REMOVABLE SETS FOR LIPSCHITZ HARMONIC FUNCTIONS ON CARNOT GROUPS

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ABSTRACT. Let $\mathbb{G}$ be a Carnot group with homogeneous dimension $Q \geq 3$ and let $\mathcal{L}$ be a sub-Laplacian on $\mathbb{G}$. We prove that the critical dimension for removable sets of Lipschitz $\mathcal{L}$-harmonic functions is $(Q-1)$. Moreover we construct self-similar sets with positive and finite $\mathcal{H}^{Q-1}$ measure which are removable.

1. INTRODUCTION

A compact set K in the complex plane is called *removable* for bounded analytic functions if for any open set Ω containing K any bounded analytic function on $\Omega \setminus K$ has an analytic extension to Ω. It is easily seen that points are removable while closed disks are not. Already at the end of the 19th century, Painlevé proved that sets of zero length are removable. He naturally raised the question of geometrically characterizing removable sets. In 1947 Ahlfors in [1] gave a potential-theoretic characterization of removable sets by defining the celebrated notion of analytic capacity. In passing we note that Vitushkin, see e.g. [40], used analytic capacity and a close variant, the so called continuous analytic capacity, to study problems of uniform rational approximation on compact sets of the complex plane. Although it was known by then that the critical dimension for removable sets is 1 very few things were known about sets with critical dimension. The following question arose: *is it true that a compact K is non-removable if and only $\mathcal{H}^1(K) > 0$?* Here, $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

The negative answer to the above question was obtained by Vitushkin [39] in the 1960's. Vitushkin constructed a removable compact set K with $0 < \mathcal{H}^1(K) < \infty$. Later on, Garnett [16] and Ivanov [18] proved that the familiar 1-dimensional 4-corners Cantor set is in fact removable for bounded analytic functions. The "irregular" geometric structure of these examples led Vitushkin to conjecture that: *a compact set K is removable if and only if it is purely unrectifiable.* Recall that a set K is called rectifiable if there exist countably many Lipschitz curves Γ_i such that $\mathcal{H}^1(K \setminus \cup_i \Gamma_i) = 0$. On the other hand a set is called purely unrectifiable if it intersects any rectifiable curve in a set of $\mathcal{H}^1$ measure zero. Although Vitushkin's conjecture is false in full generality (this was proved in an astonishing way by Mattila in [24]) it turns out that it holds if we restrict attention to sets of finite length. The latter result is due to David [9].

The proof of Vitushkin's conjecture has a long and interesting history which is deeply related to the geometric study of singular integrals. See [38], [25] or [37] for extensive treatments. We first remark that the "if" part in the restricted conjecture of Vitushkin follows from Calderón's theorem on the L^2 boundedness of the Cauchy transform on Lipschitz graphs with small Lipschitz constant. It is of interest that Calderón studied this problem in connection with partial differential equations with minimal smoothness conditions not being aware with the connections to removability. In subsequent years the topic was studied extensively and several deep

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contributions were made, see e.g. [8],[19] and [11]. Nevertheless it was Melnikov's discovery in [28] of the relation of the Cauchy kernel to the so-called Menger curvature that triggered many advances during the 1990's, which eventually led to the complete resolution of Vitushkin's conjecture. In [26] Mattila, Melnikov and Verdera proved Vitushkin's conjecture in the particular case where the set K is 1 Ahlfors–David regular, or in short 1-AD-regular. A Radon measure μ is s-AD-regular, $s > 0$, if

$$\frac{r^s}{C} \leq \mu(B(z,r)) \leq Cr^s \text{ for } z \in \operatorname{spt}\mu \text{ and } 0 < r < \operatorname{diam}(\operatorname{spt}(\mu)),$$

for some fixed constant C. A set K is s-AD regular if the measure $\mathcal{H}^s\lfloor K$ is s-AD regular. A few years later David characterized in [9] the removable sets of bounded analytic functions among sets of finite length and Tolsa gave a complete Menger curvature integral characterization in [36] of all removable sets of bounded analytic functions. We mention that all these results depend on the deep geometric study of the Cauchy singular integral.

A compact set $K \subset \mathbb{R}^n$ is said to be removable for Lipschitz harmonic functions if whenever D is an open set containing K and $f : D \to \mathbb{R}$ is a Lipschitz function which is harmonic in $D \setminus K$, then f is harmonic in D. David and Mattila in [10] characterized planar removable sets with finite length: finite length removable sets for either bounded analytic or Lipschitz harmonic functions are precisely the purely 1-unrectifiable sets. This is one of the various reasons why Lipschitz harmonic functions are a natural class to study. Very recently Nazarov, Tolsa and Volberg [31] extended the result of David and Mattila in $\mathbb{R}^n$ by proving that a compact set $K \subset \mathbb{R}^n$ with $\mathcal{H}^{n-1}(K) < \infty$ is removable for Lipschitz harmonic functions if and only if it is purely $(n-1)$-unrectifiable. We should mention here that both results depend heavily on singular integrals. The result of David and Mattila is based on intricate Tb theorems for non-doubling measures and the Cauchy transform. Nazarov, Tolsa and Volberg base their proof on their earlier very deep work [30], where they prove that if μ is an $(n-1)$-AD regular measure, then the Riesz kernel $x/|x|^n, x \in \mathbb{R}^n \setminus \{0\}$, defines bounded singular operators in $L^2(\mu)$ if and only if μ is $(n-1)$-uniformly rectifiable. Uniform rectifiability can be thought as a quantitative version of rectifiability. The Riesz kernels arise naturally in the study of removable sets for Lipschitz harmonic functions, as one readily sees that $\nabla\Gamma_n = x/|x|^n, x \in \mathbb{R}^n \setminus \{0\}$, where $\Gamma_n = c_n|x|^{2-n}$ denotes the fundamental solution of the Laplacian for $n \geq 3$.

Recently, significant effort has been made towards the extension of classical Euclidean analysis and geometry into general non-Riemannian spaces, including Carnot groups and more abstract metric measure spaces. In particular, potential theory related to sub-Laplacians in Carnot groups is an active research field with many recent developments, see [4] and the references given there. In [6] the problem of removability for Lipschitz $\mathcal{L}$-harmonic functions in the Heisenberg group $\mathbb{H}^n$ was considered. It was established there that, in accordance with the Euclidean case, the critical removability dimension is $Q-1$, where $Q = 2n+2$ denotes the Hausdorff dimension of the Heisenberg group. Moreover, examples of separated self-similar removable sets with positive and finite $(Q-1)$-measure were given. An essential ingredient in order to establish the existence of such sets was the proof of a general criterion for unboundedness of singular integrals on self similar sets of metric groups.

The aim of the present paper is to extend the results from [6] to general Carnot groups. Our first result reads as follows.

Theorem 1.1. *Let C be a compact subset of a Carnot group $\mathbb{G}$ and denote by Q the homogeneous dimension of $\mathbb{G}$. Let $\mathcal{L}$ be the sub-Laplacian in $\mathbb{G}$.*

(i) *If $\mathcal{H}^{Q-1}(C) = 0$, C is removable for Lipschitz $\mathcal{L}$-harmonic functions.*
(ii) *If $\dim C > Q-1$, C is not removable for Lipschitz $\mathcal{L}$-harmonic functions.*

The proof of Theorem 1.1 is similar to the proof from [6]. Nevertheless we decided for the convenience of the reader to provide all of the details, although in some places the arguments are identical to those in [6]. As in [6] the proof of Theorem 1.1 relies on a representation theorem for Lipschitz $\mathcal{L}$-harmonic functions (Theorem 3.1). The analogue of Theorem 3.1 in [6] uses the divergence theorem of Franchi, Serapioni and Serra-Cassano [15] which is known to be true only for step two Carnot groups.

In the case of general Carnot groups, we overcome this obstacle using the Euclidean regularity of the domains appearing in the proof of Theorem 3.1. In fact, we have finite unions of bounded sets with smooth boundary, that are sets of finite perimeter in the Euclidean sense. On the other hand, we have also to detect the Euclidean reduced boundary, which we accomplish by perturbing a given piecewise smooth boundary and using the classical Sard's theorem. Then joining the Euclidean divergence theorem for finite perimeter sets, [12], with area-type formulae for the sub-Riemannian spherical Hausdorff measure of smooth sets, [20], [23], we reach the sub-Riemannian divergence formula in this special class of domains.

Additional technical difficulties arise from the fact that, while the fundamental solution Γ of the sub-Laplacian in the Heisenberg group has an explicit formula, the corresponding fundamental solution for general sub-Laplacians in general Carnot groups admits no such formula. Nevertheless the fundamental solution is always $(2-Q)$-homogeneous and this fact is essential in our proofs.

We also study the critical case (dimension $Q-1$). It is easy to construct nonremovable sets of positive and finite $\mathcal{H}^{Q-1}$ measure (see Remark 4.13). Our second main theorem reads as follows.

Theorem 1.2. *There exist sets $K \subset \mathbb{G}$ with $0 < \mathcal{H}^{Q-1}(K) < \infty$ which are removable for Lipschitz $\mathcal{L}$-harmonic functions.*

In [6] such sets were constructed in the Heisenberg group $\mathbb{H}^n$ based on Strichartz-type tilings, see [35]. However in general Carnot groups such tilings do not exist, and we provide an alternate constructive argument involving separated self-similar Cantor subsets in vertical subgroups of $\mathbb{G}$. As in the Euclidean case we need to consider singular integrals with respect to the kernel $k = \nabla_{\mathbb{G}}\Gamma$, which is $(1-Q)$-homogeneous. Roughly speaking, if one is able to prove that a certain singular integral is unbounded on $L^2(\mathcal{H}^{Q-1}\lfloor K)$, then the set K is removable. Our idea is to construct a separated self similar set K, with $0 < \mathcal{H}^{Q-1}(K) < \infty$, which lives on a dilation cone where at least one coordinate of the kernel k keeps constant sign. Moreover the set K is constructed in such a way that it has a fixed point at the origin. These properties enable us to apply directly the unboundedness criterion for singular integrals on self similar sets from [6] (reproduced in this paper as Theorem 4.11).

Removability of sets can be studied for other partial differential equations, and in other regularity classes. In [7], quantitative estimates on the size of removable sets for solutions of a wide variety of partial differential equations in Carnot groups are given.

The paper is organised as follows. In section 2 we lay down the necessary background in Carnot groups as well as some basic properties of their sub-Laplacians. In section 3 we prove a representation theorem for Lipschitz $\mathcal{L}$-harmonic functions outside some compact set K, namely Theorem 3.1, and this leads to the proof of Theorem 1.1. In section 4 we provide examples of removable sets with positive and finite $\mathcal{H}^{Q-1}$-measure.

2. Definitions and notation

A *Carnot group* is a connected, simply connected and nilpotent Lie group $\mathbb{G}$, with graded Lie algebra

$$\mathfrak{g} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_s ,$$

such that $[\mathfrak{v}_1, \mathfrak{v}_i] = \mathfrak{v}_{i+1}$ for $i = 1, 2, \ldots, s-1$ and $[\mathfrak{v}_1, \mathfrak{v}_s] = 0$. Under these conditions the exponential mapping $\exp : \mathfrak{g} \to \mathbb{G}$ is bianalytic, hence we can canonically identify elements $\mathfrak{g}$, namely left invariant vector fields, with elements of $\mathbb{G}$. The integer $s \geq 1$ is the *step* of $\mathbb{G}$. We denote the group law in $\mathbb{G}$ by $\cdot$ and the identity element of $\mathbb{G}$ by 0.

We fix an inner product $\langle \ , \ \rangle$ in $\mathfrak{v}_1$ and let $X_1, \ldots, X_m$ be an orthonormal basis for $\mathfrak{v}_1$ relative to this inner product. Using this basis, we construct the *horizontal subbundle* $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ with fibers $H_p\mathbb{G} = \operatorname{span}\{X_1(p), \ldots, X_m(p)\}$, $p \in \mathbb{G}$. A left-invariant vector field X on $\mathbb{G}$ is *horizontal* if it is a section of $H\mathbb{G}$. The inner product on $\mathfrak{v}_1$ defines a left invariant family of inner products on the fibers of the horizontal subbundle.

