Area of submanifolds in homogeneous groups

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Outline

The origin

Homogeneous groups are a family of Lie groups that arise from different fields of Mathematics:

Harmonic Analysis and Differential Geometry

Harmonic Analysis

E. M. Stein, [Int. Congress of Mathematics, Nice, 1970], proposed to develop "Analysis on the boundary", observing that these boundaries can be special classes of *nilpotent Lie groups*.

Folland, [Ark. Mat. 1975], proposed the notion of stratified group that is a special class of homogeneous groups where a "subelliptic regularity theory" can be developed with the appropriate function spaces.

A key result in this setting was Hörmander's hypoellipticity of the sub-Laplacian, proved in Acta Math. 1967.

Then many works followed these directions. We mention papers by L. P. Rothschild and E. M. Stein, J.-M. Bony, F. Ricci, N. Garofalo and many others.

Differential Geometry

G. D. Mostow, [Publ. Math. I. H. E. S., 1968], proved an important *rigidity theorem for compact Riemannian manifolds with constant negative curvature*.

Pansu, [Ann. Math. 1989], extended the Mostow rigidity to specific simple groups, introducing the *class of Carnot groups and proved a Rademacher's type theorem*.

Different theories in Harmonic Analysis and in Differential Geometry independently arrived at the **same special class of nilpotent Lie groups** (that we introduce later):

called *stratified group*, after the work of G. B. Folland,

called *Carnot group*, after the work of P. Pansu

Analysis and Geometric Measure theory in metric spaces

The paper by P. Hajłasz and P. Koskela, titled "Sobolev meets Poincaré", C. R. Acad. Sci. Paris, 1995, is the born of Analysis in metric spaces.

E. De Giorgi, [Atti Sem Mat. Fis. Univ. Modena, 1995,]

proposed a theory of currents in metric spaces.

Currents generalize in the distributional sense the notion of oriented surface, using *differential forms*.

L. Ambrosio and B. Kirchheim, [Acta Math., 2000],

completed the De Giorgi's theory by solving the Plateau problem in metric spaces.

Hausdorff measure in a metric space

We introduce a general notion of surface measure in any metric space: the *Hausdorff measure*.

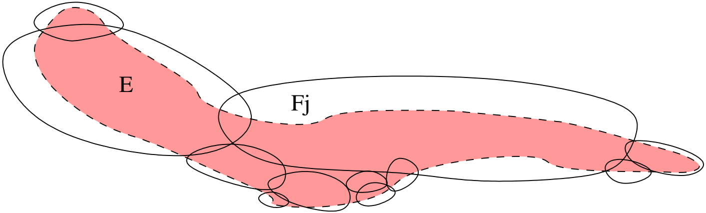
Let $\alpha \geq 0$ and let (X, d) be a metric space.

The α-dimensional Hausdorff measure $\mathcal{H}^\alpha$ is defined on any set $E \subset X$ as follows

$$\mathcal{H}^\alpha(E) = \sup_{t>0} \inf_{\{F_j\} \subset \mathcal{F}_t} \sum_{j=1}^{\infty} \beta_\alpha \mathrm{diam}(F_j)^\alpha$$

where $\mathcal{F}_t = \left\{ \{F_j\}_{j \in \mathbb{N}} : E \subset \bigcup_{j \in \mathbb{N}} F_j, \mathrm{diam}(F_j) \leq t \right\}$ and $\beta_\alpha > 0$ is a geometric constant to be properly chosen.

We cover the set E with elements F_j whose diameter is less than t.



As $t > 0$ becomes smaller and smaller, the approximation of the area of the set improves.

