H-CONVEX DISTRIBUTIONS IN STRATIFIED GROUPS

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Abstract. We extend Dudley's characterization of convex functions to the framework of h-convex distributions on stratified groups. Precisely, we prove that every distribution with nonnegative horizontal Hessian is defined by an h-convex function.

1. Introduction

In Euclidean spaces, the first distributional characterization of convexity goes back to L. Schwartz in [16], who proved that a distribution in $\mathbb{R}$ is a convex function if and only if its second derivative is a nonnegative Radon measure. Bakel'man showed that all second order distributional derivatives of a convex function in $\mathbb{R}^n$ are signed Radon measures, [1]. Subsequently, Reshetnyak established that a locally summable function is defined by a convex function if and only if its distributional Hessian is nonnegative, [14]. This characterization has reached its full generality thanks to a result by Dudley, who proved that every distribution with nonnegative Hessian in the distributional sense is defined by a convex function, [5].

In the framework of stratified groups, the natural notion of convexity is that of *h-convexity*. An h-convex function $u : \Omega \to \mathbb{R}$ defined on an open set Ω of a stratified group $\mathbb{G}$ satisfies the property of being classically convex along all horizontal lines contained in Ω. More information on convexity in stratified groups can be found in the seminal papers by Danielli, Garofalo and Nhieu [4] and by Juutinen, Lu, Manfredi and Stroffolini [9], [10].

In all stratified groups, every h-convex function has nonnegative *horizontal Hessian* in the distributional sense, as observed in [4] and [9]. Surprisingly, the converse of this statement, namely, establishing whether a distribution with nonnegative horizontal Hessian is given by an h-convex function has not been addressed yet. This note gives a full answer to this question. Our first result in this direction extends Reshetnyak's characterization to all stratified groups.

Theorem 1.1. *If $\mu \in \mathcal{D}'(\Omega)$ is a Radon measure, then μ is defined by an h-convex function if and only if it is an h-convex distribution.*

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Our scheme is elementary, although it differs from the standard approach: we consider the group convolution of the measure μ, but instead of computing its horizontal Hessian by direct differentiation, we consider its distributional version. This respects the noncommutativity of the convolution operator. As a byproduct of Theorem 1.1, we have the following basic fact.

Corollary 1.2. *If $u \in L^1_{loc}(\Omega)$ is h-convex in the distributional sense, then outside a negligible set it coincides with a locally Lipschitz continuous h-convex function on Ω.*

This result shows that the condition vi) in Theorem 3.1 of [10] implies the condition i) of the same theorem, along with the stronger local Lipschitz continuity, although the upper semicontinuity assumption of Theorem 3.1 is not used. The reason of this resides in the theory of subharmonic functions of [2], that is applied to the characterizations in Theorem 3.1 of [10] and that requires the upper semicontinuity as an underlying assumption.

To reach the complete distributional characterization of h-convexity, we combine Corollary 1.2 and Lemma 2.3 below. By means of the fundamental solution of the sub-Laplacian Δ_H in stratified groups, [6], this lemma shows that an h-convex distribution T is the sum of a Δ_H-harmonic function and a locally summable function. Since Δ_H-harmonic functions are smooth by Hörmander's theorem, [8], we can conclude that T is given by a function in $L^1_{loc}(\Omega)$ and then Corollary 1.2 applies.

We should remark that this approach might be of interest also in the classical context, since it gives a different proof of Dudley's theorem, [5]. Recall that Dudley's argument uses a "simplex construction" that cannot be extended to general stratified groups, although the rich properties of Heisenberg groups allow to carry out Dudley's argument in these groups, see [17]. We have now arrived at our main result.