We denote by d the *Carnot–Carathéodory metric* on $\mathbb{G}$, defined by infimizing the lengths of horizontal paths joining two fixed points, where the horizontal length is computed using the aforementioned inner product. More specifically we define:

Definition 2.1. An absolutely continuous curve $\gamma : [0, T] \to \mathbb{G}$ will be called sub-unit, with respect to the vector fields $X_1, \ldots, X_m$, if there exist real measurable functions $a_j : [0, T] \to \mathbb{R}$, with $j = 1, \ldots, m$, such that $\sum_{j=1}^m a_j(t)^2 \leq 1$ for a.e. $t \in [0, T]$ and

$$\dot{\gamma}(t) = \sum_{j=1}^m a_j(t) X_j(\gamma(t)) \quad \text{for a.e. } t \in [0, T].$$

Definition 2.2. For $p, q \in \mathbb{G}$ their Carnot-Carathéodory distance is

$$\begin{aligned} d(p, q) = \inf\{T > 0 : \text{there is a sub-unit curve } \gamma : [0, T] \to \mathbb{G} \\ \text{such that } \gamma(0) = p \text{ and } \gamma(T) = q\}. \end{aligned}$$

It follows by Chow's theorem that the above set of curves joining p and q is not empty and hence d is a metric on $\mathbb{G}$. The closed and open balls with respect to d will be denoted by $B(p, r)$ and $U(p, r)$ respectively.

For each $t > 0$, we define $\delta_t : \mathfrak{g} \to \mathfrak{g}$ by setting $\delta_t(X) = t^i X$ if $X \in \mathfrak{v}_i$ and extending the mapping by linearity. The identification of the Lie algebra with the Lie group via the exponential mapping allows us to introduce dilations on $\mathbb{G}$, that we also denote by δ_t. Then $(\delta_t)_{t>0}$ is the one-parameter family of *dilations* of $\mathbb{G}$ satisfying $d(\delta_t(p), \delta_t(q)) = td(p, q)$ for $p, q \in \mathbb{G}$. Another family of automorphisms in $\mathbb{G}$ are the *left translations* $\tau_q : \mathbb{G} \to \mathbb{G}$ defined by $\tau_q(x) = q \cdot x, x \in \mathbb{G}$, for all $q \in \mathbb{G}$. We note also that the metric d is left invariant, i.e., $d(q \cdot p_1) = d(q \cdot p_2)$ for $q, p_1, p_2 \in \mathbb{G}$.

The Jacobian determinant of δ_t (with respect to Haar measure) is everywhere equal to t^Q, where

$$Q = \sum_{i=1}^s i \dim \mathfrak{v}_i$$

is the *homogeneous dimension* of $\mathbb{G}$. **In this paper, we always assume $Q \geq 3$.**

A measurable function f on $\mathbb{G}$ will be called λ-*homogeneous*, or homogeneous of degree λ, if $f \circ \delta_t = t^\lambda f$ for all $t > 0$. A continuous function $\| \cdot \| : \mathbb{G} \to [0, \infty)$ is called a *homogeneous norm* if $\|\delta_t(p)\| = t\|p\|$ for all $t > 0$ and $p \in \mathbb{G}$ and $\|p\| > 0$ for all $p \neq 0$. A typical example of a homogeneous norm is the function

$$\|p\|_{cc} := d(p, 0).$$

All homogeneous norms in $\mathbb{G}$ are equivalent: recall that two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ are said to be equivalent if there exists a positive constant c such that

$$c^{-1}\|p\|_2 \leq \|p\|_1 \leq c\|p\|_2 \qquad \text{for all } p \in \mathbb{G}. \tag{2.1}$$

Proofs of these facts, as well as other properties of homogeneous norms, can be found in [4].

Since $\mathbb{G}$ is identified with the linear space $\mathfrak{g}$, we can fix a *graded basis* of $\mathfrak{g}$, hence we can identify elements of $\mathbb{G}$ with elements of $\mathbb{R}^N$, where $N = \sum_{i=1}^{s} \dim \mathfrak{v}_i$. A graded basis in $\mathfrak{g}$ respects the grading, that is there exists s ordered subsets of the basis that are in turn bases of the single layers $\mathfrak{v}_i$. One can check that translations with respect to graded coordinates preserve the Lebesgue measure in $\mathbb{R}^N$. As a consequence, the Haar measure on $\mathbb{G}$ can be obtained from the Lebesgue measure on $\mathbb{R}^N$. It also agrees (up to a constant) with the Q-dimensional Hausdorff measure in the metric space $(\mathbb{G}, d)$.

In this paper we will denote the Haar measure of a set $E \subset \mathbb{G}$ by $|E|$, and we will write integrals with respect to this measure as $\int_E f(x)\,dx$ or $\int_E f$. We refer the reader to [29], [4] or [5] for further information on Carnot groups and their metric geometry.

In particular a fixed basis $X_1, \dots, X_m$ of the first layer $\mathfrak{v}_1$ is fixed. This is the so-called horizontal frame, that linearly spans all of the horizontal directions. If f is a real function defined on an open set of $\mathbb{G}$ its $\mathbb{G}$-gradient is given by

$$\nabla_{\mathbb{G}} f = (X_1 f, \dots, X_m f).$$

The $\mathbb{G}$-divergence of a function $\phi = (\phi_1, \dots, \phi_m) : \mathbb{G} \to \mathbb{R}^m$ is defined as

$$\operatorname{div}_{\mathbb{G}} \phi = \sum_{i=1}^{m} X_i \phi_i.$$

Remark 2.3. For our purposes, a sub-Riemannian divergence theorem is necessary. We will deal with regular domains comprised of finite unions of smooth open and bounded sets. Let div denote the standard divergence in $\mathbb{R}^N$ and let X be a C^1 smooth vector field on $\mathbb{R}^N$ with

$$X = (a_1, \dots, a_N) \sim a_1 \partial_{x_1} + \cdots a_N \partial_{x_N}.$$

If Ω is a bounded set of finite perimeter and f is a C^1 smooth real valued function on an open neighborhood of $\overline{\Omega}$, then

$$\int_\Omega Xf = \int_{\mathcal{F}^*\Omega} f \, \langle X, \nu \rangle \, d\|\partial\Omega\| - \int_\Omega f \operatorname{div} X, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product, ν is the generalized outer normal to Ω, $\mathcal{F}^*\Omega$ is the reduced boundary and $\|\partial\Omega\|$ is the perimeter measure of Ω, [12]. The validity of (2.2) is seen from the following equalities:

$$\begin{aligned}
\int_\Omega Xf &= \int_\Omega \sum_{l=1}^{N} a_l \, \partial_{x_l} f = \int_\Omega \sum_{l=1}^{N} \Big(\partial_{x_l}(a_l f) - f \partial_{x_l} a_l \Big) \\
&= \int_\Omega \operatorname{div}(fX) - \int_\Omega f \operatorname{div} X \\
&= \int_{\mathcal{F}^*\Omega} f \, \langle X, \nu \rangle \, d\|\partial\Omega\| - \int_\Omega f \operatorname{div} X.
\end{aligned}$$

All the left invariant vector fields X of a Carnot group satisfy $\operatorname{div} X = 0$. As a corollary of (2.2), it follows therefore that

$$\int_\Omega \operatorname{div}_{\mathbb{G}} F = \int_{\mathcal{F}^*\Omega} \sum_{j=1}^{m} f_j \, \langle X_j, \nu \rangle \, d\|\partial\Omega\| = \int_{\mathcal{F}^*\Omega} \langle F, \nu_{\mathbb{G}} \rangle \, d\|\partial\Omega\| \,, \tag{2.3}$$

where $\nu_{\mathbb{G}} = \big(\langle X_1, \nu \rangle, \dots, \langle X_m, \nu \rangle \big)$ is the *non-normalized horizontal normal.*

The sub-Laplacian in $\mathbb{G}$ is given by

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2$$

or equivalently

$$\mathcal{L} = \operatorname{div}_{\mathbb{G}} \nabla_{\mathbb{G}}.$$

Definition 2.4. Let $D \subset \mathbb{G}$ be an open set. A real valued function $f \in C^2(D)$ is called $\mathcal{L}$-harmonic, or simply harmonic, on D if $\mathcal{L}f = 0$ on D.

We shall consider removable sets for Lipschitz solutions of the sub-Laplacian:

Definition 2.5. A compact set $C \subset \mathbb{G}$ will be called removable, or $\mathcal{L}$-removable for Lipschitz $\mathcal{L}$-harmonic functions, if for every domain D with $C \subset D$ and every Lipschitz function $f : D \to \mathbb{R}$,

$$\mathcal{L}f = 0 \text{ in } D \setminus C \text{ implies } \mathcal{L}f = 0 \text{ in } D.$$

As usual we denote for any $D \subset \mathbb{G}$ and any function $f : D \to \mathbb{R}$,

$$\operatorname{Lip}(f) := \sup_{x,y \in D} \frac{|f(x) - f(y)|}{d(x,y)},$$

and we will also use the following notation for the upper bound for the Lipschitz constants in Carnot-Carathéodory balls:

$$\operatorname{Lip}_{\mathrm{B}}(f) := \sup\{\operatorname{Lip}(f|_{U_c(p,r)}) : p \in D, r > 0, U_c(p,r) \subset D\}.$$

The following proposition is known. It follows, for example, from the Poincaré inequality, see Theorem 5.16 in [5] and the arguments for its proof on pages 106-107. A simple direct proof which applies directly in our setting can be found in [6].

Proposition 2.6. Let $D \subset \mathbb{G}$ be a domain and let $f \in C^1(D)$. Then $\operatorname{Lip}_{\mathrm{B}}(f) < \infty$ if and only if $\|\nabla_{\mathbb{G}} f\|_\infty < \infty$. More precisely, there is a constant $c(\mathbb{G})$ depending only on $\mathbb{G}$ such that

$$\|\nabla_{\mathbb{G}} f\|_\infty \leq \operatorname{Lip}_{\mathrm{B}}(f) \leq c(\mathbb{G}) \|\nabla_{\mathbb{G}} f\|_\infty. \tag{2.4}$$

Fundamental solutions for sub-Laplacians in homogeneous Carnot groups are defined in accordance with the classical Euclidean setting.

Definition 2.7 (Fundamental solutions)**.** A function $\Gamma : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a fundamental solution for $\mathcal{L}$ if:

(i) $\Gamma \in C^\infty(\mathbb{R}^N \setminus \{0\})$,
(ii) $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $\lim_{\|p\|_{cc} \to \infty} \Gamma(p) \to 0$,
(iii) for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \Gamma(p) \mathcal{L}\varphi(p)\, dp = -\varphi(0).$$

It also follows easily, see Theorem 5.3.3 and Proposition 5.3.11 of [4], that for every $p \in \mathbb{G}$,

$$\Gamma * \mathcal{L}\varphi(p) = -\varphi(p) \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{2.5}$$

Convolutions are defined as usual by

$$f * g(p) = \int f(q^{-1} \cdot p) g(q)\, dq$$

for $f, g \in L^1$ and $p \in \mathbb{G}$.

A very general result due to Folland [13] guarantees the existence of a fundamental solution for each sub-Laplacian on a homogeneous Carnot group with homogeneous dimension $Q \geq 3$. The following proposition gathers some well-known properties of such fundamental solutions. Proofs can be found in [4].

Proposition 2.8 (Properties of Γ)**.** Let Γ be the fundamental solution of $\mathcal{L}$. Then for all $p \in \mathbb{G} \setminus \{0\}$ and all $t > 0$:

(i) (*Symmetry*) $\Gamma(p^{-1}) = \Gamma(p)$,
(ii) (*δ_t-homogeneity*) $\Gamma(\delta_t(p)) = t^{2-Q}\Gamma(p)$,
(iii) (*Positivity*) $\Gamma(p) > 0$.

The function

$$\|p\|_\Gamma = \begin{cases} \Gamma(p)^{\frac{1}{2-Q}} & \text{if } p \in \mathbb{G} \setminus \{0\} \\ 0 & \text{if } p = 0. \end{cases}$$

is a symmetric homogeneous norm which is C^∞ away from the origin. Let

$$d_\Gamma(p, q) = \|p^{-1} \cdot q\|_\Gamma$$

be the quasi-distance defined by $\| \cdot \|_\Gamma$. We will denote the corresponding open and closed balls by $U_\Gamma(p, r)$ and $B_\Gamma(p, r)$ respectively. Note also that by (2.1) d and d_Γ are globally equivalent.