We will precisely use the spherical measure $\mathcal{S}^\alpha$, that differs from the Hausdorff measure since the sets in the covering are replaced by metric balls. However they are equivalent measures

$$2^{-\alpha}\,\mathcal{S}^\alpha \le \mathcal{H}^\alpha \le \mathcal{S}^\alpha .$$

Rectifiable and unrectifiable sets

Definition

We say that a subset $S \subset X$ of a metric space (X, d) is *k-rectifiable* if there exists a subset $A \subset \mathbb{R}^k$ and a

$$\text{Lipschitz mapping } f : A \to X \text{ such that } S = f(A).$$

Definition

We say that a subset $S \subset X$ of a metric space (X, d) is *purely k-unrectifiable* if for every subset $A \subset \mathbb{R}^k$ and every

$$\text{Lipschitz mapping } f : A \to X$$

there holds

$$\mathcal{H}^k(f(A) \cap S) = 0.$$

Failure of De Giorgi's theory on homogeneous groups

The following papers

L. Ambrosio, B. Kirchheim, Math. Ann. 2000

The first Heisenberg group $\mathbb{H}$, that is the simplest noncommutative stratified group, is purely k-unrectifiable for every $k \geq 2$.

V.M., Arch. Math. 2004

All purely k-unrectifiable homogeneous groups can be algebraically characterized.

prove that *for many homogeneous groups*

there are no *"k-rectifiable sets"*

to which we can apply

the Ambrosio–De Giorgi–Kirchheim's theory of currents in metric spaces.

The theory of metric currents is also related to the following metric area formula.

B. Kirchheim, Proc. Amer. Math. Soc., 1994

Consider an injective Lipschitz map $F : A \to X$ from $A \subset \mathbb{R}^k$ to a metric space X. Then we have

$$\mathcal{H}^k(F(A)) = \int_A JF(x)dx,$$

where the crucial point is the a.e. metric differentiability of Lipschitz mappings, which leads to a natural definition of metric Jacobian $JF(x)$.

Problem

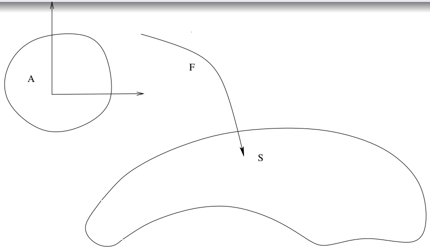
The previous Kirchheim's area formula does not apply to surfaces in homogeneous groups.
So in homogeneous groups, **we do not have a theory of currents** and **the classical tools to compute the Hausdorff measure of sets do not work**.

These problems are the **main motivations** to study the area formula **noncommutative homogeneous groups**.

In short terms: can we compute the **surface area** *of some special classes of* **purely k-unrectifiable sets***?*

k-dimensional surfaces in $\mathbb{R}^n$

Before introducing homogeneous groups, we review some basic facts about area formula in Euclidean spaces.

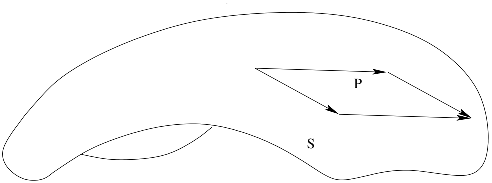


For a Lebesgue measurable set $A \subset \mathbb{R}^k$ and an injective Lipschitz mapping F on A, the classical definition of *k-dimensional area* is

$$\mathcal{H}^k(S) = \int_A JF(x)\,dx\,.$$

Infinitesimal k-dimensional area in $\mathbb{R}^n$

Heuristically Jacobian $JF(x)dx_1 dx_2 \cdots dx_k$ can be seen as the k-dimensional **infinitesimal surface area of the approximating k-dimensional parallelepiped** $dF(x)(dQ)$, where dQ is an "infinitesimal cube".



$JF(x)$ corresponds to the k-volume of the parallelepiped $P = dF(x)(Q)$, where Q is the unit cube in $\mathbb{R}^k$.

