(\xi)$ we obtain that $J_q F(\xi) = c$ for any $\xi \in \mathbb{R}^q$. We know that $F = \exp \circ L$, where $L(\xi) = \sum_{i=1}^{q} \xi_j W_j$ and (W_j) is an orthonormal basis of $\mathcal{G}$. Since the map $dF(0) = d\exp(0) \circ L = L$ has jacobian equal to one, then $c = 1$ and the thesis follows. □

Motivated by the previous proposition in the sequel we will also adopt the simpler notation $v_g(A) = |A|$ for every measurable subset $A \subset \mathbb{G}$.

Definition 1.6 (Dilations)**.** Let $\mathbb{G}$ be a stratified group. For every $r > 0$ we define the *dilation* $\delta_r : \mathbb{G} \longrightarrow \mathbb{G}$ by

$$\delta_r p = \exp\Big(\sum_{j=1}^{\iota} r^j\, \mathfrak{p}_{Hj}\Big(\exp^{-1}(p)\Big)\Big).$$

Consider a system of graded coordinates $F : \mathbb{R}^q \longrightarrow \mathbb{G}$. The *coordinate dilation* $\Lambda_r : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ associated to F is defined by $\Lambda_r = F^{-1} \circ \delta_r \circ F$ and can be written as

$$\Lambda_r(x) = \sum_{j=1}^{q} r^{d_j} x_j e_j, \tag{2}$$

where (e_j) is the canonical basis of $\mathbb{R}^q$ and d_j is the degree of the coordinate x_j.

In view of relation (2) it is easy to see that $\mathcal{L}^q(\Lambda_r(A)) = r^Q \mathcal{L}^q(A)$ for any measurable subset $A \subset \mathbb{R}^q$, where we have defined

$$Q = \sum_{k=1}^{q} d_k = \sum_{j=1}^{\iota} j \, \dim(H^j). \tag{3}$$

Thus, by definition of coordinate dilation and by Proposition 1.5, for any $r > 0$ and any measurable subset $E \subset \mathbb{G}$ we have

$$v_g(\delta_r E) = v_g\left(F(\Lambda_r(F^{-1}(E)))\right) = \mathcal{L}^q\left(\Lambda_r(F^{-1}(E))\right) = r^Q \, \mathcal{L}^q\left(F^{-1}(E)\right) = r^Q \, v_g(E).$$

Finally, the left invariance of v_g yields the formula

$$v_g(l_p \circ \delta_r(E)) = r^Q \, v_g(E) \tag{4}$$

for any $p \in \mathbb{G}$ and any $r > 0$. Observing that $B_{p,r} = l_p(\delta_r B_1)$ we have in particular

$$v_g(B_{p,r}) = r^Q \, v_g(B_1).$$

This implies that the number Q defined in (3) is the Hausdorff dimension of the group and that the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ built with respect a homogeneous distance is finite on bounded sets and it is proportional to v_g.

The *horizontal subbundle* $H\mathbb{G}$ is defined by the collection of all subspaces

$$H_p\mathbb{G} = \{X(p) \mid X \in H^1\}$$

where $p \in \mathbb{G}$. These are the so-called *horizontal spaces*. We denote by $H\Omega$ the subfamily of horizontal spaces $H_p\mathbb{G}$ where $p \in \Omega$ and Ω is an open subset of $\mathbb{G}$. A different way to introduce horizontal spaces is the following

$$dl_p(H_e\mathbb{G}) = H_p\mathbb{G} \tag{5}$$

for every $p \in \mathbb{G}$, where e is the unit element of the group.

By definition of stratified group, the Lie algebra spanned by H^1 coincides with $\mathcal{G}$, then the well known Chow theorem implies that any two points of $\mathbb{G}$ can be connected by at least one absolutely continuous curve a.e. tangent to the horizontal subbundle, [10]. These curves are the so-called *horizontal curves* which permit us to introduce the *Carnot-Carathéodory distance*. Let $p, p' \in \mathbb{G}$ and consider the infimum among lengths of all horizontal curves connecting p with p', where the length is computed with respect to the graded metric of the group. This infimum is the Carnot-Carathéodory distance between p and p' and it is denoted by $\rho(p, p')$. The left invariance of the graded metric implies that the Carnot-Carathéodory distance is left invariant, namely, $\rho(p'p, p's) = \rho(p, s)$ for every $p, p', s \in \mathbb{G}$ and it is also homogeneous with respect to dilations, $\rho(\delta_r p, \delta_r s) = r\rho(p, s)$ for every $r > 0$.

Next we will use the usual Euclidean norm on $\mathbb{G}$. To do this, there is not a unique choice and we will refer to a fixed system of graded coordinates $F : \mathbb{R}^q \longrightarrow \mathbb{G}$. This choice will not affect our arguments because if $G : \mathbb{R}^q \longrightarrow \mathbb{G}$ is another system of graded coordinates, the change of variable $F \circ G^{-1} : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is an isomorphism, hence the Euclidean norm with respect to G is equivalent to the one with respect to F.

With this convention we state an important relation between the Euclidean distance and the Carnot-Carathéodory distance on stratified groups

$$|x-y| \le \rho(x,y) \le C\,|x-y|^{1/\iota} \quad \text{for any } x,y \in K \subset \mathbb{G}\,, \tag{6}$$

where K is compact and C is a dimensional constant depending on K. Note that ι is the step of the group. Estimates (6) can be proved in more general Carnot-Carathéodory spaces, [58]. In particular the Carnot-Carathéodory distance is continuous and it induces the same topology of $\mathbb{G}$.

Definition 1.7 (Homogeneous distance)**.** A continuous map $d : \mathbb{G} \times \mathbb{G} \longrightarrow [0, +\infty)$ is a *homogeneous distance* if it satisfies the axioms of an abstract distance, it is left invariant, $d(p'p, p's) = d(p, s)$ for any $p, p', s \in \mathbb{G}$, and it is homogeneous with respect to dilations, $d(\delta_r p, \delta_r s) = r\, d(p, s)$ for every $r > 0$.

Clearly the Carnot-Carathéodory distance is the foremost homogeneous distance, but also other useful examples of homogeneous distances can be considered. For instance, in Proposition 2.19 we will study symmetry properties of the homogeneous distance d_∞ introduced in [29]. It is worth observing that homogeneous distances are biLipschitz equivalent in the following sense. If d_1 and d_2 are homogeneous distances, then there exists a constant $C > 1$ such that for every $p, s \in \mathbb{G}$ we have

$$C^{-1}\, d_1(p, s) \le d_2(p, s) \le C\, d_1(p, s).$$

This fact can be seen using the classical argument used for norms of finite dimensional spaces, that actually are a particular case. In the sequel we will always refer to a fixed homogeneous distance.

Definition 1.8 (Metric ball)**.** The open ball of radius $r > 0$ and center $p \in \mathbb{G}$ will be denoted by $B_{p,r}$. If we wish to emphasize the particular choice of the homogeneous distance d we write $B^d_{p,r}$. Open balls with radius r and centered at the unit element of the group are denoted by B_r. We use similar conventions for closed balls $D_{p,r}$ of center p and radius r.

Definition 1.9 (Hausdorff measures)**.** Let d be a homogeneous distance of $\mathbb{G}$ and let $a \ge 0$. For each subset $E \subset \mathbb{G}$ we define the *a-dimensional spherical Hausdorff measure*

$$\mathcal{S}^a(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_{j=1}^{\infty} \frac{\operatorname{diam}(D_{x_i,t_i})^a}{2^a} \mid E \subset \bigcup_{i=1}^{\infty} D_{x_i,t_i}\,,\ t_i \le \varepsilon \Big\}$$

and the *a-dimensional Hausdorff measure* of E as

$$\mathcal{H}^a(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_{j=1}^{\infty} \omega_a \frac{\operatorname{diam}(F_i)^a}{2^a} \mid E \subset \bigcup_{i=1}^{\infty} F_i\,,\ \operatorname{diam}(F_i) \le \varepsilon \Big\}$$

where $\{F_i\}$ are subsets of $\mathbb{G}$ and $\operatorname{diam}(A) = \sup_{(x,y)\in A\times A} d(x, y)$ for any $A \subset \mathbb{G}$. The dimensional constant ω_a is defined as follows

$$\omega_a = \frac{\pi^{a/2}}{\Gamma(1 + a/2)} \quad \text{and} \quad \Gamma(s) = \int_0^\infty r^{s-1} e^{-r}\, dr\,.$$

Remark 1.10. Note that we have used the dimensional constant ω_a only for the definition of the Hausdorff measure $\mathcal{H}^a$. In fact, our formulae involving the spherical Hausdorff measure $\mathcal{S}^a$ will contain the metric factor θ^g_{Q-1} (Definition 2.17) that naturally replaces the constant ω_a.