Let $k = \nabla_{\mathbb{G}}\Gamma$, then $k = (k_1, \dots, k_m) : \mathbb{G} \setminus \{0\} \to \mathbb{R}^m$, and

$$k(p) = \nabla_{\mathbb{G}}\Gamma(p) = \nabla_{\mathbb{G}}(\|p\|_\Gamma^{2-Q}) = (2 - Q)\frac{\nabla_{\mathbb{G}}\|p\|_\Gamma}{\|p\|_\Gamma^{Q-1}} := \frac{\Omega(p)}{\|p\|_\Gamma^{Q-1}}$$

for $p \in \mathbb{G} \setminus \{0\}$. Furthermore Ω is smooth in $\mathbb{G} \setminus \{0\}$ and δ_t-homogeneous of degree zero, which in particular implies that k is $(1 - Q)$-homogeneous and

$$|k(p)| \lesssim \|p\|_\Gamma^{1-Q} \tag{2.6}$$

for $p \in \mathbb{G} \setminus \{0\}$. Notice also that

$$k_i(p) = \frac{\Omega_i(p)}{\|p\|_\Gamma^{Q-1}}, \qquad p \in \mathbb{G} \setminus \{0\}, \tag{2.7}$$

where $\Omega = (\Omega_1, \dots, \Omega_m)$ and every function Ω_i is smooth and homogeneous of degree zero.

We denote by $\mathcal{H}^s, s \geq 0$, the s-dimensional Hausdorff measure obtained from the Carnot-Caratheodory metric d, i.e. for $E \subset \mathbb{G}$ and $\delta > 0$, $\mathcal{H}^s(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E)$, where

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_i \operatorname{diam}(E_i)^s : E \subset \bigcup_i E_i, \operatorname{diam}(E_i) < \delta\right\}.$$

In the same manner the s-dimensional spherical Hausdorff measure for $E \subset \mathbb{G}$ is defined as $\mathcal{S}^s(E) = \sup_{\delta>0} \mathcal{S}^s_\delta(E)$, where

$$\mathcal{S}^s_\delta(E) = \inf\left\{\sum_i r_i^s : E \subset \bigcup_i B(p_i, r_i), r_i \leq \delta, p_i \in \mathbb{G}\right\}.$$

We will denote by $\mathcal{H}^s_\Gamma$ and $\mathcal{S}^s_\Gamma$ the Hausdorff and spherical Hausdorff measures with respect to d_Γ. Since homogeneous norms are equivalent it follows that the measures $\mathcal{H}^s, \mathcal{S}^s$, $\mathcal{H}^s_\Gamma$ and $\mathcal{S}^s_\Gamma$ are all mutually absolutely continuous with bounded Radon-Nikodym derivatives.

3. The critical dimension for $\mathcal{L}$-removable sets

We first prove a representation theorem for Lipschitz harmonic functions outside compact sets of finite $\mathcal{H}^{Q-1}$ measure.

Theorem 3.1. *Let C be a compact subset of $\mathbb{G}$ with $\mathcal{H}^{Q-1}(C) < \infty$ and let $D \supset C$ be a domain in $\mathbb{G}$. Suppose $f : D \to \mathbb{R}$ is a Lipschitz function such that $\mathcal{L}f = 0$ in $D \setminus C$. Then there exist a bounded domain G, $C \subset G \subset D$, a Borel function $h : C \to \mathbb{R}$ and an $\mathcal{L}$-harmonic function $H : G \to \mathbb{R}$ such that*

$$f(p) = \int_C \Gamma(q^{-1} \cdot p) h(q)\, d\mathcal{H}^{Q-1}(q) + H(p) \text{ for } p \in G \setminus C$$

and $\|h\|_{L^\infty(\mathcal{H}^{Q-1} \lfloor C)} + \|\nabla_{\mathbb{G}} H\|_\infty \lesssim 1$.

Proof. Let D_1 be a domain such that $C \subset D_1 \subset D$, $\bar{D_1}$ is compact and $\operatorname{dist}(\overline{D_1}, \mathbb{G} \setminus D) > 0$. For every $m = 1, 2, \ldots$ there exists a finite number of balls $U_{m,j} := U_\Gamma(p_{m,j}, r_{m,j})$, $j = 1, \ldots, j_m$, such that $U_{m,j} \cap C \neq \emptyset$,

$$C \subset \bigcup_{j=1}^{j_m} U_{m,j} \subset D_1, \quad r_{m,j} \leq \frac{1}{m}, \tag{3.1}$$

and

$$\sum_{j=1}^{j_m} r_{m,j}^{Q-1} \leq \mathcal{S}_\Gamma^{Q-1}(C) + \frac{1}{m}. \tag{3.2}$$

Temporarily fix $m \in \mathbb{N}$, and for simplicity let $p_j := p_{m,j}$ and $r_j := r_{m,j}$. The boundary of the union of the balls, $\bigcup_j U_\Gamma(p_j, r_j)$, is contained in the union of the boundaries, and hence has (Euclidean) dimension at most $N - 1$. We want to show that the overlap set

$$\bigcup_{j \neq i} \partial U_\Gamma(p_i, r_i) \cap \partial U_\Gamma(p_j, r_j)$$

is a null set for the Euclidean Hausdorff $(N-1)$-measure, in order to ensure that it is negligible for the classical divergence theorem. This follows from Sard's theorem, provided we adjust the radii slightly.

Since C is compact and the balls $U_\Gamma(p_j, r_j)$ are open, we have room to decrease the radii slightly while still covering C.

Lemma 3.2. *Assume the centers p_j are distinct, and fix intervals $J_j = [r_j - \epsilon, r_j]$ for some $\epsilon > 0$. Then there exist values $r'_j \in J_j$ so that*

$$\dim_E \left(\bigcup_{i \neq j} \partial U_\Gamma(p_i, r'_i) \cap \partial U_\Gamma(p_j, r'_j) \right) \leq N - 2.$$

Consequently, $\cup_{j \neq i} \partial U_\Gamma(p_i, r'_i) \cap \partial U_\Gamma(p_i, r'_i)$ is a null set for the measure $\mathcal{H}_E^{N-1}$.

Here $\dim_E$ refers to the dimension in the underlying Euclidean metric of $\mathbb{R}^N$.

Proof. It suffices to assume $j_m = 2$. We wish to show that

$$\dim_E(\partial U_\Gamma(p_1, r'_1) \cap \partial U_\Gamma(p_2, r'_2)) \leq N - 2$$

for some $r'_1 \in J_1$, $r'_2 \in J_2$. Consider the map $F : \mathbb{G} \to \mathbb{R}^2$ given by

$$F(p) = (d_\Gamma(p, p_1), d_\Gamma(p, p_2)).$$

Then F is C^∞, and $F^{-1}(r_1', r_2') = \partial U_\Gamma(p_1, r_1') \cap \partial U_\Gamma(p_2, r_2')$ for $r_1', r_2' > 0$. According to Sard's theorem [32], the set of critical values of F has measure zero in $\mathbb{R}^2$. Since $J_1 \times J_2$ has positive measure, there exist $r_1' \in J_1$, $r_2' \in J_2$ so that (r_1', r_2') is a regular value of F, i.e., $\operatorname{rank} DF(p) = 2$ for all $p \in F^{-1}(r_1', r_2')$. Moreover, the set $F^{-1}(r_1', r_2')$ is a smooth submanifold whose (Euclidean) dimension is at most $\dim_E \mathbb{G} - \dim_E \mathbb{R}^2 = N - 2$. □

The balls $U_\Gamma(p_j, r_j')$ continue to cover C and satisfy (3.1) and (3.2). In view of the above, we can assume without loss of generality that the conclusion of the lemma holds for the original balls $U_\Gamma(p_j, r_j)$ (i.e., we relabel r_j' as r_j).

The Dimension Comparison Theorem in Carnot groups (see Theorem 2.4 and Proposition 3.1 in [3]), in codimension one, implies that the spherical Hausdorff measure $\mathcal{S}_\Gamma^{Q-1}$ constructed from the metric d_Γ for a fixed homogeneous distance Γ is bounded above (up to a constant) by the Euclidean measure $\mathcal{H}_E^{N-1}$. It follows from this and Lemma 3.2 that the overlap set is also a null set for the spherical Hausdorff measure $\mathcal{S}_\Gamma^{Q-1}$.

Let $G_m = \cup_{j=1}^{j_m} U_{m,j}$ and

$$0 < \varepsilon_m < \min\{1, \operatorname{dist}(C, \mathbb{G} \setminus G_m), \operatorname{dist}(G_m, \mathbb{G} \setminus D_1)\}.$$

By the Whitney-McShane Extension Lemma there exists a Lipschitz function $F : \mathbb{G} \to \mathbb{R}$ such that $F|_D = f$ and F is bounded.

If $d_0 = 1 + \max_{z \in \overline{D_1}} d(z, 0)$, then the condition $d(y, 0) + d(z, 0) \le d_0$ gives

$$d(y^{-1} \cdot z, z) \le c(d_0) d(y, 0)^{1/s}, \tag{3.3}$$

due to [21, 3.18]. Let $\Phi \in C_0^\infty(\mathbb{R}^N)$, $\Phi \ge 0$, such that $\operatorname{spt} \Phi \subset U(0, 1)$ and $\int \Phi = 1$. For any $\delta > 0$ let $\Phi_\delta(x) = \delta^{-Q} \Phi(\delta_{1/\delta}(x))$. We consider the sequence of mollifiers

$$f_m(x) := F * \Phi_{\delta_m}(x) = \int F(y) \Phi_{\delta_m}(x \cdot y^{-1})\, dy = \int_{U(0,\delta_m)} F(y^{-1} \cdot x) \Phi_{\delta_m}(y)\, dy \tag{3.4}$$

for $x \in \mathbb{G}$ and $\delta_m = (\frac{\varepsilon_m}{2c(d_0)})^s$. Since F is bounded and uniformly continuous,

$$\|f_m - F\|_\infty \to 0$$

on compact sets of $\mathbb{G}$. Furthermore for all $m \in \mathbb{N}$, we have that

(i) $f_m \in C^\infty$,
(ii) $\|\nabla_{\mathbb{G}} f_m\|_\infty \le \|\nabla_{\mathbb{G}} F\|_\infty < \infty$.

For $\delta > 0$ and $S \subset \mathbb{G}$ let

$$L(S, \delta) = \{p \in S : \operatorname{dist}(p, S^c) > \delta\}.$$

If $x \in L(D_1 \setminus C, \varepsilon_m)$, $y \in B(0, \delta_m)$ and $z \in C$, by (3.3) we obtain

$$d(y^{-1} \cdot x, z) \ge d(x, z) - d(y^{-1} \cdot x, x) > \varepsilon_m - c(d_0) d(y, 0)^{1/s} > 0.$$

In particular $y^{-1} \cdot x \notin C$ and in the same way $y^{-1} \cdot x \notin \mathbb{G} \setminus D_1$. Therefore every mollifier f_m is harmonic in D_{ε_m}. We continue by choosing another domain D_2 such that $G_m \subset D_2 \subset L(D_1, \varepsilon_m)$ for all $m = 1, 2, \ldots$, and an auxiliary function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi = \begin{cases} 1 & \text{in } D_2 \\ 0 & \text{in } \mathbb{G} \setminus \overline{D}_1. \end{cases}$$

For $m = 1, 2, \ldots$ set $g_m := \varphi f_m$ and notice that $g_m \in C_0^\infty(\mathbb{R}^N)$ and

$$\|\nabla_{\mathbb{G}} g_m\|_\infty \le A_1$$

where A_1 does not depend on m. It follows by (2.5) that for all $m \in \mathbb{N}$,

$$-g_m(p) = \Gamma * \mathcal{L}g_m(p) \text{ for all } p \in \mathbb{G}. \tag{3.5}$$

Notice that

(i) $g_m = 0$ in $\mathbb{G} \setminus \overline{D_1}$,
(ii) $g_m = f_m$ in $D_2 \setminus G_m$ and hence $\mathcal{L}g_m = \mathcal{L}f_m = 0$ in $D_2 \setminus G_m$.