Jacobian through multi-linear algebra

In analytic terms, we consider the space $\Lambda_k(\mathbb{R}^n)$ of k-vectors in $\mathbb{R}^n$, equipped with the scalar product

$$\langle v_1 \wedge \cdots \wedge v_k, u_1 \wedge \cdots \wedge u_k \rangle = \det \left[\left(\langle v_i, u_j \rangle \right)_{i,j=1,\ldots,k} \right]$$

where $v_i, u_j \in \mathbb{R}^n$. Then the Jacobian is

$$JF(x) = \| dF(x)(e_1) \wedge dF(e_2) \wedge \cdots \wedge dF(e_k) \| \, .$$

k-vectors and k-planes

The k-vector

$$dF(x)(e_1) \wedge dF(e_2) \wedge \cdots \wedge dF(e_k)$$

represents the *oriented k-parallelepiped P* spanned by the vectors $dF(x)(e_1), \ldots, dF(e_k)$ of $\mathbb{R}^n$.

We can write the previous expression as

$$\sum_{1 \le i_1 < \cdots < i_k \le n} M^{i_1 \cdots i_k}(DF(x))\, b_{i_1} \wedge \cdots b_{i_k}$$

where

$$M^{i_1 \cdots i_k}(DF(x)) = \det\left[\left(\partial_{x_j} F^{i_l}\right)_{j=1,\ldots,k}^{l=1,\ldots,k}\right]$$

and $(b_1, \ldots, b_n)$ is the canonical orthonormal basis of $\mathbb{R}^n$.

The minors $M^{i_1 \cdots i_k}(DF(x))$ for the mapping f on an open set of $\mathbb{R}^k$ play the same role of the components of the velocity vector of a curve on an open interval.

The formula

$$JF(x) = \|dF(x)(e_1) \wedge dF(e_2) \wedge \cdots \wedge dF(e_k)\| .$$

for the Jacobian gives a higher dimensional version of Pythagoras Theorem for the k-dimensional plane $DF(x)(\mathbb{R}^k) \subset \mathbb{R}^n$:

$$JF(x) = \sqrt{\sum_{1 \le i_1 < \cdots < i_k \le n} M^{i_1 \cdots i_k}(Df(x))^2}.$$

Area formula for k-surfaces in $\mathbb{R}^n$

Let $\mathbb{B}_k$ denote the Euclidean unit ball in $\mathbb{R}^k$ and set $\beta_k = \text{Vol}_k(\mathbb{B})/2^k$ in the definition of $\mathcal{H}^k$, when the ambient metric space is $\mathbb{R}^n$ with $n \geq k$.

Theorem (Area formula)

Let $F : A \longrightarrow \mathbb{R}^n$ be an injective Lipschitz mapping, where $A \subset \mathbb{R}^k$ is a measurable subset of $\mathbb{R}^k$. Then we have

$$\mathcal{H}^k(F(A)) = \int_A JF(x)\, dx\,.$$

Homogeneous Lie groups

A *homogeneous Lie group* $\mathbb{G}$, topologically identified with $\mathbb{R}^q$ is defined through the following properties

1. a polynomial operation $xy = p(x, y)$,
2. dilations $\delta_r : \mathbb{R}^q \to \mathbb{R}^q$, $r > 0$ that satisfy $\delta_r(xy) = (\delta_r x)(\delta_r y)$,
3. a continuous distance $d : \mathbb{R}^q \times \mathbb{R}^q \to [0, +\infty)$,

with the compatibility properties:

1. $d(\delta_r x, \delta_r y) = r\, d(x, y)$, (namely it is homogeneous)
2. $d(xz, xy) = d(z, y)$ (namely it is left invariant).

d is called the *homogeneous distance* of $\mathbb{G}$.

Comparison with the Euclidean distance

If d is a homogeneous distance on $\mathbb{G}$, for any compact set $K \subset \mathbb{G} \approx \mathbb{R}^{\mathrm{q}}$, we have

$$C^{-1}|x-y| \le d(x,y) \le C\,|x-y|^{1/\iota}$$

whenever $x, y \in K$ and $C > 0$ depends on K.