When we want to specify the use of the Euclidean distance we will write $\mathcal{H}^a_{|\cdot|}$ and $\mathcal{S}^a_{|\cdot|}$ and for the case of the Carnot-Carathéodory distance we will write $\mathcal{H}^a_\rho$ and $\mathcal{S}^a_\rho$. Throughout the paper the symbol Ω will denote an open subset of $\mathbb{G}$. We will denote by $\Gamma(H\Omega)$ the space of smooth sections of $H\Omega$, i.e. the *horizontal vector fields*. The space $\Gamma_c(H\Omega)$ denotes the family of horizontal vector fields compactly supported in Ω.

Definition 1.11 (Horizontal divergence)**.** Let $(X_1, \dots, X_m)$ be a basis of left invariant vector fields of H^1. The *horizontal divergence* (in short H-*divergence*) of the section $\varphi = \sum_{i=1}^m \varphi^j X_j \in \Gamma(H\Omega)$ is defined by $\operatorname{div}_H \varphi = \sum_{j=1}^m X_j \varphi^j$.

Notice that the previous definition does not depend on the choice of the basis of H^1. Now we recall the notion of "image" of a vector field through a diffeomorphism. Let $f : M \longrightarrow N$ be a C^1 diffeomorphism of differentiable manifolds and let X be a vector field of M. The *image of* X *under* f is the vector field of N defined by

$$f_* X(p) = df(f^{-1}(p)) \left(X \left(f^{-1}(p) \right) \right)$$

for every $p \in N$. A vector field $X \in H^1$ is read in $\mathbb{R}^q$ through a system of graded coordinates $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ when it is defined as $\tilde{X} = F_*^{-1} X$. In the sequel we will use this notation to denote vector fields of $\mathcal{G}$ with respect to graded coordinates of $\mathbb{R}^q$. With this notation the horizontal divergence can be written as follows

$$(\operatorname{div}_H \varphi) \circ F = \sum_{j=1}^m \tilde{X}_j \tilde{\varphi}^j \,, \tag{7}$$

where $\varphi \in \Gamma(H\mathbb{G})$ and $\tilde{\varphi} = \varphi \circ F$. Another useful formula involving left invariant vector fields in $\mathbb{R}^q$ is the following

$$\tilde{W}_k = \partial_{x_k} + \sum_{j=m_{d_k}+1}^{q} a_{kj}(x_1, \dots, x_{j-1}) \, \partial_{x_j} \,, \tag{8}$$

where $\tilde{W}_k = F_*^{-1} W_k$ with $k = 1, \dots, q$, the integers m_i and d_j are introduced in Definition 1.4, the maps a_{kj} are homogeneous polynomials with respect to coordinate dilations and the graded basis $(W_1, W_2, \dots, W_q)$ of $\mathcal{G}$ is associated to the system of graded coordinates F, see also p.621 of [66]. Note that in formula (8) we have used the standard representation of vector fields as first order differential operators.

Let $u : \Omega \longrightarrow \mathbb{R}$ be a C^1 map, $p = F(x) \in \Omega$ and $\tilde{u} = u \circ F$. We consider the map $y \longrightarrow \tilde{u}(x \cdot y)$. By left invariance of vector fields $\tilde{W}_j$ on $\mathbb{R}^q$ we have the useful formula

$$\left. \frac{\partial \tilde{u}}{\partial y_j} \right|_{y=0} = \tilde{W}_j \tilde{u}(x) = W_j u(p). \tag{9}$$

In the following definition the symbol $|\cdot|$ will denote the norm induced by the Riemannian metric on tangent spaces. We will use this notation whenever its meaning

will be clear from the context. The space $\Gamma_c(H\Omega)$ denotes the family of smooth sections of the subbundle $H\mathbb{G}$ having compact support.

Definition 1.12 (Perimeter measure)**.** We say that a measurable set $E \subset \Omega$ is of *H-finite perimeter* on the open subset $\Omega \subset \mathbb{G}$ if

$$|\partial E|_H(\Omega) = \sup\left\{ \int_E \operatorname{div}_H \phi \, dv_g \;\middle|\; \phi \in \Gamma_c(H\Omega),\ |\phi| \leq 1 \right\} < \infty\,.$$

If E has H-finite perimeter on every open subset U compactly contained in Ω we say that E has *locally H-finite perimeter* in Ω. We will denote by $|\partial E|_H$ the associated Radon measure.

By virtue of Riesz representation theorem there exists a measurable section ν_E of $H\mathbb{G}$ such that

$$\int_E \operatorname{div}_H \phi \, dv_g = -\int_{\mathbb{G}} \langle \phi, \nu_E \rangle \, d\,|\partial E|_H\,, \tag{10}$$

for any $\phi \in \Gamma_c(H\mathbb{G})$. We say that ν_E is the *generalized inward normal* to E. By the standard polar decomposition (Corollary 1.29 of [3]) we have that $|\nu_E(p)| = 1$ for $|\partial E|_H$-a.e. $p \in \Omega$.

2. Horizontal set of C^1 submanifolds

In this section we introduce the notion of horizontal point for a k-codimensional submanifold and we prove that the set of all horizontal points is $\mathcal{H}^{Q-k}$-negligible. The symbols $\mathbb{G}$ and $\mathbb{M}$ denote stratified groups and Ω stands for an open subset of $\mathbb{G}$. We will show the relationship between the intrinsic notion of P-differentiability and the notion of horizontal point. Recall that P-differentiability of maps between stratified groups was successfully introduced by Pansu in order to study rigidity properties, [59].

Definition 2.1 (G-linear map)**.** We say that $L : \mathbb{G} \longrightarrow \mathbb{M}$ is an G-*linear map* if it is a group homomorphism and $L(\delta_r p) = \delta'_r L(p)$ for every $p \in \mathbb{G}$ and every $r > 0$, where δ_r and δ'_r are dilations of the stratified groups $\mathbb{G}$ and $\mathbb{M}$, respectively.

Definition 2.2 (P-differentiable map)**.** Let $f : \Omega \longrightarrow \mathbb{M}$, where $\mathbb{M}$ is a stratified group. We say that f is P-*differentiable* at $p \in \Omega$ if there exists a G-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ such that

$$\frac{\rho'\left(f(p)^{-1} f(s), L(p^{-1}s)\right)}{\rho(p,s)} \longrightarrow 0 \quad \text{as} \quad s \to p\,, \tag{11}$$

where ρ and ρ' are the Carnot-Carathéodory distances of $\mathbb{G}$ and $\mathbb{M}$, respectively. The unique G-linear map which satisfies (11) is called the P-*differential* of f at p and it is denoted by $d_H f(p)$. By the exponential map $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ we will also read the P-differential as a linear map between the Lie algebras of $\mathbb{G}$ and of $\mathbb{M}$. In this case we will use the same notation $d_H f(p)$.

Definition 2.3 (C^1_H map)**.** We denote by $C^1_H(\Omega, \mathbb{M})$ the class of maps $f : \Omega \longrightarrow \mathbb{M}$ which are P-differentiable at every point of Ω and whose P-differential $p \longrightarrow d_H f(p)$ is continuous. In the case $\mathbb{M} = \mathbb{R}$ we simply write $C^1_H(\Omega)$.

In the present paper we work with C^1 maps and C^1 submanifolds, but mainly using the notion of P-differentiability. The next proposition ensures that this is possible since C^1 maps in the usual sense are also C^1_H.