Theorem 1.3. *Let Ω be an open set of $\mathbb{G}$. If $T \in \mathcal{D}'(\Omega)$ is h-convex, then T is defined by an h-convex function on Ω.*

Recall that all measurable h-convex functions are locally Lipschitz continuous, [15]. Thus, Theorem 1.3 shows that the class of h-convex measurable functions coincides with that of h-convex distributions that are locally Lipschitz continuous h-convex functions. Although we still do not know whether one can find h-convex functions in higher step groups that are nonmeasurable, these functions in any case would not be included in the family of h-convex distributions. This should clarify that either measurability or local boundedness from above have to be included in the definition of h-convexity. In fact, both these conditions imply that any h-convex function is indeed Lipschitz continuous. Finally, this note should be thought of as a further contribution in showing the interesting interplay between the well developed theory of subharmonic functions in stratified groups and the theory of h-convex functions, that is still far from being completely understood.

2. H-convex distributions

A finite dimensional connected and simply connected nilpotent Lie group can be thought of as a vector space $\mathbb{G}$ equipped with a polynomial group operation, that is given in turn by the Baker-Campbell-Hausdorff formula. Let $\mathcal{G}$ be the Lie algebra of $\mathbb{G}$ and set $n = \dim \mathcal{G}$. We say that $\mathbb{G}$ is *stratified* if $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$, where $V_j = [V_1, V_{j-1}]$, for all $1 < j \le \iota$ and $V_j = \{0\}$ if and only if $j > \iota$. Thus, $\mathbb{G}$ is canonically equipped with a structure of graded vector space, with $\mathbb{G} = H_1 \oplus H_2 \oplus \cdots \oplus H_\iota$ and $H_j = \{v \in \mathbb{G} : v = X(0) \,\text{and}\, X \in V_j\}$. We can define a natural family of dilations $\delta_r : \mathbb{G} \to \mathbb{G}$ compatible with the group operation, see [7]. We fix $(X_1, \ldots, X_m)$ as a basis of V_1. Open balls with respect to a fixed homogeneous distance d will be denoted by $B_{x,r}$ and we simply write B_r when x is the origin.

We denote by H_x the left translation of H_1 by x, namely $H_x = xH_1$. For every $h \in H_1$ we define the *horizontal segment* $\{th,\ t \in [0,1]\}$ through the short notation $[0,h]$. Moreover for every $x \in \mathbb{G}$ we set $x \cdot [0,h] = \{x\delta_t h,\ t \in [0,1]\}$.

Definition 1. We say that $u : \Omega \to \mathbb{R}$ is *h-convex* if for every $x, y \in \Omega$ such that $x \in H_y$ and $x \cdot [0, x^{-1}y] \subset \Omega$, we have $u(x\delta_\lambda(x^{-1}y)) \le \lambda u(y) + (1-\lambda)u(x)$, for all $\lambda \in [0,1]$.

Recall that $\mathcal{D}(\Omega)$ corresponds to $C_c^\infty(\Omega)$ topologized in the standard way, where Ω is an open set of a stratified group thought of as a differentiable manifold. We denote by $\mathcal{D}'(\Omega)$ the topological vector space of distributions on Ω.

Remark 1. Let $(X_1, \ldots, X_m)$ denote an orthonormal basis of the first layer V_1 and let $T \in \mathcal{D}'(\Omega)$. The vector fields X_j have formal adjoint $X_j^\star = -X_j$, so this justifies the following definition

$$\langle X_j T, \phi \rangle := -\langle T, X_j \phi \rangle, \quad \phi \in \mathcal{D}(\Omega).$$

Definition 2. Let $T \in \mathcal{D}'(\Omega)$. The *distributional Hessian of* T is the matrix valued distribution $\langle D_H^2 T,\ \psi \rangle := \langle T,\ \nabla_H^2 \psi \rangle$ with entries

$$\left\langle \frac{1}{2} \left(X_j X_i + X_i X_j \right) T,\ \psi \right\rangle := \left\langle T,\ \frac{1}{2} \left(X_i X_j + X_j X_i \right) \psi \right\rangle$$

for every $i, j = 1, \ldots, m$ and every $\psi \in \mathcal{D}(\Omega)$.

Definition 3. We say that the distributional Hessian of $T \in \mathcal{D}'(\Omega)$ is *nonnegative* if for every nonnegative test function $\psi \in \mathcal{D}(\Omega)$ the matrix $\langle T,\ \nabla_H^2 \psi \rangle$ is nonnegative. In this case we write $D_H^2 T \ge 0$ and say that the distribution T is *h-convex*.