The exponent $1/\iota$ cannot be improved to a larger number.

Due to this metric anisotropy, for “many” smooth submanifolds $S \subset \mathbb{G}$ there holds

$$\dim_{\mathrm{d}} S > \dim_{|\cdot|} S.$$

So also these surfaces S are unrectifiable and the Kirchheim's metric area formula **cannot** be used.

Example: the Heisenberg group

The Heisenberg group is a simple example of noncommutative homogeneous group.

We consider on $\mathbb{R}^3$ a skew-symmetric bilinear map

$$b\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = (0, 0, x_1 y_2 - x_2 y_1)$$

and define the group operation

$$xy = x + y + b(x, y).$$

Dilations here are

$$\delta_r x = (r x_1, r x_2, r^2 x_3) \quad \text{for any } r > 0.$$

This defines the well known *Heisenberg group* $\mathbb{H} \approx \mathbb{R}^3$.

On $\mathbb{H}$ we can introduce the so-called Cygan-Koranyi norm

$$\|x\| = \sqrt[4]{(x_1^2 + x_2^2)^2 + x_3^2}.$$

Homogeneity holds $\|\delta_r x\| = r\|x\|$ for all $r > 0$.
Triangle inequality holds

$$\|xy\| \leq \|x\| + \|y\|.$$

A homogeneous distance is then automatically defined

$$d(x, y) = \|x^{-1}y\|.$$

Metric anisotropy

The polynomial operation $p(x, y)$ of the homogeneous group $\mathbb{G}$ is associated to a grading

$$\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota \approx \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_\iota}$$

and the anisotropic dilations $\delta_r : \mathbb{G} \to \mathbb{G}$ are

$$\delta_r(x) = r^j x \quad \text{for} \quad j = 1, \ldots, \iota \quad \text{and} \quad x \in H^j.$$

By homogeneity, for any $x \in H^j$ we have

$$d(\delta_r x, 0) = d(r^j x, 0) = r\, d(x, 0) \quad \text{for all } r > 0, \quad \text{hence}$$

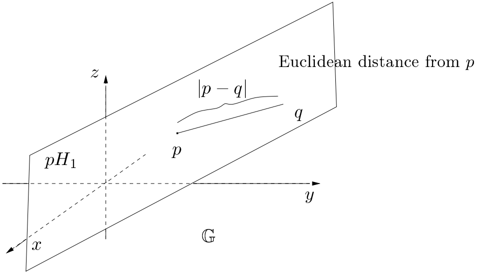
$$d(x, 0) = d\left((\sqrt[j]{|x|})^j \frac{x}{|x|}, 0 \right) = \sqrt[j]{|x|}\, d\left(\frac{x}{|x|}, 0 \right)$$

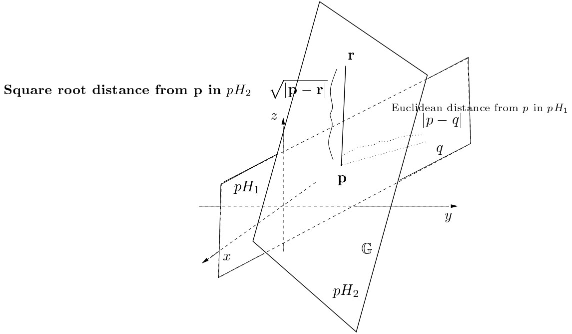
and so $\quad d(x, 0) \approx \sqrt[j]{|x|} \quad$ for $x \in H^j$.