Proposition 2.4 (P-differentiability)**.** *The inclusion $C^1(\Omega,\mathbb{R}^k) \subset C^1_H(\Omega,\mathbb{R}^k)$ holds and for every $u \in C^1(\Omega,\mathbb{R}^k)$ and every $p \in \Omega$ we have the formula $d_H u(p)(Y(p)) = du(p)(\mathfrak{p}_H(Y)(p))$ whenever $Y \in \mathcal{G}$. The map $\mathfrak{p}_H : \mathcal{G} \longrightarrow H^1$ denotes the canonical projection introduced in Definition 1.2.*

Proof. Let us fix a system of graded coordinates $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ with respect to a graded basis $(W_1, W_2, \dots, W_q)$ of $\mathcal{G}$. Define $\tilde{u} = u \circ F \in C^1(\tilde{\Omega},\mathbb{R}^k)$, where $\tilde{\Omega} = F^{-1}(\Omega)$. We define $x = F^{-1}(p)$ where $p \in \Omega$. Using graded coordinates together with coordinate dilations and the notion of P-differentiability our proof boils down to show the existence of the limit

$$\lim_{r\to 0^+} \frac{\tilde{u}\left(x \cdot \Lambda_r(y)\right) - \tilde{u}(x)}{r}$$

uniformly as y varies in some bounded neighbourhood of the origin. Let us define the map

$$r \longrightarrow \tilde{u}\left(x \cdot \Lambda_r(y)\right) = \tilde{u}\Big(x \cdot \Big(\sum_{j=1}^{q} r^{d_j} y_j e_j\Big)\Big) = \psi(r,y).$$

Since the map ψ is C^1 we can write

$$\frac{\psi(r,y) - \psi(0,y)}{r} = r^{-1} \int_0^r \partial_t \psi(\tau, y)\, d\tau,$$

hence formula (9) implies that

$$\frac{\psi(r,y) - \psi(0,y)}{r} = r^{-1} \sum_{j=1}^{q} \int_0^r d_j\, \tau^{d_j - 1} y_j\, \tilde{W}_j \tilde{u}\left(x \cdot \Lambda_\tau y\right)\, d\tau$$

$$= \sum_{j=1}^{m} y_j \int_0^1 \tilde{W}_j \tilde{u}\left(x \cdot \Lambda_{r\tau} y\right)\, d\tau + \sum_{j=m+1}^{q} d_j\, r^{d_j - 1} y_j \int_0^1 \tau^{d_j - 1} \tilde{W}_j \tilde{u}\left(x \cdot \Lambda_{r\tau} y\right)\, d\tau,$$

where the degree d_j of the coordinate y_j is greater than one if and only if $j > m$. By the continuity of $\tilde{W}_j \tilde{u}_j$ for every $j = 1, \dots, q$, formula (9) and the last equality we have proved that

$$\lim_{r\to 0^+} \frac{\tilde{u}\left(x \cdot \Lambda_r(y)\right) - \tilde{u}(x)}{r} = \sum_{j=1}^{m} y_j \tilde{W}_j\, \tilde{u}(x) = \sum_{j=1}^{m} y_j\, W_j u(p). \tag{12}$$

The previous expression yields a G-linear map with respect to graded coordinates y

$$d_H u(p)\Big(\sum_{j=1}^{q} y_j W_j(p)\Big) = \sum_{j=1}^{m} y_j\, W_j u(p).$$

By the explicit formula for the P-differential and the continuity of W_j for every $j = 1, \dots, q$ it follows the continuity of $d_H u(p)$ with respect to p. Finally, observing that

$$\mathfrak{p}_H\Big(\sum_{j=1}^{q} y_j W_j\Big)(p) = \sum_{j=1}^{m} y_j W_j(p) \in H_p\mathbb{G}$$

for every $Y \in \mathcal{G}$ we achieve the relation $d_H u(p)(Y(p)) = du(p)(\mathfrak{p}_H(Y)(p))$. □

Proposition 2.5. *Let $u \in C^1(\Omega)$ and $p \in \Omega$. Then there exists a unique horizontal vector denoted by $\nabla_H u(p) \in H_p\mathbb{G}$, such that $d_H u(p)(V) = \langle \nabla_H u(p), V\rangle_p$ for any $V \in T_p\mathbb{G}$ and $\nabla_H u(p)$ is the the orthogonal projection of $\nabla u(p)$ onto $H_p\mathbb{G}$.*

Proof. The linear map $d_H u(p)$ can be represented by a vector $W \in T_p\mathbb{G}$ such that $d_H u(p)(V) = \langle W, V\rangle_p$ for any $V \in T_pG$. We first prove that $W \in H_p\mathbb{G}$. We first observe that $T_p\mathbb{G} = H_p^1\mathbb{G} \oplus \cdots \oplus H_p^\iota\mathbb{G}$ where we have defined for any $j = 1, \dots, \iota$

$$H_p^j\mathbb{G} = \{X(p) \in T_p\mathbb{G} \mid X \in H^j\}.$$

Then we can write $V = \sum_{j=1}^{\iota} V_j$ and $W = \sum_{j=1}^{\iota} W_j$, where $V_j, W_j \in H_p^j\mathbb{G}$. By virtue of Proposition 2.4 we know that $d_H u(p)(V) = du(p)(V_1) = \langle \nabla u(p), V_1\rangle$, hence

$$\langle \nabla u(p), V_1\rangle_p = \langle W, V\rangle_p = \langle W, V_1\rangle.$$

Due to the fact that we are using a graded metric the subspaces $\{H_p^j\mathbb{G} \mid j = 1, \dots, \iota\}$ are orthogonal each other, then the arbitrary choice of V implies that $W_j = 0$ for every $2 \le j \le \iota$ and

$$\langle \nabla u(p) - W_1, V_1\rangle = 0$$

for every $V_1 \in H_p\mathbb{G}$. The last property characterizes W_1 as the orthogonal projection of $\nabla u(p)$ onto $H_p\mathbb{G}$. In particular W_1 is the projection of W onto $H_p\mathbb{G}$. Thus, defining $\nabla_H u(p) = W_1$ the proof is achieved. □

Definition 2.6 (Horizontal normal)**.** Let $\Sigma \subset \Omega$ be a C^1 submanifold of codimension one and let $p \in \Sigma$. We denote by $\nu(p)$ a unit normal to Σ at p with respect to a fixed graded metric. We say that the orthogonal projection of $\nu(p)$ onto $H_p\mathbb{G}$ is a *horizontal normal* of Σ at p and we denote it by $\nu_H(p)$.

Remark 2.7. Note that the horizontal normal should be considered up to its sign, because we do not require oriented submanifolds in the definition of horizontal normal and all functions depending on the horizontal normal do not depend on its sign. In more rigorous terms, one should identify ν_H and $-\nu_H$ as equivalent relation in $H_p\mathbb{G}$ and consider the corresponding quotient space.

Lemma 2.8. *Let $u : \Omega \longrightarrow \mathbb{R}$ be a C^1 map. Suppose that $u^{-1}(0) \neq \emptyset$ and assume that $\nabla u(p) \neq 0$ for every $p \in \Omega$. Then, for every $p \in u^{-1}(0)$ and every $Z \in T_p\mathbb{G}$ we have*

$$d_H u(p)(Z) = |\nabla u(p)|\, \langle \nu_H(p), Z\rangle_p \quad \text{and} \quad \nu_H(p) = \frac{\nabla_H u(p)}{|\nabla u(p)|}\,, \tag{13}$$

where $\nu_H(p)$ is the horizontal normal of the submanifold $u^{-1}(0)$ at the point p.

Proof. We observe that $\nu(p) = \nabla u(p)/|\nabla u(p)|$ is a unit normal to $u^{-1}(0)$ at p. By definition of horizontal normal, the orthogonal projection of $\nu(p)$ onto $H_p\mathbb{G}$ is $\nu_H(p)$, then Proposition 2.5 concludes the proof. □

Definition 2.9 (Horizontal gradient)**.** Let $u \in C^1(\Omega)$. In view of Proposition 2.5 we can define the *horizontal gradient* of u as the unique C^1 horizontal vector field $\nabla_H u$ such that

$$d_H u(p)(Y) = \langle \nabla_H u(p), Y\rangle_p$$

for every $p \in \Omega$ and every $Y \in T_p\mathbb{G}$.