Therefore for all $m \in \mathbb{N}$ and $p \in D_2 \setminus G_m$,

$$-f_m(p) = \int_{G_m} \Gamma(q^{-1} \cdot p)\mathcal{L}g_m(q)\, dq + \int_{\overline{D_1}\setminus D_2} \Gamma(q^{-1} \cdot p)\mathcal{L}g_m(q)\, dq \tag{3.6}$$

by (3.5). For $m \in \mathbb{N}$ set $H_m : D_2 \to \mathbb{R}$ to be

$$H_m(p) = -\int_{\overline{D_1}\setminus D_2} \Gamma(q^{-1} \cdot p)\mathcal{L}g_m(q)\, dq \tag{3.7}$$

and $I_m : D_2 \setminus G_m \to \mathbb{R},\ m = 1, 2, \ldots$ to be

$$I_m(p) = -\int_{G_m} \Gamma(q^{-1} \cdot p)\mathcal{L}g_m(q)\, dq. \tag{3.8}$$

Since the functions $\mathcal{L}g_m$ are uniformly bounded in $\overline{D_1} \setminus D_2$, for all $m \in \mathbb{N}$

(i) H_m is harmonic in D_2,
(ii) $\|\nabla_{\mathbb{G}} H_m\|_\infty \lesssim 1$, since $\nabla_{\mathbb{G}}\Gamma$ is locally integrable.

The functions H_m are C^∞ by Hörmander's theorem, see for example Theorem 1 in Preface of [4]. Thus we can apply Proposition 2.6 and conclude from (ii) that $\mathrm{Lip}_{\mathrm{B}}(H_m) \lesssim 1$.

The functions I_m can be expressed as

$$I_m(p) = -\int_{G_m} \mathrm{div}_{\mathbb{G},q}(\Gamma(q^{-1} \cdot p)\nabla_{\mathbb{G}} g_m(q))\, dq + \int_{G_m} \langle \nabla_{\mathbb{G}}\Gamma(p^{-1} \cdot q), \nabla_{\mathbb{G}} g_m(q)\rangle\, dq, \tag{3.9}$$

where $\mathrm{div}_{\mathbb{G},q}$ stands for the $\mathbb{G}$-divergence with respect to the variable q and we also used the left invariance of $\nabla_{\mathbb{G}}$ and the symmetry of Γ to get that

$$\nabla_{\mathbb{G},q}(\Gamma(q^{-1} \cdot p)) = \nabla_{\mathbb{G},q}(\Gamma(p^{-1} \cdot q)) = \nabla_{\mathbb{G}}\Gamma(p^{-1} \cdot q).$$

By (2.3) one has the identity

$$\int_\Omega \mathrm{div}_{\mathbb{G}} F = \int_{\mathcal{F}^*\Omega} \langle F, \nu_{\mathbb{G}}\rangle\, d\|\partial\Omega\|$$

for every C^1 horizontal vector field F and bounded C^1 smooth domain Ω, where $\nu_{\mathbb{G}}$ denotes the non-normalized horizontal normal introduced in Remark 2.3. For instance, we may take $\Omega = U_\Gamma(p_{j,m}, r_{j,m})$ as above for each m and j. In fact, in this case Lemma 3.2 implies that the overlap of the boundaries is a null set for the $\mathcal{H}_E^{N-1}$ measure, whence

$$\mathcal{F}^*\Omega = \partial\Omega \setminus N\,,$$

where $\mathcal{H}_E^{N-1}(N) = 0$ and the generalized outer normal ν coincides with the classical outer normal of Ω at smooth points of $\partial\Omega$. Since the restriction of the perimeter measure to the reduced boundary is the $(N-1)$-dimensional Hausdorff measure, it follows that

$$\int_\Omega \mathrm{div}_{\mathbb{G}} F = \int_{\partial\Omega} \langle F, \nu_{\mathbb{G}}\rangle\, d\mathcal{H}_E^{N-1}\,. \tag{3.10}$$

Next we want to show that the identity

$$|\nu_{\mathbb{G}}|\, \mathcal{H}_E^{N-1} \llcorner \partial\Omega = \alpha\, \mathcal{S}^{Q-1} \llcorner \partial\Omega \tag{3.11}$$

holds for such piecewise C^1 domains Ω, where α is a Borel function on $\partial\Omega$ and $\mathcal{S}_\Gamma^{Q-1}$ is the spherical Hausdorff measure with respect to a fixed homogeneous distance d_Γ. Since $\partial\Omega$ is not C^1, we cannot apply directly the area formula of [23] to represent the spherical Hausdorff measure, as that formula is restricted to C^1 domains and arbitrary auxiliary Riemannian metrics.

We proceed as follows. The overlap of the boundaries is $\mathcal{H}_E^{N-1}$ negligible, hence it also $\mathcal{S}_\Gamma^{Q-1}$ negligible due to Proposition 3.1 of [3]. The restriction of (3.11) to the smooth parts of $\partial\Omega$ where $\nu_{\mathbb{G}}$ vanishes easily follows from the $Q-1$-dimensional negligibility of characteristic points, see [22]. In fact, these points are characterized by the vanishing of the horizontal normal $\nu_{\mathbb{G}}$. Finally, we consider the Borel subset B_0 of $\partial\Omega$ that does not intersect both the overlap set and the characteristic set. From the measure theoretic area formula of [20], joined with the blow-up theorem at non-characteristic points [23], we obtain

$$|\nu_{\mathbb{G}}|\,\mathcal{H}_E^{N-1}\llcorner B_0 = \alpha\,\mathcal{S}_\Gamma^{Q-1}$$

for some Borel function α defined on B_0, that extends by zero at points of $\partial\Omega\setminus B_0$. Consequently, (3.10) implies the identity

$$\int_\Omega \operatorname{div}_{\mathbb{G}} F = \int_{\partial\Omega} \left\langle F, \frac{\nu_{\mathbb{G}}}{|\nu_{\mathbb{G}}|} \right\rangle \alpha\, d\mathcal{S}_\Gamma^{Q-1} \tag{3.12}$$

for every C^1 horizontal vector field F and bounded piecewise C^1 domain Ω for which the overlap set is a null set for the boundary measure. The integrand in (3.12) is undefined on the characteristic set, but this is irrelevant since it is a null set for the measure. The important fact is that, from Theorem 5.4 of [23], we obtain two geometric constants $c_1, c_2 > 0$, independent of Ω, such that $c_1 \le \alpha \le c_2$ at $\mathcal{S}_\Gamma^{Q-1}$ a.e. point of B_0. Switching from a general homogeneous distance to the Carnot-Carathéodory distance d and corresponding spherical Hausdorff measure $\mathcal{S}^{Q-1}$, we obtain

$$\begin{aligned}&\int_{G_m} \operatorname{div}_{\mathbb{G},q}\Big(\Gamma(q^{-1}\cdot p)\nabla_{\mathbb{G}} g_m(q)\Big)\, dq \\ &\qquad = \int_{\partial G_m} \Gamma(q^{-1}\cdot p)\Big\langle \nabla_{\mathbb{G}} g_m(q), \frac{\nu_m(q)}{|\nu_m(q)|}\Big\rangle\, b_m(q)\, d\mathcal{S}^{Q-1}(q),\end{aligned} \tag{3.13}$$

for some $b_m \in L^\infty(\mathcal{S}^{Q-1}\llcorner \partial G_m)$, where ν_m is the non-normalized horizontal normal of G_m and $c_1 \le b_m \le c_2$ at $\mathcal{S}^{Q-1}$-a.e. point of ∂G_m and for every m. (Note that the Radon–Nikodym derivative of $\mathcal{S}_\Gamma^{Q-1}$ with respect to $\mathcal{S}^{Q-1}$, which is bounded away from zero and infinity, is included in the weight function b_m.)

By (3.2), $|G_m| \to 0$, therefore for $p \in D_2 \setminus C$,

$$\lim_{m\to\infty} \left| \int_{G_m} \langle \nabla_{\mathbb{G}}\Gamma(p^{-1}\cdot q), \nabla_{\mathbb{G}} g_m(q)\rangle\, dq \right| \to 0, \tag{3.14}$$

since $|\nabla_{\mathbb{G}} g_m|$ is uniformly bounded in D_2 and $\nabla_{\mathbb{G}}\Gamma$ is locally integrable.

Notice that the signed measures,

$$\sigma_m = \Big\langle \nabla_{\mathbb{G}} g_m(\cdot), \frac{\nu_m(\cdot)}{|\nu_m(\cdot)|}\Big\rangle\, b_m\, \mathcal{S}^{Q-1}\lfloor \partial G_m, \tag{3.15}$$

have uniformly bounded total variations $\|\sigma_m\|$. This follows by (3.2), as

$$\begin{aligned}\|\sigma_m\| &\leq \|\nabla_{\mathbb{G}} g_m\|_\infty \|b_m\|_{L^\infty(\mathcal{S}^{Q-1})} \mathcal{S}^{Q-1}(\partial G_m)\\ &\lesssim \sum_j \mathcal{S}_\Gamma^{Q-1}(\partial U_{m,j}) \lesssim \sum_j r_{m,j}^{Q-1}\\ &\lesssim \mathcal{S}_\Gamma^{Q-1}(C) + \frac{1}{m}.\end{aligned} \tag{3.16}$$

Therefore, by a general compactness theorem, see e.g. [2], we may extract a weakly converging subsequence $(\sigma_{m_k})_{k\in\mathbb{N}}$ such that $\sigma_{m_k} \to \sigma$. Furthermore $\operatorname{spt}\sigma := \operatorname{spt}|\sigma| \subset C$ and by (3.16)

$$\|\sigma\| \leq \liminf_{k\to\infty} \|\sigma_{m_k}\| \lesssim \mathcal{S}^{Q-1}(C). \tag{3.17}$$

Finally combining (3.9)—(3.15) we get that for $p \in D_2 \setminus C$,

$$\lim_{k\to\infty} I_{m_k}(p) = \int_C \Gamma(q^{-1}\cdot p)\, d\sigma(q)$$

and by (3.6)—(3.8)

$$f(p) = \int_C \Gamma(q^{-1}\cdot p)\, d\sigma(q) + \lim_{k\to\infty} H_{m_k}(p).$$

Since the sequence of harmonic functions (H_{m_k}) is equicontinuous on compact subsets of D_2, the Arzelà-Ascoli theorem implies that there exists a subsequence $(H_{m_{k_l}})$ which converges uniformly on compact subsets of D_2. From the Mean Value Theorem for sub-Laplacians and its converse, see [4], Theorems 5.5.4 and 5.6.3, we deduce that $(H_{m_{k_l}})$ converges to a function H which is harmonic in D_2. Therefore for $p \in D_2 \setminus C$,

$$f(p) = \int_C \Gamma(q^{-1}\cdot p)\, d\sigma q + H(p).$$

Furthermore the function H is C^∞ in D_2 with $\mathrm{Lip}_{\mathrm{B}}(H) \lesssim 1$, therefore by Proposition 2.6

$$\|\nabla_{\mathbb{G}} H\|_\infty \lesssim 1.$$

In order to complete the proof it suffices to show that

$$\sigma \ll \mu \text{ and } h := \frac{d\sigma}{d\mu} \in L^\infty(\mu), \tag{3.18}$$

where $\mu = \mathcal{S}^{Q-1}\lfloor C$. The measure-theoretic proof of (3.18) can be found in [6]. □

Lemma 3.3. *For $p_1, p_2 \neq q \in \mathbb{G}$*

$$|\Gamma(q^{-1}\cdot p_1) - \Gamma(q^{-1}\cdot p_2)| \lesssim d(p_1,p_2)(d(q,p_1)^{1-Q} + d(q,p_2)^{1-Q}).$$

Proof. Let $p_1, p_2 \neq q \in \mathbb{G}$. Without loss of generality assume that $d(p_1,q) \leq d(p_2,q)$. We are going to consider two cases.