Then the sharp metric comparison estimates appear as follows:

$$
\begin{array}{lll}
C^{-1}|x-y| \le d(x,y) \le C\,|x-y| & \text{if} & x^{-1}y \in H_1 \\
C^{-1}\sqrt{|x-y|} \le d(x,y) \le C\,\sqrt{|x-y|} & \text{if} & x^{-1}y \in H_2 \\
\vdots \quad \vdots \quad \vdots & & \\
C^{-1}|x-y|^{1/\iota} \le d(x,y) \le C\,|x-y|^{1/\iota} & \text{if} & x^{-1}y \in H_\iota
\end{array}
$$

whenever $x, y \in K$ and $C > 0$ depends on K.

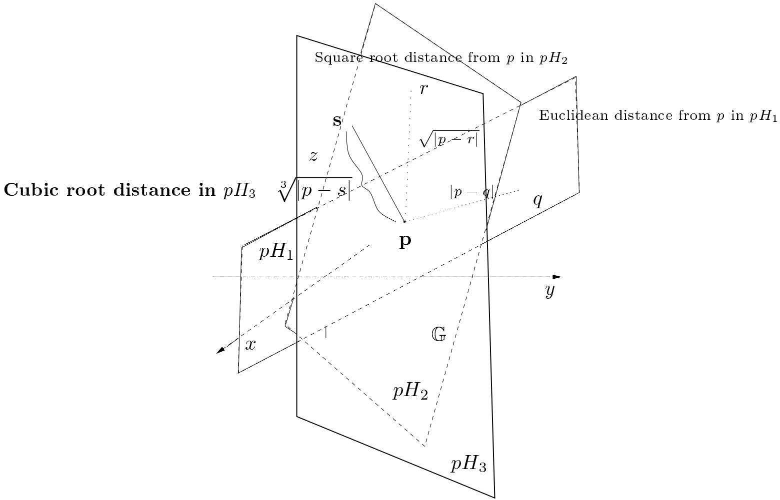




Along directions of *degree two*, namely $x^{-1}y \in H_2$, we have

$$d(x,y)^2 = d(x^{-1}y,0)^2 \approx |x-y|.$$

For a smooth curve moving along these directions we get its *locally finite and positive* 2-dimensional Hausdorff measure with respect to d.


Square root distance from p in pH_2
r
Euclidean distance from p in pH_1
s
$\sqrt{|p-r|}$
z
Cubic root distance in pH_3
$\sqrt[3]{|p-s|}$
$|p-q|$
q
p
pH_1
y
$\mathbb{G}$
x
pH_2
pH_3

Anisotropy in algebraic terms

The "magic fact" is that such a metric anisotropy has an underlying algebraic structure.

Let us recall the grading of the group

$$\mathbb{G} = H^1 \oplus \cdots \oplus H^\iota \approx \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_\iota}.$$

We assign a degree to the single vectors of the subspaces H^j:

$$\begin{aligned} \deg v &= 1 \quad \text{for all } v \in H^1 \approx \mathbb{R}^{n_1} \times \{0\} \times \cdots \times \{0\} \\ \deg v &= 2 \quad \text{for all } v \in H^2 \approx \{0\} \times \mathbb{R}^{n_2} \times \cdots \times \{0\} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \deg v &= \iota \quad \text{for all } v \in H^\iota \approx \{0\} \times \cdots \times \{0\} \times \mathbb{R}^{n_\iota}. \end{aligned}$$

Anisotropic directions in the Heisenberg group

For $(x_1, x_2, x_3), (y_1, y_2, y_3)$ in the Heisenberg group $\mathbb{H} \approx \mathbb{R}^3$, we recall the group operation
$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1+y_1, x_2+y_2, x_3+y_3+x_1y_2-x_2y_1)$.

We have $\mathbb{H} \approx \mathbb{R}^2 \times \mathbb{R} = H^1 \oplus H^2$, where
$H^1 = \mathbb{R}^2 \times \{0\}$ are the directions of *degree 1*, and
$H^2 = \{0\} \times \mathbb{R}$ are the directions of *degree 2*.