Definition 2.10 (Horizontal point)**.** Let $\Sigma \subset \Omega$ be a C^1 submanifold of codimension k and let $p \in \Sigma$. We say that $p \in \Sigma$ is a *horizontal point* of Σ if

$$\dim(H_p\mathbb{G}) - \dim\left(H_p\mathbb{G} \cap T_p\Sigma\right) \leq k-1. \tag{14}$$

The *horizontal set* of Σ is the subset of all horizontal points and it is denoted by $C(\Sigma)$.

The previous definition is inspired by Lemma 2.11 and Theorem 2.13 in the following sense. Let Σ be a submanifold of codimension one defined as a level set of a C^1 map $u : \Omega \longrightarrow \mathbb{R}$. Theorem 2.13 singles out a class of "intrinsic" singular points of Σ, corresponding to the subset of Σ where $d_H u$ vanishes, (this subset is well defined due to Proposition 2.4). In view of Lemma 2.11 this subset exactly corresponds to the subset of horizontal points of Σ. The singularity of u at $p \in \Sigma$ can be also expressed by the condition $\dim(H_p\mathbb{G}) = \dim\left(H_p\mathbb{G} \cap T_p\Sigma\right)$, namely, $d_H u(p)$ is vanishing. For submanifolds of higher codimension condition (14) amounts to the non surjectivity of $d_H u(p) : \mathbb{G} \longrightarrow \mathbb{R}^k$, because $\dim\left(d_H u(p)(H_p\mathbb{G})\right) = \dim(H_p\mathbb{G}) - \dim\left(H_p\mathbb{G} \cap T_p\Sigma\right)$, although the map u does not appear in (14). In the case $k = 1$, one can interpret inequality (14) by the inclusion $H_p\mathbb{G} \subset T_p\Sigma$, that coincides with condition $\dim(H_p\mathbb{G}) = \dim(H_p\mathbb{G} \cap T_p\Sigma)$. It is clear that when $k > 1$ condition (14) means that the horizontal subspace $H_p\mathbb{G}$ is allowed to intersect $T_p\Sigma$ without necessarily being contained in it. Thus in higher codimension, a point p such that $H_p\mathbb{G} \subset T_p\Sigma$ is a horizontal point, but the horizontal set $C(\Sigma)$ includes a larger class of points.

Lemma 2.11 (Singular points)**.** *Let O be an open subset of $\mathbb{G}$ with the unit element e contained in O and let $u \in C^1(O, \mathbb{R}^k)$ such that $u(e) = 0$ and the differential map $du(p) : T_p\mathbb{G} \longrightarrow \mathbb{R}^k$ is surjective for any $p \in O$. Then, defining $\Sigma = u^{-1}(0)$, we represent the horizontal set of Σ as follows*

$$C(\Sigma) = \left\{ p \in \Sigma \mid d_H u(p) : \mathcal{G} \longrightarrow \mathbb{R}^k \text{ is not surjective} \right\}.$$

Proof. We choose $p \in \Sigma \subset O$. Since the submanifold Σ coincides with the level set $u^{-1}(0)$ we have $\mathrm{Ker}(du(p)) = T_p\Sigma$. Let us define the subspace

$$S(p) = \{X \in H^1 \mid X(p) \in H_p\mathbb{G} \cap T_p\Sigma\} \subset H^1.$$

We wish to prove that

$$\mathrm{Ker}(d_H u(p)) = S(p) \oplus H^2 \oplus \cdots \oplus H^\iota. \tag{15}$$

We will use the formula

$$d_H u(p)(Y(p)) = du(p)\left(\mathfrak{p}_H(Y)(p)\right) \quad \text{for any } Y \in \mathcal{G}, \tag{16}$$

proved in Proposition 2.4. Let $Y \in S(p) \oplus H^2 \oplus \cdots \oplus H^\iota$. By definition of $S(p)$ we have that $\mathfrak{p}_H(Y)(p) \in H_p\mathbb{G} \cap T_p\Sigma$ and by (16) we obtain that $d_H u(p)(H^j) = \{0\}$ for every $j = 2, \ldots, \iota$, then $du(p)((Y)(p)) = 0$. As a result, we have proved the first inclusion

$$S(p) \oplus H^2 \oplus \cdots \oplus H^\iota \subset \mathrm{Ker}(d_H u(p)).$$

Conversely, assume that $Y \in \mathrm{Ker}(d_H u(p))$. By (16) it follows that $\mathfrak{p}_H(Y)(p) \in \mathrm{Ker}(du(p))$, then $\mathfrak{p}_H(Y)(p) \in H_p\mathbb{G} \cap T_p\Sigma$. We have proved that $\mathfrak{p}_H(Y) \in S(P)$, hence $Y \in S(p) \oplus H^2 \oplus \cdots \oplus H^\iota$. The arbitrary choice of Y implies the opposite inclusion

$$\mathrm{Ker}(d_H u(p)) \subset S(p) \oplus H^2 \oplus \cdots \oplus H^\iota,$$

therefore (15) follows. The equality (15) and the relation $\dim(S(p)) = \dim(H_p\mathbb{G} \cap T_p\Sigma)$ yield

$$\dim(d_H u(p)(\mathcal{G})) = \dim(\mathcal{G}) - \dim(\mathrm{Ker}(d_H u(p))) = \dim(H_p\mathbb{G}) - \dim(H_p\mathbb{G} \cap T_p\Sigma)\,. \tag{17}$$

Now assume that $p \in C(\Sigma)$. By definition of horizontal point we know that

$$\dim(H_p\mathbb{G}) - \dim(H_p\mathbb{G} \cap T_p\Sigma) \leq k - 1,$$

therefore formula (17) implies that $\dim(d_H u(p)(\mathcal{G})) \leq k-1$ and that $d_H u(p) : \mathcal{G} \longrightarrow \mathbb{R}^k$ cannot be surjective. Conversely, if $d_H u(p) : \mathcal{G} \longrightarrow \mathbb{R}^k$ is not surjective, then by (17) it follows that $\dim(H_p\mathbb{G}) - \dim(H_p\mathbb{G} \cap T_p\Sigma) \leq k - 1$. This concludes the proof. □

Proposition 2.12. *Let Σ be a submanifold of $\mathbb{G}$. Then for any $p \in \mathbb{G}$ we have*

$$l_p(C(\Sigma)) = C(l_p(\Sigma))\,.$$

Proof. The left translation $l_p : \mathbb{G} \longrightarrow \mathbb{G}$ is a diffeomorphism, therefore $l_p(\Sigma)$ is another submanifold of $\mathbb{G}$. In view of (5) and using the chain rule it follows that $dl_p(s)\,(H_s\mathbb{G}) = H_{ps}\mathbb{G}$ for any $s \in \mathbb{G}$, hence

$$dl_p(s)\,(H_s\mathbb{G} \cap T_s\Sigma) = H_{ps}\mathbb{G} \cap dl_p(s)(T_s\Sigma) = H_{ps}\mathbb{G} \cap T_{ps}\,(l_p\Sigma)\,. \tag{18}$$

As a consequence, for any $s \in \mathbb{G}$ we have

$$\dim\,(H_s\mathbb{G} \cap T_s\Sigma) = \dim\,(H_{ps}\mathbb{G} \cap T_{ps}\,(l_p\Sigma))\,.$$

By the definition of horizontal point, the last equality implies that $s \in C(\Sigma)$ if and only if $ps \in C(l_p\Sigma)$. □

An essential tool to achieve Theorem 2.16 is the following Sard-type theorem, which corresponds to Theorem 2.7 of [47].