Case I. $d(p_1,p_2) \geq \frac{1}{2} d(p_1,q)$. In this case, since d_Γ is globally equivalent to d we have

$$\begin{aligned}|\Gamma(q^{-1}\cdot p_1) - \Gamma(q^{-1}\cdot p_2)| &\lesssim \frac{1}{d(p_1,q)^{Q-2}} + \frac{1}{d(p_2,q)^{Q-2}}\\ &\lesssim \frac{1}{d(p_1,q)^{Q-2}} \lesssim \frac{d(p_1,p_2)}{d(p_1,q)^{Q-1}}\end{aligned}$$

Case II. $d(p_1,p_2) < \frac{1}{2}d(p_1,q)$. In this case, by the definition of the Carnot-Carathéodory metric there exists a sub-unit curve $\gamma : [0, d(p_1,p_2)] \to \mathbb{G}$ such that $\gamma(0) = q^{-1}\cdot p_1$ and $\gamma(d(p_1,p_2)) = q^{-1}\cdot p_2$. Furthermore,

$$\gamma([0,d(p_1,p_2)]) \subset B(q^{-1}\cdot p_1, d(p_1,p_2)). \tag{3.19}$$

Hence for every $t \in [0, d(p_1,p_2)]$

$$\begin{aligned} \|\gamma(t)\| &\gtrsim d(0,\gamma(t)) \geq d(0,q^{-1}\cdot p_1) - d(\gamma(t), q^{-1}\cdot p_1) \\ &\geq d(q,p_1) - d(p_1,p_2) \geq \frac{1}{2}d(q,p_1) \end{aligned} \tag{3.20}$$

since $d(\gamma(t), q^{-1}\cdot p_1) \leq d(p_1,p_2)$ by (3.19). Therefore, with $T := d(p_1,p_2)$ we have

$$\begin{aligned} |\Gamma(q^{-1}\cdot p_1) - \Gamma(q^{-1}\cdot p_2)| &= |\Gamma(\gamma(0)) - \Gamma(\gamma(T))| = \left|\int_0^T \frac{d}{dt}(\Gamma(\gamma(t))\,dt\right| \\ &\leq \int_0^T \left(\sum_{j=1}^m (X_j\Gamma(\gamma(t)))^2\right)^{\frac{1}{2}} dt = \int_0^T |\nabla_{\mathbb{G}}\Gamma(\gamma(t))|\,dt \\ &\lesssim \int_0^T \frac{dt}{\|\gamma(t)\|^{Q-1}} \lesssim \frac{d(p_1,p_2)}{d(p_1,q)^{Q-1}} \end{aligned}$$

where we used (2.6) and (3.20) respectively. □

We are now able to prove Theorem 1.1 which as discussed earlier is also valid for Lipschitz harmonic functions in $\mathbb{R}^n$, with Q replaced by n.

Proof of Theorem 1.1. The first statement follows from Theorem 3.1. To see this let $D \supset C$ be a subdomain of $\mathbb{G}$. Applying Theorem 3.1 and recalling that C is a null set for the measure $\mathcal{H}^{Q-1}$, we deduce that if $f : D \to \mathbb{R}$ is Lipschitz in D and $\mathcal{L}$-harmonic in $D\setminus C$, then there exists an $\mathcal{L}$-harmonic function H in a domain G, $C \subset G \subset D$, such that

$$f(p) = H(p) \text{ for } p \in G \setminus C.$$

This implies that $f = H$ in G. Hence f is harmonic in G, and so also in D. Therefore C is removable.

In order to prove (ii) let $Q-1 < s < \dim C$. By Frostman's lemma in compact metric spaces, see [25], there exists a nonvanishing Borel measure μ with $\operatorname{spt}\mu \subset C$ such that

$$\mu(B(p,r)) \leq r^s \text{ for } p \in \mathbb{G}, r > 0.$$

We define $f : \mathbb{G} \to \mathbb{R}^+$ as

$$f(p) = \int \Gamma(q^{-1}\cdot p)d\mu(q).$$

It follows that f is a nonconstant function which is C^∞ in $\mathbb{G}\setminus C$ and

$$\mathcal{L}f = 0 \text{ on } \mathbb{G}\setminus C.$$

Furthermore f is Lipschitz. Indeed, for $p_1, p_2 \in \mathbb{G}$ we may use Lemma 3.3 to obtain

$$\begin{aligned} |f(p_1) - f(p_2)| &= \left|\int \Gamma(q^{-1}\cdot p_1)\,d\mu(q) - \int \Gamma(q^{-1}\cdot p_2)\,d\mu(q)\right| \\ &\lesssim d(p_1,p_2)\left(\int \frac{1}{d(p_1,q)^{Q-1}}\,d\mu(q) + \int \frac{1}{d(p_2,q)^{Q-1}}\,d\mu(q)\right) \\ &\lesssim d(p_1,p_2). \end{aligned}$$

To prove the last inequality let $p \in \mathbb{G}$, and consider two cases. If $\operatorname{dist}(p, C) > \operatorname{diam}(C)$,

$$\int \frac{1}{d(p,q)^{Q-1}}\, d\mu(q) \le \frac{\mu(C)}{\operatorname{diam}(C)^{Q-1}} \lesssim 1.$$

If $\operatorname{dist}(p, C) \le \operatorname{diam}(C)$, then $C \subset B(p, 2\operatorname{diam}(C))$. Let $A = 2\operatorname{diam}(C)$, then

$$\begin{aligned}\int \frac{1}{d(p,q)^{Q-1}}\, d\mu(q) &\le \sum_{j=0}^{\infty} \int_{B(p,2^{-j}A)\setminus B(p,2^{-(j+1)}A)} \frac{d\mu(q)}{d(p,q)^{Q-1}} \\ &\le \sum_{j=0}^{\infty} \frac{\mu(B(p,2^{-j}A))}{(2^{-(j+1)}A)^{Q-1}} \\ &\le 2^{Q-1}A^{s-(Q-1)} \sum_{j=0}^{\infty} (2^{s-(Q-1)})^{-j} \\ &\lesssim 1.\end{aligned}$$

Assume, by way of contradiction, that f is $\mathcal{L}$-harmonic on $\mathbb{G}$. Since $f \ge 0$, by a Liouville-type theorem for sub-Laplacians, see e.g. Theorem 5.8.1 of [4], we deduce that f is constant. Hence we have reached a contradiction and consequently C is not removable. □

4. Removable sets with positive and finite $\mathcal{H}^{Q-1}$ measure

In this section we shall construct a self-similar Cantor set K in $\mathbb{G}$ which is $\mathcal{L}$-removable despite having positive $\mathcal{H}^{Q-1}$ measure. As noted earlier our proof is rather different than the one in [6]. Nevertheless, note that in Theorem 4.1 there is also one piece $S_0(K)$ of K which is well separated from the others. This fact allows for a straightforward application of the condition in Theorem 4.11.

We let $\mathbb{S} := \{p \in \mathbb{G} : ||p||_{cc} = 1\}$ be the unit sphere centered at the origin in this norm. The norm $||\cdot||_{cc}$ is comparable to any other homogeneous norm on $\mathbb{G}$, in particular, to the homogeneous norm

$$|||p||| := |p_1| + |p_2|^{1/2} + \cdots + |p_s|^{1/s}, \qquad p = (p_1, p_2, \ldots, p_s).$$

Definition 4.1. For a set $A \subset \mathbb{S}$, we define the *dilation cone* over A to be the set

$$\widehat{A} := \{\delta_r(p) : r > 0, p \in A\}.$$

We will prove the following theorem.

Theorem 4.1. *Let $U \subset \mathbb{S}$ be a nonempty open set. There exists a self-similar iterated function system $\mathcal{F} = \{S_i : i = 0, 1, \ldots, M\}$ with invariant set K such that the following conditions are satisfied:*

(i) *the map S_0 has fixed point 0,*
(ii) *$K \subset \widehat{U}$,*
(iii) *the pieces $S_0(K), \ldots, S_M(K)$ are pairwise disjoint, and*
(iv) *$0 < \mathcal{H}^{Q-1}(K) < \infty$.*

Fix a horizontal vector $\vec{v} \in \mathfrak{v}_1$ and denote by $\mathbb{V} := \{\exp(t\vec{v}) : t \in \mathbb{R}\}$ the corresponding horizontal one-parameter subgroup of $\mathbb{G}$. Denote by $\mathbb{W} := \exp(\vec{v}^{\perp} \times \mathfrak{v}_2 \times \cdots \times \mathfrak{v}_s)$ the corresponding complementary vertical subgroup, and by $\mathbb{W}_a$, $a \in \mathbb{V}$, the coset $a * \mathbb{W}$ of $\mathbb{W}$. We may choose $\vec{v}$ and a so that $U \cap \mathbb{W}_a \neq \emptyset$. In what follows we will assume that $\vec{v}$ and a have been so chosen.

Lemma 4.2. *There exists a self-similar iterated function system $\mathcal{F}' = \{S_i : i = 1, \dots, M\}$ with invariant set K' such that the following conditions are satisfied:*

(i) *the fixed points of each of the maps S_i, $1 \le i \le M$, lie in $\widehat{U} \cap \mathbb{W}_a$,*
(ii) *$K' \subset \widehat{U} \cap \mathbb{W}_a$,*
(iii) *the pieces $S_1(K'), \dots, S_M(K')$ are pairwise disjoint, and*
(iv) *$0 < \mathcal{H}^t(K') < \infty$, where t is the Hausdorff dimension of K'.*

Remark 4.3. In both Theorem 4.1 and Lemma 4.2, condition (iv) follows from condition (iii), by results of Schief, see [33, Theorem 2.5].

In the proofs we will use the following elementary algebraic fact.

Lemma 4.4. *There exists a constant $C_0 \ge 1$ so that*

$$d(\delta_r(q), q) \le C_0 ||q||_{cc} \tag{4.1}$$

for all $q \in \mathbb{G}$ and $0 \le r \le 1$.

Proof. By the 1-homogeneity of both sides of the desired inequality (4.1), it suffices to establish the result for points q with $||q||_{cc} = 1$. Since the function $(q, r) \mapsto d(\delta_r(q), q)$ is continuous from $\mathbb{G} \times [0, 1] \to \mathbb{R}$, the conclusion follows from compactness of the CC unit sphere. □

Proof of Lemma 4.2. We first observe that the coset $\mathbb{W}_a$, equipped with the restriction of the Carnot-Carathéodory metric, is AD $(Q-1)$-regular. This can be proved in several ways. For instance, we may observe that each such coset $\mathbb{W}_a$ is isometric to the vertical subgroup $\mathbb{W}$, and that the Haar measure on $\mathbb{W}$ is AD $(Q-1)$-regular.

Let B be a Carnot-Carathéodory ball centered at a point of $U \cap \mathbb{W}_a$ such that $(1+2C_0)B \subset \widehat{U}$ and $\operatorname{diam} B \le 2$, where C_0 is as above. For $\epsilon > 0$, let $p_1, \dots, p_M \in B \cap \mathbb{W}_a$ be a maximal collection of points with mutual distance at least $\epsilon \operatorname{diam} B$. By the Ahlfors regularity of $\mathbb{W}_a$,

$$\frac{1}{C_1}\epsilon^{1-Q} \le M \le C_1 \epsilon^{1-Q}, \tag{4.2}$$

where $C_1 \ge 1$ is independent of ϵ. The choice of ϵ will be made later in the proof, but we note here that we may choose ϵ small enough that $M \ge 2^{Q-1}$.

Let $r > 0$ be such that

$$r = \frac{\epsilon \operatorname{diam} B}{2C_1^{1/(Q-1)}(10 + 50C_0(\operatorname{diam} B))}. \tag{4.3}$$

Note that $r < M^{1/(1-Q)}$ by (4.2). In particular, $r < \frac{1}{2}$.

We consider the self-similar iterated function system $\mathcal{F}' = \{S_i : i = 1, \dots, M\}$, where S_i is the contraction mapping of $\mathbb{G}$ with fixed point p_i and contraction ratio r. Explicitly, $S_i : \mathbb{G} \to \mathbb{G}$ is given by

$$S_i(p) = p_i * \delta_r(p_i^{-1} * p), \quad i = 1, \dots, M.$$

Let K' be the invariant set for $\mathcal{F}'$. Condition (i) is true by construction. The inclusion $K' \subset \mathbb{W}_a$ is true since K' is the closure of the full orbit of the set of fixed points and the coset $\mathbb{W}_a$ is invariant under each of the maps $S_1, \dots, S_M$.