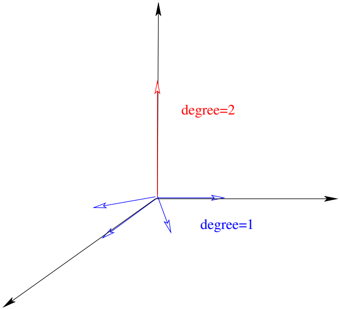
From the group operation, starting from the bases (e_1, e_2) of H^1 and e_3 of H^2, we get the left invariant vector fields

$$X_1 = \partial_{x_1} - x_2\partial_{x_3}, \quad X_2 = \partial_{x_2} + x_1\partial_{x_3} \quad \text{and} \quad X_3 = \partial_{x_3}.$$

The only non trivial bracket relation is

$$[X_1, X_2] = X_1X_2 - X_2X_1 = 2X_3.$$

In the Heisenberg group $\mathbb{H}$ we have:



In the Heisenberg group, at the origin, the vertical direction $(0, 0, 1)$ has degree two and all *horizontal directions* $(\alpha, \beta, 0)$ have degree one.

We translate the previous notion of degree for vectors of $T_0\mathbb{G}$ to any point $p \in \mathbb{G}$, using left translations.

We left translate any direction v at the origin a point $p \in \mathbb{G}$, finding a set of left invariant vector fields

$$X_1(p), \ldots, X_{n_1}(p), X_{n_1+1}(p), \ldots, X_{n_1+n_2}(p), \ldots, X_{\mathrm{q}}(p),$$

$$\begin{cases} \deg(X_j) = 1 & 1 \le j \le n_1 \\ \deg(X_j) = 2 & n_1 + 1 \le j \le n_1 + n_2 \\ \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \\ \deg(X_j) = \iota & n_1 + \cdots + n_{\iota-1} + 1 \le j \le \mathrm{q} \end{cases}$$

Definition

Let $X_{j_i} \wedge \cdots \wedge X_{j_k}$ be a k-plane. Then we set

$$\deg\left(X_{j_1} \wedge \cdots \wedge X_{j_k}\right) = \deg(X_{j_1}) + \cdots + \deg(X_{j_k})$$

Example

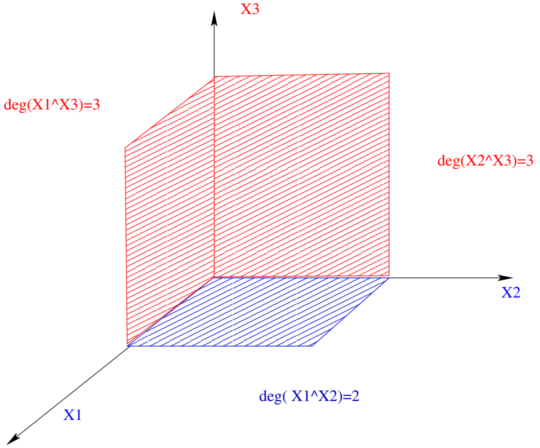
The Heisenberg group $\mathbb{H}$ has three left invariant vector fields (X_1, X_2, X_3) where the horizontal ones satisfy

$$\deg(X_1) = \deg(X_2) = 1$$

and the vertical one satisfies $\deg(X_3) = 2$. We also have

$$\deg(X_1 \wedge X_2) = 2,\ \deg(X_1 \wedge X_3) = 3,\ \deg(X_2 \wedge X_3) = 3.$$