Theorem 2.13 (Sard-type theorem)**.** *Let $\mathbb{G}$ and $\mathbb{M}$ be stratified groups of Hausdorff dimension Q and P, respectively, with $Q \geq P$. Let $A \subset \mathbb{G}$ be a measurable set. Consider a Lipschitz map $\tau : A \longrightarrow \mathbb{M}$ and define the set of singular points*

$$S = \{p \in A \mid d_H\tau(p) \text{ exists and it is not surjective}\}.$$

Then, for $\mathcal{H}^P$-a.e. $\xi \in \mathbb{M}$ it follows $\mathcal{H}^{Q-P}\,(S \cap \tau^{-1}(\xi)) = 0$.

Remark 2.14. Due to the general Eilenberg inequality proved in 2.10.25 of [23] the fact that points where u is not P-differentiable are negligible implies that they are $\mathcal{H}^{Q-P}$-negligible in $\mathcal{H}^P$-a.e. level set. In other words, for $\mathcal{H}^P$-a.e. level set of u the subset of points p where $d_H\tau(p)$ exists has full measure.

We will apply Theorem 2.13 to the case $\mathbb{M} = \mathbb{R}^k$, $P = k$ and $\tau \in C^1(U, \mathbb{R}^k)$, where U is an open subset of $\mathbb{G}$. Due to Proposition 2.4, any map of $C^1(U, \mathbb{R}^k)$ also belongs to $C^1_H(U, \mathbb{R}^k)$, then it is everywhere P-differentiable, with continuous P-differential. The existence everywhere of the P-differential allows us to divide the points of $\tau^{-1}(t)$ into two disjoint subsets for every $t \in \mathbb{R}^k$. The first one is the subset of points $p \in \tau^{-1}(t)$ such that $d_H\tau(p) : \mathcal{G} \longrightarrow \mathbb{R}^k$ is not surjective and the second one is the complement. It is clear that C^1_H maps are locally Lipschitz with respect to the Carnot-Carathéodory

distance, therefore Theorem 2.13 can be applied to $u_{|K}$, where K is a compact subset of U, getting

$$\mathcal{H}^{Q-k}\Big(\Big\{p \in K \mid \tau(p) = t \text{ and } d_H\tau(p) \text{ is not surjective}\Big\}\Big) = 0\,. \tag{19}$$

for a.e. $t \in \mathbb{R}^k$. If we take a countable family of compact subsets $\{K_j\}$ whose union is equal to U and we apply (19) to any of these compact subsets, it follows that for a.e. $t \in \mathbb{R}^k$ we have

$$\mathcal{H}^{Q-k}\Big(\Big\{p \in U \mid \tau(p) = t \text{ and } d_H\tau(p) \text{ is not surjective}\Big\}\Big) = 0\,. \tag{20}$$

This proves the following corollary of Theorem 2.13.

Corollary 2.15. *Let $\tau : U \longrightarrow \mathbb{R}^k$ be a C^1 map. Then $\tau \in C^1_H(U, \mathbb{R}^k)$ and for a.e. $t \in \mathbb{R}^k$ equation (20) holds.*

The next theorem is the main result of this paper.

Theorem 2.16 (Negligibility)**.** *Let $\Sigma \subset \Omega$ be a C^1 submanifold of codimension k. Then we have $\mathcal{H}^{Q-k}(C(\Sigma)) = 0$.*

Proof. According to Definition 1.4, we fix a graded basis $(W_1, \dots, W_q)$ and the associated graded coordinates given by $F : \mathbb{R}^q \longrightarrow \mathbb{G}$. Let us fix a point $p \in \Sigma$. By definition of C^1 submanifold there exists an open neighbourhood O_p of p, a map $u_p \in C^1(O_p, \mathbb{R}^k)$ and some integers $1 \le j_1 < j_2 < \cdots < j_k \le q$ such that for any $s \in O_p$ the vectors $(W_{j_1}u_p(s), W_{j_2}u_p(s), \dots, W_{j_k}u_p(s))$ are linearly independent and $\Sigma \cap O_p = u_p^{-1}(0)$. Proposition 2.12 permits us to translate p to the unit element $e \in \mathbb{G}$.

We define $O = l_{p^{-1}}O_p$ and $u : O \longrightarrow \mathbb{R}^k$ as $u(s) = u_p(l_p s)$ for each $s \in O$. The left invariance of vector fields W_j gives $W_j u(s) = W_j u_p(l_p s)$ for every $s \in O$ and every $j = 1, \dots, q$. Then for every $s \in O$ the vectors $(W_{j_1}u(s), W_{j_2}u(s), \dots, W_{j_k}u(s))$ are linearly independent. We define the translated submanifold $\Sigma_p = l_{p^{-1}}\Sigma$, the open set $\tilde{O} = F^{-1}(O) \subset \mathbb{R}^q$ and $\tilde{\Sigma} = F^{-1}(\Sigma_p) \subset \mathbb{R}^q$, observing that $\tilde{u}^{-1}(0) = \tilde{\Sigma} \cap \tilde{O}$ where $\tilde{u} = u \circ F : \tilde{O} \longrightarrow \mathbb{R}^k$ is C^1. We have

$$\partial_{x_j}\tilde{u}(0) = du \circ \partial_{x_j}F(0) = du \circ W_j(e) = W_j u(e)$$

for any $j = 1, \dots, q$. Then the vectors $(\partial_{x_{j_1}}\tilde{u}, \dots, \partial_{x_{j_k}}\tilde{u})$ are linearly independent in an open neighbourhood $O_1 \subset \tilde{O}$ of the origin. Let us define the subspace

$$\Pi = \Big\{x \in \mathbb{R}^q \mid x_{j_1} = x_{j_2} = \cdots = x_{j_k} = 0\Big\}\,.$$

By the implicit function theorem there exist an open subset $A \subset \Pi$ containing the origin and a C^1 map $\varphi : A \longrightarrow \mathbb{R}^k$ such that $\tilde{u}(\xi, \varphi(\xi)) = 0$ for any $\xi \in A$. Our notation precisely means

$$(\xi, \varphi(\xi)) = \sum_{i \notin \{j_1, \dots, j_k\}} \xi_i\, e_i + \sum_{i \in \{j_1, \dots, j_k\}} \varphi^i(\xi)\, e_{j_i}$$

where (e_j) is the canonical basis of $\mathbb{R}^q$. The map $\Phi : A \longrightarrow O_1$ is defined by $\Phi(\xi) = (\xi, \varphi(\xi)) \in \tilde{\Sigma}$ for any $\xi \in A$. We introduce the C^1 map $\Psi : \mathbb{R}^k \times A \longrightarrow \mathbb{G}$,

$$\Psi(t, \xi) = \exp t_1 W_{j_1} \cdot \exp t_2 W_{j_2} \cdot \cdots \cdot \exp t_k W_{j_k} \cdot F(\Phi(\xi))$$

for every $(t,\xi)\in\mathbb{R}^k\times A$. We have used the dot symbol to denote the group operation. We have

$$\partial_{t_i}\Psi(0)=W_{j_i}(e)\in T_e\mathbb{G}\quad\text{and}\quad\partial_{\xi_j}\Psi(0)=W_j(e)+\sum_{i=1}^{k}\varphi_{\xi_j}^{j_i}(0)\,W_{j_i}(e)\,,$$

for every $i=1,\dots,k$ and every $j\notin\{j_1,\dots,j_k\}$. It is easily recognized that vectors

$$(\partial_{t_1}\Psi(0),\dots,\partial_{t_q}\Psi(0))$$

are a basis of $T_e\mathbb{G}$, hence there exist $\varepsilon>0$ and open sets $A_1\subset A$ and $U\subset\mathbb{G}$ such that $\Psi\left((-\varepsilon,\varepsilon)^k\times A_1\right)=U$, $0\in A_1$, $e\in U$ and the function

$$\Psi:(-\varepsilon,\varepsilon)^k\times A_1\longrightarrow U$$

is invertible. Let us consider the projection $\mathfrak{p}:\mathbb{R}^q\longrightarrow\mathbb{R}^k$ defined by

$$\mathfrak{p}(x)=(x_{j_1},x_{j_2},\dots,x_{j_k})$$

for any $x\in\mathbb{R}^q$ and define the C^1 map $\tau:U\longrightarrow(-\varepsilon,\varepsilon)^k$ as

$$\tau(s)=\mathfrak{p}\left(\Psi^{-1}(s)\right),\qquad\text{for any } s\in U.$$

Thus, Corollary 2.15 implies that for a.e. $t\in(-\varepsilon,\varepsilon)^k$ we have

$$\mathcal{H}^{Q-k}\Big(\Big\{s\in U\mid\tau(s)=t\text{ and }d_H\tau(s)\text{ is not surjective}\Big\}\Big)=0. \tag{21}$$