In the smooth case, a simple computation shows that a nonnegative horizontal Hessian characterizes h-convexity, see for instance [12].

Definition 4. Let $\phi \in \mathcal{D}(\mathbb{G})$ be a nonnegative function, whose support is contained in the unit open ball of $\mathbb{G}$ with respect to the fixed homogeneous norm and such that $\int_{\mathbb{G}} \phi(y)\, dy = 1$. For every $\varepsilon > 0$, we set $\phi_\varepsilon(x) = \varepsilon^{-Q} \phi(\delta_{\frac{1}{\varepsilon}} x)$. We say that $\{\phi_\varepsilon\}_{\varepsilon > 0}$ is a *family of mollifiers*.

Throughout, we denote by $d(x)$ the distance $d(x,0)$ for every $x\in\mathbb{G}$.

Lemma 2.1 (Remark 3.10 of [13])**.** *Let $\mathbb{G}$ be a stratified group of step ι. Let $w,h\in\mathbb{G}$ and let $\nu>0$ be such that $d(w),d(h)\le\nu$. Then there exits a constant $C(\nu)$, also depending on $\mathbb{G}$, such that $d(w^{-1}hw)\le C(\nu)d(h)^{\frac{1}{\iota}}$.*

We will use the notation $\Omega_{-r}=\{x\in\Omega:\operatorname{dist}(x,\Omega^c)>r\}$ for any $r>0$.

Proof of Theorem 1.1. We first suppose that Ω is bounded. Let $h>0$ be such that Ω_{-h} is nonempty and notice that

$$\mu_\varepsilon(x)=\int_\Omega \phi_\varepsilon(xy^{-1})\,d\mu(y)$$

is well defined on Ω for all $\varepsilon>0$. For any $\psi\in\mathcal{D}(\Omega_{-h})$, we get

$$\begin{aligned}\int_{\Omega_{-h}}\nabla_H^2\psi(x)\;\mu_\varepsilon(x)\,dx &= \int_\Omega\left(\int_{\Omega_{-h}}\phi_\varepsilon(xy^{-1})\;\nabla_H^2\psi(x)\,dx\right)d\mu(y)\\ \big(\,x=zy\,\big)\quad &= \int_\Omega\left(\int_{B_\varepsilon}\phi_\varepsilon(z)\;\nabla_H^2\psi(zy)\,dz\right)d\mu(y)\\ &= \int_{B_\varepsilon}\phi_\varepsilon(z)\left(\int_\Omega\nabla_H^2[\psi(zy)]\,d\mu(y)\right)dz\\ &\ge\; 0\,,\end{aligned}$$

since $y\to\psi(zy)$ is smooth and compactly supported in Ω. In fact, let $\omega\in\Omega^c$, then Lemma 2.1, with $\nu=C(\Omega)=\sup_{x\in\Omega}d(x)$, yields

$$d(y,\omega)\ge d(zy,\omega)-d(y^{-1}zy)>h-C(\Omega)\varepsilon^{\frac{1}{\iota}}>0.$$

for all $\varepsilon^{\frac{1}{\iota}}<\frac{h}{C(\Omega)}$. Then it follows from Proposition 5.1 in [12] that μ_ε is h-convex. Furthermore, for every compact set K contained in Ω_{-h} we have the uniform estimate

$$\int_K|\mu_\varepsilon(x)|\,dx\le|\mu(K)|<+\infty\,.$$

Then the L^∞-estimates in Theorem 9.2 of [4] joined with the classical Ascoli-Arzelà compactness theorem and the arbitrary choice of $h>0$ imply the local uniform convergence of μ_ε to an h-convex function defined in Ω. In the case Ω is any open set, we fix an arbitrary $M>0$ and consider the set $\Omega\cap B_M$. The proof proceeds as in the previous case and by the arbitrary choice of $M>0$, the conclusion follows. □

We say that a distribution $\tau\in\mathcal{D}'(\Omega)$ is homogeneous of degree α if for every $\phi\in\mathcal{D}(\Omega)$ and $r>0$ we have $\langle\tau,\phi\circ\delta_r\rangle=r^{-\alpha-Q}\langle\tau,\phi\rangle$, where $Q=\sum_{i=1}^{\iota} i\dim V_i$ is the so called homogeneous dimension of $\mathbb{G}$.