2-planes in $\mathbb{H}$ at the origin



From k-vectors to the degree of submanifolds

Let Σ be an n-dimensional C^1 smooth submanifold of a stratified group, let $p \in \Sigma$ and consider the unit n-vector

$$\tau = \tau_1 \wedge \cdots \wedge \tau_{\mathrm{n}}$$

where $(\tau_1, \ldots, \tau_{\mathrm{n}})$ is an orthonormal basis of $T_p\Sigma$. Then

$$\tau(p) = \sum_{1 \le i_1 < \cdots < i_{\mathrm{n}} \le \mathrm{q}} \tau^{i_1 \cdots i_{\mathrm{n}}}(p) \, (X_{i_1} \wedge \cdots \wedge X_{i_{\mathrm{n}}})(p)$$

is the *tangent k-vector* of Σ at p. The *(pointwise) degree* of Σ at p is the integer

$$\deg_\Sigma(p) = \max\left\{ \deg\left(X_{i_1} \wedge \cdots \wedge X_{i_{\mathrm{n}}}\right) : \tau^{i_1 \cdots i_{\mathrm{n}}} \neq 0 \right\}.$$

The *degree* of Σ is the integer

$$\deg(\Sigma) := \max_{p \in \Sigma} \deg_\Sigma(p).$$

Gromov, 1996, (using an equivalent definition of degree) pointed out that

$$\deg(\Sigma) = d\text{-Hausdorff dimension of } \Sigma. \qquad (1)$$

for “many" submanifolds Σ, namely for a dense class with respect to smooth perturbations.

Conjecture

Formula (1) holds for **all** sufficiently smooth submanifolds.

Points of maximum degree

The study of this dimension problem is by a blow-up procedure at specific points of the submanifold.

We consider *points of maximum degree* of the submanifold Σ, namely points $p \in \Sigma$ such that

$$d_\Sigma(p) = \deg \Sigma.$$

Points of maximum degree are the "good points" of the submanifold with respect to the geometry of the group.

Related open question

Can we get also an integral formula for the spherical measure at least for all sufficiently smooth submanifold?

Intrinsic area of submanifolds

The notion of intrinsic area requires suitable projections of the tangent multivector to a submanifold $\Sigma \subset \mathbb{G}$.

Let $p \in \Sigma$ with degree N and consider the tangent n-vector at p:

$$\tau(p) = \sum_{1 \leq i_1 < \cdots < i_\mathrm{n} \leq \mathrm{q}} \tau^{i_1 \cdots i_\mathrm{n}}(p)\,\big(X_{i_1} \wedge \cdots \wedge X_{i_\mathrm{n}}\big)(p).$$

We define the **projection** $\pi_\mathrm{N}(\tau(p)) \in \Lambda_\mathrm{n}(\mathrm{Lie}(\mathbb{G}))$ as

$$\pi_\mathrm{N}(\tau(p)) = \sum_{\deg\left(X_{i_1} \wedge \cdots \wedge X_{i_\mathrm{n}}\right) = \mathrm{N}} \tau^{i_1 \cdots i_\mathrm{n}}(p)\, X_{i_1} \wedge \cdots \wedge X_{i_\mathrm{n}}.$$

It is the projection of $\tau(p)$ on the "directions of maximum degree" N.

Intrinsic area of submanifolds

The idea is that the *length* of the projected tangent space

$$\|\pi_{\mathrm{N}}(\tau(p))\| = \sqrt{\sum_{\deg(X_{i_1}\wedge\cdots\wedge X_{i_{\mathrm{n}}})=\mathrm{N}} (\tau^{i_1\cdots i_{\mathrm{n}}}(p))^2}$$

at points of maximum degree $p \in \Sigma$ is the **density of the intrinsic area.**

V. M. and D. Vittone, J. Reine Ang. Math., 2008

If τ_Σ is a *unit* tangent n-vector with respect to a Riemannian metric g and N is the degree if of Σ, then the *intrinsic measure* of the n-dimensional submanifold Σ is

$$\mu_\Sigma(A) = \int_A \|\pi_{\mathrm{N}}(\tau_\Sigma)\| \, d\sigma_g$$

for any Borel set $A \subset \Sigma$, where σ_g is the n-dimensional Riemannian volume measure on Σ.