By the fact that Ψ is invertible it follows that $d\tau(s)$ is surjective for every $s\in U$, then the subset $S_t=\tau^{-1}(t)\subset U$ is a C^1 submanifold for any t. We can apply Lemma 2.11, obtaining that

$$C\left(S_t\right)=\Big\{s\in S_t\mid d_H\tau(s)\text{ is not surjective}\Big\}.$$

In view of (21) we get $\mathcal{H}^{Q-k}\left(C(S_t)\right)=0$ for a.e. $t\in(-\varepsilon,\varepsilon)^k$. By definition of τ we know that

$$\begin{aligned}S_t=\tau^{-1}(t)&=\Big\{s\in U\mid\Psi^{-1}(s)\in\{t\}\times A_1\Big\}\\&=\exp t_1W_{j_1}\cdot\exp t_2W_{j_2}\cdot\dots\cdot\exp t_kW_{j_k}\cdot F\left(\Phi(A_1)\right)\\&=l_{s_{t_1}}(l_{s_{t_2}}(\cdots(l_{s_{t_k}}F\left(\Phi(A_1)\right))\cdots),\end{aligned}$$

where $t=(t_1,t_2,\dots,t_k)$ and $s_{t_i}=\exp t_iW_{j_i}$ for every $i=1,2,\dots,k$. For $t=0\in\mathbb{R}^k$ the previous equations yield $S_0=F\left(\Phi(A_1)\right)\subset\Sigma_p$, then

$$S_t=l_{s_{t_1}}(l_{s_{t_2}}(\cdots(l_{s_{t_k}}S_0))\cdots).$$

The previous formula and the $\mathcal{H}^{Q-1}$-negligibility of $C(S_t)$ for a.e. $t\in(-\varepsilon,\varepsilon)^k$ yield

$$\begin{aligned}0=\mathcal{H}^{Q-k}\left(C(S_t)\right)&=\mathcal{H}^{Q-k}\Big(C(l_{s_{t_1}}(l_{s_{t_2}}(\cdots(l_{s_{t_k}}S_0))\cdots))\Big)\\&=\mathcal{H}^{Q-k}\left(l_{s_{t_k}}\left(l_{s_{t_{k-1}}}\cdots\left(l_{s_{t_1}}\left(C(S_0)\right)\cdots\right)\right)\right)=\mathcal{H}^{Q-k}\left(C(S_0)\right).\end{aligned}$$

In the third equality we have used Proposition 2.12 and in the fourth one we have used the fact that left translations are isometries with respect to any homogeneous distance.

Note that S_0 contains the unit element e and it is an open subset of Σ_p. We know that $\Sigma_p = l_{p^{-1}}\Sigma$, hence defining $S_p = l_p S_0$ we have that $S_p \subset \Sigma$ is an open neighbourhood of p in the topology of Σ. Thus, reasoning as before we obtain

$$\mathcal{H}^{Q-k}(S_p) = \mathcal{H}^{Q-k}\left(C(l_p S_0)\right) = \mathcal{H}^{Q-k}\left(l_p(C(S_0)\right) = \mathcal{H}^{Q-k}\left(C(S_0)\right) = 0.$$

The arbitrary choice of p implies that we can find a countable family of open subsets

$$\left\{S_{p_j} \mid S_{p_j} \subset \Sigma, j \in \mathbb{N}\right\}.$$

such that $\Sigma = \bigcup_{j\in\mathbb{N}} S_{p_j}$ and $\mathcal{H}^{Q-k}\left(C(S_{p_j})\right) = 0$ for every $j \in \mathbb{N}$. Finally, the thesis follows by the equality $C(\Sigma) = \bigcup_{j\in\mathbb{N}} C(S_{p_j})$. □

A first important consequence of the previous theorem occurs in codimension one, where we obtain the representation of the $Q-1$ dimensional spherical Hausdorff measure of C^1 hypersurfaces. To see this, we will use the notion of "metric factor", [48].

Definition 2.17 (Metric factor)**.** Let B_1 be the open unit ball with respect to a fixed homogeneous distance d of $\mathbb{G}$. Consider $\nu \in H^1 \setminus \{0\}$ along with its orthogonal hyperplane $\tilde{\mathcal{L}}(\nu)$ in $\mathcal{G}$ and define $\mathcal{L}(\nu) = \exp\tilde{\mathcal{L}}(\nu) \subset \mathbb{G}$. Let $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ represent a system of graded coordinates and define

$$\theta^g_{Q-1}(\nu) = \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}(\nu) \cap B_1)\right). \tag{22}$$

The map $\nu \longrightarrow \theta^g_{Q-1}(\nu)$ is called the *metric factor* of the homogeneous distance d with respect to the direction ν and the graded metric g.

With a slight abuse of notation, it will be useful in the sequel to define the metric factor also for horizontal vectors. We define

$$\theta^g_{Q-1}(v) = \theta^g_{Q-1}(V)$$

where $v \in H_p\mathbb{G}$ and $V \in H^1 \setminus \{0\}$ is the unique vector field such that $V(p) = v$. Note that a more rigorous, but unnecessary formulation should consider the metric factor θ^g_{Q-1} as a function on the tangent bundle $T\mathbb{G}$ and which is constant along left invariant vector fields.

Remark 2.18. As we have seen in Lemma 1.10 of [48], the above definition does not depend on the choice of graded coordinates. So the number $\theta^g_{Q-1}(\nu)$ depends only on the homogeneous distance d, the direction of ν and the graded metric g. By the symmetry of Euclidean balls and taking two of them $B^{|\cdot|}_{r_1}$ and $B^{|\cdot|}_{r_2}$ such that $B^{|\cdot|}_{r_1} \subset F^{-1}(B_1) \subset B^{|\cdot|}_{r_2}$, we notice that the intersection $F^{-1}\left(\mathcal{L}(\nu)\right) \cap B^{|\cdot|}_{r_i}$ for $i = 1, 2$ is independent of the direction ν. As as result, for any $\nu \in H^1 \setminus \{0\}$ we have

$$\omega_{q-1}\, r_1^{q-1} \leq \theta^g_{Q-1}(\nu) \leq \omega_{q-1}\, r_2^{q-1},$$

then $\nu \longrightarrow \theta^g_{Q-1}(\nu)$ is uniformly bounded from above and from below by positive constants which depend on the homogeneous distance and on the graded metric.

In Subsection 2.1 of [48] a class of stratified groups where the Carnot-Carathéodory distance yields a constant metric factor is singled out, namely, the class of $\mathcal{R}$-rotational groups, which encompasses Heisenberg groups and more general H-type groups. In the appendix of [29], Franchi, Serapioni and Serra Cassano have constructed a general

homogeneous distance d_∞ for every stratified group. Its explicit formula is stated in $\mathbb{R}^q$ with respect to graded coordinates:

$$d_\infty(x,0) = \max_{j=1,\dots,\iota} \{\varepsilon_j \, |(x^{m_{j-1}+1},\dots,x^{m_j})|^{1/j}\} \tag{23}$$

with the left invariant property $d_\infty(x,y) = d_\infty(x^{-1}\cdot y,0)$ and $\varepsilon_j \in]0,1[$ for every $j=1,\dots,q$, are suitable dimensional constants depending only on the group. The integer m_j for every $j=0,\dots,\iota$ is introduced in Definition 1.4 and the dot symbol between x^{-1} and y denotes the group operation in graded coordinates, according to the discussion below Definition 1.4. In the next proposition we prove that the metric factor with respect to this distance becomes a constant function of the horizontal direction ν.