To proceed further we introduce the terminology and notation of symbolic dynamics. Let $W = \{1, \dots, M\}$ be the symbol space, let W_m be the m-fold product of W with itself (with W_0 containing only the empty set), and let $W_* = \cup_{m \ge 0} W_m$. Elements of W_m are called *words of length m* in letters drawn from W. For $w \in W_*$, $w = w_1 w_2 \cdots w_m$, set $S_w = S_{w_1} \circ S_{w_2} \circ \cdots \circ S_{w_m}$.

We will make use of the fact that K' is the closure of the set

$$\bigcup_{w\in W_*} S_w(p_1);$$

similarly, for each i, $S_i(K')$ is the closure of the set

$$\bigcup_{w\in W_*} S_{iw}(p_1).$$

Let $w = w_1 w_2 \cdots w_m \in W_*$. Repeated application of the triangle inequality, together with the fact that S_i is a similarity with contraction ratio r, shows that $d(S_w(p_1), p_1)$ is less than or equal to

$$d(p_1, S_{w_1}(p_1)) + r\, d(p_1, S_{w_2}(p_1)) + r^2\, d(p_1, S_{w_3}(p_1)) + \cdots + r^{m-1}\, d(p_1, S_{w_m}(p_1)).$$

For any $i = 1, \ldots, M$,

$$d(p_1, S_i(p_1)) = d(p_1, p_i * \delta_r(p_i^{-1} * p_1)) = d(p_i^{-1} * p_1, \delta_r(p_i^{-1} * p_1)).$$

Applying Lemma 4.4 yields

$$d(p_1, S_i(p_1)) \le C_0 d(p_1, p_i) \le C_0 \operatorname{diam} B.$$

Consequently, since $r < \frac{1}{2}$,

$$d(S_w(p_1), p_1) \le C_0 \frac{1}{1-r} \operatorname{diam} B \le 2C_0 \operatorname{diam} B$$

and so

$$K' \subset B(p_1, 2C_0 \operatorname{diam} B) \subset (1 + 2C_0)B \subset \widehat{U}.$$

This completes the proof of condition (ii). We note in passing that

$$\operatorname{diam} K' \le 4C_0 \operatorname{diam} B. \tag{4.4}$$

In view of Remark 4.3, it remains only to check condition (iii). Note that the dimension of K', $\log M / \log(1/r)$, is strictly less than $Q - 1$ by the choice of r.

To verify (iii) we show that

$$S_i(K') \subset B(p_i, \tfrac{1}{5}\epsilon) \tag{4.5}$$

for each $i = 1, \ldots, M$. (Recall that $d(p_i, p_{i'}) \ge \epsilon$ for all $i \ne i'$.) Following a similar argument as above and using (4.4), we conclude that

$$d(S_{iw}(p_1), p_i) = d(S_{iw}(p_1), S_i(p_i)) = r d(S_w(p_1), p_i) \le (\operatorname{diam} K')r \le 4C_0(\operatorname{diam} B)r.$$

By the choice of r,

$$4C_0(\operatorname{diam} B)r < \frac{1}{5}\epsilon.$$

The proof of (4.5) is complete. □

Remark 4.5. We record the following consequence of (4.5) and the definition of ϵ:

$$\operatorname{dist}(S_i(K'), S_{i'}(K')) \ge \frac{3}{5}\epsilon \qquad \text{for all } 1 \le i, i' \le M,\ i \ne i'.$$

Proof of Theorem 4.1. Let $S_1, \ldots, S_M$ be selected as in the proof of Lemma 4.2, define $r_0 > 0$ by the equation

$$r_0^{Q-1} + Mr^{Q-1} = 1,$$

and let S_0 be the contraction mapping $S_0(p) = \delta_{r_0}(p)$. Then condition (i) is satisfied. In view of Remark 4.3, it suffices to verify conditions (ii) and (iii).

We will employ symbolic dynamics as introduced in the preceding proof to both iterated function systems $\mathcal{F}'$ and $\mathcal{F}$. In order to distinguish between these two systems, we continue to denote by $W = \{1, \ldots, M\}$ the word space for the IFS $\mathcal{F}'$. We let $V = \{0, \ldots, M\}$ be the symbol space for the IFS $\mathcal{F}$, we let V_m be the m-fold product of V with itself, and we let $V_* = \cup_{m\geq 0} V_m$. For $v \in V_*$, $v = v_1 v_2 \cdots v_m$, we set $S_v = S_{v_1} \circ S_{v_2} \circ \cdots \circ S_{v_m}$. We make use of the fact that K is the closure of the set

$$\bigcup_{v\in V_*} S_v(K');$$

similarly, for each $i = 1, \ldots, M$, $S_i(K)$ is the closure of the set

$$\bigcup_{v\in V_*} S_{iv}(K').$$

Each element $v \in V_*$ of length m can be uniquely written in the form

$$v = u_{k_0} w_{\ell_0} u_{k_1} w_{\ell_1} \cdots u_{k_{T-1}} w_{\ell_{T-1}} u_{k_T}$$

where u_k is a word consisting of k copies of the letter 0, $w_\ell \in W_\ell$, $k_0, \ldots, k_T \geq 0$, $\ell_0, \ell_1, \ldots, \ell_{T-1} \geq 1$, and

$$k_0 + \ell_0 + k_1 + \ell_1 + \cdots + k_{T-1} + \ell_{T-1} + k_T = m.$$

Words in V_* with initial letter i, in the above representation, are precisely words for which $k_0 = 0$ and w_{ℓ_0} begins with the letter i. We analyze the image of K' under such words.

For $\delta > 0$ and $S \subset \mathbb{G}$, we denote by $N(S, \delta) = \{p \in \mathbb{G} : \operatorname{dist}(p, S) < \delta\}$ the δ-neighborhood of S.

Lemma 4.6. *There exists a constant $C > 0$ so that if $w \in W_\ell$ and $k, \ell \in \mathbb{N}$, then*

$$(S_w \circ S_0^k)(K') \subset N(S_w(K'), r^\ell(1 + 5C_0(\operatorname{diam} B))).$$

Proof. Recalling (4.4), we note that it suffices to prove that

$$d(S_w(S_0^k(p)), S_w(p)) \leq r^\ell(1 + 5C_0(\operatorname{diam} B)).$$

for all $p \in K'$. Since S_w has contraction ratio r^ℓ, this is equivalent to proving that

$$d(S_0^k(p), p) \leq 1 + 5C_0(\operatorname{diam} B)$$

By Lemma 4.4,

$$d(S_0^k(p), p) \leq C_0 ||p||_{cc}.$$

Using the fact that $B \cap K' \neq \emptyset$ and (4.4), we obtain

$$||p||_{cc} \leq 1 + \operatorname{diam} B + \operatorname{diam} K' \leq 1 + (1 + 4C_0)(\operatorname{diam} B) \leq 1 + 5C_0(\operatorname{diam} B),$$

completing the proof. □

In a geodesic metric space (e.g., $\mathbb{G}$ equipped with the Carnot–Carathéodory metric), we have

$$N(N(S,\delta),\epsilon) = N(S, \delta+\epsilon) \qquad \text{for any set } S \text{ and any } \delta, \epsilon > 0.$$

This fact and an easy inductive argument leads to the following result.

Lemma 4.7. *If* $w_{\ell_0} \in W_{\ell_0}$, $w_{\ell_1} \in W_{\ell_1}$, $\dots$, $w_{\ell_{T-1}} \in W_{\ell_{T-1}}$ *and* $k_1, \dots, k_T \in \mathbb{N}$, *then*

$$(S_{w_{\ell_0}} \circ S_0^{k_1} \circ S_{w_{\ell_1}} \circ S_0^{k_2} \circ \cdots \circ S_{w_{\ell_{T-1}}} \circ S_0^{k_T})(K') \subset N(S_{w_{\ell_0}}(K'), \rho)$$

where

$$\rho = (r^{\ell_0} + r^{\ell_0+\ell_1} + \cdots + r^{\ell_0+\ell_1+\cdots+\ell_{T-1}})(1 + 5C_0(\operatorname{diam} B))).$$

We now conclude the proof of Theorem 4.1. Since $r < \frac{1}{2}$ and $\ell_0 \geq 1$, we deduce from Lemma 4.7 that

$$r^{\ell_0} + r^{\ell_0+\ell_1} + \cdots + r^{\ell_0+\ell_1+\cdots+\ell_{T-1}} \leq \frac{r^{\ell_0}}{1-r} \leq 2r$$

and hence that

$$(S_{w_{\ell_0}} \circ S_0^{k_1} \circ S_{w_{\ell_1}} \circ S_0^{k_2} \circ \cdots \circ S_{w_{\ell_{T-1}}} \circ S_0^{k_T})(K') \subset N(S_{w_{\ell_0}}(K'), 2(1 + 5C_0(\operatorname{diam} B))r).$$

In particular, if the first letter of w_{ℓ_0} is i, then

$$(S_{w_{\ell_0}} \circ S_0^{k_1} \circ S_{w_{\ell_1}} \circ S_0^{k_2} \circ \cdots \circ S_{w_{\ell_{T-1}}} \circ S_0^{k_T})(K') \subset N(S_i(K'), 2(1 + 5C_0(\operatorname{diam} B))r). \tag{4.6}$$

As discussed above, this means that all sets of the form $S_v(K')$, where $v \in V_*$ has initial letter i, are contained in the set on the right hand side of (4.6), so

$$S_i(K) \subset N(S_i(K'), 2(1 + 5C_0(\operatorname{diam} B))r).$$

By the choice of r, $2(1 + 5C_0(\operatorname{diam} B))r < \frac{1}{5}\epsilon$ and so

$$S_i(K) \subset N(S_i(K'), \frac{1}{5}\epsilon). \tag{4.7}$$

In view of Remark 4.5, the sets $S_1(K), \dots, S_M(K)$ are disjoint.

Next, we want to show that $S_0(K) \cap S_i(K) = \emptyset$ for $1 \leq i \leq M$. To this end, we consider projection $\pi_{\mathbb{V}}$ into the horizontal subgroup $\mathbb{V}$. The set $\mathbb{V}$ can be isometrically identified with $\mathbb{R}$; we denote by $P_{\mathbb{V}} : \mathbb{G} \to \mathbb{R}$ the composition of $\pi_{\mathbb{V}}$ with this identification. There exists a self-similar contraction $T_i : \mathbb{R} \to \mathbb{R}$ so that $T_i \circ P_{\mathbb{V}} = P_{\mathbb{V}} \circ S_i$. Explicitly,

$$T_0(t) = r_0 t \qquad \text{and} \qquad T_i(t) = a + r(t-a) \qquad \text{for } i = 1, \dots, M.$$

It suffices to prove that

$$P_{\mathbb{V}}(S_0(K)) \cap P_{\mathbb{V}}(S_i(K)) = \emptyset,$$

i.e.,

$$T_0(P_{\mathbb{V}}(K)) \cap T_i(P_{\mathbb{V}}(K)) = \emptyset.$$

Since $P_{\mathbb{V}}(K) \subset [0, a]$, the latter condition holds provided

$$r_0 + r < 1. \tag{4.8}$$

Recalling that r and r_0 are related by $r_0^{Q-1} + Mr^{Q-1} = 1$, we rewrite (4.8) as

$$r < 1 - (1 - Mr^{Q-1})^{1/(Q-1)}. \tag{4.9}$$

We observe that

$$\begin{aligned} Mr^{Q-1} &\geq \frac{M\epsilon^{Q-1}(\operatorname{diam} B)^{Q-1}}{2^{Q-1}C_1(10 + 50C_0 \operatorname{diam} B)^{Q-1}} \\ &\geq \frac{(\operatorname{diam} B)^{Q-1}}{2^{Q-1}C_1^2(10 + 50C_0 \operatorname{diam} B)^{Q-1}} \end{aligned} \tag{4.10}$$

In view of (4.3) and (4.10), we see that (4.9) is satisfied provided that

$$\frac{\epsilon \operatorname{diam} B}{C_1^{1/(Q-1)}(10+50C_0(\operatorname{diam} B))} < 1 - \left(1 - \frac{(\operatorname{diam} B)^{Q-1}}{2^{Q-1}C_1^2(10+50C_0 \operatorname{diam} B)^{Q-1}}\right)^{1/(Q-1)}.$$

The latter inequality is true provided ϵ is chosen sufficiently small. This completes the proof that $S_0(K)$ is disjoint from each of the sets $S_i(K)$, $i = 1, \dots, M$, and hence completes the proof of (iii).