The problem of the area formula

If a smooth submanifold $\Sigma \subset \mathbb{G}$ of degree N is given, then two measures are available:

1. the spherical measure $\mathcal{S}_d^{\mathrm{N}} \llcorner \Sigma$
   (d is the homogeneous distance of $\mathbb{G}$)
2. the intrinsic measure μ_Σ (previously introduced)

Under specific assumptions, it is possible to show that

$$\mathcal{S}_d^{\mathrm{N}} \llcorner \Sigma = \mu_\Sigma,$$

but the general validity is a **largely open question.**

V. M., Calc. Var. 2019

Under specific assumptions of the submanifold Σ, we have the area formula

$$\int_\Sigma \beta_d(A_p\Sigma)\, d\mathcal{S}_d^{\mathrm{N}}(p) = \int_\Sigma \left\| \pi_{\mathrm{N}}\left(\tau_\Sigma(p)\right)\right\| \, d\sigma_g(p)$$

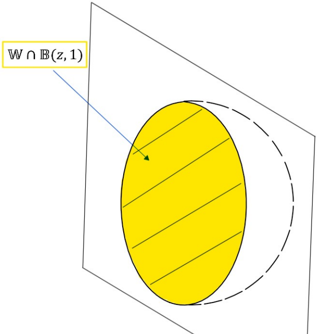
where we have defined

1. $A_p\Sigma$ is the **homogeneous tangent space at p**,
2. σ_g is the **Riemannian volume measure on Σ** with respect to the Riemannian metric g,
3. $\beta_p(A_p\Sigma)$ is the **spherical factor**, that takes into account the symmetries of the distance d.
   Actually we can define the spherical factor for any subspace $\mathbb{W}$ of $\mathbb{G}$.

Spherical factor

Let $\mathbb{W}$ be a p-dimensional subspace of $\mathbb{G}$. Then we define the **spherical factor** of $\mathbb{W}$ with respect to d as

$$\beta_d(\mathbb{W}) = \max_{z \in \mathbb{B}(0,1)} \mathcal{H}_E^p(\mathbb{W} \cap \mathbb{B}(z,1)).$$



The key metric tool to prove the formula

The following theorem **replaces the Kirchheim's metric area formula** to compute the area of subsets in **noncommutative homogeneous groups**.

Theorem [V. M., P. Royal Soc. Ed., 2015]

Under rather general (but technical) regularity assumptions on the metric space (X, d) and on a Borel measure μ, if $\Sigma \subset X$ is a Borel set and $\mu << \mathcal{S}^\alpha \llcorner \Sigma$, then

$$\mu(B) = \int_B \mathfrak{s}^\alpha(\mu, \cdot)\, d\mathcal{S}^\alpha \llcorner \Sigma \tag{2}$$

for any Borel set $B \subset \Sigma$, where $\mathfrak{s}^\alpha(\mu, p)$ is the *spherical Federer α-density* that has the following explicit formula

$$\mathfrak{s}^\alpha(\mu, p) = \inf_{\varepsilon>0} \sup \left\{ \frac{2^\alpha\, \mu(\mathbb{B})}{\operatorname{diam}(\mathbb{B})^\alpha} : \mathbb{B} \text{ is a closed ball}, p \in \mathbb{B}, \operatorname{diam}\mathbb{B} \le \varepsilon \right\}.$$

V. M., J. Geom. Analysis, 2022

Under additional symmetry conditions on the distance d,

we have the simpler formula

$$\mathcal{S}^{\mathrm{N}}(\Sigma) = \int_{\Sigma} \|\pi_{\mathrm{N}}(\tau_{\Sigma}(p))\| \, d\sigma_g(p),$$

where $\beta_p(A_p\Sigma) = \omega_{\mathrm{N}}$ is constant for every $p \in \Sigma$, due to the symmetries of d and we have defined $\mathcal{S}^{\mathrm{N}} = \omega_{\mathrm{N}} \mathcal{S}_d^{\mathrm{N}}$.

ご清聴ありがとうございました