Proposition 2.19. *Let θ^g_{Q-1} represent the metric factor with respect to the distance d_∞. Then there exists $\alpha_{Q-1}>0$ such that for every $\nu \in H^1 \setminus \{0\}$ we have $\theta^g_{Q-1}(\nu) = \alpha_{Q-1}$.*

Proof. For any couple of horizontal vectors $\nu,\mu \in H\setminus\{0\}$ we can find an isometry $\tau : \mathcal{G} \longrightarrow \mathcal{G}$ such that $\tau(\nu)=\mu$ and $\tau(\tilde{\mathcal{L}}(\nu)) = \tilde{\mathcal{L}}(\mu)$, where $\tilde{\mathcal{L}}(\nu)$ and $\tilde{\mathcal{L}}(\mu)$ are the orthogonal spaces to ν and μ, respectively. We read these orthogonal spaces in $\mathbb{G}$ defining $\mathcal{L}(\nu) = \exp\left(\tilde{\mathcal{L}}(\nu)\right)$ and $\mathcal{L}(\mu) = \exp\left(\tilde{\mathcal{L}}(\mu)\right)$. Let $(W_1,\dots,W_q)$ be a graded basis of $\mathcal{G}$, let $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ be the associated system of graded coordinates and define $I(x) = \sum_{j=1}^q x^j\, W_j \in \mathcal{G}$ for every $x\in\mathbb{R}^q$. By the expression of d_∞ it is easy to see that the unit ball $\tilde{B}_1 \subset \mathbb{R}^q$ with respect to d_∞ is preserved under the family of isometries $\tilde{\tau}$, i.e. $\tilde{\tau}(\tilde{B}_1) = \tilde{B}_1$. Thus, taking into account that $F = \exp\circ I$ and that $\tilde{\tau} = I^{-1}\circ\tau\circ I : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is an Euclidean isometry

$$\begin{aligned} &F^{-1}(\mathcal{L}(\mu))\cap\tilde{B}_1 = I^{-1}(\tilde{\mathcal{L}}(\mu))\cap\tilde{B}_1 = I^{-1}(\tau(\tilde{\mathcal{L}}(\nu)))\cap\tilde{B}_1 \\ &= \tilde{\tau}\left(I^{-1}(\tilde{\mathcal{L}}(\nu))\cap\tilde{\tau}^{-1}(\tilde{B}_1)\right) = \tilde{\tau}\left(I^{-1}(\tilde{\mathcal{L}}(\nu))\cap\tilde{B}_1\right) = \tilde{\tau}\left(F^{-1}(\mathcal{L}(\nu))\cap\tilde{B}_1\right), \end{aligned}$$

then, by Definition 2.17 it follows that

$$\begin{aligned} \theta^g_{Q-1}(\mu) &= \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}(\mu))\cap\tilde{B}_1)\right) = \mathcal{H}^{q-1}_{|\cdot|}\left(\tilde{\tau}\left(F^{-1}(\mathcal{L}(\nu))\cap\tilde{B}_1)\right)\right) \\ &= \mathcal{H}^{q-1}_{|\cdot|}\left(F^{-1}(\mathcal{L}(\nu))\cap\tilde{B}_1)\right) = \theta^g_{Q-1}(\nu). \end{aligned}$$

This concludes the proof. □

In view of Theorem 2.16 in the case $k=1$, the statement of Theorem 2.3 in [48] holds for any C^1 submanifold of codimension one. As a consequence, the relation between the Riemannian measure σ_g induced by the graded metric g restricted to a C^1 submanifold Σ of codimension one and $\mathcal{S}^{Q-1}\llcorner\Sigma$ is established on arbitrary stratified groups.

Theorem 2.20. *Let Σ be a C^1 submanifold of codimension one in Ω. Then we have*

$$\theta^g_{Q-1}(\nu_H)\,\mathcal{S}^{Q-1}\llcorner\Sigma = |\nu_H|\;\sigma_g\llcorner\Sigma\,, \tag{24}$$

$$\mathcal{S}^{Q-1}\llcorner\Sigma = \frac{|\nu_H|}{\theta^g_{Q-1}(\nu_H)}\;\sigma_g\llcorner\Sigma\,, \tag{25}$$

where σ_g is the measure induced by the graded metric g restricted to Σ.

3. $(\mathbb{G}, \mathbb{R}^k)$-RECTIFIABILITY AND PERIMETER MEASURE

In this section we present some applications of Theorem 2.16. We start introducing families of regular sets with respect to the geometry of stratified groups.

Definition 3.1 ($(\mathbb{G}, \mathbb{R}^k)$-regular set)**.** A subset $\Sigma \subset \Omega$ is a $(\mathbb{G}, \mathbb{R}^k)$-*regular set* if there exists a map $f \in C^1_H(\Omega, \mathbb{R}^k)$ such that $\Sigma = f^{-1}(0)$ and the P-differential

$$d_H f(p) : \mathbb{G} \longrightarrow \mathbb{R}^k$$

is a surjective G-linear map for any $p \in \Sigma$.

Definition 3.2 ($(\mathbb{G}, \mathbb{R}^k)$-rectifiability)**.** We say that $S \subset \Omega$ is $(\mathbb{G}, \mathbb{R}^k)$-*rectifiable*, if there exists a sequence of $(\mathbb{G}, \mathbb{R}^k)$-regular sets $\{\Sigma_j\}$ such that

$$\mathcal{H}^{Q-k}\Big(S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) = 0.$$

The $(\mathbb{G}, \mathbb{R}^k)$-rectifiability turns out to have a "fractal nature" with respect to the Euclidean viewpoint, since $(\mathbb{G}, \mathbb{R}^k)$-regular sets may have Euclidean Hausdorff dimension strictly greater than their topological dimension, then they cannot be rectifiable in the Euclidean sense, [41]. In the following definition we recall the classical notion of rectifiability, utilizing countable unions of C^1 surfaces, instead of Lipschitz images.

Definition 3.3 (Rectifiability)**.** We say that a subset $S \subset \mathbb{G}$ is *rectifiable* of codimension k or simply $(q-k)$-*rectifiable* if there exists a sequence of C^1 surfaces $\{\Sigma_j\}$ of dimension $(q-k)$ such that

$$\mathcal{H}^{q-k}_{|\cdot|}\Big(S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) = 0\,.$$

Remark 3.4. Notice that $(\mathbb{G}, \mathbb{R}^k)$-rectifiability coincides with rectifiability when $\mathbb{G} = \mathbb{R}^q$ and $q > k$. Then the notion of $(\mathbb{G}, \mathbb{R})$-rectifiability allows us to state in a unified way the De Giorgi rectifiability theorem in stratified groups. By results of [19], [27], [29] we know that if $|\partial E|_H(\mathbb{G}) < \infty$, where $\mathbb{G}$ is either an Euclidean space or a stratified group of step two, then the H-reduced boundary $\partial_{*H} E$ is $(\mathbb{G}, \mathbb{R})$-rectifiable. Presently, this result for stratified groups of step higher than two is an open issue.

In order to complete the picture, we briefly mention a further extension of the notion of $(\mathbb{G}, \mathbb{R}^k)$-rectifiability. In fact, we can replace $\mathbb{R}^k$ with another stratified group $\mathbb{M}$.

Definition 3.5 ($(\mathbb{G}, \mathbb{M})$-regular set)**.** A subset $\Sigma \subset \Omega$ is a $(\mathbb{G}, \mathbb{M})$-*regular surface* if there exist $f \in C^1_H(\Omega, \mathbb{M})$ such that $f^{-1}(e) = \Sigma$ and

$$d_H f(p) : \mathbb{G} \longrightarrow \mathbb{M}$$

is a surjective G-linear map for any $p \in \Sigma$.