It remains to verify (ii). We first record the identity

$$K = \bigcup_{k \geq 0} S_0^k \left(\bigcup_{i=1}^{M} S_i(K) \right).$$

In view of (4.7) and the choice of the data, $S_1(K) \cup \dots \cup S_M(K) \subset \widehat{U}$. Since $\widehat{U}$ is a dilation cone and S_0 is a dilation, it follows that $K \subset \widehat{U}$ as desired. The proof of the theorem is complete. □

Remark 4.8. It follows easily, see e.g. [33, Theorem 2.9] and [17, Theorem 5.3.1], that if K is the separated set of Theorem 4.1 the measure $\mathcal{H}^{Q-1}\lfloor K$ is $(Q-1)$-AD regular.

In the following we fix some notation.

Notation 4.9. For a signed Borel measure σ set

$$T_\sigma(p) := \int k(q^{-1} \cdot p)\, d\sigma(q), \text{ whenever it exists,}$$

$$T_\sigma^\varepsilon(p) := \int_{\mathbb{G} \setminus B(p,\varepsilon)} k(q^{-1} \cdot p)\, d\sigma(q)$$

and

$$T_\sigma^*(p) := \sup_{\varepsilon > 0} |T_\sigma^\varepsilon(p)|.$$

The proof of the following lemma is rather similar to that of Lemma 5.4 in [27].

Lemma 4.10. *Let σ be a signed Borel measure in $\mathbb{G}$ and A_σ a positive constant such that $|\sigma|(B(p,r)) \leq A_\sigma r^{Q-1}$ for $p \in \mathbb{G}, r > 0$. Then*

$$|T_\sigma^*(p)| \leq \|T_\sigma\|_\infty + A_T \text{ for } p \in \mathbb{G},$$

where A_T is a constant depending only on σ.

Proof. We can assume that $L = \|T_\sigma\|_\infty < \infty$. The constants that will appear in the following depend only on n and σ. For $\varepsilon > 0$ and $p \in \mathbb{G}$,

$$\begin{aligned}
\frac{1}{|(B(p,\varepsilon/4))|} \int_{B(p,\varepsilon/4)} \int_{B(p,\varepsilon)} \frac{1}{\|q^{-1} \cdot z\|^{Q-1}}\, d|\sigma|(q)\, dz \\
&\approx \varepsilon^{-Q} \int_{B(p,\varepsilon/4)} \int_{B(p,\varepsilon)} \frac{1}{\|q^{-1} \cdot z\|^{Q-1}}\, d|\sigma|(q)\, dz \\
&\leq \varepsilon^{-Q} \int_{B(p,\varepsilon)} \int_{B(q,2\varepsilon)} \frac{dz}{\|q^{-1} \cdot z\|^{Q-1}}\, d|\sigma|(q) \\
&\approx \varepsilon^{1-Q} |\sigma|(B(p,\varepsilon)) \leq A_\sigma
\end{aligned}$$

where we used Fubini and the fact that

$$\int_{B(q,2\varepsilon)} \frac{dz}{\|q^{-1} \cdot z\|^{Q-1}} \lesssim \int_{B(q,2\varepsilon)} \frac{dz}{d(q,z)^{Q-1}} \approx \varepsilon,$$

which is easily checked by summing over the annuli $B(q,2^{1-i}\varepsilon)\setminus B(q,2^{-i}\varepsilon), i=0,1,\ldots$.

Now because of the inequality established above we can choose $z\in B(p,\varepsilon/4)$ with $|T_\sigma(z)|\le L$ such that

$$\int_{B(p,\varepsilon)}|k(q^{-1}\cdot z)|\,d|\sigma|(q)\lesssim\int_{B(p,\varepsilon)}\frac{1}{\|q^{-1}\cdot z\|^{Q-1}}\,d|\sigma|(q)\le L_1.$$

Therefore,

$$\begin{aligned}|T_\sigma^\varepsilon(p)-T_\sigma(z)|&=\left|\int_{\mathbb{G}\setminus B(p,\varepsilon)}k(q^{-1}\cdot p)d|\sigma|(q)-\int k(q^{-1}\cdot z)d|\sigma|(q)\right|\\&\le\int_{\mathbb{G}\setminus B(p,\varepsilon)}|k(q^{-1}\cdot p)-k(q^{-1}\cdot z)|d|\sigma|(q)+\int_{B(p,\varepsilon)}|k(q^{-1}\cdot z)|d|\sigma|(q)\\&\le\int_{\mathbb{G}\setminus B(p,\varepsilon)}|k(q^{-1}\cdot p)-k(q^{-1}\cdot z)|d|\sigma|(q)+L_1.\end{aligned}$$

Since k is a C^∞, $(1-Q)$-homogeneous function on $\mathbb{G}\setminus\{0\}$ by [14, Proposition 1.7]

$$|k(X\cdot Y)-k(X)|\le C\|Y\|_{cc}\|X\|_{cc}^{-Q}\quad\text{for all }\|Y\|_{cc}\le\|X\|_{cc}/2. \tag{4.11}$$

Therefore if $z\in B(p,\varepsilon/4)$ and $q\in B(p,\varepsilon)^c$, letting $X=q^{-1}\cdot z, Y=z^{-1}\cdot p$ we have that

$$\|X\|_{cc}=d(q,z)\ge d(q,p)-d(p,z)>3\varepsilon/4\ge 3d(z,p)=3\|Y\|_{cc}$$

and

$$\int_{\mathbb{G}\setminus B(p,\varepsilon)}|k(q^{-1}\cdot p)-k(q^{-1}\cdot z)|\,d|\sigma|(q)\lesssim\int_{\mathbb{G}\setminus B(p,\varepsilon)}\frac{d(p,z)}{d(z,q)^Q}\,d|\sigma|(q).$$

Since

$$\begin{aligned}\int_{\mathbb{G}\setminus B(p,\varepsilon)}\frac{d(p,z)}{d(z,q)^Q}\,d|\sigma|(q)&\le\frac{\varepsilon}{2}\sum_{j=0}^{\infty}\int_{B(z,2^j\varepsilon)\setminus B(z,2^{j-1}\varepsilon)}\frac{1}{d(p,q)^Q}\,d|\sigma|(q)\\&\le\frac{\varepsilon}{2}\sum_{j=0}^{\infty}\frac{|\sigma|(B(p,2^j\varepsilon))}{(2^{j-1}\varepsilon)^Q}\\&\le A_\sigma\frac{\varepsilon}{2}\sum_{j=0}^{\infty}\frac{(2^j\varepsilon)^{Q-1}}{(2^{j-1}\varepsilon)^Q}\\&=L_2,\end{aligned}$$

we deduce that

$$\int_{\mathbb{G}\setminus B(p,\varepsilon)}|k(q^{-1}\cdot p)-k(q^{-1}\cdot z)|\,d|\sigma|(q)\le L_2.$$

Therefore

$$|T_\sigma^\varepsilon(p)|\le|T_\sigma^\varepsilon(p)-T_\sigma(z)|+|T_\sigma(z)|\le L_1+L_2+L.$$

The lemma is proven. □

Theorem 4.11. *Let K be the separated self similar set obtained in Theorem 4.1 and let k_i be any of the coordinate kernels of k. If there exists $x=S_w(x)\in K$, $w\in V_*$, such that*

$$\int_{K\setminus S_w(K)}k_i(x^{-1}\cdot y)\,d\mathcal{H}^{Q-1}(y)\ne 0,$$

then the maximal operator $T^*_{\mathcal{H}^{Q-1}\lfloor K}$ is unbounded in $L^2(\mathcal{H}^{Q-1}\lfloor K)$.

The previous theorem was proved in [6] in the abstract setting of complete metric groups with dilations. Here we have formulated a version tailored to our setting. We will also need the following Lemma which compares usual maximal singular integrals to maximal symbolic singular integrals on separated self-similar sets. The proof can be found in [6, Lemma 2.4].

Lemma 4.12. *Let K be the separated self similar set obtained in Theorem 4.1 and let k_i be any of the coordinate kernels of k. Then*

(i) *there exists a constant $\alpha_K > 0$, depending only on the set K, such that*

$$\operatorname{dist}(S_v(K), K \setminus S_v(K)) \geq \alpha_K \operatorname{diam}(S_v(K))$$

for every $v \in V_*$, and

(ii) *there is a constant A_C, depending only on the set K and the kernel k_i, such that*

$$\left| \int_{S_w(K)\setminus S_v(K)} k_i(p^{-1} \cdot y) \, d\mathcal{H}^{Q-1}(y) \right| \leq \left| \int_{B(p,2\operatorname{diam}(S_w(K)))\setminus B(p,2\operatorname{diam}(S_v(K)))} k_i(p^{-1} \cdot y) \, d\mathcal{H}^{Q-1}(y) \right| + A_K$$

for all $w, v \in V$ and $p \in \mathbb{G}$ for which $S_v(K) \subset S_w(K)$ and

$$\operatorname{dist}(p, S_v(K)) \leq \frac{\alpha_K}{2} \operatorname{diam}(S_v(K)).$$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. There exists some $i = 1, \dots, m$ such that k_i is not identically zero in $\mathbb{G} \setminus \{0\}$. Therefore, since Ω_i is continuous in $\mathbb{S}$, there exists some open set $U \subset \mathbb{S}$ such that, without loss of generality, $\Omega_i(p) > 0$ for all $p \in U$. In particular k_i is positive for all $p \in \widehat{U} \setminus \{0\}$.

Now let K be the separated self similar set that we obtain from Theorem 4.1 for $\widehat{U}$ as above. Notice that since $K \subset \widehat{U}$, and k_i is positive on $\widehat{U} \setminus \{0\}$

$$\int_{K\setminus S_0(K)} k_i(y) \, d\mathcal{H}^{Q-1}(y) > 0. \tag{4.12}$$

Since 0 is a fixed point of K, Theorem 4.11 implies that $T^*_{\mathcal{H}^{Q-1}\lfloor K}$ is unbounded in $L^2(\mathcal{H}^{Q-1}\lfloor K)$.

Suppose that K is not removable. Then there exists a domain $D \supset K$ and a Lipschitz function $f : D \to R$ which is $\mathcal{L}$-harmonic in $D \setminus K$ but not in D. By Theorem 3.1 there exists a domain $G, K \subset G \subset D$, a Borel function $h : C \to \mathbb{R}$ and an $\mathcal{L}$-harmonic function $H : G \to \mathbb{R}$ such that

$$f(p) = \int_K \Gamma(q^{-1} \cdot p) h(q) \, d\mathcal{H}^{Q-1}(q) + H(p) \text{ for } p \in G \setminus K$$

and $\|h\|_{L^\infty(\mathcal{H}^{Q-1}\lfloor K)} + \|\nabla_{\mathbb{G}} H\|_\infty \lesssim 1$. Let $\sigma = h\mathcal{H}^{Q-1}\lfloor K$. In this case by the left invariance of $\nabla_{\mathbb{G}}$ as in (3.9) and recalling Notation 4.9

$$T_\sigma(p) = \nabla_{\mathbb{G}} f(p) - \nabla_{\mathbb{G}} H(p) \text{ for all } p \in G \setminus K$$

which implies that

$$|T_\sigma(p)| \lesssim 1 \quad \text{for all} \quad p \in G \setminus K. \tag{4.13}$$

Let $\delta = \operatorname{dist}(K, \mathbb{G} \setminus G) > 0$. Then for $p \in \mathbb{G} \setminus G$,

$$|T_\sigma(p)| \lesssim \int \frac{1}{\|q^{-1} \cdot p\|^{Q-1}} \, d|\sigma|(q) \leq \frac{|\sigma|(K)}{\delta^{Q-1}} \lesssim 1. \tag{4.14}$$

By (4.13) and (4.14) we deduce that $T_\sigma \in L^\infty$. Hence, recalling Remark 4.8, the measure $\mathcal{H}^{Q-1}\lfloor K$ is $(Q-1)$-AD regular and we can apply Lemma 4.10 to conclude that T_σ^* is bounded.