Definition 5 ([6])**.** Consider the sub-Laplacian $\Delta_H=\sum_{i=1}^m X_i^2$ on $\mathbb{G}$. A distribution Γ on Ω is a fundamental solution for Δ_H if :
(i) $\Gamma\;\in\;C^\infty(\mathbb{G}\setminus\{0\})$,

(ii) Γ is homogeneous of degree $2-Q$,
(iii) $\Delta_H \Gamma = -\delta_0$ in the sense of distributions.

Remark 2. Notice that $\Gamma \in L^1_{loc}(\mathbb{G})$. This follows by conditions (i) and (ii) in the previous definition, see also Corollary 1.7 in [6].

Theorem 2.2 ([6], Theorem 2.1)**.** *Let $\mathbb{G}$ be a Carnot group of homogeneous dimension $Q > 2$, then there exists a fundamental solution Γ for Δ_H.*

Lemma 2.3. *Let $\Omega \subset \mathbb{G}$ be an open set and let $\Omega_1 \subset \Omega$ be a bounded open set such that $\bar{\Omega}_1 \subset \Omega$. If $T \in \mathcal{D}'(\Omega)$ satisfies $\Delta_H T \geq 0$, then its restriction to Ω_1 is given by a function in $L^1_{loc}(\Omega_1)$.*

Proof. Since $\Delta_H T \geq 0$, we know that there exists a nonnegative Radon measure μ on Ω such that $\Delta_H T = \mu$. Let Γ be as in Definition 5, and consider the function

$$v(x) = -\int_{\Omega_1} \Gamma(y^{-1}x) d\mu(y).$$

Since Γ is locally integrable on $\mathbb{G}$, for every compact set $K \subset \Omega_1$ we have

$$\int_K |v(x)| dx \leq \int_{\Omega_1} \int_K |\Gamma(y^{-1}x)| dx d\mu(y) \leq \mu(\Omega_1) \sup_{y \in \bar{\Omega}_1} \int_K |\Gamma(y^{-1}x)| dx < +\infty.$$

Let us show that v satisfies the distributional equality $\Delta_H v = \mu_{|\Omega_1}$. In fact, for every $\phi \in \mathcal{D}(\Omega_1)$ we have

$$\langle \Delta_H v, \phi \rangle = -\int_{\Omega_1} \int_{\Omega_1} \Gamma(y^{-1}x) \Delta_H \phi(x) dx \, d\mu(y).$$

Thus, Γ being the fundamental solution for Δ_H, we get

$$\langle \Delta_H v, \phi \rangle = \int_{\Omega_1} \phi(y) d\mu(y).$$

Hence $\Delta_H (T - v) = 0$ in $\mathcal{D}'(\Omega_1)$. Since Δ_H is hypoelliptic, [8], the function $T - v$ coincides a.e. with a smooth Δ_H-harmonic function h on Ω_1. Finally, we can conclude that T is represented by an L^1_{loc} function on Ω_1. □

We wish to point out that this lemma follows the same lines used to prove representation formulae for upper semicontinuous subharmonic functions, see Theorem 9.4.4 in [3]. Combining Lemma 2.3 and Corollary 1.2, we establish the proof of our main result.

Proof of Theorem 1.3. Let $\bar{\Omega}_n \subset \Omega$, $n \in \mathbb{N}$ be an increasing sequence of open and bounded sets. By hypothesis, $D^2_H T \geq 0$ and in particular $\Delta_H T \geq 0$. Hence, Lemma 2.3 implies that T is represented by an L^1_{loc} function on Ω_n. Finally we can conclude, by Corollary 1.2, that T is defined by an h-convex function on Ω_n. Since $\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega$, the theorem follows. □

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