It is apparent that the notion of $(\mathbb{G}, \mathbb{M})$-regularity in higher codimension allows us a certain amount of freedom in the choice of $\mathbb{M}$, but not all codomains are "good" to be considered. For instance, the family of $(\mathbb{H}^n, \mathbb{H}^m)$-regular sets is empty whenever $n > m$. This follows by the fact that there are no surjective G-linear maps between $\mathbb{H}^n$ onto $\mathbb{H}^m$, as it can be checked using the definition of G-linearity and the group operation in the Heisenberg group, see also Theorem 2.8 of [47]. As soon as we have

a surjective G-linear map $L : \mathbb{G} \longrightarrow \mathbb{M}$ a canonical example of $(\mathbb{G}, \mathbb{M})$-regular set can be given by choosing the subgroup $N = L^{-1}(e) \subset \mathbb{G}$ which is $(\mathbb{G}, \mathbb{M})$-regular. In view of Proposition 1.12 of [47] the Hausdorff dimension of N is $Q-P$, where Q and P are the Hausdorff dimensions of $\mathbb{G}$ and $\mathbb{M}$, respectively. These considerations lead to the following definition.

Definition 3.6 ($(\mathbb{G}, \mathbb{M})$-rectifiability)**.** We say that $S \subset \Omega$ is $(\mathbb{G}, \mathbb{M})$-*rectifiable*, if there exists a sequence of $(\mathbb{G}, \mathbb{M})$-regular sets $\{\Sigma_j\}$ such that

$$\mathcal{H}_{\rho}^{Q-P}\Big(S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) = 0\,,$$

where ρ is the Carnot-Carathéodory distance of $\mathbb{G}$.

Theorem 3.7. *Let $\Sigma \subset \Omega$ be a connected submanifold of class C^1 and codimension k. Then Σ is $(\mathbb{G}, \mathbb{R}^k)$-rectifiable.*

Proof. By Lemma 2.11 we see that $C(\Sigma)$ is a closed subset of Σ, then $\Sigma \setminus C(\Sigma)$ is a countable union of connected pieces that can be represented locally as level sets of C^1 maps with surjective P-differential. Theorem 2.16 implies $\mathcal{H}^{Q-k}\left(C(\Sigma)\right) = 0$, hence the thesis follows. □

Theorem 3.8. *Every rectifiable set of codimension k is $(\mathbb{G}, \mathbb{R}^k)$-rectifiable.*

Proof. Let S be a rectifiable set of codimension k and let $\{\Sigma_j \mid j \in \mathbb{N}\}$ be a family of C^1 submanifolds of codimension k such that $\mathcal{H}_{|\cdot|}^{q-k}\Big(S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) = 0$. By Proposition 4.4 of [29], for every $\alpha > 0$ and every $R > 0$ there exists $C_R > 0$ such that

$$\mathcal{H}^{Q-q+\alpha}(E) \leq C_R\, \mathcal{H}_{|\cdot|}^{\alpha}(E), \tag{26}$$

whenever $E \subset B_R$. Using the estimate (26) for $\alpha = q - k$ we obtain

$$\mathcal{H}^{Q-k}\Big(B_R \bigcap S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) \leq \mathcal{H}_{|\cdot|}^{q-k}\Big(B_R \bigcap S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) = 0$$

for every $R > 0$, then $\mathcal{H}^{Q-k}\Big(S \setminus \bigcup_{j\in\mathbb{N}} \Sigma_j\Big) = 0$. By Theorem 3.7 we know that Σ_j is $(\mathbb{G}, \mathbb{R}^k)$-rectifiable for any $j \in \mathbb{N}$. This concludes the proof. □

Finally, we discuss an application of Theorem 2.16 related to a question raised by Danielli, Garofalo and Nhieu in [15]. In this paper the authors prove that there exist constants $c, C > 0$ such that for every relatively compact open set E with C^2 boundary the estimates

$$c\, \mathcal{S}^{Q-1}(\partial E) \leq |\partial E|_H(\mathbb{H}^n) \leq C\, \mathcal{S}^{Q-1}(\partial E)\,, \tag{27}$$

hold, where $Q = 2n + 2$ is the Hausdorff dimension of the Heisenberg group $\mathbb{H}^n$. Here the authors conjecture the validity of (27) for arbitrary stratified groups. Due to Theorem 2.5 of [49], see also Theorem 3.5 of [28], we have the following formula

$$|\partial E|_H = \theta_{Q-1}^{g}(\nu_H)\, \mathcal{S}^{Q-1} \llcorner \partial E\,, \tag{28}$$

under the assumption that the characteristic set is $\mathcal{S}^{Q-1}$-negligible. Then Theorem 2.16 makes (28) true for any set E with C^1 boundary. Thus, formula (28) immediately

extends the validity of (27) to any open set with C^1 boundary contained in an arbitrary stratified group, obtaining that

$$\underline{\theta}^g_{Q-1}\ \mathcal{S}^{Q-1}(\partial E \cap \Omega)\ \leq |\partial E|_H(\Omega) \leq\ \overline{\theta}^g_{Q-1}\ \mathcal{S}^{Q-1}(\partial E \cap \Omega)\,, \tag{29}$$

where $\Omega \subset \mathbb{G}$ is an arbitrary bounded open set. The constants $\underline{\theta}^g_{Q-1}$, $\overline{\theta}^g_{Q-1}$, are defined as follows

$$\underline{\theta}^g_{Q-1} = \inf_{\nu \in H^1} \theta^g_{Q-1}(\nu) \quad \text{and} \quad \overline{\theta}^g_{Q-1} = \sup_{\nu \in H^1} \theta^g_{Q-1}(\nu).$$

In Remark 2.18 we have shown that the function $\nu \longrightarrow \theta^g_{Q-1}(\nu)$ is bounded from above and from below by positive constants, therefore estimates (29) are nontrivial.

As last remark, we wish to show that the constants of (29) are actually optimal. To do this, we need of the following definition.

Definition 3.9. Let $p \in \mathbb{G}$ and let $\nu \in H^1 \setminus \{0\}$. The *vertical half spaces* with respect to ν are defined as follows

$$S^+(\nu) = \exp\left(\left\{Y \in \mathcal{G} \;\middle|\; \langle \nu, Y\rangle > 0\right\}\right)$$
$$S^-(\nu) = \exp\left(\left\{Y \in \mathcal{G} \;\middle|\; \langle \nu, Y\rangle < 0\right\}\right)\,.$$

The *vertical half spaces* with respect to ν and *centered at* $p \in \mathbb{G}$ are defined as follows

$$l_p\left(S^+(\nu)\right) = S^+(p,\nu) \quad \text{and} \quad l_p\left(S^-(\nu)\right) = S^-(p,\nu).$$

Let us denote by $\mathcal{O}$ the family of nonempty bounded open sets of $\mathbb{G}$ and by $\mathcal{E}$ the family of open sets with C^1 boundary. We easily see that

$$\begin{aligned}
\underline{\theta}^g_{Q-1} &\leq \inf\left\{\frac{|\partial E|_H(\Omega)}{\mathcal{S}^{Q-1}(\partial E \cap \Omega)} \;\middle|\; \Omega \in \mathcal{O},\ E \in \mathcal{E}\right\} \\
&\leq \inf\left\{\frac{|\partial S^+(\nu)|_H(\Omega)}{\mathcal{S}^{Q-1}(\partial S^+(\nu) \cap \Omega)} \;\middle|\; \Omega \in \mathcal{O},\ \nu \in H^1 \setminus \{0\}\right\} \\
&= \inf\{\theta^g_{Q-1}(\nu) \,|\, \nu \in H^1 \setminus \{0\}\}\ = \underline{\theta}^g_{Q-1},
\end{aligned}$$

where the first equality of the last line follows from (28). We also have

$$\begin{aligned}
\overline{\theta}^g_{Q-1} &\geq \sup\left\{\frac{|\partial E|_H(\Omega)}{\mathcal{S}^{Q-1}(\partial E \cap \Omega)} \;\middle|\; \Omega \in \mathcal{O},\ E \in \mathcal{E}\right\} \\
&\geq \sup\left\{\frac{|\partial S^+(\nu)|_H(\Omega)}{\mathcal{S}^{Q-1}(\partial S^+(\nu) \cap \Omega)} \;\middle|\; \Omega \in \mathcal{O},\ \nu \in H^1 \setminus \{0\}\right\} \\
&= \sup\{\theta^g_{Q-1}(\nu) \,|\, \nu \in H^1 \setminus \{0\}\} = \overline{\theta}^g_{Q-1}.
\end{aligned}$$

This proves our claim.

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