Furthermore since f is not harmonic in K, $h \neq 0$ in a set of positive $\mathcal{H}^{Q-1}$ measure. Therefore there exists a point $p \in K$ of approximate continuity (with respect to $\mathcal{H}^{Q-1}\lfloor K$) of h such that $h(p) \neq 0$. Let $w_n \in \{0,1,\dots,M\}^n$, where M is as in Theorem 4.1, be such that $p \in S_{w_n}(K)$. Then by the approximate continuity of h,

$$r^{(1-Q)n}(S_{w_n}^{-1})_\sharp(\sigma\lfloor S_{w_n}(K)) \rightharpoonup h(p)\mathcal{H}^{Q-1}\lfloor K \text{ as } n \to \infty.$$

We can now check that the boundedness of T_σ^* implies that $T^*_{\mathcal{H}^{Q-1}\lfloor K}$ is bounded. Let $z \in \mathbb{G} \setminus (K \cup \bigcup_{n=1}^\infty S_{w_n}^{-1}(K))$. If $\operatorname{dist}(z,K) > \frac{\alpha_K}{2}\operatorname{diam}(K)$, then

$$|T_{\mathcal{H}^{Q-1}\lfloor K}(z)| \lesssim 1. \tag{4.15}$$

Therefore we can assume that $\operatorname{dist}(z,K) \leq \frac{\alpha_K}{2}\operatorname{diam}(K)$. Hence for any $w \in V_*$,

$$\begin{aligned}\operatorname{dist}(S_w(z), S_w(K)) &= r^{|w|}\operatorname{dist}(z,K)\\ &\leq r^{|w|}\frac{\alpha_K}{2}\operatorname{diam}(K)\\ &= \frac{\alpha_K}{2}\operatorname{diam}(S_w(K)).\end{aligned} \tag{4.16}$$

Notice that the 0-homogeneity of k_i implies that $k_i(S_{w_n}^{-1}(q)^{-1}\cdot z) = r^{(Q-1)n}k_i(q^{-1}\cdot S_{w_n}(z))$. Therefore,

$$\begin{aligned}h(p)T_{\mathcal{H}^{Q-1}\lfloor K}(z) &= \lim_{n\to\infty} r^{(1-Q)n}\int k_i(q^{-1}\cdot z)\,d(S_{w_n}^{-1})_\sharp(\sigma\lfloor S_{w_n}(K))(q)\\ &= \lim_{n\to\infty} r^{(1-Q)n}\int_{S_{w_n}(K)} k_i(S_{w_n}^{-1}(q)^{-1}\cdot z)\,d\sigma(q)\\ &= \lim_{n\to\infty}\int_{S_{w_n}(K)} k_i(q^{-1}\cdot S_{w_n}(z))\,d\sigma(q)\\ &= \lim_{n\to\infty}\left(\int_K k_i(q^{-1}\cdot S_{w_n}(z))\,d\sigma(q) - \int_{K\setminus S_{w_n}(K)} k_i(q^{-1}\cdot S_{w_n}(z))\,d\sigma(q)\right).\end{aligned}$$

Since $z \notin \bigcup_{n=1}^\infty S_{w_n}^{-1}(K)$,

$$\left|\int_K k_i(q^{-1}\cdot S_{w_n}(z))\,d\sigma(q)\right| \leq \|T_\sigma^*\|_\infty.$$

Furthermore by Lemma 4.12 and (4.16) we get that,

$$\left|\int_{K\setminus S_{w_n}(K)} k_i(q^{-1}\cdot S_{w_n}(z))\,d\sigma(q)\right| \leq 2\|T_\sigma^*\|_\infty + A_K.$$

Therefore,

$$|h(p)T_{\mathcal{H}^{Q-1}\lfloor K}(z)| \leq 3\|T_\sigma^*\|_\infty + A_K,$$

and since

$$\left|K \cup \bigcup_{n=1}^\infty S_{w_n}^{-1}(K)\right| = 0$$

we get that $T_{\mathcal{H}^{Q-1}\lfloor C_{Q-1}} \in L^\infty$. Hence by Lemma 4.10 $T^*_{\mathcal{H}^{Q-1}\lfloor K}$ is bounded in $L^2(\mathcal{H}^{Q-1}\lfloor K)$ and we have reached a contradiction. The proof of the theorem is complete. □

Remark 4.13. Vertical hyperplanes of the form $\{(x,t) \in \mathbb{G} : x \in W, t \in \mathbb{R}\}$, where W is a linear hyperplane in $\mathbb{R}^m$, are homogeneous subgroups of $\mathbb{G}$, that is, they are closed subgroups invariant under the dilations δ_r. Their Hausdorff dimension is $Q-1$. If V is any such vertical hyperplane and σ denotes the $(Q-2)$-dimensional Lebesgue measure on V it follows by [34, Theorem 4, page 623 and Corollary 2, page 36] that T^*_σ is bounded in $L^2(\sigma)$. This implies, for example by the methods used in [27], that positive measure subsets of vertical hyperplanes are not removable for Lipschitz harmonic functions.

5. Concluding comments and questions

As in the Euclidean case the study of removable sets for Lipschitz $\mathcal{L}$-harmonic functions with positive and finite $\mathcal{H}^{Q-1}$-measure heavily depends on the study of the singular integral $T(f) = (T_1(f), \dots, T_m(f))$ where formally

$$T_i(f)(p) = \int k_i(p^{-1} \cdot q) f(q)\, d\mathcal{H}^{Q-1}(q)$$

and $k = (k_1, \dots, k_m) = \nabla_{\mathbb{G}}\Gamma$.

Our understanding of such singular integrals is extremely limited even when the fundamental solution of the sub-Laplacian, and hence the kernel k, have explicit formulas as in the Heisenberg group. There are two natural directions one could pursue in order to extend our knowledge of the topic. First of all it is not known what regularity and smoothness assumptions are needed for a $(Q-1)$-AD regular set M in order the operator T to be bounded in $L^2(\mathcal{H}^{Q-1}\lfloor M)$. Recall that sets which define L^2-bounded operators can be seen to be non-removable, cf. Remark 4.13. Second it is not known how much we can extend the range of removable $(Q-1)$-dimensional self-similar sets. We are not aware of any self-similar sets where the condition in Theorem 4.11 fails for all its fixed points. Nevertheless due to the changes in sign of the kernel checking that the integral in Theorem 4.11 does not vanish could be technically very complicated.

References

[1] Ahlfors, L. V. Bounded analytic functions. *Duke Math. J.*, 14 (1947), 1–11.

[2] Ambrosio, L., Fusco, N., and Pallara, D. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. Oxford University Press, New York, 2000.

[3] Balogh, Z. M., Tyson, J. T., and Warhurst, B. Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups. *Adv. Math. 220*, 2 (2009), 560–619.

[4] Bonfiglioli, A., Lanconelli, E., and Uguzzoni, F. *Stratified Lie groups and potential theory for their sub-Laplacians.* Springer Monographs in Mathematics. Springer, Berlin, 2007.

[5] Capogna, L., Danielli, D., Pauls, S. D., and Tyson, J. T. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, vol. 259 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 2007.

[6] Chousionis, V., and Mattila, P. Singular integrals on self-similar sets and removability for Lipschitz harmonic functions in Heisenberg groups. *J. Reine Angew. Math.*. to appear.

[7] Chousionis, V., and Tyson, J. T. Removable sets for homogeneous linear PDE in Carnot groups. Preprint, 2013.

[8] Coifman, R., McIntosh, A., and Y., M. L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes. *Ann. of Math. 116*, 2 (1982), 361–387.

[9] David, G. Unrectifiable 1-sets have vanishing analytic capacity. *Rev. Mat. Iberoamericana 14*, 2 (1998), 369–479.

[10] David, G., and Mattila, P. Removable sets for Lipschitz harmonic functions in the plane. *Rev. Mat. Iberoamericana 16*, 1 (2000), 137–215.

[11] David, G., and Semmes, S. *Analysis of and on uniformly rectifiable sets*, vol. 38 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1993.

[12] De Giorgi, E. Nuovi teoremi relativi alle misure (r-1) dimensionali in uno spazio ad r dimensioni. *Ricerche Mat. 4* (1955), 95–113.

[13] Folland, G. B. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat. 13*, 2 (1975), 161–207.
[14] Folland, G. B., and Stein, E. M. *Hardy spaces on homogeneous groups*, vol. 28 of *Mathematical Notes*. Princeton University Press, Princeton, New Jersey, 1982.
[15] Franchi, B., Serapioni, R., and Serra Cassano, F. Rectifiability and perimeter in the Heisenberg group. *Math. Ann. 321* (2001), 479–531.
[16] Garnett, J. B. Positive length but zero analytic capacity. *Proc. Amer. Math. Soc. 24* (1970), 696–699.
[17] Hutchinson, J. E. Fractals and self similarity. *Indiana Univ. Math. J. 30*, 5 (1981), 713–747.
[18] Ivanov, L. D. On sets of analytic capacity zero. In *Linear and Complex Analysis. Problem Book 3*, vol. 1574 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994, pp. 150–153.
[19] Jones, P. W. Rectifiable sets and the traveling salesman problem. *Invent. Math. 102*, 1 (1990), 1–15.
[20] Magnani, V. On a measure theoretic area formula. Preprint, 2013.
[21] Magnani, V. Towards differential calculus in stratified groups. *J. Aust. Math. Soc.*. to appear.
[22] Magnani, V. Characteristic points, rectifiability and perimeter measure on stratified groups. *J. Eur. Math. Soc. (JEMS) 8*, 4 (2006), 585–609.
[23] Magnani, V. Non-horizontal submanifolds and coarea formula. *J. Anal. Math. 106* (2008), 95–127.
[24] Mattila, P. Smooth maps, null sets for integralgeometric measure and analytic capacity. *Ann. of Math. 123*, 2 (1986), 303–309.
[25] Mattila, P. *Geometry of sets and measures in Euclidean spaces*, vol. 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.
[26] Mattila, P., Melnikov, M., and Verdera, J. The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. of Math. (2) 144*, 1 (1996), 127–136.
[27] Mattila, P., and Paramonov, P. On geometric properties of harmonic Lip_1-capacity. *Pacific. J. Math. 171*, 2 (1995), 469–491.
[28] Melnikov, M. Analytic capacity: a discrete approach and the curvature of measure (Russian). *Mat. Sb. 186*, 6 (1995), 57–76.
[29] Montgomery, R. *A tour of subriemannian geometries, their geodesics and applications*, vol. 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
[30] Nazarov, F., Tolsa, X., and Volberg, A. On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1. Preprint, 2012.
[31] Nazarov, F., Tolsa, X., and Volberg, A. The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions. Preprint, 2012.
[32] Sard, A. The measure of the critical values of differentiable maps. *Bull. Amer. Math. Soc. 48* (1942), 883–890.
[33] Schief, A. Self-similar sets in complete metric spaces. *Proc. Amer. Math. Soc. 124*, 2 (1996), 481–490.
[34] Stein, E. M. *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals*, vol. 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1993.
[35] Strichartz, R. S. Self-similarity on nilpotent Lie groups. In *Geometric analysis (Philadelphia, PA, 1991)*, vol. 140 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1992, pp. 123–157.
[36] Tolsa, X. Painlevé's problem and the semiadditivity of analytic capacity. *Acta Math. 190*, 1 (2003), 105–149.
[37] Tolsa, X. *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón–Zygmund theory*, vol. 307 of *Progress in Mathematics*. Birkhaüser, Basel, 2014.
[38] Verdera, J. Removability, capacity and approximation. In *Complex potential theory (Montreal, PQ, 1993)*, vol. 439 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* Kluwer Acad. Publ., Dordrecht, 1994, pp. 419–473.
[39] Vitushkin, A. G. Example of a set of positive length but of zero analytic capacity (russian). *Dokl. Akad. Nauk SSSR*, 127 (1959), 241–249.
[40] Vitushkin, A. G. Analytic capacity of sets in problems of approximation theory. *Uspeikhi Mat. Nauk. 22*, 6 (1967), 141–199.

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